ON THE MATHIEU CONJECTURE FOR $SU(2)$

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Abstract. We study the Mathieu Conjecture for $SU(2)$ using the matrix elements of its unitary irreducible representations. We state a conjecture for the particular case $SU(2)$ implying the Mathieu Conjecture for $SU(2)$.

1. Introduction

Conjecture 1.1 (Mathieu [6]). *Let G be a compact connected Lie group and let f be complex-valued G-finite function on G such that $\int_G f^P(g)\,dg = 0$ for every $P \in \mathbb{N}_{>0}$. Then for any complex-valued G-finite function h on G we have $\int_G f^P(g)h(g)\,dg = 0$ for $P \gg 0$.*

The Mathieu Conjecture 1.1 dates back to 1997 and is closely related to the Jacobian conjecture, since it actually implies the Jacobian conjecture, see [6]. See van den Essen [2], Smale [7] for more information on the history of the Jacobian conjecture. The Mathieu Conjecture 1.1 was proved for abelian compact groups by Duistermaat and Van der Kallen [1] in 1998. We study the Mathieu Conjecture 1.1 for the case $G = SU(2)$. Using explicit formulas for the Haar measure and known representation theoretic properties of $SU(2)$ we make the Mathieu Conjecture 1.1 more explicit. In particular, we use the fact that $SU(2)$-finite functions are finite linear combinations of matrix elements of finite dimensional irreducible representations of $SU(2)$ and that the matrix elements behave well under a subgroup $K \cong U(1)$ according to suitable characters. Note that the Mathieu Conjecture 1.1 is linear in the G-finite function h, but not in the G-finite function f. By the Peter-Weyl theorem, any $SU(2)$-finite function is the finite linear combination of matrix elements of irreducible representations. After recalling the necessary results on $SU(2)$ in Section 2, we show in Section 3 the validity of the Mathieu Conjecture 1.1 for f a single matrix element or a sum of two matrix elements. For the sum of three matrix elements there is a partial result. These considerations lead to Conjecture 4.1, and Theorem 4.2 shows that this conjecture implies the Mathieu Conjecture 1.1 for $SU(2)$. Conjecture 4.1 describes the condition $\int_{SU(2)} f(g)^P\,dg = 0$ for all $P > 0$ in terms of a support condition on the characters of the abelian subgroup $U(1)$ of $SU(2)$ acting from the left and right on the individual matrix elements occurring in f.

We note that the Mathieu Conjecture 1.1 for bi-K-invariant functions is settled by Francoise et al. [3, Cor. 4.1], since the bi-K-invariant $SU(2)$-finite functions are the polynomials on $[-1, 1]$.

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2. $SU(2)$

We briefly recall some required notions of $SU(2)$. Details can be found in e.g. [8], [9]. Let $k(\phi) = \begin{pmatrix} e^{\frac{i}{2}\phi} & 0 \\ 0 & e^{-\frac{i}{2}\phi} \end{pmatrix}$ and $a(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$ be elements of $SU(2)$, then any element $g \in \mathrm{SU}(2)$ can be expressed in terms of Euler angles $g = k(\phi)a(\theta)k(\psi)$ with $\phi \in [0, 2\pi)$, $\theta \in (0, \pi)$, $\psi \in [-2\pi, 2\pi)$. In terms of the Euler angles the Haar integral is, cf [8, III, §6.1, (5)],

$$\int_{SU(2)} f(g)\, dg = \frac{1}{16\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_{-2\pi}^{2\pi} F(\phi,\theta,\psi) \sin\theta \, d\psi \, d\theta \, d\phi, \tag{2.1}$$

where $F(\phi,\theta,\psi) = f\big(k(\phi)a(\theta)k(\psi)\big)$. Denote the subgroup $K \cong U(1)$ generated by $k(\phi)$. For a function f transforming by a non-trivial K-character under left- or right multiplication by K, we have $\int_{SU(2)} f(g)\, dg = 0$ by (2.1). The subgroup generated by $a(\theta)$ is the group $SO(2)$.

The finite-dimensional irreducible representations are labeled by the spin $\ell \in \frac{1}{2}\mathbb{N}$ and are of dimension $2\ell + 1$. The standard basis for the representation space is labeled as $\{-\ell, -\ell + 1, \ldots, \ell\}$, and the corresponding matrix elements $t^\ell_{m,n}$ are $SU(2)$-finite functions, and any $SU(2)$-finite function is a finite linear combination of matrix elements of irreducible finite-dimensional representations. The matrix-elements $t^\ell_{m,n}$ behave well according to left and right action by K, cf. [8, III, §3.3, (3)]

$$t^\ell_{m,n}(k(\phi)g) = e^{-im\phi}\, t^\ell_{m,n}(g), \qquad t^\ell_{m,n}(gk(\psi)) = e^{-in\psi}\, t^\ell_{m,n}(g). \tag{2.2}$$

In particular, $t^0_{0,0}(g) = 1$, and the algebra of bi-K-invariant $SU(2)$-finite functions consists of finite linear combinations of $t^\ell_{0,0}$, $\ell \in \mathbb{N}$. For $\ell \in \mathbb{N}$ we have $t^\ell_{0,0}(a(\theta)) = P_\ell(\cos\theta)$, cf. [8, III, §3.9, (5)] where P_ℓ is the Legendre polynomial in its standard normalisation $P_\ell(1) = 1$, [4, §4.5], [5, §1.8.3], which is real-valued on $[-1, 1]$. The Legendre polynomials are orthogonal on $[-1, 1]$ with respect to the uniform measure; $\int_{-1}^1 P_n(x)P_m(x)\, dx = \delta_{n,m} 2/(2n+1)$. Moreover, the Schur orthogonality relations are, [8, III, §6.2, (1)]

$$\int_{SU(2)} t^{\ell_1}_{m,n}(g) \overline{t^{\ell_2}_{p,q}(g)}\, dg = \frac{1}{2\ell_1 + 1} \delta_{\ell_1,\ell_2} \delta_{m,p} \delta_{n,q}, \tag{2.3}$$

which in case $m = n = p = q = 0$ give the orthogonality for the Legendre polynomials.

3. The Mathieu Conjecture for $SU(2)$ for simple f

We start using some simple observations related to the condition in the Mathieu Conjecture 1.1 for $G = SU(2)$. Firstly, by the Schur orthogonality relations (2.3)

$$\int_{SU(2)} t^\ell_{m,n}(g)\, dg \neq 0 \iff \ell = 0. \tag{3.1}$$

Secondly, by the left and right K-behaviour of the matrix elements (2.2) and the Haar measure in Euler angles (2.1) we see

$$\int_{SU(2)} \big(t^{\ell_1}_{m_1,n_1}\big)^{\alpha_1}(g) \cdots \big(t^{\ell_k}_{m_k,n_k}\big)^{\alpha_k}(g)\, dg \neq 0 \quad \Longrightarrow \quad \sum_{i=1}^k \alpha_i m_i = 0 = \sum_{i=1}^k \alpha_i n_i \tag{3.2}$$

for $\alpha_i \in \mathbb{N}$, $\ell_i \in \frac{1}{2}\mathbb{N}$ and $m_i, n_i \in \{-\ell_i, \ldots, \ell_i\}$.

Lemma 3.1. *$\int_{SU(2)} \left(t^\ell_{m,n}\right)^P (g)\, dg = 0$ for all integer $P > 0$ if and only if $m \neq 0$ or $n \neq 0$.*

Proof. The implication $\Leftarrow$ follows from (3.2). To prove the other implication, we observe that for $\ell \in \mathbb{N}$

$$\int_{SU(2)} \left(t^\ell_{0,0}(g)\right)^2 dg = \frac{1}{2}\int_0^\pi \left(P_\ell(\cos\theta)\right)^2 \sin\theta\, d\theta = \frac{1}{2}\int_{-1}^1 \left(P_\ell(x)\right)^2 dx > 0. \qquad \square$$

Now we can verify the Mathieu Conjecture 1.1 in the case f consists of one matrix element.

Proposition 3.2. *The Mathieu Conjecture 1.1 is true for $G = SU(2)$ with f a single matrix element $f = t^\ell_{m,n}$.*

Proof. Since all non-negative powers of f integrate to zero, Lemma 3.1 shows that $m \neq 0$ or $n \neq 0$, so in particular $\ell \neq 0$. Let $h = t^{\ell_0}_{a,b}$. We assume $m \neq 0$, the case $n \neq 0$ being similar. By (3.2) we see that $Pm + a \neq 0$ implies $\int_{SU(2)} \big(f(g)\big)^P h(g)\, dg = 0$, which is the case for $P > |a|/|m|$. $\square$

The same strategy can also be employed to deal with $f = A_1 t^{\ell_1}_{m_1,n_1} + A_2 t^{\ell_2}_{m_2,n_2}$, where $A_i \in \mathbb{C}$, assuming $A_1 \neq 0 \neq A_2$ and $(\ell_1, m_1, n_1) \neq (\ell_2, m_2, n_2)$. Note

$$\int_{SU(2)} \big(f(g)\big)^P dg = \sum_{\alpha=0}^P \binom{P}{\alpha} A_1^\alpha A_2^{P-\alpha} \int_{SU(2)} \big(t^{\ell_1}_{m_1,n_1}\big)^\alpha(g) \big(t^{\ell_2}_{m_2,n_2}\big)^{P-\alpha}(g)\, dg. \tag{3.3}$$

Lemma 3.3. *Let f be as above with at least one of (m_1, m_2, n_1, n_2) non-zero, then*

$$\exists P > 0 : \quad \int_{SU(2)} \big(f(g)\big)^P dg \neq 0 \quad \Longleftrightarrow \quad \det\begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} = 0 \wedge m_1 m_2 \leq 0 \wedge n_1 n_2 \leq 0.$$

Remark 3.4. Note that the condition in Lemma 3.3 means that $(0, 0)$ is on the line segment from (m_1, n_1) to (m_2, n_2).

Proof. $\Rightarrow$: Since at least one term in the right hand side of (3.3) has to be non-zero, (3.2) shows that $m_1\alpha + m_2(P - \alpha) = 0 = n_1\alpha + n_2(P - \alpha)$, which gives the result.

$\Leftarrow$: Note that $\dim\operatorname{Ker}\begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} = 1$. Pick a solution $(\alpha, \beta) \in \mathbb{N}^2$ to $m_1\alpha + m_2\beta = 0 = n_1\alpha + n_2\beta$, and put $M = \alpha + \beta$. Then

$$\int_{SU(2)} \big(f(g)\big)^M dg = \binom{\alpha+\beta}{\alpha} A_1^\alpha A_2^\beta \int_{SU(2)} \big(t^{\ell_1}_{m_1,n_1}\big)^\alpha(g) \big(t^{\ell_2}_{m_2,n_2}\big)^\beta(g)\, dg, \tag{3.4}$$

using (3.2), since for $\gamma \neq 0$

$$\begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} \begin{pmatrix} \alpha+\gamma \\ \beta-\gamma \end{pmatrix} = \gamma \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

since the kernel is one-dimensional. The integrand on the right hand side of (3.4) is a bi-K-invariant function, so that by (2.1) we can restrict to the integral over $g = a(\theta)$, $\theta \in [0, \pi]$. By [8, III, §3,(3),(4)] the integrand in $a(\theta)$ is real-valued. In case the integral is non-zero we are done. Otherwise, we put $P = 2M$, and then in the same way there is again at most one non-zero integral in the right hand side of (3.3), namely for $(2\alpha, 2\beta)$. The integral can be restricted to $SO(2)$ as before. Since this is the integral of a square, since the function $\big(t^{\ell_1}_{m_1,n_1}\big)^{\alpha}(a(\theta))\big(t^{\ell_2}_{m_2,n_2}\big)^{\beta}(a(\theta))$ is real, the integral is non-zero. □

Proposition 3.5. *The Mathieu Conjecture 1.1 is true for $G = SU(2)$ with f a sum of two matrix element $f = A_1 t^{\ell_1}_{m_1,n_1} + A_2 t^{\ell_2}_{m_2,n_2}$, where $A_1 \neq 0 \neq A_2$ and $(\ell_1, m_1, n_1) \neq (\ell_2, m_2, n_2)$.*

Proof. It suffices to take $h = t^{\ell}_{a,b}$ and to assume that $\int_{SU(2)} (f(g))^P \, dg = 0$ for all $P > 0$. We need to show that $\int_{SU(2)} (f(g))^P t^{\ell}_{a,b}(g) \, dg$ vanishes for sufficiently large P.

First assume that not all of m_i's and n_i's are zero, then by Lemma 3.3 we have $m_1 m_2 > 0$ or $n_1 n_2 > 0$ or $\det \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} \neq 0$. Consider the last case, then by (3.2), (3.3) we see that $\int_{SU(2)} (f(g))^P t^{\ell}_{a,b}(g) \, dg$ can only be non-zero if

$$m_1\alpha + m_2\beta = -a, \quad n_1\alpha + n_2\beta = -b, \qquad \alpha + \beta = P, \ \alpha, \beta \in \mathbb{N}.$$

The first two equations have a unique solution $(\alpha_0, \beta_0) \in \mathbb{Q}^2$. In case $(\alpha_0, \beta_0) \in \mathbb{N}^2$, we see that for all $P > \alpha_0 + \beta_0$ the integral is zero. In case $m_1 m_2 > 0$, we consider $m_1\alpha + m_2\beta + a = 0$. In case $\mathrm{sgn}(m_1) = \mathrm{sgn}(a)$, we have no solution $(\alpha, \beta) \in \mathbb{N}^2$, so that integral is zero using (3.2), (3.3). In case $\mathrm{sgn}(m_1) = -\mathrm{sgn}(a)$, we see that the integral is zero for $P > |a| / \min(|m_1|, |m_2|)$. The case $n_1 n_2 > 0$ is dealt with analogously.

In case $m_1 = m_2 = n_1 = n_2 = 0$, f is a bi-K-invariant function, and

$$\int_{SU(2)} (f(g))^P \, dg = \frac{1}{2} \int_{-1}^{1} \big(A_1 P_{\ell_1}(x) + A_2 P_{\ell_2}(x)\big)^P \, dx.$$

By Boyarchenko's result, see [3, Cor. 4.1], there is no polynomial f such that $\int_{-1}^{1} (f(x))^P \, dx = 0$ for all $P > 0$, so the Mathieu Conjecture 1.1 is trivially valid in this case. □

The fact that at most one term in the binomial expansion leads to a non-zero integral is typical for f a linear combination of two matrix elements. For a combination of three matrix-elements it gets more complicated.

Proposition 3.6. *Let $f = \sum_{i=1}^{3} A_i t^{\ell_i}_{m_i,n_i}$ with $A_i \neq 0$ for all i and let (ℓ_i, m_i, n_i) be mutually different. Assume that $M = \begin{pmatrix} 1 & 1 & 1 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$ has $\mathrm{rank}(M) \neq 2$. Then the Mathieu Conjecture 1.1 is valid for f.*

Proof. The analogue of (3.3) is the trinomial expansion

$$\int_{SU(2)} f^P(g) \, \mathrm{d}g = \sum_{\substack{\alpha_1+\alpha_2+\alpha_3 = P \\ \alpha_i \in \mathbb{N}}} \binom{P}{\alpha_1, \alpha_2, \alpha_3} \prod_{i=1}^{3} A_i^{\alpha_i} \int_{SU(2)} \prod_{i=1}^{3} \big(t^{l_i}_{m_i n_i}\big)^{\alpha_i}(g) \, dg. \tag{3.5}$$

As before, it suffices to consider the case $h = t^{\ell}_{a,b}$. We have to consider the cases $\mathrm{rank}(M) = 1$ and $\mathrm{rank}(M) = 3$. In the first case $m_i = m$ and $n_i = n$ for all i, and the integral in (3.5) is zero if $m \neq 0$ or $n \neq 0$ by (3.2). In case $m \neq 0$ we see that

$$\int_{SU(2)} f^P(g) t^\ell_{a,b}(g)\,\mathrm{d}g = \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=P \\ \alpha_i \in \mathbb{N}}} \binom{P}{\alpha_1,\alpha_2,\alpha_3} \prod_{i=1}^3 A_i^{\alpha_i} \int_{SU(2)} \prod_{i=1}^3 \left(t^{l_i}_{m_i n_i}\right)^{\alpha_i}(g) t^\ell_{a,b}(g)\,dg \tag{3.6}$$

can only be non-zero if $Pm + a = 0$, so that for $P > |a|/|m|$ the integral is zero. The case $n \neq 0$ is analogous. In case $m = n = 0$, we see that the condition in the Mathieu Conjecture is not valid using [3, Cor. 4.1] as in the proof of Proposition 3.5.

In case $\mathrm{rank}(M) = 3$, M is invertible with M^{-1} having rational entries. In particular, for each $P \in \mathbb{N}$ there is at most one term in the right hand side of (3.5) which can be non-zero, namely for $\overrightarrow{\alpha}_P = \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = M^{-1}\begin{pmatrix}P\\0\\0\end{pmatrix}$ under the additional condition $\overrightarrow{\alpha}_P \in \mathbb{N}^3$. Assuming that this is the case, we see that, analogous to the proof of Proposition 3.5, $\int_{SU(2)} f^{2P}(g)\,\mathrm{d}g \neq 0$.

So we need to consider the case that $\overrightarrow{\alpha}_P \notin \mathbb{N}^3$ for all $P > 0$. Then the integral in (3.6) can only be non-zero in case

$$M^{-1}\begin{pmatrix}P\\-a\\-b\end{pmatrix} = \frac{1}{\det(M)}\left(P\begin{pmatrix}m_2n_3 - m_3n_2\\ m_3n_1 - m_1n_3\\ m_1n_2 - m_2n_1\end{pmatrix} - a\begin{pmatrix}n_2 - n_3\\ n_3 - n_1\\ n_1 - n_2\end{pmatrix} - b\begin{pmatrix}m_3 - m_2\\ m_1 - m_3\\ m_2 - m_1\end{pmatrix}\right) \in \mathbb{N}^3$$

Since $\overrightarrow{\alpha}_P$ corresponds to the first term, i.e. $a = b = 0$, and $\overrightarrow{\alpha}_P \notin \mathbb{N}^3$ for all $P > 0$ we have $\det(M)^{-1}(m_i n_{i+1} - m_{i+1} n_i) < 0$ for some $i \in \{1,2,3\}$ with convention $m_4 = m_1$, $n_4 = n_1$. Then for $P > \big(|a||n_i - n_{i+1}| + |b||m_{i+1} - m_i|\big)/|m_i n_{i+1} - m_{i+1} n_i|$ the i-th coefficient is negative, so that the integral in (3.6) is zero. □

Remark 3.7. In case $\mathrm{rank}(M) = 1$ the convex hull C of $\big((m_i, n_i)\big)_{i=1}^3$ equals $\{(m,n)\}$, and in case $\mathrm{rank}(M) = 3$ we have $(0,0) \in C$ if and only if $\exists\, \overrightarrow{\alpha} \in \mathbb{Q}^3_{\geq 0}$ with $M\overrightarrow{\alpha} = (1,0,0)^t$. From the proof of Proposition 3.6 we see that $\int_{SU(2)} f(g)^P\,dg = 0$ for all $P > 0$ precisely when $(0,0) \notin C$ in the cases $\mathrm{rank}(M) \neq 2$. In case $\mathrm{rank}(M) = 2$ the integral in (3.5) can have more than one non-zero term, and we have no control on possible cancellations. However, one expects that these cancellations cannot occur for all multiples of P as well. The techniques of Francoise et al. [3] might be useful in this regard considering it as polynomial identies in the A_i's.

4. An alternative conjecture for the Mathieu Conjecture for $SU(2)$

Consider an arbitrary $SU(2)$-finite function $f = \sum_{i=1}^k A_i t^{\ell_i}_{m_i,n_i}$ with $A_i \neq 0$ for every $1 \leq i \leq k$, then applying the multinomial theorem shows that if

$$\int_{SU(2)} (f(g))^P\,dg = \sum_{\alpha_i \in \mathbb{N}, \sum_{i=1}^k \alpha_i = P} \binom{P}{\alpha_1,\ldots,\alpha_k} \prod_{i=1}^k A_i^{\alpha_i} \int_{SU(2)} \prod_{i=1}^k \left(t^{\ell_i}_{m_i,n_i}\right)^{\alpha_i}(g)\,dg \neq 0 \tag{4.1}$$

for some $P > 0$, then for some $(\alpha_1, \dots, \alpha_k)$ we have $\sum_{i=1}^k \frac{\alpha_i}{P} m_i = \sum_{i=1}^k \frac{\alpha_i}{P} n_i = 0$ by (3.2), so $(0,0)$ is in the convex hull C of $\big((m_i, n_i)\big)_{i=1}^k$ over $\mathbb{Q}$.

Conjecture 4.1. *For any $SU(2)$-finite function $f = \sum_{i=1}^k A_i t^{\ell_i}_{m_i,n_i}$, $A_i \neq 0$ for all $1 \leq i \leq k$, we have that $\int_{SU(2)} (f(g))^P \, dg = 0$ for all $P \in \mathbb{N}_{>0}$ if and only if $(0,0)$ is not contained in the closed convex hull of $\big((m_i, n_i)\big)_{i=1}^k$.*

Lemma 3.1, Remarks 3.4, 3.7 support Conjecture 4.1.

Theorem 4.2. *Assume Conjecture 4.1 holds, then the Mathieu Conjecture 1.1 for $SU(2)$ holds.*

Proof. It suffices to show that $\int_{SU(2)} (f(g))^P t^{\ell}_{a,b}(g) \, dg = 0$ for P sufficiently large assuming that $(0,0)$ is not contained in the closed convex hull C of $\big((m_i, n_i)\big)_{i=1}^k$. Using (4.1) we see that $\int_{SU(2)} (f(g))^P t^{\ell}_{a,b}(g) \, dg$ can only be non-zero if $(-\frac{a}{P}, -\frac{b}{P}) \in C$. Since $(0,0) \notin C$, we see that for P sufficiently large this is not the case and the integral is zero. □

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