Some remarks on the Jacobian conjecture and polynomial endomorphisms

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Abstract

In this paper, we first show that homogeneous Keller maps are injective on lines through the origin. We subsequently formulate a generalization, which is that under some conditions, a polynomial endomorphism with r homogeneous parts of positive degree does not have r times the same image point on a line through the origin, in case its Jacobian determinant does not vanish anywhere on that line. As a consequence, a Keller map of degree r does not take the same values on $r > 1$ collinear points, provided r is a unit in the base field.

Next, we show that for invertible maps $x + H$ of degree d, such that $\ker \mathcal{J} H$ has $n - r$ independent vectors over the base field, in particular for invertible power linear maps $x + (Ax)^{*d}$ with $\operatorname{rk} A = r$, the degree of the inverse of $x + H$ is at most d^r.

Keywords. Jacobian conjecture, polynomial map, Drużkowski map.

1 Introduction

Throughout this paper, we will write K for any field and $K[x] = K[x_1, x_2, \ldots, x_n]$ for the polynomial algebra over K with n indeterminates $x = x_1, x_2, \ldots, x_n$.

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Let $F = (F_1, F_2, \ldots, F_m) : K^n \to K^m$ be a polynomial mapping, i.e. $F_i \in K[x]$ for all $i \le m$. Denote by $\mathcal{J} F$ the $(m \times n)$-matrix

$$\mathcal{J} F = \begin{pmatrix} \frac{\partial}{\partial x_1} F_1 & \frac{\partial}{\partial x_2} F_1 & \cdots & \frac{\partial}{\partial x_n} F_1 \\ \frac{\partial}{\partial x_1} F_2 & \frac{\partial}{\partial x_2} F_2 & \cdots & \frac{\partial}{\partial x_n} F_2 \\ \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial x_1} F_m & \frac{\partial}{\partial x_2} F_m & \cdots & \frac{\partial}{\partial x_n} F_m \end{pmatrix}$$

The well-known Jacobian conjecture (JC), raised by O.H. Keller in 1939 in [Kel], states that in case the characteristic $\operatorname{chr} K$ of K is zero, a polynomial mapping $F : K^n \to K^n$ is invertible if the Jacobian determinant $\det \mathcal{J} F$ is a nonzero constant.[1] This conjecture has been attacked by many people from various research fields, but is still open, even for $n = 2$. Only the case $n = 1$ is obvious, but the map $x_1 \mapsto x_1 - x_1^q$ over $\mathbb{F}_q$ (which is the zero map) shows that $\operatorname{chr} K = 0$ is required. For more information about the wonderful 70-year history, see [BCW], [vdE1], and the references therein. For more recent developments, see the second author's Ph.D. thesis [dB] and the survey article [vdE2].

Among the vast interesting and valid results, a satisfactory positive result was obtained by S.S.S. Wang in 1980 in [Wan], which is that the Jacobian conjecture holds when the degree of the concerned polynomial map is equal to two. A more simple proof of Wang's result was obtained by showing that a quadratic Keller map is injective over the algebraic closure of the base field $K \ni \frac{1}{2}$, since it had already been shown that that is sufficient for concluding invertibility, see [Gro, Prop. 17.9.6], [Yag, Lm. 3], [CyRus] and [Rud]. There are other authors that can be added to this list, but some authors only proved bijectivity.

The proof in [BCW, (2.4)] and [vdE1, Prop. 4.3.1] of Wang's injectivity result is due to S. Oda and is roughly as follows. Assume the opposite, say that $F(a) = F(b)$. The first step is reducing to the case that $b = 0$. The second step is showing that $a = 0$ in case $F(a) = F(0)$. The second step can easily be generalized in the sense that for Keller maps with terms of degree 0, 1, and d only, we have $a = 0$ in case $F(a) = F(0)$, where $d \ge 2$ is arbitrary. But the next proposition shows a stronger statement for the line Ka through a and the origin.

Proposition 1.1. *Assume $F : K^n \to K^n$ is a Keller map with terms of degree 0, 1, and d only, where $d > 1$ is a unit in K, and $a \in K^n$. Then $F|_{Ka}$ is injective.*

Proof. By replacing K by its algebraic closure, we may assume that K is algebraically closed. Write $F = F^{(0)} + F^{(1)} + F^{(d)}$, where $F^{(i)} \in K[x]^n$ has terms of degree i only. By Euler's theorem for homogeneous functions, we have

[1] The actual formulation by Keller, also known as Keller's conjecture, was over $\mathbb{Z}$ instead of K, with $\det \mathcal{J} F \in \{-1, 1\}$ a unit in $\mathbb{Z}$, but Keller's conjecture and the JC are equivalent when they are quantified over all dimensions.

$F_i^{(d)} = d^{-1} \sum_{j=1}^n x_j \frac{\partial}{\partial x_j} F_i^{(d)} = d^{-1} \mathcal{J} F_i^{(d)} \cdot x$ for all i. Hence $F(a) = F(b)$, if and only if

$$F^{(0)} + F^{(1)}(a) + d^{-1}(\mathcal{J} F^{(d)})|_a \cdot a = F^{(0)} + F^{(1)}(b) + d^{-1}(\mathcal{J} F^{(d)})|_b \cdot b$$

Assume $F(a) = F(\lambda a)$ for some $\lambda \in K$. By subtracting $F^{(0)}$ and substituting $b = \lambda a$, we get

$$\Big(\mathcal{J} F^{(1)}|_a + d^{-1}(\mathcal{J} F^{(d)})|_a\Big) \cdot a = \Big(\lambda \mathcal{J} F^{(1)}|_a + d^{-1}\lambda^d(\mathcal{J} F^{(d)})|_a\Big) \cdot a$$

where $|_f$ means substituting f for x. Since $\mathcal{J} F^{(1)}|_a = \mathcal{J} F^{(1)}$, this is equivalent to

$$\Big((1-\lambda) \mathcal{J} F^{(1)} + d^{-1}(1-\lambda^d)(\mathcal{J} F^{(d)})|_a\Big) \cdot a = 0$$

Notice that $d^{-1}(1-\lambda^d) = d^{-1}(1+\lambda+\cdots+\lambda^{d-1}) \cdot (1-\lambda)$, and $\mathcal{J} F^{(d)}$ is homogeneous of degree $d-1$. If we define $\mu := \sqrt[d-1]{d^{-1}(1+\lambda+\cdots+\lambda^{d-1})}$, then the above is equivalent to

$$(1-\lambda)\Big(\mathcal{J} F^{(1)} + (\mathcal{J} F^{(d)})|_{\mu a}\Big) \cdot a = 0$$

i.e. $(\mathcal{J} F)|_{\mu a} \cdot (\lambda-1)a = 0$. Since $\det(\mathcal{J} F)|_{\mu a}$ is a nonzero constant, $(\lambda-1)a = 0$ is the only possibility. Hence $b = a$, as desired. $\square$

The map $F = x_1 - x_1^q$ over $\mathbb{F}_q$ shows that the condition that d is a unit in K is required in proposition 1.1. We will generalize proposition 1.1 in section 2. In the above proof of 1.1, we do not need that $\det \mathcal{J} F$ is a nonzero constant: it is sufficient that $\det \mathcal{J} F$ does not vanish on Ka, provided we can take $(d-1)$-th roots in K. This is expressed in theorem 2.2 in section 2.

Although it is sufficient to show injectivity over the algebraic closure, it is not true that injective polynomial maps are automatically invertible: take for instance the map $F = x_1 + x_1^3$ over $\mathbb{R}$. Injective quadratic maps do not need to be invertible either: take for instance the map $F = (x_1 + x_2x_3, x_2 - x_1x_3, x_3)$ over $\mathbb{R}$, which is injective because for fixed x_3 it corresponds to the linear map $\tilde{F} = (x_1+x_2x_3, x_2-x_1x_3)$ over $\mathbb{R}[x_3]$, which has Jacobian determinant $1+x_3^2 \neq 0$ just as F itself. It is however true that injective polynomial maps over $\mathbb{R}$ are automatically surjective, see [BbRo].

Quadratic polynomial maps over a field $K \ni \frac{1}{2}$, such that Jacobian determinant does not vanish anywhere, are always injective. This follows from both Oda's proof of Wang's theorem and the proof of proposition 1.1. In the proof of proposition 1.1, the Keller condition is used to replace K by its algebraic closure without affecting that the Jacobian determinant is nonzero everywhere, but since a $(d-1)$-th root is taken, there is no need to replace K by its algebraic closure when $d = 2$. For twodimensional Keller maps over an algebraically closed field of characteristic zero, there is a nice result due to J. Gwoździewicz in [Gwo], namely that they are invertible when they are injective on one single line (any line). His short proof makes use of the Abhyankar-Moh-Suzuki theorem, see e.g. [vdE1, Th. 5.3.5] for the characteristic zero case which is used in [Gwo].

However in [Pin], S. Pinchuk constructed a twodimensional polynomial map over $\mathbb{R}$ of degree 25 with nonzero Jacobian everywhere, which is *not injective*[2] The counterexample of Pinchuk can be transformed to a polynomial map of degree three in larger dimension by way of reduction techniques that are indicated below. In that manner, in 1999 E. Hubbers constructed a non-injective cubic polynomial map of a very special form in dimension 1999 over $\mathbb{R}$ with nonzero Jacobian everywhere, namely a so-called Drużkowski map. This result has not been published, though.

After having solved the quadratic case of the JC, it is a natural question what happens when the degree of the polynomial map is equal to three. This case of the JC has not been solved yet, but instead, it is proved that the JC holds in general in case it holds for cubic maps of the form $x + H$, where $x = (x_1, x_2, \ldots, x_n)$ is the identity map in dimension n, and $H = (H_1, H_2, \ldots, H_n)$ is homogeneous of degree three, i.e. each H_i is either homogeneous of degree three or zero.

In fact, the JC holds in general if for some $d \geq 3$ (any $d \geq 3$), the JC holds for maps of the form $x + H$, where $x = (x_1, x_2, \ldots, x_n)$ is the identity map in dimension n, and $H = (H_1, H_2, \ldots, H_n)$ is homogeneous of degree d, and proposition 1.1 shows that Keller maps of this form are injective on lines through the origin.

A subsequent reduction is due to L.M. Drużkowski, which asserts that the JC is true in general if it is true for maps of the form $x+H$ such that H_i is either a third power of a linear form or zero for each i. Therefore, a polynomial map $x+H$ as such is called a *Drużkowski map.* If $A \in \mathrm{Mat}_n(K)$, then we can take the matrix product Ax of A with the column vector x, and Ax is a column vector of linear forms. Next, we can take the Hadamard product $(Ax) * (Ax) * (Ax)$, which is called the third Hadamard power of Ax and denoted as $(Ax)^{*3}$ by many authors[3]. Thus Drużkowski showed that the JC is true in general if it is true for maps of the form $x + (Ax)^{*3}$.

Similarly, one can define $x + (Ax)^{*d}$, and just as above, the JC is true in general in case it is true for maps of the form $x + (Ax)^{*d}$, where d is one's favorite integer larger than two. If F is an invertible polynomial map of degree d in dimension n, then the degree of its inverse is at most d^{n-1}. This has been proved in [BCW, Cor. (1.4)], which is a direct consequence of a more or less similar result about birational maps in projective space by O. Gabber, see [BCW, Th. (1.5)]. But if F is of the form $x + (Ax)^{*d}$ and $\operatorname{rk} A = r$, then the degree of the inverse of F is at most d^r. We will prove this in section 3.

[2]Pinchuk claims to have constructed a map of degree 40, but does not give it explicitly. Inspection of his proofs leads to a map of degree ≥ 25 which can be reduced to degree $= 25$ by way of an elementary automorphism.

[3]In fact, it is popular to denote $\alpha^{\diamond d}$ for the 'composition' of d copies of some object α by means of applying any commutative binary operator $\diamond$ exactly $d-1$ times, such as $K^{\times n}$ for the n-dimensional vector space over K, and in an invertible context, the JC is about the existence of $F^{\circ(-1)}$.

2 Polynomial maps that take the same values on several collinear points

At first, a lemma with a generalized Vandermonde matrix which is assumed to have full rank. The powers in the Vandermonde matrix correspond to the degrees of the homogeneous parts of the polynomial map in theorem 2.2, the main theorem of this section.

Lemma 2.1. *Let K be a field and assume that $G : K \to K^m$ is given by*

$$G(t) = C(t^{d_1}, t^{d_2}, \ldots, t^{d_r}, t^{d_{r+1}})$$

where $C \in \operatorname{Mat}_{m,r+1}(K)$. Suppose that $G(a_1) = G(a_2) = \cdots = G(a_r) = 0$ and that $\operatorname{rk} A^{(r,r)} = r$, where $A^{(s,r)}$ is the generalized Vandermonde matrix

$$A^{(s,r)} := \begin{pmatrix} a_1^{d_1} & a_2^{d_1} & \cdots & a_r^{d_1} \\ a_1^{d_2} & a_2^{d_2} & \cdots & a_r^{d_2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{d_s} & a_2^{d_s} & \cdots & a_r^{d_s} \end{pmatrix} \tag{1}$$

Then $\operatorname{rk} C \le 1$ and the last column of C is nonzero in case $C \ne 0$.

Proof. Since $\operatorname{rk} A^{(r+1,r)} = r$, there is only one relation between the rows of A up to scalar multiplication, say $v^{\mathrm{t}} A^{(r+1,r)} = 0$ for some nonzero vector $v \in K^{r+1}$. By assumption, we have $CA^{(r+1,r)} = 0$, whence each row of C is a scalar multiple of v^{t}. Thus $\operatorname{rk} C \le 1$.

If $C \ne 0$ and the last column of C is zero, then $CA^{(r+1,r)} = 0$ would imply $\operatorname{rk} A^{(r,r)} < r$, which is a contradiction. □

Theorem 2.2. *Let K be a field and assume $F : K^n \to K^m$ is a polynomial map such that the degree of each term of F is contained in $\{d_1, d_2, \ldots, d_r, d_{r+1}\} \ni 0$. Assume furthermore that for all $c \in K^r$,*

$$\frac{\partial}{\partial t}(t^{d_{r+1}} + c_r t^{d_r} + \cdots + c_2 t^{d_2} + c_1 t^{d_1})$$

has a root in K.

Take $b \in K^n$ nonzero and assume $F(a_1 b) = F(a_2 b) = \cdots = F(a_r b)$ and $\operatorname{rk} A^{(r,r)} = r$ for certain $a_i \in K$, where $A^{(r,r)}$ is as in (1). Then there is an $a_{r+1} \in K$ such that $(\mathcal{J} F)|_{a_{r+1} b} \cdot b = 0$. In particular, $\operatorname{rk}(\mathcal{J} F)|_{a_{r+1} b} < n$.

Proof. In this proof, we will write $|_{x=f}$ and $|_{t=f}$ for the substitution of f for x and t respectively. Take $G(t) = F(tb) - F(a_1 b)$. Then $G(a_i) = F(a_i b) - F(a_1 b) = 0$ for each $i \le r$, and the degree of each term of $G(t)$ is contained in $\{d_1, d_2, \ldots, d_r, d_{r+1}\}$. Hence there exists a C as in lemma 2.1, and by the chain rule

$$(\mathcal{J} F)|_{x=tb} \cdot b = \mathcal{J}_t\, F(tb) = \mathcal{J}_t\, G(t) = C\, \mathcal{J}_t(t^{d_1}, t^{d_2}, \ldots, t^{d_r}, t^{d_{r+1}}) \tag{2}$$

If $\mathcal{J}_t\, G(t) = 0$, then we can take a_{r+1} arbitrary. Hence assume that $\mathcal{J}_t\, G(t) \neq 0$, say that

$$\frac{\partial}{\partial t} G_i(t) = C_i\, \mathcal{J}_t(t^{d_1}, t^{d_2}, \ldots, t^{d_r}, t^{d_{r+1}}) \neq 0$$

Since $\operatorname{rk} C = 1$ and the last column of C is nonzero, every column of C is dependent of the last one. Therefore, $C_i \neq 0$ refines to $C_{i(r+1)} \neq 0$. By assumption, $C_{i(r+1)}^{-1} \frac{\partial}{\partial t} G_i(t)$ has a root $a_{r+1} \in K$. Thus

$$C_i\big(\, \mathcal{J}_t(t^{d_1}, t^{d_2}, \ldots, t^{d_r}, t^{d_{r+1}})\big)\Big|_{t=a_{r+1}} = C_{i(r+1)} \Big(C_{i(r+1)}^{-1} \frac{\partial}{\partial t} G_i(t)\Big)\Big|_{t=a_{r+1}} = 0$$

Using that $C_i \neq 0$ and again that $\operatorname{rk} C = 1$, we see that every row of C is dependent of C_i, which gives $C(\mathcal{J}_t(t^{d_1}, t^{d_2}, \ldots, t^{d_r}, t^{d_{r+1}}))|_{t=a_{r+1}} = 0$. Substituting $t = a_{r+1}$ in (2) subsequently gives $(\mathcal{J}\, F)|_{x=a_{r+1}b} \cdot b = 0$, as desired. □

Corollary 2.3. *Let K be a field and assume $F : K^n \to K^n$ is a polynomial map such that the degree of each term of F is contained in $\{d_1, d_2, \ldots, d_r, d_{r+1}\} \ni 0$, where $\operatorname{chr} K \nmid d_{r+1} \neq 1$.*

Assume $b \in K^n$ is nonzero, such that $F(a_1 b) = F(a_2 b) = \cdots = F(a_r b)$ and $\operatorname{rk} A^{(r,r)} = r$ for certain $a_i \in K$, where $A^{(r,r)}$ is as in (1). Then $\det \mathcal{J}\, F \notin K^*$.

Proof. By replacing K by its algebraic closure, we may assume that K is algebraically closed. The condition $\operatorname{chr} K \nmid d_{r+1} \neq 1$ tells us that for all $c \in K^r$,

$$\frac{\partial}{\partial t}(t^{d_{r+1}} + c_r t^{d_r} + \cdots + c_2 t^{d_2} + c_1 t^{d_1})$$

has a root in K. Hence by theorem 2.2, there exists an $a_{r+1} \in K$ such that $(\mathcal{J}\, F)|_{a_{r+1}b}$ does not have full rank. Consequently, $x = a_{r+1}b$ is a root of $\det \mathcal{J}\, F$. This gives the desired result. □

If we take $r = 2$ in corollary 2.3, then $d_3 > 1$ must be a unit in K and $0 \in \{d_1, d_2\}$. Furthermore, the conclusion of corollary 2.3 is trivial when $1 \notin \{d_1, d_2\}$, thus we may assume that $d_1 = 0$ and $d_2 = 1$. Since the Vandermonde matrix $A^{(r,r)}$ always has full rank when $r = 2$ and $a_1 \neq a_2$, proposition 1.1 is the case $r = 2$ of corollary 2.3.

In corollary 2.4 below, we show that a Keller map of degree r does not take the same values on r collinear points, provided $r > 1$ is a unit in the base field.

Corollary 2.4. *Let K be a field and assume $F : K^n \to K^n$ is a polynomial map such that $F(p_1) = F(p_2) = \cdots = F(p_r)$ for distinct collinear $p_i \in K^n$. If $\operatorname{chr} K \nmid r \neq 1$ and $r \geq \deg F$, then $\det \mathcal{J}\, F \notin K^*$.*

Proof. By replacing F by $F(x - p_1)$, we may assume that $p_i = a_i b$ for distinct $a_i \in K$, where $b = p_2 - p_1 \neq 0$. Take $d_i = i - 1$ for all $i \leq r + 1$. Then $A^{(r,r)}$ is a regular Vandermonde matrix and $\operatorname{rk} A^{(r,r)} = r$, where $A^{(r,r)}$ is as in (1). Hence the desired result follows from corollary 2.3. □

Remark 2.5. Just as with proposition 1.1, the map $f = x_1 - x_1^q$ with $r = 2$ and $(d_1, d_2, d_3) = (0, 1, q)$ shows that the condition that $\operatorname{chr} K \nmid d_{r+1}$ is required in corollary 2.3. When we avoid the use of the condition that $\operatorname{chr} K \nmid d_{r+1}$ by taking $r = 3$ and $(d_1, d_2, d_3, d_4) = (0, 1, q, q + 1)$, the condition that the Vandermonde matrix $A^{(r,r)}$ has full rank appears necessary in corollary 2.3.

The condition $\operatorname{chr} K \nmid d_{r+1}$ is also required if we take $r \geq \deg F$ and $d_i = i-1$ for all i in corollary 2.3, which we did in the proof of corollary 2.4. This is why the map $f = x_1 - x_1^q$ shows that the condition $\operatorname{chr} K \nmid r \neq 1$ is required in corollary 2.4 as well.

3 The degree of the inverse of a polynomial map

We first show that the degree bound d^{n-1} on the inverse of a polynomial map F of degree d in dimension n is reached when $F = x + (Ax)^{*d}$ is power linear such that $\operatorname{rk} A = n - 1$.

Proposition 3.1. *Assume $F = x + (Ax)^{*d}$ is power linear of degree $d \geq 1$ such that $\operatorname{rk} A = n-1$. If $\det \mathcal{J} F = 1$, then F is invertible and its inverse has degree d^{n-1}.*

Proof. The case $d = 1$ is trivial, so assume that $d \geq 2$. In [Che1, Th. 1] and [Dru2, Th. 2.1], it is shown that F is linearly triangularizable in case $d = 2$ and tame in case $d = 3$ respectively, but inspection of the proofs tells us that in both cases, there exists a $T \in \mathrm{GL}_n(K)$ such that $T^{-1}F(Tx) = x + (Bx)^{*d}$, where B is lower triangular with zeroes on the diagonal. Furthermore, the case $d \geq 4$ follows in a similar manner as the case $d = 3$, see also [TdB, Th. 4]. Therefore, we may assume that A is lower triangular with zeroes on the diagonal.

Let G be the inverse of F. Then $G_1 = x_1$, and substituting $x = G$ in $x_i = F_i - (A_{i1}x_1 + \cdots + A_{i(i-1)}x_{i-1})^d$ gives $G_i = x_i - (A_{i1}G_1 + \cdots + A_{i(i-1)}G_{i-1})^d$, which is an inductive formula for G. Since $\operatorname{rk} A = n - 1$, we see that the entries $A_{i(i-1)}$ on the subdiagonal of A are all nonzero, and $\deg G_i = d \deg G_{i-1}$ follows for all i by induction. Hence $\deg G_n = d^{n-1} \deg G_1 = d^{n-1}$. ☐

Before we prove our theorem, we first show that by way of a linear conjugation, we can reduce to the case that $\ker \mathcal{J}(F - x) \cap K^n = \{0\}^r \times K^{n-r}$.

Lemma 3.2. *Let K be a field and assume $F : K^n \to K^n$ is an invertible polynomial map. If $\ker \mathcal{J}(F - x) \cap K^n$ has dimension $n - r$ as a K-space, then there exists a $T \in \mathrm{GL}_n(K)$ such that for $G := T^{-1}F(Tx)$, we have*

$$\ker \mathcal{J}(G - x) \cap K^n = \{0\}^r \times K^{n-r}$$

Furthermore, G is invertible and the degree of its inverse is the same as that of F.

Proof. Take $T \in \mathrm{GL}_n(K)$ such that the last $n - r$ columns of T are a basis of the K-space $\ker \mathcal{J}(F - x) \cap K^n$, and set $G := T^{-1}F(Tx)$. Then

$$G - x = T^{-1}F(Tx) - T^{-1}Tx = T^{-1}(F - x)|_{Tx}$$

and by the chain rule, $\mathcal{J}(G-x) = T^{-1} \cdot (\mathcal{J}(F-x))|_{Tx} \cdot T$. Hence

$$\begin{aligned}
\ker \mathcal{J}(G-x) \cap K^n &= T^{-1}\big(\ker \big(T^{-1} \cdot (\mathcal{J}(F-x))|_{Tx}\big)\big) \cap K^n \\
&= T^{-1}(\ker \mathcal{J}(F-x))|_{Tx} \cap T^{-1}(K^n)|_{Tx} \\
&= T^{-1}(\ker \mathcal{J}(F-x) \cap K^n)|_{Tx} \\
&= T^{-1}(KTe_{r+1} + KTe_{r+2} + \cdots + KTe_n)|_{\cdots} \\
&= Ke_{r+1} + Ke_{r+2} + \cdots + Ke_n = \{0\}^r \times K^{n-r}
\end{aligned}$$

where e_i is the i-th standard basis unit vector. If $\tilde{F}$ is the inverse of F, then $\tilde{G} := T^{-1}\tilde{F}(Tx)$ is the inverse of G and $\tilde{G}$ has the same degree as $\tilde{F}$, as desired. $\square$

Remark 3.3. Write $\tilde{x} = (x_1, x_2, \ldots, x_r)$. If we take for B the first r rows of T^{-1} and for C the first r columns of T, then we get $(G_1, G_2, \ldots, G_r) = B(F(Tx))$, and since $G_i \in K[\tilde{x}]$ for all $i \le r$, the substitution $x_{r+1} = \cdots = x_n = 0$ has no effect, and $(G_1, G_2, \ldots, G_r) = B(F(C\tilde{x}))$ follows. Furthermore, BC is a leading principal minor matrix of $T^{-1}T$ and hence equal to I_r, and since $T^{-1} \ker B = \ker BT = \{0\}^r \times K^{n-r} = T^{-1}(\ker \mathcal{J}(F-x) \cap K^n)|_{\cdots}$, we have $\ker B = \ker \mathcal{J}(F-x) \cap K^n$. Consequently, $(G_1, G_2, \ldots, G_r)$ and F are paired in the sense of [GZ] when $r < n$, where the condition $\ker B = \ker A$ is replaced by $\ker B = \ker \mathcal{J}(F-x) \cap K^n$, to allow maps F that are not of the form $x + (Ax)^{*d}$ as well.

Theorem 3.4. *Let K be a field with $\operatorname{chr} K = 0$ and assume $F = x + H : K^n \to K^n$ is an invertible polynomial map of degree d. If $\ker \mathcal{J} H \cap K^n$ has dimension $n - r$ as a K-space, then the inverse polynomial map of F has degree at most d^r.*

Proof. From lemma 3.2, it follows that by replacing F by $T^{-1}F(Tx)$ for a suitable $T \in \mathrm{GL}_n(K)$, we may assume that $\ker \mathcal{J} H \cap K^n = \{0\}^r \times K^{n-r}$. This means that the last r columns of $\mathcal{J} H$ are zero. Hence $H_i \in K[x_1, x_2, \ldots, x_r]$ for all i. Let $x - G$ be the inverse of F at this stage. Then $G(F) = H$, because $x = (x - G)|_F = x + H - G(F)$.

Let $L = K(x_1, x_2, \ldots, x_r)$. Since $F_i, H_i \in L$ for all $i \le r$ and $F_{r+1}, F_{r+2}, \ldots, F_n$ are algebraically independent over L, we obtain from $G_i(F) = H_i$ that $G_i \in K[x_1, x_2, \ldots, x_r]$ for all $i \le r$. Hence $(x_1 - G_1, x_2 - G_2, \ldots, x_r - G_r)$ is the inverse polynomial map of $(F_1, F_2, \ldots, F_r)$. Therefore, $\deg(x_1 - G_1, x_2 - G_2, \ldots, x_r - G_r) \le d^{r-1}$ on account of [BCW, Cor. (1.4)].

Using that F is the inverse of $x - G$, we obtain by substituting $x = x - G$ in $G_i(F) = H_i$ that $G_i = H_i(x - G)$ for all i. Since $\deg H_i \le d$ and $H_i \in K[x_1, x_2, \ldots, x_r]$ for all i, we get

$$\deg G_i \le \deg H_i \cdot \deg(x_1 - G_1, x_2 - G_2, \ldots, x_r - G_r) \le d \cdot d^{r-1}$$

for all i, which gives the desired result. $\square$

Corollary 3.5. *Let K be a field with $\operatorname{chr} K = 0$ and assume $F : K^n \to K^n$ is an invertible polynomial map of the form $F = x + H$, where H is power linear* *of degree d. If $\operatorname{rk} \mathcal{J} H = r$, then the inverse polynomial map of F has degree at most d^r.*

Proof. Say that $H = (Ax)^{*d}$. Then we have $\mathcal{J} H = d \operatorname{diag}((Ax)^{*(d-1)}) \cdot A$, where $\operatorname{diag}(v)$ stands for a square matrix with diagonal v and zeroes elsewhere. Consequently, $\ker \mathcal{J} H = \ker A$ and $\operatorname{rk} A = \operatorname{rk} \mathcal{J} H = r$. Since A is a matrix over K, the dimension of $\ker A \cap K^n$ as a K-space is $n - r$. Hence the desired result follows from theorem 3.4. □

Now that we have a better estimate of the degree of the inverse of a polynomial map, it is interesting to know some cases that the polynomial map has an inverse on account of the Keller condition. The following result was proved by A. van den Essen in [vdE1, Prop. 2.9] and C. Cheng in [Che2, Th. 2] for homogeneous maps and power linear maps.

Theorem 3.6. *Let K be a field with $\operatorname{chr} K = 0$ and write $\tilde{x} = (x_1, x_2, \ldots, x_r)$. Assume that Keller maps $\tilde{F} = \tilde{x} + \tilde{H}$ in dimension r over K such that $\tilde{H}$ is (homogeneous) of degree d are invertible. Then Keller maps $F = x + H$ in dimension n over K such that H is (homogeneous) of degree d and $\ker \mathcal{J} H \cap K^n$ has dimension $n - r$ are invertible as well.*

A similar result holds when we replace 'invertible' by 'linearly triangularizable'.

Proof. Let F be as above. Just as in the proof of theorem 3.4, we may assume that $H_i \in K[\tilde{x}]$ for all i. Since F is a Keller map in addition, the leading principal minor determinant of size r of $\mathcal{J} F$ is a nonzero constant. Now the leading principal minor matrix of size r of $\mathcal{J} F$ is of the form $\mathcal{J}_{\tilde{x}} \tilde{F}$ with $\tilde{F} = (F_1, F_2, \ldots, F_r)$, whence $\tilde{F}$ is a Keller map. By assumption, the inverse $\tilde{x} - \tilde{G}$ of $\tilde{F}$ exists, and $\tilde{F} - \tilde{G}(\tilde{F}) = \tilde{x}$. If we define $G_i = \tilde{G}_i$ for all $i \le r$ and $G_i = H_i(\tilde{x} - \tilde{G})$ for all $i \ge r + 1$, then substituting $x = F$ in $x_i - G_i$ gives

$$F_i - G_i(F) = \begin{cases} \tilde{F}_i - \tilde{G}_i(\tilde{F}) = x_i & \text{if } i \le r \\ x_i + H_i - H_i(\tilde{F} - \tilde{G}(\tilde{F})) = x_i & \text{if } i \ge r + 1 \end{cases}$$

Hence $x - G$ is the inverse of F.

In case $\tilde{F}$ is linearly triangularizable, say that $\tilde{T}^{-1}\tilde{F}(\tilde{T}\tilde{x})$ has a lower triangular Jacobian, then $T^{-1}F(Tx)$ has a lower triangular Jacobian as well if we define

$$T = \begin{pmatrix} \tilde{T} & \emptyset \\ \emptyset & I_{n-r} \end{pmatrix}$$

which completes the proof of theorem 3.6. □

Corollary 3.7. *Let K be a field with $\operatorname{chr} K = 0$ and assume that $F = x + H$ is a Keller map in dimension n over K such that $\ker \mathcal{J} H \cap K^n$ has dimension $n - r$. Then F is linearly triangularizable in the following cases:*

i) $r \le 3$ and H is homogeneous of degree $d \ge 2$,

ii) $r = 4$ and H is homogeneous of degree 2.

Furthermore, F is invertible as well in the following cases:

iii) $r = 2$ *and* $\deg H \leq 101$,

iv) $r = 3$ *and* $\deg H \leq 3$,

v) $r = 4$ *and* H *is homogeneous of degree* 3.

Proof. This follows from theorem 3.6 and what is known about the Jacobian conjecture in small dimensions r. For i), see [dBvdE]. For ii), use [MO] and [vdE1, Th. 7.4.4]. For the rest, see [vdE1] and the references therein. $\square$

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