The E_8 moduli 3-stack of the C-field in M-theory

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Abstract

The higher gauge field in 11-dimensional supergravity – the C-field – is constrained by quantum effects to be a cocycle in some twisted version of differential cohomology. We argue that it should indeed be a cocycle in a certain twisted nonabelian differential cohomology. We give a simple and natural characterization of the full smooth moduli 3-stack of configurations of the C-field, the gravitational field/background, and the (auxiliary) E_8-field. We show that the truncation of this moduli 3-stack to a bare 1-groupoid of field configurations reproduces the differential integral Wu structures that Hopkins-Singer had shown to formalize Witten's argument on the nature of the C-field. We give a similarly simple and natural characterization of the moduli 2-stack of boundary C-field configurations and show that it is equivalent to the moduli 2-stack of anomaly free heterotic supergravity field configurations. Finally we show how to naturally encode the Hořava-Witten boundary condition on the level of moduli 3-stacks, and refine it from a condition on 3-forms to a condition on the corresponding full differential cocycles.

Contents

1 Introduction

The higher gauge fields appearing in string theory (such as the B-field and the RR-fields) and in 11-dimensional M-theory (the C-field) have local presentations by higher degree differential forms that generalize the "vector potential" 1-form familiar from ordinary electromagnetism. However, just as Dirac charge quantization asserts that *globally* the field of electromagnetism is of a more subtle nature, namely given by a connection on a circle bundle, the higher gauge fields in string theory are globally of a more subtle nature: they are cocycles in *differential cohomology* (see for instance [Fr]). Moreover, even this refined statement is strictly true only when each of these fields is considered in isolation. In the full theory they all interact with each other and "twist" or "shift" each other. As a result, generally the higher gauge fields of string theory are modeled by cocycles in some notion of *twisted differential cohomology.* See [HS, Fr, Sch] for mathematical background and [DFM, BM, SSS09b, FScSt, FSaSc] for applications in this context. In this article we discuss the differential cohomology of the C-field in 11-dimensional supergravity, twisted by the field of gravity in the bulk of spacetime, as well as by the E_8-gauge field on Hořava-Witten boundaries [HW] and on M5-branes.

The general theory of *twisted differential cohomology* and its characterization of higher gauge fields in string theory it to date only partially understood. For instance, it has been well established that the underlying bare cohomology that controls the interaction of the B-field in type II string theory with the Chan-Paton gauge bundles on D-branes is *twisted K-theory*, and that for trivial B-field the corresponding differential cohomology theory is *differential K-theory*, but a mathematical construction of fully fledged *twisted differential K-theory* has not appeared yet in the literature (see, however, [CMW, KV]). Similarly, partial results apply to the lift of this configuration from type II to M-theory. It is clear that the C-field *in isolation* is modeled by cocycles in degree-4 ordinary differential cohomology, just as the B-field in isolation is modeled by degree-3 differential cohomology, and the electromagnetic field by degree-2 differential cohomology. Less is known about the interaction of the C-field with the degrees of freedom on branes, which here are M5-branes. In our companion article [FSaSc] we investigated aspects of this interaction. The present article provides a detailed discussion of the mathematical model of the C-field, as used there.

The C-field experiences a subtle twist already by its interaction with the field of gravity, via the Spin-structure on spacetime. This was first argued in [Wi97] (we review the argument in section 3.2): *the degree-4 integral class* $[2G]$ *of the C-field is constrained to equal the first fractional Pontrjagin class of the Spin structure modulo the addition of an integral class divisible by 2.* The interpretation of division by 2 in the flux quantization is given in [Sa10b] and related to Wu structures in [Sa11a, Sa11c]. The flux quantization condition can be viewed as defining a twisted String structure [SSS09b]. Dependence of the partition function in M-theory on the Spin structure is investigated in [Sa12a]. Anomalies of M-theory and string theory on manifolds with String structures via E_8 gauge theory is discussed in [Sa11b], and the relation to gerbes is discussed in [Sa10a]. The $\mathbb{Z}_2$-twist of the C-field for a fixed background Spin structure has been formalized in [HS], following an argument in [Wi96, Wi97], by a kind of twisted abelian differential cohomology (which we review in section 3.3). However, two questions remain:

1. On Hořava-Witten boundaries as well as on M5-branes, the C-field interacts with *nonabelian* and in fact *higher* nonabelian gauge fields. *What is the proper refinement of the corresponding twisted differential cohomology* to *non-abelian* differential cohomology?

2. More generally, already the field of gravity, in the first-order formulation relevant for supergravity, is a cocycle in *nonabelian* differential cohomology (a Poincaré-connection decomposing into a vielbein and a Spin connection). If we do not fix a gravitational background configuration / Spin structure as in the above model: *what is the nonabelian differential cohomology that unifies gravity, the C-field and its boundary coupling to* E_8*-gauge fields?*

In previous work [SSS09a, SSS09b, FScSt] we have developed a more general theory of *nonabelian* differential cohomology (see [Sch] for a comprehensive account), and have shown that various phenomena in string theory, such as Green-Schwarz anomaly cancellation, find their full description (technically: the full higher moduli stacks of field configurations without any background fields held fixed) in this theory. Moreover, in [FSaSc] we have analyzed aspects of the nonabelian 2-form field on M5-branes using this machinery, while briefly sketching related aspects of the C-field. Here we provide further details.

We construct and then analyze a model for the C-field in nonabelian differential cohomology. We show that it reproduces the relevant properties of previous models, mainly [DMW, DFM, FM, Sa10b], and refines them in the following ways.

1. All three gauge fields are dynamical (gravity, C-field, E_8-field), none is fixed background. In particular, where in previous models the fixed gravitational background is perceived of as a twist of the dynamical C-field, here the twisting is democratic, and in effect the whole construction yields a single *twisted differential String structure* as introduced in [SSS09b].

2. As a result, the whole construction is outside the scope of abelian differential cohomology and necessarily lives in higher nonabelian differential cohomology. Only truncations and reductions where the Spin connection is held fixed and the E_8-field is reduced to its instanton sector sit in the purely abelian sector, as previously conceived.

3. The full moduli 3-stack of field configurations is produced by a simple and natural homotopy pullback construction. This means that not only the gauge transformations, but also their gauge-of-gauge transformations as well as their higher gauge transformations, are accounted for. Moreover, the *smooth structure* on all this is retained. In summary, this means that the smooth moduli 3-stack that we produce integrates the relevant (off-shell) BRST Lie 3-algebroid of field configurations (gravity, C-field, E_8-field), involving the appropriate ghosts, ghosts-of-ghosts and third order ghosts. This is the correct starting point for any actual quantization of the system (as an effective low-energy gravitational higher gauge theory, as it were, but conceivably of relevance also to the full "M-theory").

4. A similarly simple and natural further homotopy pullback gives the boundary field moduli 2-stack of the C-field. We demonstrate that this is equivalent to the moduli 2-stack of anomaly free heterotic field configurations as found in [SSS09b].

5. We lift the Hořava-Witten boundary condition on the C-Field from 3-forms to differential cocycles and further to the level of moduli 3-stacks, there combining it with the flux quantization condition. This involves a generalization of string orientifolds to what we call *membrane orientifolds*.

In section 2 we give an informal discussion of central ideas of our constructions. In section 3 we recollect and set up the mathematical machinery needed. Then in section 4 discuss our model and analyze its properties.

2 Informal overview

The following sections are written in formal mathematical style. But in order to provide the pure physicist reader with a working idea of what the formalization is about, and in order to help the pure mathematician reader get a working idea of the physical meaning of the homotopy-theoretic constructions, we give in this section an informal discussion of some central ideas and of our main construction (see also the Introduction of [FSaSc]) .

The ambient theory in which higher gauge theory is naturally formulated is the combination of differential geometry with homotopy theory: *higher differential geometry*.[1] With hindsight, this has its very roots in gauge theory. A *BRST complex* with its ghost fields and ghosts-of-ghosts and so forth, up to ghosts or order n is secretly a *Lie n-algebroid*, the higher analog of a Lie algebra.[2] Whereas a Lie algebra encodes an

[1] See section 1.2 of [Sch] for a gentle introduction and section 3.3 for a detailed account.
[2] See section 1.3.5 of [Sch] for a gentle introduction, and section 3.4 for a detailed account.

infinitesimal symmetry of a single object, a BRST complex encodes several objects – the gauge field configurations – together with the infinitesimal symmetries – the gauge transformations – between them, together with the symmetries of symmetries between those, and so on. Just as a Lie algebra is the approximation to a finite smooth object, a Lie group, so a Lie n-algebroid is the approximation to a finite smooth object: this is called a *smooth n-groupoid* or, equivalently, a *smooth n-stack*. For instance, for G a Lie group and X a smooth manifold, there is a smooth stack of G-gauge fields on X, which we denote $[X, \mathbf{B}G_{\mathrm{conn}}]$, and which is the finite version of the BRST-complex of (off-shell) G-Yang-Mills theory on X. If we forget the smooth structure on this, we write $\mathbf{H}(X, \mathbf{B}G_{\mathrm{conn}})$ for the underlying *groupoid* of field configurations: it contains, as its *objects*, the gauge field configurations, and, as its *morphisms*, all the gauge transformations between these. By quotienting out the gauge transformations we obtain the plain set

$$\hat{H}^1(X, G) := H(X, \mathbf{B}G_{\mathrm{conn}})$$

of gauge equivalence classes, which physically is the set of gauge equivalence classes of G-gauge field configurations on X, and which mathematically is the degree-1 *nonabelian differential cohomology* on X with coefficients in G.

The simplest example of interest is obtained for $G = U(1)$, in which case $\mathbf{H}(X, \mathbf{B}U(1)_{\mathrm{conn}})$ is the groupoid of Maxwell field configurations on X. The examples of interest to us are $G = E_8$, the largest exceptional simple Lie group, and $G = \mathrm{Spin}$. In the first case, $\mathbf{H}(X, (\mathbf{B}E_8)_{\mathrm{conn}})$ is the groupoid of E_8-gauge fields as they live, for instance, on a Hořava-Witten boundary of 11-dimensional spacetime. In the second case $G = \mathrm{Spin}$, $\mathbf{H}(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}})$ is the groupoid of Spin-connections on X, which, in the first-order formulation of gravity that is of relevance in supergravity, encodes part of the field of gravity itself.

All these examples admit *higher analogs*. For instance, for every natural number n, there is a moduli n-stack of *n-form gauge fields*, which we write $\mathbf{B}^n U(1)_{\mathrm{conn}}$. This is such that $[X, \mathbf{B}^n U(1)_{\mathrm{conn}}]$ is the Lie integration of the BRST complex of (off-shell) n-form field configurations. Then $\mathbf{H}(X, \mathbf{B}^n U(1)_{\mathrm{conn}})$ is the underlying *n-groupoid* of field configurations. Its objects are, locally on patches $U \hookrightarrow X$, given by differential n-forms C_U. Its gauge transformations between fields C_U and C'_U are locally given by $(n-1)$-forms B_U, such that

$$C'_U = C_U + dB_U \, .$$

Its gauge-of-gauge transformations between gauge transformations $\{B_U, \ B'_U\}$ are $(n-2)$-forms A_U, such that

$$B'_U = B_U + dA_U \, .$$

The pattern continues in a similar fashion. The global structure is more intricate, but is essentially given by gluing such local data on intersections of patches by precisely such higher gauge transformations.

It is clear from the above discussion that the supergravity C-field is bound to be essentially an object in $\mathbf{H}(X, \mathbf{B}^3 U(1)_{\mathrm{conn}})$. But the situation is slightly more involved, because there is a quantum constraint on the C-field. All we have to do is add this constraint to the picture, making sure this is done in the proper gauge theoretic way. More precisely, the C-field interacts with the field of gravity, whose configurations are $\mathbf{H}(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}})$, and, over Hořava-Witten boundaries, with an E_8-gauge field in $\mathbf{H}(\partial X, (\mathbf{B}E_8)_{\mathrm{conn}})$; this extends to the bulk, at least at the level of the underlying principal bundles in $\mathbf{H}(X, \mathbf{B}E_8)$. Moreover, every Spin connection and every E_8-connection induces associated *Chern-Simons circle 3-bundles* via maps[3] of 3-stacks denoted

$$\tfrac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

and

$$\hat{\mathbf{a}} : (\mathbf{B}E_8)_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}} \, .$$

[3] Details are in [FScSt]. See also section 4.1 of [Sch].

The quantum constraint on these three fields (reviewed below in section 3.2) is on – integral cohomology classes ("instanton sectors") – given by the equation (see [Wi97])

$$[2G_4] = \tfrac{1}{2}p_1 + 2a\,, \tag{1}$$

where $[G_4]$ is the class of the C-field. It is useful to encode that equation graphically: the set of triples of gauge equivalence classes of fields that satisfies this equation is the *fiber product* or *pullback* of the maps on cohomology sets

$$\tfrac{1}{2}\hat{p}_1 + 2a : H(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8)_{\mathrm{conn}}) \to H(X, \mathbf{B}^3U(1)_{\mathrm{conn}})$$

and the map

$$2G_4 : H(X, \mathbf{B}^3U(1)_{\mathrm{conn}}) \to H(X, \mathbf{B}^3U(1))$$

that simply forgets the underlying connection data. Namely, the solution set of (1) on cohomology is the set that universally completes, in the top left corner, this square of functions between sets:

$$\begin{tikzcd}
{} \arrow[r] \arrow[d] & H(X, \mathbf{B}^3U(1)_{\mathrm{conn}}) \arrow[d, "2G_4"] \\
H(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8) \arrow[r, "\frac{1}{2}p_1 + 2a"] & H(X, \mathbf{B}^3U(1))
\end{tikzcd} \quad .$$

From this perspective it might seem as if imposing the quantization condition simply restricts the set of possible field configurations to the subset of those triples that satisfy the quantization condition. But a moment of reflection shows that this is wrong: physically, because for *quantization* we must not be working with sets of gauge equivalence classes of field configurations. Instead, we need to retain at least the full BRST complexes of fields, and better yet, as we do here, retain also the finite gauge transformations, hence consider the n-groupoids of field configurations. Mathematically, the reason is that forming an ordinary fiber product in homotopy theory breaks the universal property of the pullback and hence makes it useless, in fact meaningless.

We find that, in either case, implementing the above quantum constraint equation means forming a universal square as above, but using the higher groupoids $\mathbf{H}(X, -)$ of field configurations, gauge transformations, and higher gauge transformation, instead of just the gauge equivalence classes $H(X, -)$. Doing so gives what mathematically is called forming a *homotopy pullback* square: a square diagram

$$\begin{tikzcd}
{} \arrow[r] \arrow[d] & \mathbf{H}(X, \mathbf{B}^3U(1)_{\mathrm{conn}}) \arrow[d, "2\mathbf{G}_4"] \arrow[dl, Rightarrow, "\simeq"'] \\
\mathbf{H}(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8) \arrow[r, "\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}"'] & \mathbf{H}(X, \mathbf{B}^3U(1))
\end{tikzcd}$$

of maps of higher groupoids, where now everything holds only *up to gauge transformations* or *up to homotopy*, as indicated by the double arrow now filling this diagram. This is the most natural thing to do physically: if condition (1) is to hold for gauge equivalence classes of fields then, clearly, on the actual fields there is a gauge transformation exhibiting the equivalence.

The mathematics of homotopy theory provides a calculus for handling such constructions up to gauge transformations. Homotopy theory is precisely the formalism for dealing with gauge systems and higher gauge systems, and this is what we use in the following. Accordingly, *all* square diagrams as above appearing later in this paper are implicitly filled by a gauge transformation, even if we will usually suppress this from the notation. Moreover, in this construction the choice of X is not essential. We may in full generality ask for the *universal smooth moduli n-stack* of C-field configurations, to be denoted **CField**. This is to be such that for any manifold X, morphisms of smooth higher stacks

$$X \to \mathbf{CField}$$

correspond precisely to triples of fields (gravity, C-field, E_8-field) on X, satisfying the quantization condition (1) up to a specified gauge equivalence, and such that homotopies between such maps correspond precisely to compatible gauge transformations between such triples of field configurations. By a basic but fundamental fact of higher geometry, this universal moduli 3-stack is necessarily characterized as completing the analogous diagram as above, now consisting of fully fledged morphisms of higher smooth stacks. In other words, the moduli 3-stack **CField** is to be this homotopy pullback of higher moduli stacks:

$$\begin{tikzcd} \mathbf{CField} \arrow[r] \arrow[d] & \mathbf{B}^3U(1)_{\mathrm{conn}} \arrow[d, "2\mathbf{G}_4"] \arrow[dl, Rightarrow, "\simeq"'] \\ \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 \arrow[r, "\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}"'] & \mathbf{B}^3U(1) \end{tikzcd} . \qquad (2)$$

In summary, this is the straightforward translation of the constraint equation (1) from gauge equivalence classes to genuine higher gauge field configurations. And this is the model for the C-field that we present here. We show in the following sections that this construction reproduces all the relevant properties of previous proposals and refines them from 1-groupoids of fields and gauge-of-gauge equivalence classes of gauge transformations to the full 3-groupoid of field configurations and further to the full smooth moduli 3-stack of field configurations.

The boundary data of C-field configurations in section 4.3 is constructed analogously: the two physical conditions (that the E_8 gauge field becomes dynamical and that the C-field class trivializes) have straightforward translation into homotopy pullback diagrams. We show in the final prop. 4.3.1 that the moduli 2-stack of C-field boundary conditions obtained this way is precisely that of anomaly-free heterotic field configurations as found in [SSS09b].

3 Ingredients

Before we come to our main constructions in section 4 we briefly lay some foundations. First we recall in section 3.1 basics of smooth moduli stacks, of the refinement of ordinary *abelian* differential cohomology to moduli stacks, and then of those aspects of *nonabelian* differential cohomology that we need in the following sections. Then we recall in section 3.2 the origin of the factor of 2, that governs the whole discussion here, from quadratic refinement of higher abelian Chern-Simons functionals. Finally, in section 3.3, we first review the formalization in [HS] of this situation in terms of *differential integral Wu classes* and then show how this refines to nonabelian differential cohomology. This leads over seamlessly to the model of the C-field introduced further below in section 4.

3.1 Abelian and nonabelian differential cohomology

We give a list of the basic definitions and properties of

1. smooth higher groupoids / smooth higher stacks,
2. abelian differential cohomology refined to smooth moduli stacks,
3. nonabelian differential cohomology,

that we invoke below in section 4. This list is necessarily somewhat terse. For a comprehensive account we refer the reader to [Sch]. Much of the necessary technology is spelled out in [FScSt], and much of the relation to phenomena in string theory is discussed in [SSS09b] and in the companion article [FSaSc].

Differential geometry can be viewed as the geometry modeled on the following *site*.

Definition 3.1.1. Let CartSp be the category whose objects are the Cartesian spaces $\mathbb{R}^n$ for $n \in \mathbb{N}$, and whose morphisms are the smooth functions between these. A family of morphism $\{U_i \to U\}$ in CartSp is called a *good cover* if each all non-empty finite intersections of the U_i in U is diffeomorphic to a Cartesian space. This defines a *coverage* (pretopology) and we regard CartSp as a site equipped with this coverage.

Higher differential geometry takes place in the ∞-*topos* over this site.[4]

Definition 3.1.2. Write

$$\mathrm{Smooth}\infty\mathrm{Grpd} := \mathrm{Sh}_\infty(\mathrm{CartSp})$$

for the ∞-category of higher stacks over CartSp. As a simplicial category, this is the *simplicial localization* L_W of the category of simplicial presheaves $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ over CartSp, at the set W of morphisms which are stalkwise weak homotopy equivalences of simplicial sets:

$$\mathrm{Smooth}\infty\mathrm{Grpd} \simeq L_W[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$$

Remark 3.1.1. The localization formally inverts the morphisms in W and is analogous to the possibly more familiar localization at *quasi-isomorphisms* that yields the derived categories of topological branes for the topological string. Here we are dealing with a non-abelian generalization and refinement of this process. Instead of just quasi-isomorphisms between chain complexes we have more generally weak homotopy equivalences between simplicial sets, and the formal inverses that we add are just *homotopy inverses*, but we also add the relevant homotopies, the relevant homotopies between homotopies, and so on.

Usually we write

$$\mathbf{H} := \mathrm{Smooth}\infty\mathrm{Grpd}$$

for short, which is suggestive in view of the following

Definition 3.1.3. For $X, A \in \mathbf{H}$ any two higher stacks, the hom-∞-groupoid between them is denoted $\mathbf{H}(X, A)$. We also call this the *cocycle ∞-groupoid* for cocycles on X with coefficients in A. For its set of connected components we write

$$H(X, A) := \pi_0 \mathbf{H}(X, A)$$

and speak of the *smooth nonabelian cohomology* or just *cohomology* set, for short, on X with coefficients in A.

Example 3.1.1. The following differential geometric objects are naturally embedded into $\mathbf{H}$:

1. smooth manifolds;
2. smooth orbifolds;
3. more general Lie groupoids / differentiable stacks;
4. diffeological spaces, such as smooth mapping spaces $C^\infty(\Sigma, X)$ between manifolds (e.g. sigma models);
5. smooth moduli stacks $\mathbf{B}G$ of G-principal bundles, for G a Lie group;
6. smooth moduli stacks $\mathbf{B}G_{\mathrm{conn}}$ and $\mathrm{Loc}(G) \simeq \mathbf{B}G_{\mathrm{flat}}$ of G-principal bundles with connection and with flat connection, respectively.

For the last two items see also Example 3.1.2 below.

There are many more and "higher" examples. Some of these we describe in detail in the following.

We will need only some basic facts of ∞-category theory.[5] One fundamental fact is the existence of all ∞-*pullbacks / homotopy pullbacks* in $\mathbf{H}$. In section 2.1.4.2 of [Sch] is a discussion of explicit constructions of these, which many of our computations in the following rely on. Another fundamental fact that we will use frequently is

[4] See section 2.1.4 and 3.3 in [Sch].

[5] Such as summarized in sections 2.1.1 and 2.1.2 of [Sch], see the references provided there.

Proposition 3.1.1 (pasting law for homotopy pullbacks)**.** *Let*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

be a diagram in $\mathbf{H}$ *and suppose that the right square is a homotopy pullback. Then the left square is a homotopy pullback precisely if the outer rectangle is.*

Using this alone there is induced a notion of higher group objects in $\mathbf{H}$.

Definition 3.1.4. Write $\infty\mathrm{Grp}(\mathbf{H})$ for the ∞-category of group ∞-stacks over CartSp (grouplike smooth A_∞-spaces). We call these *smooth ∞-groups*.[6] Write $\mathbf{H}^{*/}$ for the ∞-category of pointed objects in $\mathbf{H}$. Write

$$\Omega : \mathbf{H}^{*/} \to \infty\mathrm{Grp}(\mathbf{H})$$

for the ∞-functor that sends a pointed object $* \to A$ to its loop space object $\Omega_* A$, defined to be the homotopy pullback

$$\begin{tikzcd} \Omega_* A \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & A \end{tikzcd} .$$

Proposition 3.1.2. $\mathbf{H}$ *has* homotopy dimension *0, hence every connected object* A *has a point* $* \to A$ *(necessarily essentially unique).*

Theorem 3.1.1 (Lurie)**.** *Looping induces an equivalence of ∞-categories*

$$\infty\mathrm{Grp}(\mathbf{H}) \underset{\mathbf{B}}{\overset{\Omega}{\underset{\simeq}{\leftrightarrows}}} \mathbf{H}^{*/}_{\geq 1}$$

between smooth ∞-groups and pointed connected *objects. The homotopy inverse functor* $\mathbf{B}$ *we call the* delooping *functor.*

Definition 3.1.5. If an ∞-stack A is an n-fold loop object, we write $\mathbf{B}^n A$ for its n-fold delooping. For X any other object we write

$$H^n(X, A) := \pi_0 \mathbf{H}(X, \mathbf{B}^n A)$$

and speak of the *degree n cohomology* on X with coefficients in A.

Example 3.1.2. Every Lie group G is naturally also a smooth ∞-group. Its delooping $\mathbf{B}G$ is the *moduli stack* of G-principal bundles: for any smooth manifold X, the cocycle groupoid

$$\mathbf{H}(X, \mathbf{B}G) \simeq G\mathrm{Bund}(X)$$

is the groupoid of smooth G-principal bundles and smooth gauge transformations between them, on X. The corresponding nonabelian smooth cohomology

$$H^1(X, G) := H(X, \mathbf{B}G)$$

coincides with degree-1 nonabelian Čech cohomology on X with coefficients in the sheaf of smooth G-valued functions.

[6] See section 2.3.2 of [Sch].

If G is an *abelian* Lie group, such as $G = U(1)$, the delooping moduli stack $\mathbf{B}U(1)$ is itself again canonically an ∞-group, called the *circle 2-group*. In fact for all $n \in \mathbb{N}$ the n-fold delooping $\mathbf{B}^n U(1)$ exists. This is the *moduli n-stack for circle n-bundles*. Morphisms $X \to \mathbf{B}^2 U(1)$ may be identified with *bundle gerbes* on X (circle 2-bundles), morphism $X \to \mathbf{B}^3 U(1)$ with *bundle 2-gerbes* (circle 3-bundles) and so on. The smooth cohomology

$$H^n(X, U(1)) := H(X, \mathbf{B}^n U(1))$$

coincides with degree-n Čech cohomology with coefficients in the sheaf of smooth $U(1)$-valued functions.

Generally, for A a sheaf of abelian groups,

$$H^n(X, A) := H(X, \mathbf{B}^n A) := \pi_0 \mathbf{H}(X, \mathbf{B}^n A)$$

coincides with the *sheaf cohomology* in degree n over X with coefficients in A.

But see also the further example 3.1.3 below.

Proposition 3.1.3. *The inclusion*

$$\mathrm{Disc}: \ \mathrm{Top} \underset{\simeq}{\xrightarrow{\mathrm{Sing}}} \infty\mathrm{Grpd} \hookrightarrow \mathbf{H}$$

of topological spaces into smooth higher stacks – as the discrete *or* locally constant *smooth ∞-stacks – has a derived left adjoint*

$$|-|: \ \mathbf{H} \xrightarrow{\Pi} \infty\mathrm{Grpd} \underset{\simeq}{\xrightarrow{|-|}} \mathrm{Top} \ ,$$

called the geometric realization *of smooth higher stacks.*

Proposition 3.1.4. *If a higher group G has a presentation as a simplicial presheaf which in turn is presented by a well-pointed simplicial topological group that is degreewise paracompact, then for X any manifold, we have an isomorphism*

$$H^1(X, G) \simeq \pi_0 \mathbf{H}(X, \mathbf{B}G) \simeq \pi_0 \mathrm{Top}(X, |BG|) \simeq \pi_0 \mathrm{Top}(X, B|G|)$$

of the smooth higher nonabelian cohomology of X with coefficients in G and homotopy classes of maps into the geometric realization of the higher moduli stack.

This follows by [RoSt]. See section 3.2.2 of [Sch].

Remark 3.1.2. In terms of gauge theory this says that for G a higher group, the geometric realization $|\mathbf{B}G|$ is the classification space of the instanton sector of higher G-gauge field configurations.

Definition 3.1.6. By general facts, $\mathrm{Disc} : \infty\mathrm{Grpd} \hookrightarrow \mathbf{H}$ is also itself a derived left adjoint. For $A \in \mathbf{H}$ any object, we write $\flat A \to A$ for the counit of the corresponding adjunction.

For G an ∞-group, we call $\flat \mathbf{B}G$ the higher moduli stack of *flat G-principal ∞-connections* or of *G-local systems.*[7]

Example 3.1.3. The moduli n-stack $\flat \mathbf{B}^n U(1)$ is presented by the complex of sheaves concentrated in degree 1 on the *constant* sheaf with values $U(1)$. This may be thought of as the sheaf of functions into the *discrete* group $U(1)_{\mathrm{disc}}$ underlying the Lie group $U(1)$:

$$\mathbf{B}^n U(1)_{\mathrm{disc}} \simeq \flat \mathbf{B}^n U(1) \, .$$

The smooth cohomology with coefficients in this discrete object coincides with ordinary *singular cohomology* with coefficients in $U(1)$

$$H^n(X, U(1)_{\mathrm{disc}}) \simeq H(X, \flat \mathbf{B}^n U(1)) \simeq H^n_{\mathrm{sing}}(X, U(1)) \, .$$

[7] See 2.3.12 in [Sch].

Definition 3.1.7. For G a smooth ∞-group, we write

$$\flat_{\mathrm{dR}}\mathbf{B}G := \flat\mathbf{B}G \times_{\mathbf{B}G} *$$

for the homotopy pullback of the counit $\flat\mathbf{B}G \to \mathbf{B}G$ along the point inclusion. Smooth cohomology with coefficients in $\flat_{\mathrm{dR}}\mathbf{B}G$ we call *G-de Rham cohomology.* The canonical morphism

$$\theta_G : G \to \flat_{\mathrm{dR}}\mathbf{B}G$$

we call the *Maurer-Cartan form* on the smooth ∞-group G. Specifically for $G = \mathbf{B}^n U(1)$ the circle $(n+1)$-group, we also write

$$\mathrm{curv} := \theta_{\mathbf{B}^n U(1)} : \mathbf{B}^n U(1) \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$$

and speak of the *universal curvature characteristic map* in degree $(n+1)$.

Proposition 3.1.5. *Under the Dold-Kan correspondence* [8]*, $\flat_{\mathrm{dR}}\mathbf{B}^n U(1)$ is presented by the truncated de Rham complex of sheaves of abelian groups*

$$\Omega^1(-) \xrightarrow{d_{\mathrm{dR}}} \Omega^2(-) \xrightarrow{d_{\mathrm{dR}}} \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^n_{\mathrm{cl}}(-)\ .$$

Morover, the universal curvature characteristic curv *is presented by a correspondence of simplicial presheaves*

$$\begin{tikzcd} \mathbf{B}^n U(1)_{\mathrm{diff}} \arrow[r, "\mathrm{curv}"] \arrow[d, "\wr"] & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1) \\ \mathbf{B}^n U(1) & \end{tikzcd},$$

where $\mathbf{B}^n U(1)_{\mathrm{conn}}$ classifies circle n-bundles equipped with pseudo-connection*: they carry connection data, but gauge transformations are allowed to freely shift the connections.*

Definition 3.1.8. For $n \geq 1$, the *moduli n-stack of circle n-bundles with connection* $\mathbf{B}^n U(1)_{\mathrm{conn}}$ is the homotopy pullback of higher stacks

$$\begin{tikzcd} \mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[r] \arrow[d] & \Omega^{n+1}_{\mathrm{cl}} \arrow[d] \\ \mathbf{B}^n U(1) \arrow[r, "\mathrm{curv}"] & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R} \end{tikzcd}\ .$$

Proposition 3.1.6. *Under the Dold-Kan correspondence $\mathbf{B}^n U(1)_{\mathrm{conn}}$ is presented by the Beilinson-Deligne complex of sheaves, either in the form*

$$C^\infty(-, U(1)) \xrightarrow{d_{\mathrm{dR}}\log} \Omega^1(-) \xrightarrow{d_{\mathrm{dR}}} \Omega^2(-) \xrightarrow{d_{\mathrm{dR}}} \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^n(-)$$

or equivalently in the form

$$\mathbb{Z} \longrightarrow C^\infty(-, \mathbb{R}) \xrightarrow{d_{\mathrm{dR}}} \Omega^1(-) \xrightarrow{d_{\mathrm{dR}}} \Omega^2(-) \xrightarrow{d_{\mathrm{dR}}} \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^n(-)\ .$$

For X a smooth manifold, the corresponding cohomology

$$H(X, \mathbf{B}^n U(1)_{\mathrm{conn}}) \simeq \hat{H}^{n+1}(X)$$

is the ordinary differential cohomology *of X in degree $(n+1)$.*

[8] See 2.1.7 of [Sch]

Definition 3.1.9. For G a topological group and $[c] \in H^{n+1}(BG, \mathbb{Z})$ a universal characteristic class, we say that a *smooth refinement* of $[c]$ is a morphism of higher smooth moduli stacks of the form

$$\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n U(1)$$

such that under geometric realization this reproduces c, in that

$$|\mathbf{c}| : BG \to K(\mathbb{Z}, n+1)$$

is a representative of $[c]$. We say a further *differential refinement* of $[c]$ is a morphism of higher moduli stacks of the form

$$\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$$

such that it completes the diagram

$$\begin{tikzcd}
\flat \mathbf{B}G \arrow[r, "\flat \mathbf{c}"] \arrow[d] & \flat \mathbf{B}^n U(1) \arrow[d] \\
\mathbf{B}G_{\mathrm{conn}} \arrow[r, dashed, "\hat{\mathbf{c}}"] \arrow[d] & \mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[d] \\
\mathbf{B}G \arrow[r, "\mathbf{c}"] & \mathbf{B}^n U(1)
\end{tikzcd} .$$

Remark 3.1.3. A smooth refinement of $[c]$ is equivalently a smooth circle n-bundle / $(n-1)$-bundle gerbe on a smooth moduli stack whose higher Dixmier-Douady class is $[c]$. Similarly a differential refinement of $[c]$ is a circle n-bundle *with connection* on the differential moduli stack whose Dixmier-Douady class is $[c]$.

Using the L_∞-algebraic data provided in [SSS09a], the following was shown in [FScSt]. See also section 4.1 in [Sch].

Proposition 3.1.7. *There exists a smooth and differential refinement of the first fractional Pontrjagin class*

$$\tfrac{1}{2}p_1 \in H^4(B\mathrm{Spin}, \mathbb{Z})$$

to the smooth moduli stack of Spin connections with values in the smooth moduli 3-stack of circle 3-bundles with 3-connection

$$\tfrac{1}{2}\hat{\mathbf{p}}_1 : \ \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}} \ .$$

Proposition 3.1.8. *Let E_8 be the largest semisimple exceptional Lie group. There exists a differential refinement of the canonical class*

$$[a] \in H^4(BE_8, \mathbb{Z})$$

to the smooth moduli stack of E_8-connections with values in the smooth moduli 3-stack of circle 3-bundles with 3-connection

$$\hat{\mathbf{a}} : \ (\mathbf{B}E_8)_{\mathrm{conn}} \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}} \ .$$

Proposition 3.1.9. *Under geometric realization, prop. 3.1.3, the smooth class* $\mathbf{a}$ *becomes an equivalence*

$$|\mathbf{a}| : BE_8 \simeq_{16} B^3 U(1) \simeq K(\mathbb{Z}, 4)$$

on 16-coskeleta.

Proof. The 15-coskeleton of the topological space E_8 is a $K(\mathbb{Z},4)$. By [FScSt], $\mathbf{a}$ is a smooth refinement of the generator $[a] \in H^4(BE_8, \mathbb{Z})$. By the Hurewicz theorem this is identified with $\pi_4(BE_8) \simeq \mathbb{Z}$. Hence in cohomology $\mathbf{a}$ induces an isomorphism

$$\pi_4(BE_8) \simeq [S^4, BE_8] \simeq H^1(S^4, E_8) \xrightarrow[\simeq]{|\mathbf{a}|} H^4(S^4, \mathbb{Z}) \simeq [S^4, K(\mathbb{Z},4)] \simeq \pi_4(S^4) \ .$$

Therefore $|\mathbf{a}|$ is a weak homotopy equivalence on 16 coskeleta. □

Remark 3.1.4. We obtain the *de Rham images* of these differential classes by postcomposition with the universal 4-curvature characteristic from def. 3.1.7:

$$\begin{aligned} (\tfrac{1}{2}\mathbf{p}_1)_{\mathrm{dR}} &: \quad \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \xrightarrow{\hat{\mathbf{p}}_1} \mathbf{B}^3 U(1)_{\mathrm{conn}} \xrightarrow{\mathrm{curv}} \flat_{\mathrm{dR}} \mathbf{B}^4 U(1) \ , \\ \mathbf{a}_{\mathrm{dR}} &: \quad (\mathbf{B}E_8)_{\mathrm{conn}} \xrightarrow{\hat{\mathbf{a}}} \mathbf{B}^3 U(1)_{\mathrm{conn}} \xrightarrow{\mathrm{curv}} \flat_{\mathrm{dR}} \mathbf{B}^4 U(1) \ . \end{aligned}$$

By prop. 3.1.5 these morphisms have a presentation by correspondences of simplicial presheaves

$$\begin{tikzcd} (\mathbf{B}E_8)_{\mathrm{diff}} \arrow[r, "\mathbf{a}_{\mathrm{diff}}"] \arrow[d, "\wr"] & \mathbf{B}^3 U(1)_{\mathrm{diff}} \arrow[r, "\mathrm{curv}"] \arrow[d, "\wr"] & \flat_{\mathrm{dR}} \mathbf{B}^4 \mathbb{R} \\ \mathbf{B}E_8 \arrow[r, "\mathbf{a}"] & \mathbf{B}^3 U(1) & \end{tikzcd}$$

involving the simplicial presheaf $(\mathbf{B}E_8)_{\mathrm{diff}}$ of E_8-*pseudo-connections*. See [FScSt] for a thorough discussion.

Every morphism $\mathbf{c} : P \to B$ of higher pointed stacks with homotopy fiber $A \to P$ may be regarded as an ∞-*bundle* over B with typical fiber A. We may therefore consider the cohomology with coefficients in A but *twisted* by cocycles $\chi \in \mathbf{H}(X, B)$.[9] Such an χ-*twisted* A-cocycle is a homotopy section σ in

$$\begin{tikzcd} & P \arrow[d, "\mathbf{c}"] \\ X \arrow[ur, dashed, "\sigma"] \arrow[r] & B \ . \end{tikzcd}$$

In the special case that $\mathbf{c}$ is interpeted as a smooth universal characteristic map as above, we think of a $\mathbf{c}$-twisted A-cocycle also as a *twisted* $\mathbf{c}$-*structure*.

Definition 3.1.10. For $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n U(1)$ a smooth characteristic map in $\mathbf{H}$, define for any $X \in \mathbf{H}$ the ∞-groupoid $\mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X)$ of *twisted* $\mathbf{c}$-*structures* to be the ∞-pullback

$$\begin{tikzcd} \mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X) \arrow[r, "\mathrm{tw}"] \arrow[d] & H^n(X, U(1)) \arrow[d] \\ \mathbf{H}(X, \mathbf{B}G) \arrow[r, "\mathbf{c}"] & \mathbf{H}(X, \mathbf{B}^n U(1)) \ , \end{tikzcd}$$

where the vertical morphism on the right is the essentially unique effective epimorphism that picks a point in every connected component.

For $\chi \in \mathbf{H}(X, \mathbf{B}^n A)$ a fixed twisting cocycle, the ∞-groupoid of χ-*twisted* $\mathbf{c}$-*structures* is the homotopy fiber

$$\begin{tikzcd} \mathbf{c}\mathrm{Struc}_{[\chi]}(X) \arrow[r] \arrow[d] & * \arrow[d, "\chi"] \\ \mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X) \arrow[r, "\mathrm{tw}"] & H^n(X, U(1)) \ . \end{tikzcd}$$

[9] See section 2.3.5 of [Sch].

In [SSS09b] (see also 4.4.4 in [Sch]) there is a list of examples of such nonabelian twisted cohomology governing anomaly cancellation in string theory: twisted Spin^c structures, smooth twisted String structures, smooth twisted Fivebrane structures. The twisted String structures we re-encounter below in section 4.3 in the boundary field configurations of the C-field.

3.2 Higher abelian Chern-Simons theories with background charge

The supergravity C-field is an example of a general phenomenon of higher abelian Chern-Simons QFTs in the presence of *background charge*. This phenomenon was originally noticed in [Wi96] and then made precise in [HS]. The holographic dual of this phenomenon is that of self-dual higher gauge theories, which for the supergravity C-field is the nonabelian 2-form theory on the M5-brane [FSaSc], and in this dual form it has been studied systematically in [DFM, BM]. We now review the idea in a way that will smoothly lead over to our refinements to nonabelian higher gauge theory in section 4.

Fix some natural number $k \in \mathbb{N}$ and an oriented manifold (compact with boundary) X of dimension $4k+3$. The gauge equivalence class of a $(2k+1)$-form gauge field $\hat{G}$ on X is an element in the differential cohomology group $\hat{H}^{2k+2}(X)$. The cup product $\hat{G} \cup \hat{G} \in \hat{H}^{4k+4}(X)$ of this class with itself has a natural higher holonomy over X, denoted

$$\begin{aligned} \exp(iS(-)) : \hat{H}^{2k+2}(X) &\to U(1) \\ \hat{G} &\mapsto \exp(i \int_X \hat{G} \cup \hat{G}) \,. \end{aligned} \tag{3}$$

This is the exponentiated action functional for bare $(4k+3)$-dimensional abelian Chern-Simons theory. For $k=0$ this reduces to ordinary 3-dimensional abelian Chern-Simons theory [CS]. Notice that, even in this case, this is a bit more subtle that Chern-Simons theory for a simply-connected gauge group G. In the latter case all fields can be assumed to be globally defined forms. But in the non-simply-connected case of $U(1)$, instead the fields are in general cocycles in differential cohomology. If, however, we restrict attention to fields C in the inclusion $H^{2k+1}_{\mathrm{dR}}(X) \hookrightarrow \hat{H}^{2k+2}(X)$, then on these the above action (3) reduces to the familiar expression

$$\exp(iS(C)) = \exp(i \int_X C \wedge d_{\mathrm{dR}} C) \,.$$

Observe now that the above action functional may be regarded as a *quadratic form* on the group $\hat{H}^{2k+2}(X)$. The corresponding bilinear form is the ("secondary", since X is of dimension $4k+3$ instead of $4k+4$) *intersection pairing*

$$\begin{aligned} \langle -,- \rangle : \hat{H}^{2k+2}(X) \times \hat{H}^{2k+2}(X) &\to U(1) \\ (\hat{a}_1 \ , \ \hat{a}_2) &\mapsto \exp(i \int_X \hat{a}_1 \cup \hat{a}_2) \,. \end{aligned}$$

However, note that from $\exp(iS(-))$ we do *not* obtain a *quadratic refinement* of the pairing. A quadratic refinement is, by definition, a function

$$q : \hat{H}^{2k+2}(X) \to U(1)$$

(not necessarily homogenous of degree 2 as $\exp(iS(-))$ is), for which the intersection pairing is obtained via the polarization formula

$$\langle \hat{a}_1, \hat{a}_2 \rangle = q(\hat{a}_1 + \hat{a}_2) q(\hat{a}_1)^{-1} q(\hat{a}_2)^{-1} q(0) \,.$$

If we took $q := \exp(iS(-))$, then the above formula would yield not $\langle -,- \rangle$, but the square $\langle -,- \rangle^2$, given by the exponentiation of *twice* the integral.

The observation in [Wi96] was that for the correct holographic physics, we need instead an action functional which is indeed a genuine quadratic refinement of the intersection pairing. But since the differential classes in $\hat{H}^{2k+2}(X)$ refine *integral* cohomology, we cannot in general simply divide by 2 and pass from

$\exp(i\int_X \hat{G}\cup\hat{G})$ to $\exp(i\int_X \frac{1}{2}\hat{G}\cup\hat{G})$. The integrand in the latter expression does not make sense in general in differential cohomology. If one tried to write it out in the "obvious" local formulas one would find that it is a functional on fields which is not gauge invariant. The analog of this fact is familiar from nonabelian G-Chern-Simons theory with simply-connected G, where also the theory is consistent only at interger *levels*. The "level" here is nothing but the underlying integral class $G\cup G$. Therefore, the only way to obtain a square root of the quadratic form $\exp(iS(-))$ is to *shift it*. Here we think of the analogy with a quadratic form

$$q : x \mapsto x^2$$

on the real numbers (a parabola in the plane). Replacing this by

$$q^\lambda : x \mapsto x^2 - \lambda x$$

for some real number λ means keeping the shape of the form, but shifting its minimum from 0 to $\frac{1}{2}\lambda$. If we think of this as the potential term for a scalar field x then its ground state is now at $x = \frac{1}{2}\lambda$. We may say that there is a *background field* or *background charge* that pushes the field out of its free equilibrium. See [Fr, DFM, FM].

To lift this reasoning to the action quadratic form $\exp(iS(-))$ on differential cocycles, we need a differential class $\hat{\lambda} \in H^{2k+2}(X)$ such that for every $\hat{a} \in H^{2k+2}(X)$ the composite class

$$\hat{a}\cup\hat{a} - \hat{a}\cup\hat{\lambda} \in H^{4k+4}(X)$$

is even, hence is divisible by 2. Because then we could define a shifted action functional

$$\exp(iS^\lambda(-)) : \hat{a} \mapsto \exp\left(i\int_X \frac{1}{2}(\hat{a}\cup\hat{a} - \hat{a}\cup\hat{\lambda})\right),$$

where now the fraction $\frac{1}{2}$ in the integrand does make sense. One directly sees that if this exists, then this shifted action is indeed a quadratic refinement of the intersection pairing:

$$\exp(iS^\lambda(\hat{a}+\hat{b}))\exp(iS^\lambda(\hat{a}))^{-1}\exp(iS^\lambda(\hat{b}))^{-1}\exp(iS^\lambda(0)) = \exp(i\int_X \hat{a}\cup\hat{b}).$$

The condition on the existence of $\hat{\lambda}$ here means, equivalently, that the image of the underlying integral class vanishes under the map

$$(-)_{\mathbb{Z}_2} : H^{2k+2}(X,\mathbb{Z}) \to H^{2k+2}(X,\mathbb{Z}_2)$$

to $\mathbb{Z}_2$-cohomology:

$$(a)_{\mathbb{Z}_2}\cup(a)_{\mathbb{Z}_2} - (a)_{\mathbb{Z}_2}\cup(\lambda)_{\mathbb{Z}_2} = 0 \in H^{4k+4}(X,\mathbb{Z}_2).$$

Precisely such a class $(\lambda)_{\mathbb{Z}_2}$ does uniquely exist on every oriented manifold. It is called the *Wu class* $\nu_{2k+2} \in H^{2k+2}(X,\mathbb{Z}_2)$, and may be *defined* by this condition. Moreover, if X is a Spin-manifold, then every second Wu class, ν_{4k}, has a pre-image in integral cohomology, hence λ does exist as required above

$$(\lambda)_{\mathbb{Z}_2} = \nu_{2k+2}.$$

It is given by polynomials in the Pontrjagin classes of X (discussed in section E.1 of [HS]). For instance the degree-4 Wu class (for $k=1$) is refined by the first fractional Pontrjagin class $\frac{1}{2}p_1$

$$(\tfrac{1}{2}p_1)_{\mathbb{Z}_2} = \nu_4.$$

In the present context, this was observed in [Wi96] (see around eq. (3.3) there).

Notice that the equations of motion of the shifted action $\exp(iS^\lambda(\hat{a}))$ are no longer $\mathrm{curv}(\hat{a}) = 0$, but are now

$$\mathrm{curv}(\hat{a}) = \tfrac{1}{2}\mathrm{curv}(\hat{\lambda}).$$

We therefore think of $\exp(iS^\lambda(-))$ as the exponentiated action functional for *higher dimensional abelian Chern-Simons theory with background charge* $\frac{1}{2}\lambda$. With respect to the shifted action functional it makes sense to introduce the shifted field

$$\hat{G} := \hat{a} - \tfrac{1}{2}\hat{\lambda}\,.$$

This is simply a re-parameterization such that the Chern-Simons equations of motion again look homogenous, namely $G = 0$. In terms of this shifted field the action $\exp(iS^\lambda(\hat{a}))$ from above, equivalently, reads

$$\exp(iS^\lambda(\hat{G})) = \exp(i \int_X \tfrac{1}{2}(\hat{G} \cup \hat{G} - (\tfrac{1}{2}\hat{\lambda})^2))\,.$$

For the case $k = 1$, this is the form of the action functional for the 7d Chern-Simons dual of the 2-form gauge field on the M5-brane first given as (3.6) in [Wi96]

In the language of twisted cohomological structures, def. 3.1.10, we may summarize this situation as follows: *In order for the action functional of higher abelian Chern-Simons theory to be correctly divisible, the images of the fields in $\mathbb{Z}_2$-cohomology need to form a* twisted Wu-structure*, [Sa11c].Therefore the fields themselves need to constitute a* twisted λ*-structure. For $k = 1$ this is a* twisted String-structure *[SSS09b] and explains the quantization condition on the C-field in 11-dimensional supergravity.*

In [HS] a formalization of the above situation has been given in terms of a notion there called *differential integral Wu structures*. In the following section we explain how this follows from the notion of twisted Wu structures [Sa11c] with the twist taken in $\mathbb{Z}_2$-coefficients. Then we refine this to a formalization to *twisted differential Wu structures* with the twist taken in smooth circle n-bundles.

3.3 Twisted differential smooth Wu structures

We discuss some general aspects of smooth and differential refinements of $\mathbb{Z}_2$-valued universal characteristic classes. For the special case of *Wu classes* we show how these notions reduce to the definition of *differential integral Wu structures* given in [HS]. We then construct a refinement of these structures that lifts the twist from $\mathbb{Z}_2$-valued cocycles to smooth circle n-bundles. This further refinement of integral Wu structures is what underlies the model for the supergravity C-field in section 4.

Recall from [SSS09b, FSaSc] the characterization of Spin^c as the loop space object of the homotopy pullback

$$\begin{tikzcd} \mathbf{B}\mathrm{Spin}^c \arrow[r] \arrow[d] & \mathbf{B}U(1) \arrow[d, "\mathbf{c}_1 \bmod 2"] \\ \mathbf{B}\mathrm{SO} \arrow[r, "\mathbf{w}_2"] & \mathbf{B}^2\mathbb{Z}_2 \end{tikzcd}\,.$$

For general $n \in \mathbb{N}$ the analog of the first Chern class mod 2 appearing here is the higher Dixmier-Douady class mod 2

$$\mathbf{DD}_{\bmod 2} : \; \mathbf{B}^n U(1) \xrightarrow{\mathrm{DD}} \mathbf{B}^{n+1}\mathbb{Z} \xrightarrow{\bmod 2} \mathbf{B}^{n+1}\mathbb{Z}_2\,.$$

Let now

$$\nu_{n+1} : \mathbf{B}\mathrm{SO} \to \mathbf{B}^{n+1}\mathbb{Z}_2$$

be a representative of the universal *Wu class* in degree $n + 1$.

In the spirit of twisted structures in [Wa, SSS09b, Sa10c, Sa11a, Sa11c], def. 3.1.10, we have

Definition 3.3.1. Let $\mathrm{Spin}^{\nu_{n+1}}$ be the loop space object of the homotopy pullback

$$\begin{tikzcd} \mathbf{B}\mathrm{Spin}^{\nu_{n+1}} \arrow[r] \arrow[d, "\boldsymbol{\nu}^{\mathrm{int}}_{n+1}"] & \mathbf{B}\mathrm{SO} \arrow[d, "\nu_{n+1}"] \\ \mathbf{B}^n U(1) \arrow[r, "\bmod 2"] & \mathbf{B}^{n+1}\mathbb{Z}_2 \end{tikzcd}\,.$$

We call the left vertical morphism $\boldsymbol{\nu}_{n+1}^{\mathrm{int}}$ appearing here the *universal smooth integral Wu structure* in degree $n+1$.

A morphism of stacks

$$\boldsymbol{\nu}_{n+1} : X \to \mathbf{B}\mathrm{Spin}^{\nu_{n+1}}$$

is a choice of orientation structure on X together with a choice of smooth integral Wu structure lifting the corresponding Wu class ν_{n+1}.

Example 3.3.1. The smooth first fractional Pontrjagin class $\frac{1}{2}\mathbf{p}_1$, from prop. 3.1.7, fits into a diagram

$$\begin{tikzcd}
\mathbf{B}\mathrm{Spin} \arrow[dr, dashed, "u"] \arrow[drr, bend left] \arrow[ddr, bend right, "\frac{1}{2}\mathbf{p}_1"'] & & \\
& \mathbf{B}\mathrm{Spin}^{\nu_4} \arrow[r] \arrow[d, "\boldsymbol{\nu}_4^{\mathrm{int}}"] & \mathbf{B}\mathrm{SO} \arrow[d, "\nu_4"] \\
& \mathbf{B}^3 U(1) \arrow[r, "\mathrm{mod}\,2"] & \mathbf{B}^4\mathbb{Z}_2
\end{tikzcd} .$$

In this sense we may think of $\frac{1}{2}\mathbf{p}_1$ as being the integral and, moreover, smooth refinement of the universal degree-4 Wu class on $\mathbf{B}\mathrm{Spin}$. Using the defining property of $\frac{1}{2}\mathbf{p}_1$, this follows with the results discussed in appendix E.1 of [HS].

Proposition 3.3.1. *Let X be a smooth manifold equipped with orientation*

$$o_X : X \to \mathbf{B}\mathrm{SO}$$

and consider its Wu-class $[\nu_{n+1}(o_X)] \in H^{n+1}(X, \mathbb{Z}_2)$

$$\nu_{n+1}(o_X) : \; X \xrightarrow{o_X} \mathbf{B}\mathrm{SO} \xrightarrow{\nu_{n+1}} \mathbf{B}^{n+1}\mathbb{Z}_2 \; .$$

The n-groupoid $\hat{\mathbf{DD}}_{\mathrm{mod}2}\mathrm{Struc}_{[\nu_{2k}]}(X)$ of $[\nu_{n+1}]$-twisted differential $\mathbf{DD}_{\mathrm{mod}2}$-structures, according to def. 3.1.10, hence the homotopy pullback

$$\begin{tikzcd}
\hat{\mathbf{DD}}_{\mathrm{mod}2}\mathrm{Struc}_{[\nu_{n+1}]}(X) \arrow[r] \arrow[d] & * \arrow[d, "\nu_{n+1}(o_X)"] \\
\mathbf{H}(X, \mathbf{B}^3 U(1)_{\mathrm{conn}}) \arrow[r, "\hat{\mathbf{DD}}_{\mathrm{mod}\,2}"] & \mathbf{H}(X, \mathbf{B}^{n+1}\mathbb{Z}_2)
\end{tikzcd} ,$$

categorifies the groupoid $\hat{\mathcal{H}}^{n+1}_{\nu_{n+1}}(X)$ of differential integral Wu structures *as in def. 2.12 of [HS]: its 1-truncation is equivalent to the groupoid defined there*

$$\tau_1 \hat{\mathbf{DD}}_{\mathrm{mod}2}\mathrm{Struc}_{[\nu_{n+1}]}(X) \simeq \hat{\mathcal{H}}^{n+1}_{\nu_{n+1}}(X) \, .$$

Proof. By prop. 3.1.8, the canonical presentation of $\mathbf{DD}_{\mathrm{mod}2}$ via the Dold-Kan correspondence is given by an epimorphism of chain complexes of sheaves, hence by a fibration in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. Precisely, the composite

$$\hat{\mathbf{DD}}_{\mathrm{mod}\,2} : \; \mathbf{B}^n U(1)_{\mathrm{conn}} \longrightarrow \mathbf{B}^n U(1) \xrightarrow{\mathrm{DD}} \mathbf{B}^{n+1}\mathbb{Z} \xrightarrow{\mathrm{mod}\,2} \mathbf{B}^{n+1}\mathbb{Z}_2$$

is presented by the vertical sequence of morphisms of chain complexes

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, hook] \arrow[d] & C^\infty(-,\mathbb{R}) \arrow[r, "d_{\mathrm{dR}}\log"] \arrow[d] & \Omega^1(-) \arrow[r, "d_{\mathrm{dR}}"] \arrow[d] & \cdots \arrow[r, "d_{\mathrm{dR}}"] & \Omega^n(-) \arrow[d] \\
\mathbb{Z} \arrow[r, hook] \arrow[d] & C^\infty(-,\mathbb{R}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[d] \\
\mathbb{Z}_2 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0
\end{tikzcd}$$

We may therefore compute the defining homotopy pullback for $\hat{\mathbf{DD}}_{\mathrm{mod}2}\mathrm{Struc}_{[\nu_{n+1}]}(X)$ as an ordinary fiber product of the corresponding simplicial sets of cocycles. The claim then follows by inspection. □

Remark 3.3.1. Explicitly, a cocycle in $\tau_1\hat{\mathbf{DD}}_{\mathrm{mod}2}\mathrm{Struc}_{[\nu_{n+1}]}(X)$ is identified with a Čech cocycle with coefficients in the Deligne complex

$$\left(\mathbb{Z} \hookrightarrow C^\infty(-,\mathbb{R}) \xrightarrow{d_{\mathrm{dR}}\log} \Omega^1(-) \xrightarrow{d_{\mathrm{dR}}} \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^n(-) \right) ,$$

such that the underlying $\mathbb{Z}[n+1]$-valued cocycle modulo 2 equals the given cocycle for ν_{n+1}. A coboundary between two such cocycles is a gauge equivalence class of ordinary Čech-Deligne cocycles such that their underlying $\mathbb{Z}$-cocycle vanishes modulo 2. Cocycles of this form are precisely those that arise by multiplication with 2 or arbitrary Čech-Deligne cocycles. This is the groupoid structure discussed on p. 14 of [HS], there in terms of singular cohomology instead of Čech cohomology.

We now consider another twisted differential structure, which refines these twisting integral Wu structures to *smooth* integral Wu structures, of def. 3.3.1.

Definition 3.3.2. For $n \in \mathbb{N}$, write $\mathbf{B}^nU(1)^{\nu_{n+1}}_{\mathrm{conn}}$ for the homotopy pullback of smooth moduli n-stacks

$$\begin{tikzcd}
\hat{\mathbf{Wu}}^{\nu_{n+1}} \arrow[r] \arrow[d] & \mathbf{B}^nU(1)_{\mathrm{conn}} \arrow[d] \\
\mathbf{B}\mathrm{Spin}^{\nu_{n+1}} \times \mathbf{B}^nU(1) \arrow[r, "\boldsymbol{\nu}^{\mathrm{int}}_{n+1}+2\mathbf{DD}"] & \mathbf{B}^nU(1)
\end{tikzcd} .$$

where $\boldsymbol{\nu}^{\mathrm{int}}_{n+1}$ is the universal smooth integral Wu class from def. 3.3.1, and where $2\mathbf{DD} : \mathbf{B}^nU(1) \to \mathbf{B}^nU(1)$ is the canonical smooth refinement of the operation of multiplication by 2 on integral cohomology. We call this the smooth moduli n-stack of *smooth differential Wu structures*.

By construction, a morphism $X \to \hat{\mathbf{Wu}}^{\nu_{n+1}}$ classifies also all possible orientation structures and smooth integral lifts of their Wu structures. In applications one typically wants to fix an integral Wu structure lifting a given Wu class. This is naturally formalized by the following construction.

Definition 3.3.3. For X an oriented manifold, and

$$\boldsymbol{\nu}_{n+1} : X \to \mathbf{B}\mathrm{Spin}^{\nu_{n+1}}$$

a given smooth Wu structure, def. 3.3.1, write $\mathbf{H}_{\boldsymbol{\nu}_{n+1}}(X, \mathbf{B}^n U(1)_{\mathrm{conn}})$ for the n-groupoid of cocycles whose underlying smooth integral Wu structure is $\boldsymbol{\nu}_{n+1}$, hence for the homotopy pullback

$$\begin{tikzcd}
\mathbf{H}_{\boldsymbol{\nu}_{n+1}}(X, \mathbf{B}^n U(1)_{\mathrm{conn}}) \arrow[r] \arrow[d] & \mathbf{H}(X, \hat{\mathbf{Wu}}^{\nu_{n+1}}) \arrow[d] \\
\mathbf{H}(X, \mathbf{B}^n U(1)) \arrow[r, "(\boldsymbol{\nu}_{n+1}, \mathrm{id})"] \arrow[d] & \mathbf{H}(X, \mathbf{B}\mathrm{Spin}^{\nu_{n+1}} \times \mathbf{B}^n U(1)) \arrow[d] \\
* \arrow[r, "\boldsymbol{\nu}_{n+1}"] & \mathbf{H}(X, \mathbf{B}\mathrm{Spin}^{\nu_{n+1}})
\end{tikzcd} .$$

Proposition 3.3.2. *Cohomology with coefficients in* $\hat{\mathbf{Wu}}^{\nu_{n+1}}$ *over a given smooth integral Wu structure coincides with the corresponding differential integral Wu structures:*

$$\hat{H}^{n+1}_{\nu_{n+1}}(X) \simeq H_{\boldsymbol{\nu}_{n+1}}(X, \mathbf{B}^n U(1)_{\mathrm{conn}}) \, .$$

Proof. Let $\check{C}(\mathcal{U})$ be the Čech-nerve of a good open cover $\mathcal{U}$ of X. By prop. 3.1.8 the canonical presentation of $\mathbf{B}^n U(1)_{\mathrm{conn}} \to \mathbf{B}^n U(1)$ is a projective fibration. Since $\check{C}(\mathcal{U})$ is projectively cofibrant (it is a projectively cofibrant replacement of X) and $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ is a simplicial model category, the morphism of Čech cocycle simplicial sets

$$[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\mathcal{U}), \mathbf{B}^n U(1)_{\mathrm{conn}}) \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\mathcal{U}), \mathbf{B}^n U(1))$$

is a Kan fibration. Hence, its homotopy pullback may be computed as the ordinary pullback of simplicial sets of this map. The claim then follows by inspection.

Explicitly, in this presentation a cocycle in the pullback is a pair $\{a, \hat{G}\}$ of a cocycle a for a circle n-bundle and a Deligne cocycle $\hat{G}$ with underlying bare cocycle G, such that there is an equality of degree-n Čech $U(1)$-cocycles

$$G = \boldsymbol{\nu}_{n+1} + 2a \, .$$

A gauge transformation between two such cocycles is a pair of Čech cochains $\{\hat{\gamma}, \alpha\}$ such that $\gamma = 2\alpha$ (the cocycle $\boldsymbol{\nu}_{n+1}$ being held fixed). This means that the gauge transformations acting on a given $\hat{G}$ solving the above constraint are precisely all the Deligne cochains, but multiplied by 2. This is again the explicit description of $\hat{H}_{\nu_{n+1}}(X)$ from remark 3.3.1. □

4 The C-field

In this section we describe our model for the C-field, first for bulk fields, and then for fields in the presence of boundaries and/or M5-branes.

4.1 The moduli 3-stack of the C-field

As we have reviewed above in section 3.2, the flux quantization condition for the C-field derived in [Wi97] is the equation

$$[G_4] = \tfrac{1}{2}p_1 \bmod 2 \quad \text{in} \quad H^4(X, \mathbb{Z}) \tag{4}$$

in integral cohomology, where $[G_4]$ is the cohomology class of the C-field itself, and $\frac{1}{2}p_1$ is the first fractional Pontrjagin class of the Spin manifold X. One can equivalently rewrite (4) as

$$[G_4] = \tfrac{1}{2}p_1 + 2a \quad \text{in} \quad H^4(X, \mathbb{Z}) \, , \tag{5}$$

where a is some degree 4 integral cohomology class on X. By the discussion in section 3.3, the correct formalization of this for *fixed* Spin structure [10] is to regard the gauge equivalence class of the C-field as a differential integral Wu class relative to the integral Wu class $\nu_4^{\mathrm{int}} = \frac{1}{2}p_1$, example 3.3.1, of that Spin structure. By prop. 3.3.2 and prop. 3.1.7, the natural refinement of this to a smooth moduli 3-stack of C-field configurations and arbitrary spin connections is the homotopy pullback of smooth 3-stacks

$$\begin{tikzcd} \hat{\mathbf{Wu}}^{\nu_{n+1}} \arrow[r] \arrow[d] & \mathbf{B}^3 U(1)_{\mathrm{conn}} \arrow[d] \\ \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}^3 U(1) \arrow[r, "\frac{1}{2}\hat{\mathbf{p}}_1 + 2\mathbf{DD}"] & \mathbf{B}^3 U(1) \end{tikzcd} .$$

Here the moduli stack in the bottom left is that of the field of gravity (spin connections) together with an auxiliary circle 3-bundle / 2-gerbe. Following the arguments in [FSaSc] (the traditional ones as well as the new ones presented there), we take this auxiliary circle 3-bundle to be the Chern-Simons circle 3-bundle of an E_8-principal bundle. According to prop. 3.1.8 this is formalized on smooth higher moduli stacks by further pulling back along the smooth refinement

$$\mathbf{a} : \mathbf{B}E_8 \to \mathbf{B}^3 U(1)$$

of the canonical universal 4-class $[a] \in H^4(BE_8, \mathbb{Z})$. Therefore, we are led to formalize the *E_8-model for the C-field* as follows.

Definition 4.1.1. The *smooth moduli 3-stack of Spin connections and C-field configurations* in the E_8-model is the homotopy pullback **CField** of the moduli n-stack of smooth differential Wu structures $\mathbf{B}^n U(1)^{\nu_4}_{\mathrm{conn}}$, def. 3.3.2, to Spin connections and E_8-instanton configurations, hence the homotopy pullback

$$\begin{tikzcd} \mathbf{CField} \arrow[r] \arrow[d] & \hat{\mathbf{Wu}}^{\nu_4} \arrow[d] \\ \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 \arrow[r, "(u, \mathbf{a})"] & \mathbf{B}\mathrm{Spin}^{\nu_4} \times \mathbf{B}^3 U(1) \end{tikzcd} ,$$

where u is the canonical morphism from example 3.3.1.

Remark 4.1.1. By the pasting law, prop. 3.1.1, **CField** is equivalently given as the homotopy pullback

$$\begin{tikzcd} \mathbf{CField} \arrow[r, "\hat{\mathbf{G}}_4"] \arrow[d] & \mathbf{B}^3 U(1)_{\mathrm{conn}} \arrow[d] \\ \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 \arrow[r, "\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}"] & \mathbf{B}^3 U(1) \end{tikzcd} .$$

Spelling out this definition, a C-field configuration

$$(\nabla_{\mathfrak{so}}, \nabla_{b^2\mathbb{R}}, P_{E_8}) : X \to \mathbf{CField}$$

on a smooth manifold X is the datum of

1. *a principal* Spin*-bundle with* $\mathfrak{so}$*-connection* $(P_{\mathrm{Spin}}, \nabla_{\mathfrak{so}})$ *on* X*;*
2. *a principal* E_8*-bundle* P_{E_8} *on* X*;*
3. *a* $U(1)$*-2-gerbe with connection* $(P_{\mathbf{B}^2 U(1)}, \nabla_{\mathbf{B}^2 U(1)})$ *on* X*;*

[10] The dependence of the partition function of the C-field on the Spin structure(s) is discussed in [Sa12a].

4. *a choice of equivalence of $U(1)$-2-gerbes between between $P_{\mathbf{B}^2 U(1)}$ and the image of $P_{\mathrm{Spin}} \times_X P_{E_8}$ via* $\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}$.

It is useful to observe that there is the following further equivalent reformulation of this definition.

Proposition 4.1.1. *The moduli 3-stack* **CField** *from def. 4.1.1 is equivalently the homotopy pullback*

$$\begin{tikzcd}
\mathbf{CField} \arrow[rr] \arrow[d] & & \Omega^4_{\mathrm{cl}} \arrow[d] \\
\mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 \arrow[rr, "(\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a})_{\mathrm{dR}}"] & & \flat_{\mathrm{dR}}\mathbf{B}^4\mathbb{R}
\end{tikzcd} ,$$

where, by remark. 3.1.4, the bottom morphism of higher stacks is presented by the correspondence of simplicial presheaves

$$\begin{tikzcd}
\mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times (\mathbf{B}E_8)_{\mathrm{diff}} \arrow[r] \arrow[d, "\wr"] & \mathbf{B}\mathrm{Spin}_{\mathrm{diff}} \times (\mathbf{B}E_8)_{\mathrm{diff}} \arrow[r, "(\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a})_{\mathrm{diff}}"] \arrow[d, "\wr"] & \mathbf{B}^3 U(1)_{\mathrm{diff}} \arrow[r, "\mathrm{curv}"] \arrow[d, "\wr"] & \flat_{\mathrm{dR}}\mathbf{B}^4\mathbb{R} \\
\mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 \arrow[r] & \mathbf{B}\mathrm{Spin} \times \mathbf{B}E_8 \arrow[r, "\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}"] & \mathbf{B}^3 U(1) &
\end{tikzcd} .$$

Moreover, it is equivalently the homtopy pullback

$$\begin{tikzcd}
\mathbf{CField} \arrow[rr] \arrow[d] & & \Omega^4_{\mathrm{cl}} \arrow[d] \\
\mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 \arrow[rr, "(\frac{1}{4}\mathbf{p}_1 + \mathbf{a})_{\mathrm{dR}}"] & & \flat_{\mathrm{dR}}\mathbf{B}^4\mathbb{R}
\end{tikzcd} ,$$

where now the bottom morphism is the composite of the bottom morphism before, postcomposed with the morphism

$$\tfrac{1}{2} : \flat_{\mathrm{dR}}\mathbf{B}^4\mathbb{R} \to \flat_{\mathrm{dR}}\mathbf{B}^4\mathbb{R}$$

that is given, via Dold-Kan, by division of differential forms by 2.

Proof. By the pasting law for homotopy pullbacks, prop. 3.1.1, the first homotopy pullback above may be computed as two consecutive homotopy pullbacks

$$\begin{tikzcd}
\mathbf{CField} \arrow[r] \arrow[d] & \mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[r] \arrow[d] & \Omega^4_{\mathrm{cl}} \arrow[d] \\
\mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 \arrow[r, "\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}"] & \mathbf{B}^3 U(1) \arrow[r, "\mathrm{curv}"] & \flat_{\mathrm{dR}}\mathbf{B}^4\mathbb{R}
\end{tikzcd} ,$$

which exhibits on the right the defining pullback of def. 3.1.8, and thus on the left the one from def. 4.1.1. The statement about the second homotopy pullback above follows analogously after noticing that

$$\begin{tikzcd}
\Omega^4_{\mathrm{cl}} \arrow[r, "\frac{1}{2}"] \arrow[d] & \Omega^4_{\mathrm{cl}} \arrow[d] \\
\flat_{\mathrm{dR}}\mathbf{B}^4\mathbb{R} \arrow[r, "\frac{1}{2}"] & \flat_{\mathrm{dR}}\mathbf{B}^4\mathbb{R}
\end{tikzcd}$$

is a homotopy pullback. □

It is therefore useful to introduce labels as follows.

Definition 4.1.2. We label the structure morphism of the above composite homotopy pullback as

$$
\begin{tikzcd}
\mathbf{CField} \arrow[r, "\hat{G}_4"] \arrow[d] & \mathbf{B}^3U(1)_{\mathrm{conn}} \arrow[r, "\mathcal{G}_4"] \arrow[d, "G_4"] \arrow[dl, Rightarrow, "H_3"', "\simeq"] & \Omega^4_{\mathrm{cl}} \arrow[d] \arrow[dl, Rightarrow, "\simeq"] \\
\mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 \arrow[r, "\frac{1}{2}\mathbf{p}_2+2\mathbf{a}"'] & \mathbf{B}^3U(1) \arrow[r, "\mathrm{curv}"'] & \flat_{\mathrm{dR}}\mathbf{B}^4U(1)
\end{tikzcd}
\quad .
$$

Here $\hat{G}_4$ sends a C-field configuration to an underlying circle 3-bundle with connection, whose curvature 4-form is $\mathcal{G}_4$.

Remark 4.1.2. These equivalent reformulations point to two statements.

1. The C-field model may be thought of as containing E_8-*pseudo-connections*, remark 3.1.4. That is, there is a higher gauge in which a field configuration consists of an E_8-connection on an E_8-bundle – even though there is no dynamical E_8-gauge field in 11d supergravity – but where gauge transformations are allowed to freely shift these connections.

2. There is a precise sense in which imposing the quantization condition (5) on integral cohomology is equivalent to imposing the condition $[G_4] = \frac{1}{4}p_1 + a$ in de Rham cohomology / real singular cohomology.

Observation 4.1.1. When restricted to a fixed Spin connection, gauge equivalence classes of configurations classified by **CField** naturally form a torsor over the ordinary degree-4 differential cohomology $H^4_{\mathrm{diff}}(X)$.

Proof. By the general discussion of differential integral Wu-structures in section 3.3. $\square$

We now comment on the relation to the proposal in [DFM].

Remark 4.1.3. The first item in remark 4.1.2 finds its correspondence in equation (3.13) in [DFM], where a definition of gauge transformation of the C-field is proposed. The second item finds its correspondence in equation (3.26) there, where another model for the groupoid of C-field configurations is proposed. However, the immediate translation of equation (3.25) used there, in the language of homotopy pullbacks is given by the homotopy limit over the diagram

$$
\begin{tikzcd}
 & & & * \arrow[d, "0"] \\
* \arrow[r, "\frac{1}{2}\hat{\mathbf{p}}_1(g)"] \arrow[rrr, bend right, "\hat{W}_5"'] & \mathbf{H}(X, \mathbf{B}^3U(1)_{\mathrm{conn}}) \arrow[r, "\mathrm{curv}"] & \Omega^4_{\mathrm{cl}}(X) \arrow[r] & \mathbf{H}(X, \mathbf{B}^4U(1)_{\mathrm{conn}})
\end{tikzcd}
\quad .
$$

On gauge equivalence classes this becomes a torsor over $H^4_{\mathrm{diff}}(X)$. So, by prop. 4.1.1, for a fixed Spin connection this is equivalent to the model that we present here (which is naturally equivalent to the group of differential integral Wu structures), since any two torsors over a given group are equivalent. However, the equivalence is non-canonical, in general. More precisely, for structures parameterized over spaces as here, the equivalence is in general non-natural, in the technical sense.

4.2 The homotopy type of the moduli 3-stack

We discuss now the homotopy type of the the 3-groupoid

$$
\mathbf{CField}(X) := \mathbf{H}(X, \mathbf{CField})
$$

of C-field configurations over a given spacetime manifold X. In terms of gauge theory, its 0-th homotopy group is the set of *gauge equivalence classes* of field configurations, its first homotopy group is the set of *gauge-of-gauge equivalence classes* of auto-gauge transformations of a given configuration, and so on.

Definition 4.2.1. For X a smooth manifold, let

$$\begin{tikzcd} & \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \arrow[d] \\ X \arrow[ur, "\nabla_{\mathfrak{so}}"] \arrow[r, "P_{\mathrm{Spin}}"'] & \mathbf{B}\mathrm{Spin} \end{tikzcd}$$

be a fixed Spin structure with fixed Spin connection. The restriction of $\mathbf{CField}(X)$ to this fixed Spin connection is the homotopy pullback

$$\begin{tikzcd} \mathbf{CField}(X)_{P_{\mathrm{Spin}}} \arrow[r] \arrow[d] & \mathbf{CField}(X) \arrow[d] \\ \mathbf{H}(X, \mathbf{B}E_8) \arrow[r, "((P_{\mathrm{Spin}}, \nabla_{\mathfrak{so}}), \mathrm{id})"] & \mathbf{H}(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8) \end{tikzcd} .$$

Proposition 4.2.1. *The gauge equivalence classes of* $\mathbf{CField}(X)_{P_{\mathrm{Spin}}}$ *naturally surjects onto the differential integral Wu structures on* X*, relative to* $\frac{1}{2}p_1(P_{\mathrm{Spin}}) \bmod 2$*, (example 3.3.1):*

$$\pi_0 \mathbf{CField}(X)_{P_{\mathrm{Spin}}} \twoheadrightarrow \hat{H}^{n+1}_{\frac{1}{2}p_1(P_{\mathrm{Spin}})}(X) \; .$$

The gauge-of-gauge equivalence classes of the auto-gauge transformation of the trivial C-field configuration naturally surject onto the singular cohomology $H^2_{\mathrm{sing}}(X, U(1))$ *(see example 3.1.3):*

$$\pi_1 \mathbf{CField}(X)_{P_{\mathrm{Spin}}} \twoheadrightarrow H^2_{\mathrm{sing}}(X, U(1)) \; .$$

Proof. By def. 4.1.1 and the pasting law, prop. 3.1.1, we have a pasting diagram of homotopy pullbacks of the form

$$\begin{tikzcd} \mathbf{CField}(X)_{P_{\mathrm{Spin}}} \arrow[r, two heads] \arrow[d] & \mathbf{H}_{\frac{1}{2}\mathbf{p}_1(P_{\mathrm{Spin}})}(X, \mathbf{B}^n U(1)_{\mathrm{conn}}) \arrow[rr] \arrow[d] & & \mathbf{H}(X, \hat{\mathbf{Wu}}^{\nu_4}) \arrow[d] \\ \mathbf{H}(X, \mathbf{B}E_8) \arrow[r, two heads, "\mathbf{H}(X, \mathbf{a})"] & \mathbf{H}(X, \mathbf{B}^3 U(1)) \arrow[r, "(\nabla_{\mathfrak{so}}, \mathrm{id})"] & \mathbf{H}(X, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}^3 U(1)) \arrow[r, "(u, \mathrm{id})"] & \mathbf{H}(X, \mathbf{B}\mathrm{Spin}^{\nu_4} \times \mathbf{B}^3 U(1)) \end{tikzcd} ,$$

where in the middle of the top row we identified, by def. 3.3.3, the n-groupoid of smooth differential Wu structures lifting the smooth Wu structure $\frac{1}{2}\mathbf{p}_1(P_{\mathrm{Spin}})$. Due to prop. 3.3.2 we are, therefore, reduced to showing that the top left morphism is surjective on π_0. But the bottom left morphism is surjective on π_0, by prop. 3.1.9. Now, the morphisms surjective on π_0 are precisely the *effective epimorphisms* in ∞Grpd, and these are stable under pullback. Hence the first claim follows.

For the second, we use that

$$\pi_1 \mathbf{CField}(X)_{P_{\mathrm{Spin}}} \simeq \pi_0 \Omega \mathbf{CField}(X)_{P_{\mathrm{Spin}}}$$

and that forming loop space objects (being itself a homotopy pullback) commutes with homotopy pullbacks and with taking cocycles with coefficients in higher stacks, $\mathbf{H}(X, -)$. Therefore, the image of the left square in the above under Ω is the homotopy pullback

$$\begin{tikzcd} \Omega \mathbf{CField}(X)_{P_{\mathrm{Spin}}} \arrow[r, two heads] \arrow[d] & \mathbf{H}_{\frac{1}{2}\mathbf{p}_1(P_{\mathrm{Spin}})}(X, \hat{\mathbf{Wu}}^{\nu_4}) \arrow[d] \\ C^\infty(X, E_8) \arrow[r, two heads, "\mathbf{H}(X, \Omega \mathbf{a})"] & \mathbf{H}(X, \mathbf{B}^2 U(1)) \end{tikzcd} ,$$

where in the bottom left corner we used

$$\begin{aligned}\Omega \mathbf{H}(X, \mathbf{B}E_8) &\simeq \mathbf{H}(X, \Omega \mathbf{B}E_8)\\ &\simeq \mathbf{H}(X, E_8)\\ &\simeq C^\infty(X, E_8)\ ,\end{aligned}$$

and similarly for the bottom right corner. This identifies the bottom morphism on connected components as the morphism that sends a smooth function $X \to E_8$ to its homotopy class under the homotopy equivalence $E_8 \simeq_{15} B^2U(1) \simeq K(\mathbb{Z}, 3)$, which holds over the 11-dimensional X. Therefore the bottom morphism is again surjective on π_0, and so is the top morphism. The claim then follows with prop. 3.3.1. □

4.3 The boundary moduli

We now consider the moduli 3-stack of the C-field in the presence of a boundary (possibly with more than one component). We will consider two variants, corresponding to two different boundary conditions on the C-fiels: The first, $\mathbf{CField}^{\mathrm{bdr}}$ corresponds to the case when the field strength G_4 of the C-field vanishes on the boundary as a differential cocycle. The second, $\mathbf{CField}^{\mathrm{bdr}'}$, corresponds to the case when G_4 is zero as a cohomology class. Extensive discussion of boundary conditions can be found in [Sa12b]. See also the discussion in the companion article [FSaSc].

Let ∂X be (a neighbourhood of) the boundary of the spacetime manifold X. The condition on the boundary configurations of the supergravity fields are

1. The C-field vanishes on the boundary, as a differential cocycle (in the Hořava-Witten model [HW] this follows by arguments as recalled for instance in section 3.1 of [Fal]) or as a cohomology class;
2. the E_8 bundle becomes equipped with a connection over the boundary, and hence becomes dynamical there.

We present now a natural morphism of 3-stacks

$$\mathbf{CField}^{\mathrm{bdr}} \to \mathbf{CField}$$

into the moduli stack of bulk C-fields, def. 4.1.1, such that C-field configurations on X with the above mentioned behavior (with the strict condition $\hat{G}_4 = 0$) over ∂X correspond to the *relative twisted cohomology*, def. 3.1.10, with coefficients in this morphism, i.e., to commuting diagrams of the form

$$\begin{tikzcd} \partial X \arrow[r, "\phi_{\mathrm{bdr}}"] \arrow[d, hook] & \mathbf{CField}^{\mathrm{bdr}} \arrow[d] \\ X \arrow[r, "\phi"] & \mathbf{CField} \end{tikzcd} .$$

Definition 4.3.1. Let

$$i : \mathbf{B}(\mathrm{Spin} \times E_8)_{\mathrm{conn}} \to \mathbf{CField}$$

be the canonical morphism induced from the commuting diagram of def. 3.1.9, for the differential characteristic maps prop. 3.1.7 and prop. 3.1.8 and the universal property of the homotopy pullback defining **CField**:

$$\begin{tikzcd} \mathbf{B}(\mathrm{Spin} \times E_8)_{\mathrm{conn}} \arrow[dr, dashed, "i"] \arrow[drrr, bend left, "\frac{1}{2}\hat{\mathbf{p}}_1 + 2\hat{\mathbf{a}}"] \arrow[ddr] & & & \\ & \mathbf{CField} \arrow[rr, "\hat{\mathbf{G}}_4"] \arrow[d] & & \mathbf{B}^3U(1)_{\mathrm{conn}} \arrow[d] \\ & \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 \arrow[r] & \mathbf{B}\mathrm{Spin} \times \mathbf{B}E_8 \arrow[r, "\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}"] & \mathbf{B}^3U(1) \end{tikzcd} .$$

Remark 4.3.1. The dashed morphism i implements the condition that the E_8-bundle picks up a connection – a dynamical gauge field – on the boundary. Therefore, to naturally define $\mathbf{CField}^{\mathrm{bdr}}$ it only remains to model the condition that $\hat{G}_4$ vanishes on the boundary, either as a differential cocycle or in the underlying integral cohomology. These requirements are immediately realized by further pulling back along the sequence of morphisms $* \xrightarrow{0} \Omega^{1\leq \bullet \leq 3} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$.

Definition 4.3.2. Let the moduli 3-stacks $\mathbf{CField}^{\mathrm{bdr}}$ and $\mathbf{CField}^{\mathrm{bdr}'}$ be defined as the consecutive homotopy pullbacks in this diagram

$$\begin{tikzcd}
\mathbf{CField}^{\mathrm{bdr}} \arrow[d] \arrow[rrr, bend left] & & & * \arrow[d] \\
\mathbf{CField}^{\mathrm{bdr}'} \arrow[d] \arrow[rrr, bend left] & & & \Omega^{1\leq \bullet \leq 3} \arrow[d] \\
\mathbf{B}(\mathrm{Spin} \times E_8)_{\mathrm{conn}} \arrow[drrr, bend left, "\frac{1}{2}\hat{\mathbf{p}}_1 + 2\hat{\mathbf{a}}"] \arrow[dr, dashed] \arrow[ddr] & & & \\
 & \mathbf{CField} \arrow[rr, "\hat{\mathbf{G}}_4"] \arrow[d] & & \mathbf{B}^3 U(1)_{\mathrm{conn}} \arrow[d] \\
 & \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8 \arrow[r] & \mathbf{B}\mathrm{Spin} \times \mathbf{B}E_8 \arrow[r, "\frac{1}{2}\mathbf{p}_1 + 2\mathbf{a}"] & \mathbf{B}^3 U(1)
\end{tikzcd} .$$

A straightforward application of the pasting law, prop. 3.1.1 and inspection of the definitions then gives

Proposition 4.3.1. *We have natural equivalences*

$$\mathbf{CField}^{\mathrm{bdr}} \simeq \mathrm{String}^{-2\hat{\mathbf{a}}}_{\mathrm{conn}} ,$$

and

$$\mathbf{CField}^{\mathrm{bdr}'} \simeq \mathrm{String}^{-2\hat{\mathbf{a}}}_{\mathrm{conn}'},$$

of the moduli 3-stack of boundary C-field configurations, with that of (nonabelian) $\mathrm{String}^{\mathbf{2a}}$ *2-connections, strict or weak, respectively, according to [FSaSc].*

Remark 4.3.2. In [FScSt] we have given a detailed construction of these 2-stacks in terms of explicit differential form data. In [SSS09b] we have shown that these are the moduli 2-stacks for heterotic background fields that satisfy the Green-Schwarz anomaly cancellation condition.

4.4 Hořava-Witten boundaries and higher orientifolds

We now discuss a natural formulation of the origin of the Hořava-Witten boundary conditions [HW] in terms of higher stacks and nonabelian differential cohomology, specifically, in terms of what we call *membrane orientifolds*. From this we obtain a corresponding refinement of the moduli 3-stack of C-field configurations which now explicitly contains the twisted $\mathbb{Z}_2$-equivariance of the Hořava-Witten background.

Earlier, around prop. 3.1.5 and prop. 3.1.6, we invoked the Dold-Kan correspondence in order to construct a higher stack from a chain complex of sheaves of abelian groups. Now, in order to add a $\mathbb{Z}_2$-twist to ordinary differential cohomology, we invoke the following nonabelian generalization of the Dold-Kan correspondence. The discrete ingredients for that construction are discussed in some detail in [BHS]. As a presentation of smooth higher stacks this is discussed in section 2.1.7 of [Sch] and the concrete application to higher orientifolds is in 4.4.3 there.

Definition 4.4.1. A *crossed complex of groups* $G_\bullet^\rho$ is a complex of groups of the form

$$\cdots \xrightarrow{\delta_2} G_2 \xrightarrow{\delta_1} G_1 \xrightarrow{\delta_0} G_0$$

with $G_{k\geq 2}$ abelian (but G_1 and G_0 not necessarily abelian), together with an action ρ_k of G_0 on G_k for all $k \in \mathbb{N}$, such that

1. ρ_0 is the adjoint action of G_0 on itself;
2. $\rho_1 \circ \delta_0$ is the adjoint action of G_1 on itself;
3. $\rho_k \circ \delta_0$ is the trivial action of G_1 on G_k for $k > 1$;
4. all δ_k respect the actions of G_0.

A morphism of crossed complexes of groups $G_\bullet^\rho \to H_\bullet^{\rho'}$ is a sequence of morphisms of component groups, respecting all this structure. Write CrossedComplex for the category of crossed complexes defined this way.

Remark 4.4.1. If we write ChainComplex for the category of ordinary chain complexes of abelian groups in non-negative degree, and KanComplex $\hookrightarrow$ sSet for the category of Kan complexes, then we have a diagram of (non-full) injections

$$\begin{tikzcd} \mathrm{ChainComplex} \arrow[r] \arrow[d, "\simeq"] & \mathrm{CrossedComplex} \arrow[r] \arrow[d, "\simeq"] & \mathrm{KanComplex} \arrow[d, "\simeq"] \\ \mathrm{StrAbStr}\infty\mathrm{Grpd} \arrow[r] & \mathrm{Str}\infty\mathrm{Grpd} \arrow[r] & \infty\mathrm{Grpd} \end{tikzcd} ,$$

where in the top row we display models, and in the bottom row the corresponding abstract notions. This immediately prolongs to presheaves of complexes. Therefore every presheaf of crossed complexes $G_\bullet^\rho$ over CartSp presents a smooth ∞-stack $\mathbf{B}(G_\bullet)$ in a way that restricts to the ordinary Dold-Kan correspondence in the case that the crossed complex is just an ordinary chain complex of abelian groups.

Examples 4.4.1. The String 2-group from prop. 4.3.1 has a presentation by the crossed complex [11]

$$(G_1 \to G_0) := (\hat{\Omega}\mathrm{Spin} \stackrel{\delta}{\to} P_*\mathrm{Spin})^\rho ,$$

where $P_*\mathrm{Spin}$ is the Fréchet Lie group of based smooth paths in the Lie group Spin, where $\hat{\Omega}\mathrm{Spin}$ is the Kac-Moody central extension of the group of smooth based loops in Spin, where the morphism δ is the evident forgetful map, and where, finally, the action ρ_1 is given by a lift of the canonical conjugation action of paths on loops:

$$\mathbf{B}\mathrm{String} \simeq \mathbf{B}(\hat{\Omega}\mathrm{Spin} \to P_*\mathrm{Spin})^\rho .$$

We are now interested in the following much simpler class of examples.

Example 4.4.1. For every $n \in \mathbb{N}$, $n \geq 1$ there is a crossed complex of groups of the form

$$(U(1) \to 0 \to \cdots \to 0 \to \mathbb{Z}_2)^\rho ,$$

with $U(1)$ in degree n, with all morphisms trivial and with $\mathbb{Z}_2$ acting in the canonical way on $U(1)$ via the identification $\mathrm{Aut}(U(1)) \simeq \mathbb{Z}_2$. With a slight abuse of notation we will write

$$\mathbf{B}^{n+1}U(1)//\mathbb{Z}_2 := \mathbf{B}(U(1) \to 0 \to \cdots \to 0 \to \mathbb{Z}_2)^\rho \quad \in \mathbf{H}$$

for the corresponding moduli $(n+1)$-stacks.

[11] See the appendix of [FSaSc] for a more detailed review of this and related results.

Remark 4.4.2. The 2-stack $\mathbf{B}^2U(1)//\mathbb{Z}_2$ is that of $U(1)$-*gerbes* in the original sense of Giraud. The canonical morphism of moduli 2-stacks

$$\mathbf{B}^2U(1) \to \mathbf{B}^2U(1)//\mathbb{Z}_2$$

embeds the less general $U(1)$-*bundle gerbes* into the genuine $U(1)$-gerbes. This distinction is often not recognized in the literature, but in the following it makes all the difference. See also observation 4.4.1 further below.

Definition 4.4.2. Define on $\mathbf{B}^nU(1)//\mathbb{Z}_2$ the universal smooth characteristic map

$$\mathbf{J}_{n-1} : \mathbf{B}^nU(1)//\mathbb{Z}_2 \to \mathbf{B}\mathbb{Z}_2$$

representing a universal class

$$[J] \in H^1(B^nU(1)//\mathbb{Z}_2, \mathbb{Z}_2)\,,$$

defined by the delooping of the evident morphism of crossed complexes

$$(U(1) \to \cdots \to \mathbb{Z}_2)^\rho \to \mathbb{Z}_2\,.$$

Proposition 4.4.1. *For all $n \geq 2$ there is a fiber sequence of smooth higher stacks*

$$\mathbf{B}^nU(1) \longrightarrow \mathbf{B}^nU(1)//\mathbb{Z}_2 \xrightarrow{\mathbf{J}_{n-1}} \mathbf{B}\mathbb{Z}_2\,.$$

Proof. The canonical presentation of the morphism on the right by a morphism of simplicial presheaves is evidently a projective fibration. The claim then follows from the fact that $U(1)[n]$ is the fiber of the canonical morphism of crossed complexes $(U(1) \to \cdots \to \mathbb{Z}_2)^\rho \to (\cdots \to 0 \to \mathbb{Z}_2)$. □

Remark 4.4.3. This means that the morphism $\mathbf{B}^nU(1) \to \mathbf{B}^nU(1)//\mathbb{Z}_2$ exhibits a universal double cover / universal $\mathbb{Z}_2$-principal bundle over the n-stack $\mathbf{B}^nU(1)//\mathbb{Z}_2$.

Corollary 4.4.1. *For $X \in \mathbf{H}$ any smooth space, a cocycle $g : X \to \mathbf{B}^nU(1)//\mathbb{Z}_2$ induces*

1. *a choice of double cover $\hat{X} \to X$,*
2. *a circle n-bundle P over $\hat{X}$ equipped with a $\mathbb{Z}_2$-twisted equivariance under the canonical $\mathbb{Z}_2$-action on $\hat{X}$, such that*
3. *the restriction of P to any fiber $\hat{X}_x$ of $\hat{X}$ is equivalent to the n-group extension $(U(1) \to \cdots \to \mathbb{Z}_2)^\rho \to \mathbb{Z}_2$.*

Proof. Consider the induced pasting diagram of homotopy pullbacks, using prop. 3.1.1

$$\begin{tikzcd}
(U(1) \to \cdots \to \mathbb{Z}_2)^\rho \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & * \arrow[d] & \\
\mathbb{Z}_2 \arrow[r] \arrow[d] & \hat{X} \arrow[r] \arrow[d] & \mathbf{B}^nU(1) \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r, "x"] & X \arrow[r, "g"] & \mathbf{B}^nU(1)//\mathbb{Z}_2 \arrow[r, "\mathbf{J}_{n-1}"] & \mathbf{B}\mathbb{Z}_2
\end{tikzcd}$$

□

Observation 4.4.1. For $n = 2$ the bundle gerbe incarnation of the structures in corollary 4.4.1 have been called *Jandl bundle gerbes* [SSW] and shown to encode *orientifold* backgrounds for strings.

We will discuss the analog for membranes. First we consider the differential refinement of this situation. For this we need the following refinement of def. 4.4.1.

Definition 4.4.3. A *crossed complex of groupoids* is a diagram

$$C_\bullet = \left(\begin{array}{ccccccccc} \cdots \xrightarrow{\delta} & C_3 & \xrightarrow{\delta} & C_2 & \xrightarrow{\delta} & C_1 & \overset{\delta_t}{\underset{\delta_s}{\rightrightarrows}} & C_0 \\ & \downarrow & & \downarrow & & \downarrow{\scriptstyle \delta_s} & & \| \\ \cdots = & C_0 & = & C_0 & = & C_0 & = & C_0 \end{array} \right),$$

where $C_1 \overset{\delta_t}{\underset{\delta_s}{\rightrightarrows}} C_0$ is equipped with the structure of a groupoid, and where $C_k \to C_0$, for all $k \geq 2$, are bundles of groups, abelian for $k \geq 2$; and equipped with a groupoid action ρ_k of C_1, such that

1. the maps δ_k, $k \geq 2$ are morphisms of groupoids over C_0 compatible with the action by C_1;
2. $\delta_{k-1} \circ \delta_k = 0$; k ≥ 3;
3. ρ_1 is the conjugation action of the groupoid on its first homotopy groups;
4. $\rho_2 \circ \delta_2$ is the conjugation action of C_2 on itself;
5. $\rho_k \circ \delta_2$ is the trivial action of C_2 for $k \geq 3$.

A morphism of crossed complexes of groupoids is a sequence of morphisms of the components such that all this structure is preserved.

The nonabelian generalization of the Dold-Kan correspondence, reviewed in detail in [BHS], is now the following.

Proposition 4.4.2. *The category of crossed complexes of groupoids is equivalent to that of strict globular ∞-groupoids. Under the natural simplicial nerve operation these embed into Kan complexes.*

Example 4.4.2. For $G_\bullet^\rho$ a crossed complex of groups, def. 4.4.1, we obtain a crossed complex of groupoids of the form

$$\cdots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \rightrightarrows * \ .$$

If $G_\bullet^\rho$ is a presheaf of crossed complexes of groupoids presenting a smooth ∞-group to be denoted by the same symbols, then, under the nonabelian Dold-Kan correspondence this presheaf of crossed complexes of groupoids presents the smooth delooping ∞-stack $\mathbf{B}(G_\bullet^\rho)$.

The example that we are interested here is the following.

Definition 4.4.4. For $n \geq 2$ write $\mathbf{B}^n U(1)_{\mathrm{conn}} // \mathbb{Z}_2$ for the smooth n-stack presented by the presheaf of n-groupoids which is given by the presheaf of crossed complexes of groupoids

$$\Omega^n(-) \times C^\infty(-, U(1)) \xrightarrow{(\mathrm{id}, d_{\mathrm{dR}} \log)} \Omega^n(-) \times \Omega^1(-) \xrightarrow{(\mathrm{id}, d_{\mathrm{dR}})} \cdots \xrightarrow{(\mathrm{id}, d_{\mathrm{dR}})} \Omega^n(-) \times \Omega^{n-2}(-) \xrightarrow{(\mathrm{id}, d_{\mathrm{dR}})}$$

$$\xrightarrow{(\mathrm{id}, d_{\mathrm{dR}})} \Omega^n(-) \times \Omega^{n-1}(-) \times \mathbb{Z}_2 \rightrightarrows \Omega^n(-) \ ,$$

where

1. the groupoid on the right has as morphisms $(A, \sigma) : B \to B'$ between two n-forms B, B' pairs consisting of an $(n-1)$-form A and an element $\sigma \in \mathbb{Z}_2$, such that $(-1)^\sigma B' = B + dA$;
2. the bundles of groups on the left are all trivial as bundles;

3. the $\Omega^1(-) \times \mathbb{Z}_2$-action is by the $\mathbb{Z}_2$-factor only and on forms given by multiplication by ± 1 and on $U(1)$-valued functions by complex conjugation (regarding $U(1)$ as the unit circle in the complex plane).

Observation 4.4.2. There are evident morphisms of smooth n-stacks

$$\flat \mathbf{B}^n U(1)//\mathbb{Z}_2 \to \mathbf{B}^n U(1)_{\mathrm{conn}}//\mathbb{Z}_2 \to \mathbf{B}^n U(1)//\mathbb{Z}_2 \,,$$

where the first one includes the flat differential coefficients and the second one forgets the connection.

Remark 4.4.4. A detailed discussion of $\mathbf{B}^2 U(1)_{\mathrm{conn}}//\mathbb{Z}_2$ is in [SWI] and [SWII].

We now discuss differential cocycles with coefficients in $\mathbf{B}^n U(1)_{\mathrm{conn}}//\mathbb{Z}_2$ over $\mathbb{Z}_2$-quotient stacks / orbifolds. Let Y be a smooth manifold equipped with a smoth $\mathbb{Z}_2$-action ρ. Write $Y//\mathbb{Z}_2$ for the corresponding global orbifold and $\rho : Y//\mathbb{Z}_2 \to \mathbf{B}\mathbb{Z}_2$ for its classifying morphism, hence for the morphism that fits into a fiber sequence of smooth stacks

$$Y \longrightarrow Y//\mathbb{Z}_2 \longrightarrow \mathbf{B}\mathbb{Z}_2 \;.$$

Definition 4.4.5. An *n-orientifold* structure $\hat{G}_\rho$ on (Y, ρ) is a *ρ-twisted $\hat{\mathbf{J}}_n$-structure* on $Y//\mathbb{Z}_2$, def. 3.1.10, hence a dashed morphism in the diagram

$$\begin{tikzcd} & \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}//\mathbb{Z}_2 \arrow[d, "\hat{\mathbf{J}}_n"] \\ Y//\mathbb{Z}_2 \arrow[ur, dashed, "\hat{G}_\rho"] \arrow[r, "\rho"] & \mathbf{B}\mathbb{Z}_2 \end{tikzcd} \;.$$

Observation 4.4.3. By corollary 4.4.1, an n-orientifold structure decomposes into an ordinary $(n+1)$-form connection $\hat{G}$ on a circle $(n+1)$-bundle over Y, subject to a $\mathbb{Z}_2$-twisted $\mathbb{Z}_2$-equivariance condition

$$\begin{tikzcd} Y \arrow[r, "\hat{G}"] \arrow[d] & \mathbf{B}^{n+1}U(1)_{\mathrm{conn}} \arrow[r] \arrow[d] & * \arrow[d] \\ Y//\mathbb{Z}_2 \arrow[r, "\hat{G}_\rho"] \arrow[rr, bend right, "\rho"'] & \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}//\mathbb{Z}_2 \arrow[r, "\hat{\mathbf{J}}_n"] & \mathbf{B}\mathbb{Z}_2 \end{tikzcd} \;.$$

For $n = 1$ this reproduces, via observation 4.4.1, the *Jandl gerbes with connection* from [SSW], hence ordinary string orientifold backgrounds, as discussed there.

Observation 4.4.4. Let $U//\mathbb{Z}_2 \hookrightarrow Y//\mathbb{Z}_2$ be a patch on which a given $\hat{\mathbf{J}}_n$-structure has a trivial underlying integral class, such that it is equivalent to a globally defined $(n+1)$-form C_U on U. Then the components of this this 3-form orthogonal to the $\mathbb{Z}_2$-action are *odd* under the action. In particular, if $U \hookrightarrow Y$ sits in the fixed point set of the action, then these components vanish. This is the Hořava-Witten boundary condition on the C-field on an 11-dimensional spacetime $Y = X \times S^1$ equipped with $\mathbb{Z}_2$-action on the circle. See for instance section 3 of [Fal] for an explicit discussion of the $\mathbb{Z}_2$ action on the C-field in this context.

We therefore have a natural construction of the moduli 3-stack of Hořava-Witten C-field configurations as follows

Definition 4.4.6. Let $\mathbf{CField}_J(Y)$ be the homotopy pullback in

$$\begin{tikzcd} \mathbf{CField}_J(Y) \arrow[r] \arrow[dd] & \hat{\mathbf{J}}_2\mathrm{Struc}_\rho(Y//\mathbb{Z}_2) \arrow[d] \\ & \mathbf{H}(Y, \mathbf{B}^3U(1)_{\mathrm{conn}}) \arrow[d] \\ \mathbf{H}(Y, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8) \arrow[r, "\mathbf{H}(Y, \frac{1}{2}\mathbf{p}_1 + 2\mathbf{a})"] & \mathbf{H}(Y, \mathbf{B}^3U(1)) \end{tikzcd} \;,$$

where the top right morphism is the map $\hat{G}_\rho \mapsto \hat{G}$ from remark 4.4.3.

The objects of $\mathbf{CField}_J(Y)$ are C-field configurations on Y that not only satisfy the flux quantization condition, but also the Hořava-Witten twisted equivariance condition (in fact the proper globalization of that condition from 3-forms to full differential cocycles). This is formalized by the following.

Observation 4.4.5. There is a canonical morphism $\mathbf{CField}_J(Y) \to \mathbf{CField}(Y)$, being the dashed morphism in

$$\begin{tikzcd}
\mathbf{CField}_J(Y) \arrow[r] \arrow[d, dashed] & \hat{\mathbf{J}}_2\mathrm{Struc}_\rho(Y//\mathbb{Z}_2) \arrow[d] \\
\mathbf{CField}(Y) \arrow[r] \arrow[d] & \mathbf{H}(Y, \mathbf{B}^3 U(1)_{\mathrm{conn}}) \arrow[d] \\
\mathbf{H}(Y, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \times \mathbf{B}E_8) \arrow[r, "\mathbf{H}(Y, \frac{1}{2}\mathbf{p}_1 + 2\mathbf{a})"] & \mathbf{H}(Y, \mathbf{B}^3 U(1))
\end{tikzcd} ,$$

which is given by the universal property of the defining homotopy pullback of **CField**, remark 4.1.1.

A supergravity field configuration presented by a morphism $Y \to \mathbf{CField}$ into the moduli 3-stack of configurations that satisfy the flux quantization condition in addition satisfies the Hořava-Witten boundary condition if, as an element of $\mathbf{CField}(Y) := \mathbf{H}(Y, \mathbf{CField})$ it is in the image of $\mathbf{CField}_J(Y) \to \mathbf{CField}(Y)$. In fact, there may be several such pre-images. A choice of one is a choice of membrane orientifold structure.

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