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Ambiguities in Pauli-Villars regularization

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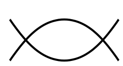
Abstract

We investigate regularization of scalar one-loop integrals in the Pauli-Villars subtraction scheme. The results depend on the number of subtractions, in particular the finite terms that survive after the divergences have been absorbed by renormalization. Therefore the process of Pauli-Villars regularization is ambiguous. We discuss how these ambiguities may be resolved by applying an asymptotically large number of subtractions, which results in a regularization that is automatically valid in any number of dimensions.

The regularization method of Pauli-Villars (PV) subtraction [1] is of long standing in quantum field theory. Although not suited to all possible problems (notably, nonabelian gauge theories), it is still used in a variety of applications [2, 3]. Essentially, PV regularization consists in pairing particle propagators with (possibly unphysical) propagators of fictitious heavy particles. In [4] these are introduced under the appelation of *unitary regulators*. If m is the mass of the physical (scalar) particle with momentum q^μ, and M that of a fictitious heavy particle, PV involves the modification of the propagator as follows:

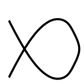
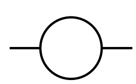
$$\frac{1}{q^2-m^2+i\epsilon} \quad \to \quad \frac{1}{q^2-m^2+i\epsilon}-\frac{1}{q^2-M^2+i\epsilon} \ , \tag{1}$$

thereby reducing the large-q behaviour from $(q^2)^{-1}$ to $(q^2)^{-2}$ and thus improving the integrability properties of diagrams. At the end of the calculation the limit $M \to \infty$ is implied. With this method, the one-loop diagram from a scalar self-interacting theory



(2)

becomes integrable in four dimensions. On the other hand, the two diagrams

 and  (3)

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are not integrable in four and six dimensions, respectively. In that case additional subtractions with a spectrum of fictitious particles are necessary.

It is tempting to perform, for a given diagram, precisely so many PV subtractions as are necessary to make the loop integral convergent: once for the diagram of Eq.(2), and twice for those of Eq.(3). But this, of course, runs counter to the idea of quantum field theory, in which Feynman diagrams themselves have no independent status but only their combination into Green's functions counts. If there is even only a single diagram that calls for a double subtraction, say, then *all* diagrams should undergo the same double subtraction, even if they are already convergent after a single one. An approach in which for each diagram precisely sufficient subytactions are made to make that diagram convergent must be considered incorrect.

This leads to the following question: how do we decide to stop making additional PV subtractions? A priori there is nothing that forbids one from making very many subtractions even if that is, strictly speaking, unnecessary.

In what follows, we show that the results of divergent one-loop integrals depend on the number of PV subtractions, and are therefore ambiguous, while loop integrals that are themselves convergent do not. We study this dependence, and point to a possible resolution of these ambiguities.

As remarked above, depending on circumstances, a single PV subtraction may not be sufficient to regularize loop integrals, and multiple subtractions become necessary. In what follows, we shall use the abbreviations $\mu = m^2$ and $\Lambda = M^2$. The k-fold PV subtraction of a propagator is defined as follows :

$$\begin{aligned} \frac{1}{s+\mu} \quad &\to \quad \left\lfloor \frac{1}{s+\mu} \right\rfloor_{PV(k)} \\ &= \quad \frac{1}{s+\mu} - \frac{\alpha_1}{s+\Lambda_1} - \frac{\alpha_2}{s+\Lambda_2} - \cdots - \frac{\alpha_k}{s+\Lambda_k} \\ &= \quad \frac{C}{(s+\mu)(s+\Lambda_1)(s+\Lambda_2)\cdots(s+\Lambda_k)} \quad . \end{aligned} \tag{4}$$

The requirement is that C be independent of s. By first considering the limit $s \to -\mu$ and then $s \to -\Lambda_r$ (assuming the Λ's to be all different) we find immediately that

$$C = \prod_{j=1}^{k}(\Lambda_j - \mu) \quad , \quad \alpha_r = \left(\prod_{j\neq r}(\Lambda_j - \mu)\right)\left(\prod_{j\neq r}(\Lambda_j - \Lambda_r)\right)^{-1} \quad . \tag{5}$$

By subtracting sufficiently many PV propagators we can achieve any desired high-momentum behaviour for the propagator.

That the above subtraction involves k different mass scales is of course lacking in elegance. We can therefore take $\Lambda_j \to \Lambda$, and then we obtain

$$\left\lfloor \frac{1}{s+\mu} \right\rfloor_{PV(k)} = \frac{\Delta^k}{(s+\mu)(s+\Lambda)^k} \quad , \tag{6}$$

with $\Delta = \Lambda - \mu$. This can also be written as

$$\left\lfloor \frac{1}{s+\mu} \right\rfloor_{PV(k)} = \frac{1}{s+\mu} - \frac{1}{s+\Lambda} - \frac{\Delta}{(s+\Lambda)^2} - \cdots - \frac{\Delta^{k-1}}{(s+\Lambda)^k} \quad . \tag{7}$$

A similar observation was made, for instance, in [5]. To understand the relation between Eq.(4) and Eq.(7) it is illustrative to consider the case $k = 2$. Taking $\Lambda_1 = \Lambda$, $\Lambda_2 = \Lambda + \delta$ with small δ, we see that

$$\begin{aligned} \left\lfloor \frac{1}{s+\mu} \right\rfloor_{PV(2)} &= \frac{1}{s+\mu} - \frac{\Lambda_2 - \mu}{\Lambda_2 - \Lambda_1} \frac{1}{s+\Lambda_1} - \frac{\Lambda_1 - \mu}{\Lambda_1 - \Lambda_2} \frac{1}{s+\Lambda_2} \\ &= \frac{1}{s+\mu} - \frac{\Delta + \delta}{\delta} \frac{1}{s+\Lambda} + \frac{\Delta}{\delta} \frac{1}{s+\Lambda+\delta} \\ &= \frac{1}{s+\mu} - \frac{1}{s+\Lambda} - \frac{\Delta}{(s+\Lambda)^2} + \mathcal{O}(\delta) \quad . \end{aligned} \tag{8}$$

For higher k analogous expansions result. Moreover the result is quite independent from the precise way in which the limit $\Lambda_j \to \Lambda$ ($j = 1, \ldots, n$) is reached. Note the following fact : if we keep subtracting without limit, we formally have

$$\begin{aligned} \lim_{k\to\infty} \left\lfloor \frac{1}{s+\mu} \right\rfloor_{PV(k)} &= \frac{1}{s+\mu} - \frac{1}{s+\Lambda} \sum_{j=0}^{\infty} \left(\frac{\Delta}{s+\Lambda} \right)^j \\ &= \frac{1}{s+\mu} - \frac{1}{s+\Lambda-\Delta} = 0 \quad . \end{aligned} \tag{9}$$

While, indeed, simply replacing every loop integral by zero makes all loop corrections trivial, this is clearly not what we want. Obviously, we must investigate the dependence of loop-integral results on the number of subtractions.

We shall restrict ourselves to one-loop computations in the context of an effective potential. Therefore no external momenta are involved, and all propagators have the form $1/(s+\mu)$, where $s = q^2$, q being the loop momentum. The (even) number of dimensions of spacetime dimensions is denoted by 2ω. After performing the Wick rotation and the angular integral of the loop momentum, the loop integral with n propagators is given by

$$J_{\omega,n} = \int_0^\infty ds\ s^{\omega-1}\ \frac{1}{(s+\mu)^n} \quad , \quad n > 0, \tag{10}$$

where we have dropped any overall factors. This integral is finite if $n > \omega$. If this is not the case, we must subtract, although strictly speaking we *ought* to be allowed to PV-subtract convergent integrals as well. We therefore should replace Eq.(10) by

$$J^{(k)}_{\omega,n} = \int_0^\infty ds\ s^{\omega-1}\ \left\lfloor \frac{1}{s+\mu} \right\rfloor^n_{PV(k)} \quad . \tag{11}$$

The integral will be convergent for finite Λ if

$$k > \frac{\omega}{n} - 1 \qquad \omega, n, k \ \text{ integers} \ . \tag{12}$$

We shall be slightly more general and define the generating function

$$\begin{aligned} H_{\omega,n}(x) &= \int\limits_0^\infty ds\ h_{\omega,n}(x,s) \ , \\ h_{\omega,n}(x,s) &= \frac{s^{\omega-1}}{(s+\mu)^n} \sum_{k\geq 1+\omega-n} \frac{\Delta^k}{(s+\Lambda)^k}\, x^k \\ &= \frac{s^{\omega-1}(x\Delta)^{1+\omega-n}}{(s+\mu)^n (s+\Lambda)^{\omega-n}(s+\Lambda-x\Delta)} \ . \end{aligned} \tag{13}$$

From the fact that all the functions h decrease as $1/s^2$ for large s we see that we have subtracted sufficiently often to make the integrals convergent for finite Λ (and hence PV-regularized). For higher values of k, we simply have additional subtractions. The integrals H have series expansions in x : the regularized integral $J^{(k)}_{\omega,n}$ is then given as the coefficient of x^{nk} in $H_{\omega,n}(x)$.

For given ω and n it is a simple matter to integrate $h_{\omega,n}(x,s)$ over s, where by construction the upper endpoint $s=\infty$ never contributes. The most important point is to note that

$$\log(\Lambda - x\Delta) \quad = \quad \log(\Lambda) + \log(1-x) + \frac{\mu}{\Lambda}\frac{x}{1-x} - \frac{1}{2}\left(\frac{\mu}{\Lambda}\frac{x}{1-x}\right)^2 + \cdots \tag{14}$$

It is already clear that at least some of the $\log(\Lambda)$ terms will be accompanied by $\log(1-x)$. Below, we give the results for $H_{\omega,n}(x)$ for the most relevant cases for 2,4, and 6 dimensions. By L we denote $\log(\Lambda/\mu)$. We have taken μ/Λ to zero wherever possible, although this has of course to be done more carefully in the case of two-loop computations.

$$\begin{aligned} H_{1,1}(x) &= \frac{x}{1-x}L + \frac{x\log(1-x)}{1-x} \ , \\ H_{1,2}(x) &= \frac{1}{1-x}\,\frac{1}{\mu} \ , \\ H_{2,1}(x) &= -x\log(1-x)\Lambda - \frac{x^2}{1-x}\mu L - \frac{x^2}{1-x}\big(\log(1-x)+1\big)\,\mu \ , \\ H_{2,2}(x) &= \frac{x}{1-x}L + \frac{x}{1-x}\big(\log(1-x)-1\big) \ , \\ H_{2,3}(x) &= \frac{1}{1-x}\,\frac{1}{2\mu} \ , \\ H_{3,1}(x) &= \big(x(1-x)\log(1-x)+x^2\big)\,\Lambda^2 + 2x^2\log(1-x)\,\mu\Lambda \\ &\quad + \frac{x^3}{1-x}\,\mu^2 L + \frac{x^3}{1-x}\left(\log(1-x)+\frac{3}{2}\right)\mu^2 \ , \end{aligned}$$

$$
\begin{aligned}
H_{3,2}(x) &= -x\log(1-x)\Lambda - \frac{2x^2}{1-x}\,\mu L - \frac{x(1+x)\log(1-x)}{1-x}\,\mu \quad , \\
H_{3,3}(x) &= \frac{x}{1-x}L + \frac{x}{1-x}\left(\log(1-x) - \frac{3}{2}\right) \quad , \\
H_{3,4}(x) &= \frac{1}{1-x}\,\frac{1}{3\mu} \quad .
\end{aligned} \tag{15}
$$

From these we can find the k-fold PV-regularized integrals $J_{\omega,n}^{(k)}$. These contain the digamma function $\psi(z)$, defined as

$$
\psi(z) \equiv \frac{d}{dz}\log\Gamma(z) \quad , \quad \psi(q) = -\gamma_{\mathrm{E}} + \sum_{\ell=1}^{q-1}\frac{1}{\ell} \quad , \quad \gamma_{\mathrm{E}} \approx 0.577\ldots \tag{16}
$$

for integer argument q. The various integrals J now read

$$
\begin{aligned}
J_{1,1}^{(k\geq 1)} &= L - \psi(k) - \gamma_{\mathrm{E}} \quad , \\
J_{1,2}^{(k\geq 0)} &= \frac{1}{\mu} \quad , \\
J_{2,1}^{(k\geq 2)} &= \frac{\Lambda}{k-1} - \mu L + \mu\left(\psi(k-1) + \gamma_{\mathrm{E}} - 1\right) \quad , \\
J_{2,2}^{(k\geq 1)} &= L - \psi(2k) - \gamma_{\mathrm{E}} - 1 \quad , \\
J_{2,3}^{(k\geq 0)} &= \frac{1}{2\mu} \quad , \\
J_{3,1}^{(k\geq 3)} &= \frac{\Lambda^2}{(k-1)(k-2)} - \frac{2\mu\Lambda}{k-2} + \mu^2 L - \mu^2\left(\psi(k-2) + \gamma_{\mathrm{E}} - \frac{3}{2}\right) \quad , \\
J_{3,2}^{(k\geq 2)} &= \frac{\Lambda}{k-1} - 2\mu L + \mu\left(\psi(2k) + \psi(2k-2) + 2\gamma_{\mathrm{E}}\right) \quad , \\
J_{3,3}^{(k\geq 1)} &= L - \psi(3k) - \gamma_{\mathrm{E}} - \frac{3}{2} \quad , \\
J_{3,4}^{(k\geq 0)} &= \frac{1}{3\mu} \quad .
\end{aligned} \tag{17}
$$

We can draw the following conclusions. The convergent integrals do not depend on the number of PV subtractions, as was to be expected. In the regularized divergent integrals the only term that does not depend on the number of subtractions, and can be considered unambiguous, is the logarithmic divergence L. Quadratic (Λ) and higher ($\Lambda^2,\ldots$) divergences *do* depend on k and are therefore ambiguous. Their coefficients approach zero with increasing number of subtractions. The finite terms are also ambiguous (as was already remarked in [6]), and increase harmonically in absolute value with k. This is evidenced by the ubiquitous $\log(1-x)/(1-x)$ in Eq.(15). Strictly speaking, therefore, the limit $k \to \infty$ is not clearly defined. We have checked that these features persist for larger values of ω.

The reason why, in the previous section, we restricted ourselves to $\omega \leq 3$ is that the scalar φ^4 theory is renormalizeable for $\omega = 2$ and superrenormalizeable for $\omega = 1$, and the φ^3 theory is renormalizeable for $\omega = 3$ and superrenormalizeable for $\omega = 1, 2$. We therefore consider the renormalization properties of the integrals J. Obviously, we must apply a sufficient number of PV subtractions to properly regularize these; but there is no obvious recipe for determining when to stop subtracting. The only reasonable approach therefore seems to consider the case of an asymptotically large number of subtractions, *i.e.* to take $k \to \infty$ in a sensible way. This has the added advantage of being applicable to theories in *any* positive dimension. We can use the fact that, asymptotically,

$$\psi(z) \approx \log(z) + \mathcal{O}\left(1/z\right) \quad . \tag{18}$$

so that $L - \psi(k) \approx \log(\Lambda/\mu k)$. In this limit we can write the regularized divergent integrals as

$$\begin{aligned}
J_{1,1} &= \log\left(\frac{\Lambda}{\mu k}\right) - \gamma_{\mathrm{E}} \quad , \\
J_{2,1} &= -\mu \log\left(\frac{\Lambda}{\mu k}\right) + \mu\left(\gamma_{\mathrm{E}} - 1\right) \quad , \\
J_{2,2} &= \log\left(\frac{\Lambda}{\mu k}\right) - \gamma_{\mathrm{E}} - \log(2) - 1 \quad , \\
J_{3,1} &= \mu^2 \log\left(\frac{\Lambda}{\mu k}\right) - \mu^2\left(\gamma_{\mathrm{E}} - \frac{3}{2}\right) \quad , \\
J_{3,2} &= -2\mu \log\left(\frac{\Lambda}{\mu k}\right) + 2\mu\left(\gamma_{\mathrm{E}} + \log(2)\right) \quad , \\
J_{3,3} &= \log\left(\frac{\Lambda}{\mu k}\right) - \gamma_{\mathrm{E}} - \log(3) - \frac{3}{2} \quad .
\end{aligned} \tag{19}$$

Under renormalization, the logarithmic term is of course absorbed (and, in the spirit of MS versus $\overline{\mathrm{MS}}$, perhaps the γ_{E} as well), and the results will be well-defined and unambiguous.

A possible objection against the above procedure might be that the logarithmic divergence still contains k, albeit in a more-or-less hidden manner. We might therefore choose to let Λ depend on the degree of PV subtraction as well, by writing

$$\Lambda = \Lambda_0 k \quad . \tag{20}$$

This choice contains a certain justice in that when we apply more PV subtractions the subtraction propagators individually become smaller. In that case the higher divergences survive, but the results are still unambiguous:

$$\begin{aligned}
J_{1,1} &= \log\left(\frac{\Lambda_0}{\mu}\right) - \gamma_{\mathrm{E}} \quad , \\
J_{2,1} &= \Lambda_0 - \mu \log\left(\frac{\Lambda_0}{\mu}\right) + \mu\left(\gamma_{\mathrm{E}} - 1\right) \quad ,
\end{aligned}$$

$$
\begin{aligned}
J_{2,2} &= \log\left(\frac{\Lambda_0}{\mu}\right) - \gamma_{\mathrm{E}} - \log(2) - 1 \quad , \\
J_{3,1} &= {\Lambda_0}^2 - 2\mu\Lambda_0 + \mu^2 \log\left(\frac{\Lambda_0}{\mu}\right) - \mu^2\left(\gamma_{\mathrm{E}} - \frac{3}{2}\right) \quad , \\
J_{3,2} &= \Lambda_0 - 2\mu\log\left(\frac{\Lambda_0}{\mu}\right) + 2\mu\left(\gamma_{\mathrm{E}} + \log(2)\right) \quad , \\
J_{3,3} &= \log\left(\frac{\Lambda_0}{\mu}\right) - \gamma_{\mathrm{E}} - \log(3) - \frac{3}{2} \quad .
\end{aligned} \tag{21}
$$

This approach works because the coefficients of Λ^n go as k^{-n} for large k. We have checked that this persists for larger values of ω.

As an example, we can apply the above strategy to *e.g.* the calculation of the electron one-loop self-energy and the one-loop vertex correction in QED. In comparison with the standard treatment as given in [7] we find that our subtraction scheme results in the following modification (in our notation):

$$
\log(\Lambda) \;\; \to \;\; \log(\Lambda_0) - \gamma_{\mathrm{E}} \tag{22}
$$

in both computations. This shows that the relation between the vertex- and the wavefunction- renormalization remains undisturbed; in particular the Ward-Takahashi identity remains valid for the regularized Green's functions.

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