# SOME REMARKS ON MODIFIED DIAGONALS

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**Abstract.** We prove a number of basic vanishing results for modified diagonal classes. We also obtain some sharp results for modified diagonals of curves and abelian varieties, and we prove a conjecture of O'Grady about modified diagonals on double covers.

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## 1. Introduction

**1.1.** Given a smooth projective variety $X$ and a base point $a$, Gross and Schoen introduced in [7] modified diagonal cycles $\Gamma^n(X,a)$ on $X^n$. For instance, $\Gamma^2(X,a) = \Delta_X - \bigl[X\times\{a\}\bigr] - \bigl[\{a\}\times X\bigr]$. In general, if $J\subset\{1,\ldots,n\}$ we define a closed subvariety $X^J\subset X^n$ by the condition that $x_i = a$ for all $i\notin J$; the modified diagonal $\Gamma^n(X,a)$ is then an alternating sum of the small diagonals on the $X^J$.

Gross and Schoen proved some vanishing results for the modified diagonals of curves, both in the Chow ring and modulo algebraic equivalence. In [3], Beauville and Voisin prove that for a K3 surface $X$ there is a distinguished point class $o_X\in\mathrm{CH}_0(X)$ and that $\Gamma^3(X,o_X)=0$ in $\mathrm{CH}_2(X^3)$. (Throughout we work with $\mathbb{Q}$-coefficients.) A consequence of this is that the intersection pairing $\mathrm{Pic}(X)^{\otimes 2}\to\mathrm{CH}_0(X)$ takes values in $\mathbb{Q}\cdot o_X$ and that $c_2(X) = 24\cdot o_X$.

Our interest in modified diagonals was sparked by the preprint [12] of O'Grady and the questions he asked. We were quickly able to answer one of these questions in the positive, proving that for a $g$-dimensional abelian variety $X$ we have $\Gamma^m(X,a)=0$ in $\mathrm{CH}_g(X^m)$ for $m>2g$ and any choice of base point; see [10]. This result is included in the present note as Theorem 4.2.

**1.2.** In this paper, we give a simple motivic description of modified diagonals, and we collect a number of basic results about them. We also introduce and study some more general classes $\gamma^n_{X,a}(\alpha)$, for any $\alpha\in\mathrm{CH}(X)$, which for $\alpha=[X]$ give back the modified diagonals $\Gamma^n(X,a)$.

We work over an arbitrary field and consider algebraic cycles (with $\mathbb{Q}$-coefficients) modulo an adequate equivalence relation $\sim$. We prove that $\Gamma^n(X,a)\sim 0$ if and only if the map $\gamma^n_{X,a}$ is zero modulo $\sim$, and that this implies the vanishing of $\gamma^{n-s}_{X,a}(\alpha)$ for all classes $\alpha$ in the image of the product map $\mathrm{CH}^{>0}(X)^{\otimes s}\to\mathrm{CH}(X)$. We also prove that if $f\colon X\to Y$ is surjective with generic fiber of dimension $r$ then $\Gamma^n(X,a)\sim 0$ implies that $\Gamma^{n-r}\bigl(Y,f(a)\bigr)\sim 0$. Further we have a precise result about what happens when we change the base point (working in the Chow ring): if $\Gamma^n(X,a)=0$ for some point $a$ then for any other base point $a'$ we have $\Gamma^{2n-2}(X,a')=0$.

In Section 4 we prove some sharp (or conjecturally sharp) vanishing results for modified diagonals of curves and abelian varieties. For a curve $C$ we use the base point $a$ to embed $C$ in its Jacobian $J$. The vanishing of $\Gamma^n(C,a)$ is then equivalent to the vanishing of some components of the class $[C]\in\mathrm{CH}_1(J)$ with respect to the Beauville decomposition of $\mathrm{CH}(J)$. This is a problem that has been studied independently of modified diagonals, notably by Polishchuk, Voisin and the second author. Our Theorem 4.2 about modified diagonals of abelian varieties, which proves a conjecture of O'Grady in [12], is an easy application of the results by Deninger and Murre [5] about motivic decompositions of abelian varieties.

Finally, in Section 5 we prove a conjecture of O'Grady about modified diagonals on double covers. We recently learned that Claire Voisin has proven a generalization of this result to covers of higher degree; this result is to appear in her forthcoming paper [15].

## 2. Definition and some basic properties of modified diagonals

Throughout, Chow groups are taken with $\mathbb{Q}$-coefficients.

**2.1.** Let $k$ be a field. Let $X$ and $Y$ be smooth projective $k$-varieties. If $X$ is connected, let $\mathrm{Corr}_i(X,Y) = \mathrm{CH}_{\dim(X)+i}(X\times Y)$. In general, write $X$ as a disjoint union of connected varieties, say $X = \coprod_\alpha X_\alpha$; then we let $\mathrm{Corr}_i(X,Y) = \oplus_\alpha \mathrm{Corr}_i(X_\alpha, Y)$. The elements of $\mathrm{Corr}_i(X,Y)$ are called correspondences from $X$ to $Y$ of degree $i$. If $Z$ is a third smooth projective $k$-variety, composition of correspondences

$$\mathrm{Corr}_i(X,Y) \times \mathrm{Corr}_j(Y,Z) \to \mathrm{Corr}_{i+j}(X,Z)$$

is defined in the usual way: $(\theta,\xi) \mapsto \mathrm{pr}_{XZ,*}\big(\mathrm{pr}_{XY}^*(\theta)\cdot \mathrm{pr}_{YZ}^*(\xi)\big)$.

We denote by $\mathsf{Mot}_k$ the category of (covariant) Chow motives over $k$. The objects are triples $(X,p,m)$ with $X$ a smooth projective $k$-variety, $p$ an idempotent in $\mathrm{Corr}_0(X,X)$, and $m \in \mathbb{Z}$. The morphisms from $(X,p,m)$ to $(Y,q,n)$ are the elements of

$$q \circ \mathrm{Corr}_{m-n}(X,Y) \circ p$$

(which is a subspace of $\mathrm{Corr}_{m-n}(X,Y)$), and composition of morphisms is given by composition of correspondences. The identity morphism on an object $(X,p,m)$ is $p\circ[\Delta_X]\circ p$, with $\Delta_X \subset X\times X$ the diagonal.

We have a covariant functor $h\colon \mathsf{SmProj}_k \to \mathsf{Mot}_k$, sending $X$ to $h(X) = (X,\Delta_X,0)$ and sending a morphism $f\colon X\to Y$ to the class of its graph $[\Gamma_f] \in \mathrm{Corr}_0(X,Y) = \mathrm{Hom}\big(h(X),h(Y)\big)$. We usually write $f_*$ instead of $[\Gamma_f]$.

There is a tensor product in $\mathsf{Mot}_k$, making it into a $\mathbb{Q}$-linear tensor category, such that $h(X)\otimes h(Y) = h(X\times Y)$. The unit object for this tensor product is the motive $\mathbf{1} = h\big(\mathrm{Spec}(k)\big)$ of a point. If $M = (X,p,m)$ is an object of $\mathsf{Mot}_k$ and $n\in\mathbb{Z}$, we let $M(n) = (X,p,m+n)$. Then $M(n) = M\otimes \mathbf{1}(n)$, and $\mathbf{1}(1)$ is the Tate motive.

The Chow groups of a motive $M$ are defined by $\mathrm{CH}_i(M) = \mathrm{Hom}_{\mathsf{Mot}_k}\big(\mathbf{1}(i),M\big)$.

**2.2.** Let $X$ be a connected smooth projective $k$-variety of dimension $d$ with a rational point $a\in X(k)$. Then

$$\pi_0 = X\times\{a\} \quad\text{and}\quad \pi_+ = [\Delta_X] - X\times\{a\}$$

are orthogonal projectors, defining a decomposition

$$h(X) = h_0(X)\oplus h_+(X)\,. \tag{2.2.1}$$

If there is a need to specify the base point, we use the notation $h_0(X,a)$ and $h_+(X,a)$.

If $f\colon X \to \mathrm{Spec}(k)$ is the structural morphism, $a \circ f$ is an idempotent endomorphism of $X$ and $\pi_0$ is just the induced endomorphism $(a\circ f)_*$ of $h(X)$. In particular, $f_*\colon h_0(X) \to h\bigl(\mathrm{Spec}(k)\bigr) = \mathbf{1}$ is an isomorphism with inverse $a_*$. On Chow groups we have $\mathrm{CH}\bigl(h_0(X)\bigr) = \mathbb{Q}\cdot[a] \subset \mathrm{CH}(X)$.

**2.3.** We have a Künneth decomposition

$$h(X^n) = \bigl[h_0(X) \oplus h_+(X)\bigr]^{\otimes n} = \bigoplus_{J\subset\{1,\ldots,n\}} h_J(X^n)\,, \tag{2.3.1}$$

where, for $J \subset \{1,\ldots,n\}$, we define

$$h_J(X^n) = h_{\nu_1}(X) \otimes \cdots \otimes h_{\nu_n}(X) \qquad \text{with} \quad \nu_i = \begin{cases} + & \text{if } i \in J \\ 0 & \text{if } i \notin J. \end{cases}$$

The summand $h_{\{1,\ldots,n\}}(X^n) = h_+(X)^{\otimes n}$ will play a special role in what follows. Identifying $X^n \times X^n$ with $(X\times X)^n$, the projector onto this summand is $\pi_+^{\otimes n}$.

**2.4.** *Definition.* — Retain the assumptions and notation of 2.2. For $n \geqslant 1$ define

$$\gamma^n_{X,a}\colon h(X) \to h(X^n)$$

by $\gamma^n_{X,a} = \pi_+^{\otimes n} \circ \Delta^{(n)}_{X,*}$, where $\Delta^{(n)}_X\colon X \to X^n$ is the diagonal morphism. We use the same notation $\gamma^n_{X,a}$ for the induced map on Chow groups $\mathrm{CH}(X) \to \mathrm{CH}(X^n)$ or on Chow groups modulo an adequate equivalence relation. Finally, we define

$$\Gamma^n(X,a) \in \mathrm{CH}_d(X^n)$$

(with $d = \dim(X)$) to be the image under $\gamma^n_{X,a}$ of the fundamental class $[X] \in \mathrm{CH}_d(X)$.

**2.5.** If $J$ is a subset of $\{1,\ldots,n\}$, we identify $X^J$ with the closed subvariety of $X^n$ given by

$$\bigl\{(x_1,\ldots,x_n) \in X^n \bigm| x_i = a \quad \text{if } i \notin J\bigr\}\,.$$

Let $\phi_J = \phi_{X,J}\colon X^J \hookrightarrow X^n$ be the corresponding closed embedding. Let $\Delta^{(J)}_X \subset X^J$ be the small diagonal of $X^J$, viewed as a cycle on $X^n$.

If $\dim(X) = 0$ then $\gamma^n_{X,a}$ is the zero map. If $d = \dim(X)$ is positive, the cycle $\Gamma^n(X,a)$ is the modified diagonal cycle introduced by Gross and Schoen in [7]. Explicitly, for $d > 0$,

$$\Gamma^n(X,a) = \sum_{\emptyset\neq J\subset\{1,\ldots,n\}} (-1)^{n-|J|}\cdot\bigl[\Delta^{(J)}_X\bigr]\,.$$

**2.6.** *Remark.* — If $\dim(X) > 0$ we can refine (2.2.1) to a decomposition

$$h(X) = h_{2d}(X) \oplus h_\star(X) \oplus h_0(X)$$

where $h_{2d}(X)$ and $h_\star(X)$ are the submotives of $h(X)$ defined by the projectors $\pi_{2d} = \{a\} \times X$ and $\pi_\star = [\Delta_X] - X \times \{a\} - \{a\} \times X$, respectively. For the study of modified diagonals this does not lead to a refinement, however, as for $n \geqslant 2$ the morphism $\gamma^n_{X,a} = \pi_+^{\otimes n} \circ \Delta^{(n)}_{X,*}$ is the same as the morphism $\pi_\star^{\otimes n} \circ \Delta^{(n)}_{X,*}$. To see this we have to show that

$$\bigl(\Delta_X^{(n)} \times \mathrm{id}_{X^n}\bigr)^* \pi_+^{\otimes n} = \bigl(\Delta_X^{(n)} \times \mathrm{id}_{X^n}\bigr)^* \pi_\star^{\otimes n}$$

in $\mathrm{CH}(X \times X^n)$. (Use [5], Proposition 1.2.1.) Abbreviating $\Delta_X^{(n)}$ to $\Delta$ and writing $p_i\colon X^n \to X$ for the $i$th projection, the difference $\bigl(\Delta_X^{(n)} \times \mathrm{id}_{X^n}\bigr)^* \bigl[\pi_+^{\otimes n} - \pi_\star^{\otimes n}\bigr]$ is a linear combination of terms

$$(\Delta \times \mathrm{id}_{X^n})^*(\beta_1 \otimes \cdots \otimes \beta_n) = (\mathrm{id}_X \times p_1)^*\beta_1 \cdots (\mathrm{id}_X \times p_n)^*\beta_n$$

where $\beta_1, \ldots, \beta_n \in \bigl\{\pi_{2d}, \pi_\star\bigr\}$ and at least one $\beta_j$ equals $\pi_{2d}$. Now note that

$$(\mathrm{id}_X \times p_i)^*\pi_{2d} \cdot (\mathrm{id}_X \times p_j)^*\pi_{2d} = 0\,, \quad \text{and} \qquad (\mathrm{id}_X \times p_i)^*\pi_{2d} \cdot (\mathrm{id}_X \times p_j)^*\pi_\star = 0$$

for all $i \neq j$.

**2.7.** *Proposition.* — *Let $f\colon X \to Y$ be a morphism of connected smooth projective $k$-varieties. Let $a \in X(k)$ and let $b = f(a)$.*

(i) *The morphism $f_*\colon h(X) \to h(Y)$ is the direct sum of two morphisms $h_0(X,a) \to h_0(Y,b)$ and $h_+(X,a) \to h_+(Y,b)$.*

(ii) *We have $\gamma^n_{Y,b} \circ f_* = f_*^{\otimes n} \circ \gamma^n_{X,a}$ for all $n \geqslant 1$.*

(iii) *Suppose $f$ is generically finite of degree $N$. Then $N \cdot \Gamma^n_{Y,b} = f_*^{\otimes n}\bigl(\Gamma^n_{X,a}\bigr)$ for all $n \geqslant 1$.*

*Proof.* For (i), if $g\colon Y \to \mathrm{Spec}(k)$ is the structural morphism then $\pi_0(Y,b) = b_* \circ g_* = f_* \circ a_* \circ g_*$ and $\pi_0(X,a) = a_* \circ g_* \circ f_*$. Hence $\pi_0(Y,b) \circ f_* = f_* \circ \pi_0(X,a)$, and because $\pi_+ = \mathrm{id} - \pi_0$ also $\pi_+(Y,b) \circ f_* = f_* \circ \pi_+(X,a)$. Part (ii) readily follows from this and (iii) follows by applying (ii) to the class $[X]$. □

## 3. Some vanishing results

**3.1.** In what follows, we consider an adequate equivalence relation $\sim$ on algebraic cycles, as in [1], Section 3.1, and we write $\mathsf{Mot}_{k,\sim}$ for the corresponding category of motives. If $M$ is an object of $\mathsf{Mot}_{k,\sim}$, let $\mathrm{A}_i(M) = \mathrm{Hom}_{\mathsf{Mot}_{k,\sim}}\bigl(\mathbf{1}(i), M\bigr)$ and $\mathrm{A}(M) = \bigoplus_{i\in\mathbb{Z}} \mathrm{A}_i(M)$. In particular, if $X$ is a smooth projective $k$-variety, $\mathrm{A}_i(X) = \mathrm{CH}_i(X)/{\sim}$.

Given a connected smooth projective $k$-variety $X$ with base point $a \in X(k)$, the decomposition (2.3.1) induces a decomposition

$$\mathrm{A}(X^n) = \bigoplus_{J\subset\{1,\ldots,n\}} \mathrm{A}_J(X^n)\,.$$

This decomposition in general depends on the chosen base point.

Define a grading $\mathrm{A}(X^n) = \mathrm{A}_{[0]}(X^n) \oplus \cdots \oplus \mathrm{A}_{[n]}(X^n)$ by letting $\mathrm{A}_{[m]}(X^n)$ be the sum of all $\mathrm{A}_J(X^n)$ with $|J| = m$. In particular, $\mathrm{A}_{[n]}(X^n) = \mathrm{A}\big(h_+(X)^{\otimes n}\big)$. This grading is not to be confused with the one given by the dimension of cycles. We have an associated descending filtration $\mathrm{Fil}^\bullet$ of $\mathrm{A}(X^n)$, given by

$$\mathrm{Fil}^r \mathrm{A}(X^n) = \bigoplus_{m=0}^{n-r} \mathrm{A}_{[m]}(X^n)\,.$$

This means that the only terms that contribute to $\mathrm{Fil}^r \mathrm{A}(X^n)$ are those coming from submotives $h_{\nu_1}(X) \otimes \cdots \otimes h_{\nu_n}(X)$ involving at least $r$ factors $h_0(X)$. Alternatively, a class in $\mathrm{A}(X^n)$ lies in $\mathrm{Fil}^r \mathrm{A}(X^n)$ if and only if it is a linear combination of classes of the form $\phi_{J,*}(\alpha)$ for subsets $J \subset \{1,\ldots,n\}$ with $n - |J| \geqslant r$. In particular, if $J \subset \{1,\ldots,n\}$ and $\beta$ is a class in $\mathrm{Fil}^s \mathrm{A}(X^J)$ then $\phi_{J,*}(\beta) \in \mathrm{Fil}^{s+n-|J|} \mathrm{A}(X^n)$.

If $f\colon X \to Y$ is a morphism of smooth connected $k$-varieties and we take $b = f(a)$ as base point on $Y$, it follows from Proposition 2.7(i) that the induced map $(f^n)_*\colon \mathrm{A}(X^n) \to \mathrm{A}(Y^n)$ is a graded map. In particular, it is strictly compatible with the associated filtrations.

**3.2.** *Remark.* — If $\alpha \in \mathrm{A}(X)$ we have a class $\Delta^{(n)}_{X,*}(\alpha) \in \mathrm{A}(X^n)$. By definition, $\gamma^n_{X,a}(\alpha)$ is the projection of this class onto the summand $\mathrm{A}_{[n]}(X^n)$. Hence $\gamma^n_{X,a}(\alpha) = 0$ in $\mathrm{A}(X^n)$ if and only if $\Delta^{(n)}_{X,*}(\alpha) \in \mathrm{Fil}^1 \mathrm{A}(X^n)$.

**3.3.** As before, let $X$ be a connected smooth projective $k$-variety with a base point $a \in X(k)$. For $n \geqslant 1$, consider the morphism $\delta^{(n)} = (\mathrm{id}_{X^{m-1}} \times \Delta_X)\colon X^n \to X^{n+1}$; so $\delta^{(n)}(x_1,\ldots,x_{n-1},x_n) = (x_1,\ldots,x_{n-1},x_n,x_n)$.

If $J \subset \{1,\ldots,n\}$ is a subset with $n \notin J$, the morphism $\delta^{(n)}_*\colon h(X^n) \to h(X^{n+1})$ induces an isomorphism $h_J(X^n) \xrightarrow{\sim} h_J(X^{n+1})$. If $n \in J$, let $\hat{J} = J \cup \{n+1\}$. In this case we have a commutative diagram

$$\begin{array}{ccccc} X^{|J|} & \xrightarrow{\sim} & X^J & \xrightarrow{\phi_J} & X^n \\ {\scriptstyle \delta^{(|J|)}}\downarrow & & \downarrow{\scriptstyle \delta^{(n)}|_{X^J}} & & \downarrow{\scriptstyle \delta^{(n)}} \\ X^{|J|+1} & \xrightarrow{\sim} & X^{\hat{J}} & \xrightarrow{\phi_{\hat{J}}} & X^{n+1}\,. \end{array}$$

It follows that $\delta^{(n)}_*\colon \mathrm{A}(X^n) \to \mathrm{A}(X^{n+1})$ respects the filtrations.

**3.4.** *Proposition.* — *Let $X$ be a connected smooth projective $k$-variety with a base point $a \in X(k)$. Let $n$ be a positive integer.*

(i) *If $\gamma^n_{X,a}(\alpha) = 0$ for some $\alpha \in \mathrm{A}(X)$ then $\gamma^{n+1}_{X,a}(\alpha) = 0$.*

(ii) *We have $\Gamma^n(X,a) = 0$ in $\mathrm{A}(X^n)$ if and only if $\gamma^n_{X,a}\colon \mathrm{A}(X) \to \mathrm{A}(X^n)$ is the zero map.*

*Proof.* (i) As remarked in 3.2, $\gamma^n_{X,a}(\alpha) = 0$ if and only if $\Delta^{(n)}_{X,*}(\alpha) \in \mathrm{Fil}^1 \mathrm{A}(X^n)$. Now use that $\Delta^{(n+1)} = \delta^{(n)} \circ \Delta^{(n)}$ and the fact just explained that $\delta^{(n)}_*$ respects the filtrations.

(ii) Assume that $\Gamma^n(X,a) = 0$ in $\mathrm{A}(X^n)$ and let $\alpha \in \mathrm{A}(X)$. Because the map $\gamma^n_{X,a}$ is linear and $\gamma^n_{X,a}[X] = \Gamma^n(X,a)$ by definition, we may assume that $\alpha \in \mathrm{A}_i(X)$ for some $i < \dim(X)$.

We know that the class of the small diagonal $\Delta_X^{(n)}$ lies in $\mathrm{Fil}^1\mathrm{A}(X^n)$; this means we can write

$$\big[\Delta_X^{(n)}\big] = \sum_{J\subsetneq\{1,\ldots,n\}} \beta_J$$

with $\beta_J \in \mathrm{A}_J(X^n)$. By definition of $\mathrm{A}_J(X^n)$ we have $\beta_J = \phi_{J,*}(b_J)$ for some class $b_J$ on $X^J$. To prove that $\Delta_{X,*}^{(n)}(\alpha) = \big[\Delta_X^{(n)}\big]\cdot \mathrm{pr}_n^*(\alpha)$ lies in $\mathrm{Fil}^1\mathrm{A}(X^n)$ we now only have to remark that

$$\beta_J\cdot \mathrm{pr}_n^*(\alpha) = \phi_{J,*}\Big(b_J\cdot(\mathrm{pr}_n\circ\phi_J)^*(\alpha)\Big)\,,$$

and that for $J\subsetneq\{1,\ldots,n\}$ any class in the image of $\phi_{J,*}$ lies in $\mathrm{Fil}^1\mathrm{A}(X^n)$. □

For the classes $\Gamma^n(X,a)$ the stability result in (i) is O'Grady's Proposition 2.4 in [12]. As we shall now show, part (ii) of the proposition can be refined. The idea is that we can view $\Gamma^{m+n}(X,a)$ as a correspondence from $X^m$ to $X^n$.

**3.5.** *Proposition. — Let $X$ be a connected smooth projective $k$-variety with base point $a\in X(k)$. Suppose $m$ and $n$ are positive integers such that $\Gamma^{m+n}(X,a)=0$ in $\mathrm{A}(X^{m+n})$. Then*

$$\sum_{\emptyset\neq K\subset\{1,\ldots,m\}} (-1)^{|K|}\cdot\gamma_{X,a}^n\Big(\Delta_X^{(K),*}(\xi)\Big) = 0 \qquad \textit{in } \mathrm{A}(X^n)$$

*for all classes $\xi\in\mathrm{CH}^{>0}(X^m)$. Here $\Delta_X^{(K)}\colon X\to X^m$ denotes the composition of the diagonal $\Delta_X\colon X\to X^K$ and the closed embedding $\phi_K\colon X^K\hookrightarrow X^m$.*

*Proof.* We may assume $\dim(X)>0$. By definition,

$$\Gamma^{m+n}(X,a) = \sum_{\emptyset\neq J\subset\{1,\ldots,m+n\}} (-1)^{m+n-|J|}\cdot\big[\Delta_X^{(J)}\big]\,.$$

Write the non-empty subsets $J\subset\{1,\ldots,m+n\}$ as $J=K\cup L$ with $K\subset\{1,\ldots,m\}$ and $L=\{m+1,\ldots,m+n\}$. Viewing $\big[\Delta_X^{(J)}\big]$ as a correspondence from $X^m$ to $X^n$, its effect on cycle classes is given by $\xi\mapsto\Delta_{X,*}^{(L)}\big(\Delta_X^{(K),*}(\xi)\big)$, where in the notation $\Delta_{X,*}^{(L)}$ we treat $L$ as a subset of $\{1,\ldots,n\}$.

If $K=\emptyset$, the map $\Delta_X^{(K)}$ is the inclusion of the point $(a,\ldots,a)$ in $X^m$; so $\Delta_X^{(K),*}(\xi)=0$ for $\xi\in\mathrm{CH}^{>0}(X^m)$. If $K\neq\emptyset$ then

$$\sum_L (-1)^{m+n-|K\cup L|}\cdot\Delta_{X,*}^{(K\cup L)}(\xi) = (-1)^{m-|K|}\cdot\gamma_{X,a}^n\Big(\Delta_X^{(K),*}(\xi)\Big)$$

and the proposition follows. □

**3.6.** *Corollary. — If $\Gamma^{m+n}(X,a)=0$ in $\mathrm{A}(X^{m+n})$ then $\gamma_{X,a}^n\colon\mathrm{A}(X)\to\mathrm{A}(X^n)$ is zero on the image of the product map $\mathrm{A}^{>0}(X)^{\otimes m}\to\mathrm{A}(X)$. In particular, if $\Gamma^{n+1}(X,a)=0$ then $\gamma_{X,a}^n(\xi)=0$ for all $\xi\in\mathrm{A}^{>0}(X)$.*

*Proof.* In the proposition, take $\xi=\xi_1\times\cdots\times\xi_m$ for classes $\xi_i\in\mathrm{CH}^{>0}(X)$. For $K\neq\{1,\ldots,m\}$ we have $\Delta_X^{(K),*}(\xi)=0$. For $K=\{1,\ldots,m\}$ we have $\Delta_X^{(K),*}(\xi)=\xi_1\cdots\xi_m$. Hence we find that $\gamma_{X,a}^n(\xi_1\cdots\xi_m)=0$. □

**3.7.** *Corollary.* — *Let $f\colon X \to Y$ be a surjective morphism of connected smooth projective $k$-varieties. Let $a \in X(k)$ and $b = f(a)$. Let $r = \dim(X) - \dim(Y)$. If $\Gamma^{n+r}(X,a) = 0$ for some $n \geqslant 1$ then $\Gamma^n(Y,b) = 0$.*

*Proof.* There exists a vector bundle $\mathscr{E}$ on $Y$ such that $X$, as a scheme over $Y$, is isomorphic to a closed subscheme of the projective bundle $\mathbb{P}(\mathscr{E})$. Let $\ell = c_1\big(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)\big)$ and write $\ell_X$ for its pull-back to $X$. We have $f_*(\ell_X^r) \in \mathrm{CH}^0(Y)$; so $f_*(\ell_X^r) = N \cdot [Y]$ for some integer $N$. By pulling back to the generic point of $Y$ we see that $N \neq 0$. So by 2.7(ii), $\Gamma^n(Y,b)$ is proportional to $f_*^{\otimes n}\big(\gamma^n_{X,a}(\ell_X^r)\big)$, which vanishes by Corollary 3.6. □

**3.8.** *Remark.* — O'Grady has made the conjecture that for a hyperkähler variety $X$ of dimension $2n$ there should exist a base point $a \in X(k)$ such that $\Gamma^{2n+1}(X,a) = 0$ in $\mathrm{CH}(X^{2n+1})$. By Corollary 3.6, if this is true then the intersection pairing $\mathrm{Pic}(X)^{\otimes 2n} \to \mathrm{CH}_0(X)$ takes values in $\mathbb{Q} \cdot [a]$. This last property is known (for a suitable choice of $a \in X(k)$) for Hilbert schemes of K3 surfaces and Fano varieties of cubic 4-folds by results of Voisin [13], Theorems 1.4(2) and 1.5, and also for generalized Kummer varieties by a result of Fu [6], Theorem 1.6.

The vanishing of $\Gamma^n(X,a)$ implies that $n > \dim(X)$; see Theorem 4.1 below. So for $X$ hyperkähler of dimension $2n$, the vanishing of $\Gamma^{2n+1}(X,a)$ is the strongest possible result. By Corollary 3.7, if O'Grady's conjecture is true then for all varieties $Y$ dominated by $X$ we again have the optimal result that $\Gamma^{\dim(Y)+1}(Y,b)$ vanishes in the Chow ring. This suggests that only very special varieties are dominated by a hyperkähler variety. Another indication of this is given by a result of Lin [8], Theorem 1.1. He takes for $X$ a Hilbert scheme of points on a complex K3 surface with infinitely many rational curves; then he proves that if $X$ dominates a variety $Y$ with $\dim(Y) < \dim(X)$, then $Y$ is rationally connected.

Part (ii) of the next lemma gives a refinement of the stability result in Proposition 3.4(i).

**3.9.** *Lemma.* — *In the situation of* 3.1, *suppose* $\big[\Delta_X^{(n)}\big] \in \mathrm{Fil}^r \mathrm{A}(X^n)$ *for some* $r \geqslant 1$.
(i) *For all $i \in \{0,\ldots,r\}$ we have* $\big[\Delta_X^{(n-i)}\big] \in \mathrm{Fil}^{r-i}\mathrm{A}(X^{n-i})$.
(ii) *For all $i \geqslant 0$ we have* $\big[\Delta_X^{(n+i)}\big] \in \mathrm{Fil}^{r+i}\mathrm{A}(X^{n+i})$.

*Proof.* In both statements, it suffices to do the case $i = 1$. Part (i) readily follows from the definitions by taking the image of $\big[\Delta_X^{(n)}\big]$ under a projection $X^n \to X^{n-1}$.

For (ii), suppose $\big[\Delta_X^{(n)}\big] \in \mathrm{Fil}^r\mathrm{A}(X^n)$ with $r \geqslant 1$. In particular, $\Gamma^n(X,a) = 0$ in $\mathrm{A}(X^n)$, which by Proposition 3.4(i) implies that $\Gamma^{n+1}(X,a) = 0$ in $\mathrm{A}(X^{n+1})$. We can write this as an identity

$$\big[\Delta_X^{(n+1)}\big] = \sum_J (-1)^{n-|J|} \cdot \big[\Delta_X^{(J)}\big]$$

in $\mathrm{A}(X^{n+1})$, where the sum runs over the non-empty subsets $J \subsetneq \{1,\ldots,n+1\}$, and where we recall that $\Delta_X^{(J)}$ is the small diagonal of $X^J$, viewed as a cycle on $X^{n+1}$. If $|J| \leqslant n-r$ then it is clear that $\Delta_X^{(J)} \in \mathrm{Fil}^{r+1}\mathrm{A}(X^{n+1})$. If not, then $n+1-r \leqslant |J| \leqslant n$ and by the assumption that $\big[\Delta_X^{(n)}\big] \in \mathrm{Fil}^r\mathrm{A}(X^n)$ together with (i) the small diagonal on $X^J$ lies in $\mathrm{Fil}^{|J|-n+r}\mathrm{A}(X^J)$. Since $\Delta_X^{(J)}$ is obtained by pushing forward this small diagonal via $\phi_J\colon X^J \hookrightarrow X^{n+1}$, it again follows that $\big[\Delta_X^{(J)}\big] \in \mathrm{Fil}^{r+1}\mathrm{A}(X^{n+1})$. □

We now investigate how changing the base point affects the vanishing of $\Gamma^n(X,a)$.

**3.10.** *Proposition.* — *Let $X$ be a connected smooth projective $k$-variety. Let $a$ and $a'$ be $k$-valued points of $X$. If $\Gamma^n(X,a) = 0$ in $\mathrm{A}(X^n)$ for some $n > 1$ then $\Gamma^{2n-2}(X,a') = 0$ in $\mathrm{A}(X^{2n-2})$.*

*Proof.* Let $\pi'_+ = [\Delta_X] - X \times \{a'\}$ be the projector that cuts out the motive $h_+(X,a')$. We write it as $\pi'_+ = \pi_+ + X \times \big[\{a\} - \{a'\}\big]$. This gives

$$(\pi'_+)^{\otimes(2n-2)} = \sum_{J \subset \{1,\ldots,2n-2\}} \pi_+^{\otimes J} \otimes \Big(X \times \big[\{a\} - \{a'\}\big]\Big)^{\otimes J'}, \tag{3.10.1}$$

where we write $J' = \{1,\ldots,2n-2\} \setminus J$.

By Corollary 3.6, the assumption that $\Gamma^n(X,a) = 0$ implies that $\gamma^m_{X,a}(a') = 0$ for all $m \geqslant n-1$. But $\gamma^m_{X,a}(a') = \big[\{a'\} - \{a\}\big]^{\otimes m}$; so in (3.10.1) we may sum only over the subsets $J \subset \{1,\ldots,2n-2\}$ of cardinality $\geqslant n$. On the other hand, by 3.9(ii) we have $\big[\Delta_X^{(2n-2)}\big] \in \mathrm{Fil}^{n-1}\mathrm{A}(X^{2n-2})$, which means that $\pi_+^{\otimes J} \otimes \big(X \times \big[\{a\} - \{a'\}\big]\big)^{\otimes J'}$ kills $\big[\Delta_X^{(2n-2)}\big]$ for all index sets $J$ with $|J| > (2n-2) - (n-1) = n-1$. Together this gives that $\big(\pi'_+\big)^{\otimes(2n-2)}\big[\Delta_X^{(2n-2)}\big] = 0$, i.e., $\Gamma^{2n-2}(X,a') = 0$. $\square$

As an example, on a K3 surface $X$ with distinguished point class $o_X$ we have $\Gamma^3(X,o_X) = 0$ by [3], Proposition 3.2. By Proposition 3.10 it follows that for any base point $a \in X(k)$ we have $\Gamma^4(X,a) = 0$, and by Corollary 3.6 we in fact find that $\Gamma^3(X,a) = 0$ if and only if $a = o_X$ in $\mathrm{CH}_0(X)$.

We finish this section by reproving Proposition 0.2 of O'Grady's paper [12], which is an easy consequence of the above.

**3.11.** *Proposition.* — *Let $X$ and $Y$ be connected smooth projective $k$-varieties with base points $a \in X(k)$ and $b \in Y(k)$. Suppose that $\Gamma^m(X,a) = 0$ in $\mathrm{A}(X^m)$ and $\Gamma^n(Y,b) = 0$ in $\mathrm{A}(Y^n)$ for some positive integers $m$ and $n$. Then $\Gamma^{m+n-1}\big(X \times Y, (a,b)\big) = 0$ in $\mathrm{A}\big((X\times Y)^{m+n-1}\big)$.*

*Proof.* By Lemma 3.9(ii) we have

$$\big[\Delta_X^{(m+n-1)}\big] \in \mathrm{Fil}^n\mathrm{A}(X^{m+n-1}), \qquad \big[\Delta_Y^{(m+n-1)}\big] \in \mathrm{Fil}^m\mathrm{A}(X^{m+n-1}).$$

This means we can write $\big[\Delta_X^{(m+n-1)}\big] = \sum_J \phi_{X,J,*}(\alpha_J)$, where the sum runs over the subsets $J \subset \{1,\ldots,m+n-1\}$ of cardinality at most $m-1$, and where $\alpha_J$ is a class on $X^J$. Similarly, $\big[\Delta_Y^{(m+n-1)}\big] = \sum_K \phi_{Y,K,*}(\beta_K)$, where the subsets $K \subset \{1,\ldots,m+n-1\}$ have cardinality at most $n-1$ and $\beta_K \in \mathrm{A}(Y^K)$.

Writing $p\colon (X\times Y)^{m+n-1} \to X^{m+n-1}$ and $q\colon (X\times Y)^{m+n-1} \to Y^{m+n-1}$ for the projections,

$$\big[\Delta_{X\times Y}^{m+n-1}\big] = p^*\big[\Delta_X^{(m+n-1)}\big] \cdot q^*\big[\Delta_Y^{(m+n-1)}\big].$$

Given subsets $J$, $K \subset \{1,\ldots,m+n-1\}$ of cardinality at most $m-1$ and $n-1$, respectively, there is an index $\nu \in \{1,\ldots,m+n-1\}$ that is not in $J \cup K$. Setting $L = \{1,\ldots,\hat{\nu},\ldots,m+n-1\}$ it is then clear that $p^*\phi_{X,J,*}(\alpha_J)\cdot q^*\phi_{Y,K,*}(\beta_K)$ is a class in the image of the push-forward under $\phi_{X\times Y,L}\colon (X\times Y)^L \hookrightarrow (X\times Y)^{m+n-1}$. Hence, $\big[\Delta_{X\times Y}^{m+n-1}\big] \in \mathrm{Fil}^1\mathrm{A}\big((X\times Y)^{m+n-1}\big)$, which means that $\Gamma^{m+n-1}(X\times Y) = 0$. $\square$

## 4. Vanishing results on curves and abelian varieties

We begin by recalling a result of O'Grady [12] about the vanishing of modified diagonals in cohomology.

**4.1.** *Theorem. — Let $X$ be a connected smooth projective $k$-variety with base point $a \in X(k)$. Let $d = \dim(X)$ and let $e$ be the dimension of the image of the Albanese map* $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$. *Then $\Gamma^n(X,a) \sim_{\mathrm{hom}} 0$ if and only if $n > d + e$.*

*Proof.* Let $H^\bullet$ denote $\ell$-adic cohomology for some prime $\ell \neq \mathrm{char}(k)$. Throughout, we view $H^\bullet(X) = \oplus_{i=0}^{2d} H^i(X)$ as a superspace; in particular, if $i$ is odd then $\mathrm{Sym}^m(H^i)$ has $\wedge^m H^i(X)$ as its underlying vector space.

The cohomology class $[\Gamma^n]$ of $\Gamma^n(X,a)$ lies in the degree $2d(n-1)$-part of $\mathrm{Sym}^n\big(H^\bullet(X)\big)$. We have

$$\mathrm{Sym}^n\big(H^\bullet(X)\big) = \bigoplus_{\substack{m=(m_0,\ldots,m_d)\\ |m|=n}} \bigotimes_{j=0}^{2d} \mathrm{Sym}^{m_j}\big(H^j(X)\big)\,,$$

where the summand $S(m) = \otimes_j \mathrm{Sym}^{m_j}\big(H^j(X)\big)$ lies in degree $\sum_{j=0}^{2d} j \cdot m_j$. By Remark 2.6 we know that the component of $[\Gamma^n]$ in $S(m)$ is zero if $m_0 > 0$ or $m_{2d} > 0$. Next consider a sequence $m = (m_0, m_1, \ldots, m_{2d})$ with $m_0 = m_{2d} = 0$. The component of $[\Gamma^n]$ in $S(m)$ is then the same as the component of the cohomology class of the small diagonal $\Delta_X^{(n)}$. If $\mu = (m_{2d}, m_{2d-1}, \ldots, m_0)$ is the reverse sequence, the intersection pairing on $\mathrm{Sym}^n\big(H^\bullet(X)\big) \subset H^\bullet(X^n)$ restricts to a perfect pairing $S(m) \times S(\mu) \to k$, and for $m' \neq \mu$ the pairing $S(m) \times S(m') \to k$ is zero. For $\beta \in S(\mu)$ we have $\big[\Delta_X^{(n)}\big] \cdot \beta = \deg\big(\Delta^*(\beta)\big)$, and we claim that this is zero whenever $m_{2d-1} > 2e$. Assuming this for a moment, the "if" statement in the theorem follows, as the highest degree we can get under the restrictions $m_0 = m_{2d} = 0$ and $m_{2d-1} \leqslant 2e$ is $2e(2d-1) + (n-2e)(2d-2) = 2e + 2nd - 2n$, so that for $n > d + e$ we cannot reach degree $2d(n-1)$.

It remains to be shown that for $i > 2e$ the multiplication map $\Delta^*\colon \mathrm{Sym}^i H^1(X) \to H^i(X)$ is zero. For this we use that $H^1(\mathrm{alb}) = \mathrm{alb}^*\colon H^1(\mathrm{Alb}_X) \to H^1(X)$ is an isomorphism. We have a commutative diagram

$$\begin{tikzcd} H^i(\mathrm{Alb}_X) = \mathrm{Sym}^i H^1(\mathrm{Alb}_X) \arrow[r, "\mathrm{Sym}^i H^1(\mathrm{alb})", "\sim"'] \arrow[dr, "H^i(\mathrm{alb})"'] & \mathrm{Sym}^i H^1(X) \arrow[d, "\Delta^*"] \\ & H^i(X) \end{tikzcd}$$

But $H^i(\mathrm{alb})$ factors through $H^i\big(\mathrm{alb}(X)\big)$, which is zero for $i > 2e$.

Finally we show that $\Gamma^{d+e}(X,a)$ is not homologically trivial, which by Proposition 3.4(i) gives the "only if" in the theorem. The only sequence $m = (m_0, m_1, \ldots, m_{2d})$ with $|m| = d + e$ and $m_0 = m_{2d} = 0$ that reaches degree $2d(d+e-1)$ is $m = (0, \ldots, 0, d-e, 2e, 0)$. With $\mu$ the reverse sequence, it suffices to produce an element $\beta \in S(\mu) = \mathrm{Sym}^{2e} H^1(X) \otimes \mathrm{Sym}^{d-e} H^2(X)$ for which $\Delta^*(\beta)$ has degree $\neq 0$. For this we take polarizations $L_1 \in H^2(\mathrm{Alb}_X)$ and $L_2 \in H^2(X)$;

then take $\beta = \mathrm{Sym}^{2e} H^1(\mathrm{alb})\big(L_1^e\big) \otimes L_2^{d-e}$. Because the map $H^{2e}(\mathrm{alb})$ is injective, $\Delta^*(\beta)$ has positive degree and we are done. $\square$

Next we turn to abelian varieties. The result we prove was conjectured by O'Grady in the first version of [12]. He also proved it for $g \leqslant 2$.

**4.2.** *Theorem. — Let $X$ be an abelian variety of dimension $g$ over a field $k$. Let $a \in X(k)$ be a base point. Then $\Gamma^n(X,a) = 0$ in $\mathrm{CH}(X^n)$ for all $n > 2g$.*

*Proof.* We give $X$ the group structure for which $a$ is the origin. For $m \in \mathbb{Z}$ let $\mathrm{mult}(m)\colon X \to X$ be the endomorphism given by multiplication by $m$. By [5], Corollary 3.2, we have a motivic decomposition $h(X) = \oplus_{i=0}^{2g} h_i(X)$ in $\mathsf{Mot}_k$ that is stable under all endomorphisms $\mathrm{mult}(m)_*$, and such that $\mathrm{mult}(m)_*$ is multiplication by $m^i$ on $h_i(X)$. (The result is stated in op. cit. for the cohomological theory but is easily transcribed into the homological language.) The relation with (2.2.1) is that $h_0(X,a) = h_0(X)$ and $h_+(X,a) = \bigoplus_{i>0} h_i(X)$.

For $n \geqslant 1$ this induces a decomposition

$$h(X^n) = \bigoplus_{\boldsymbol{i}=(i_1,\ldots,i_n)} \bigotimes_{j=1}^{n} h_{i_j}(X)\,,$$

where the sum runs over the elements $\boldsymbol{i} = (i_1,\ldots,i_n)$ in $\{0,\ldots,2g\}^n$. Under this decomposition we have

$$h_r(X^n) = \bigoplus_{|\boldsymbol{i}|=r} \bigotimes_{j=1}^{n} h_{i_j}(X)\,,$$

where the sum runs over the $n$-tuples $\boldsymbol{i}$ with $|\boldsymbol{i}| = i_1 + \cdots + i_n$ equal to $r$.

Now observe that $\big[\Delta_X^{(n)}\big] \in \mathrm{CH}\big(h_{2g}(X^n)\big)$, because $\mathrm{mult}(m)_*\big[\Delta_X^{(n)}\big] = m^{2g}\cdot\big[\Delta_X^{(n)}\big]$ for all $m$. The theorem follows, since for $n > 2g$ and $\boldsymbol{i} = (i_1,\ldots,i_n)$ in $\{0,\ldots,2g\}^n$ with $|\boldsymbol{i}| = 2g$ there is at least one index $j$ with $i_j = 0$. $\square$

Next we turn to curves. Part (i) of the next result is due to Gross and Schoen; see [7], Proposition 3.1. This result is also an immediate consequence of Theorem 4.1. Part (ii) is due to Polishchuk; see [11], Corollary 4.4(iv). Part (iii) is essentially due to Polishchuk and the first author in [9] (see especially the proof of loc. cit. Theorem 8.5) but we need to combine the calculations that are done there with some known facts about the Chow ring of the Jacobian, as we shall now explain.

**4.3.** *Theorem. — Let $C$ be a complete nonsingular curve of genus $g$ over a field $k$ with a base point $a \in X(k)$. Then*

(i) $\Gamma^n(C,a) \sim_{\mathrm{hom}} 0$ *for all* $n > 2$;

(ii) $\Gamma^n(C,a) \sim_{\mathrm{alg}} 0$ *(modulo torsion) for all* $n > \mathrm{gonality}(C)$;

(iii) $\Gamma^n(C,a) = 0$ *in* $\mathrm{CH}_1(C^n)$ *for all* $n > g+1$.

*Proof.* For curves of genus 0 the result is trivial. (Because we work modulo torsion, we may extend the ground field and assume $C = \mathbb{P}^1$; then note that the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ is rationally equivalent to $\big(\{\mathrm{pt}\} \times \mathbb{P}^1\big) + \big(\mathbb{P}^1 \times \{\mathrm{pt}\}\big)$.) Hence we may assume $g > 0$. Let $\iota\colon C \to J$ be the closed embedding associated with the base point $a$. As discussed above, $h(J) = \oplus_{i=0}^{2g} h_i(J)$. This means we can decompose $\bigl[\iota(C)\bigr] \in \mathrm{CH}_1(J)$ as

$$\bigl[\iota(C)\bigr] = \sum_{i=0}^{2g} \gamma_i$$

with $\gamma_i \in \mathrm{CH}\bigl(h_i(J)\bigr)$. In particular, for $m \in \mathbb{Z}$ we have $\mathrm{mult}(m)_*(\gamma_i) = m^i \cdot \gamma_i$. It is known that:

(a) $\gamma_i \neq 0$ only for $i \in \{2, \ldots, g+1\}$;

(b) $\gamma_i$ is torsion modulo algebraic equivalence for $i > \mathrm{gonality}(C)$;

(c) $\gamma_i$ is homologically trivial for $i \neq 2$.

In fact, (c) holds because $\mathrm{mult}(m)_*$ acts on $H^{2g-2}(J)$ as multiplication by $m^2$, (a) follows from the precise summation range in the main theorem of [2] (in the notation of [2] our $\gamma_i$ lies in $\mathrm{CH}^{g-1}_{i-2}(J)$), and (b) is a result of Colombo and van Geemen [4].

We denote by $C^{[d]}$ the $d$th symmetric power of $C$ and let $C^{[\bullet]} = \coprod_{d \geqslant 0} C^{[d]}$, which is a monoid scheme. Let $\mathrm{CH}\bigl(C^{[\bullet]}\bigr) = \oplus_{d\geqslant 0} \mathrm{CH}\bigl(C^{[d]}\bigr)$, which is a $\mathbb{Q}$-algebra for the Pontryagin product. The maps $u_d\colon C^{[d]} \to J$ give us a morphism $u\colon C^{[\bullet]} \to J$, which induces a homomorphism $u_*\colon \mathrm{CH}\bigl(C^{[\bullet]}\bigr) \to \mathrm{CH}(J)$. By [9], Theorem 3.4, there is a $\mathbb{Q}$-subalgebra $\mathbb{K} \subset \mathrm{CH}\bigl(C^{[\bullet]}\bigr)$ such that the restriction of $u_*$ to $\mathbb{K}$ gives an isomorphism $\mathbb{K} \xrightarrow{\sim} \mathrm{CH}(J)$. Further, by ibid., Lemma 8.4 and the proof of Theorem 8.5, all classes $\Gamma^n(C,a)$ lie in this subalgebra $\mathbb{K}$ and we have, for $n \geqslant 2$,

$$u_*\bigl(\Gamma^n(C,a)\bigr) = n! \cdot \sum_{i=0}^{2g} S(i,n) \cdot \gamma_i\,,$$

where $S(i,n)$ denotes the Stirling number of the second kind. Note that $S(i,n) = 0$ if $n > i$. Putting together these facts, the theorem follows from (a)–(c) above. $\square$

**4.4.** Let us now discuss to what extent the above results are sharp.

For abelian varieties, our result in Theorem 4.2 is sharp, since by Theorem 4.1 $\Gamma^{2g}(X,a)$ is not even homologically trivial. The same remark applies to part (i) of Theorem 4.3.

Part (ii) of Theorem 4.3 is conjecturally sharp for the generic curve $C$ of genus $g$. In fact, under the genericity assumption it is expected that $\gamma_i$ is not algebraically trivial for $i = \lfloor (g+3)/2 \rfloor = \mathrm{gonality}(C)$. We refer to [14] for recent results (in characteristic 0) towards this conjecture.

Finally, (iii) of Theorem 4.3 is sharp for the generic pointed curve in characteristic 0. This is proven by the second author in [16], Proposition 5.14, which gives $\gamma_{g+1} \neq 0$.

## 5. Double covers

The following result proves a conjecture made by O'Grady in [12]. We had originally hoped to extend this to more general covers, but our method leads to some non-trivial combinatorial problems. As we just learned that Claire Voisin has obtained such a more general result using a different argument, we restrict ourselves to double covers. As in Section 3, we consider an adequate equivalence relation $\sim$ and write $\mathrm{A}(X) = \mathrm{CH}(X)/{\sim}$.

**5.1.** *Theorem.* — *Let $f\colon X \to Y$ be a double cover. Let $\sigma$ be the corresponding involution of $X$. Let $a \in X(k)$ be a base point such that $a \sim \sigma(a)$, and write $b = f(a)$. If $\Gamma^n(Y,b) = 0$ in $\mathrm{A}(Y^n)$ then $\Gamma^{2n-1}(X,a) = 0$ in $\mathrm{A}(X^{2n-1})$.*

**5.2.** As a preparation for the proof we need to introduce some notation. Given an integer $m$ and a subset $J \subset \{1,\ldots,m\}$, let $Z_J \subset X^m$ denote the image of the morphism $\zeta_J\colon X \to X^m$ for which

$$\mathrm{pr}_j \circ \zeta_J = \begin{cases} \sigma & \text{if } j \in J; \\ \mathrm{id}_X & \text{if } j \notin J. \end{cases}$$

If $J'$ is the complement of $J$, we have $Z_{J'} = Z_J$. Further, $Z_\emptyset = Z_{\{1,\ldots,m\}} = \Delta_X^{(m)}$.

For $r \leqslant m$, let

$$V_r = \sum_{\substack{J\subset\{1,\ldots,m\}\\ |J|=r}} \bigl[Z_J\bigr]\,.$$

It follows from the previous remarks that $V_{m-r} = V_r$ and that $V_0 = V_m = \bigl[\Delta_X^{(m)}\bigr]$. We write $V_r^{(m)}$ if there is a need to specify $m$.

The pull-back of the class $\bigl[\Delta_Y^{(m)}\bigr]$ is $\frac{1}{2}\cdot\sum_{r=0}^m V_r$.

**5.3.** For $(i,j) \in \{1,\ldots,m\}\times\{1,\ldots,m+1\}$, consider the morphism $\phi_{i,j}\colon X^m \to X^{m+1}$ given by

$$(x_1,\ldots,x_m) \mapsto \bigl(x_1,\ldots,x_{j-1},\sigma(x_i),x_j,\ldots,x_m\bigr)\,.$$

Let $\Phi$ be the sum of the graphs of the $\phi_{i,j}$; so, $\Phi = \sum_{i,j}\bigl[\Gamma_{\phi_{i,j}}\bigr]$. This is a correspondence of degree 0 from $X^m$ to $X^{m+1}$. Again we write $\Phi^{(m)}$ if we want to specify $m$.

**5.4.** *Lemma.* — *For $r \leqslant m$ we have*

$$\Phi_*\bigl(V_r^{(m)}\bigr) = r(m+1-r)\cdot V_r^{(m+1)} + (r+1)(m-r)\cdot V_{r+1}^{(m+1)}\,.$$

*Proof.* Given $j \in \{1,\ldots,m\}$, let $\alpha_j\colon \{1,\ldots,m\} \to \{1,\ldots,m+1\}$ be the strictly increasing map such that $j$ is not in the image of $\alpha_j$. Fix some subset $K \subset \{1,\ldots,m+1\}$. We have to count the number of choices for $J \subset \{1,\ldots,m\}$ with $|J| = r$ and an index pair $(i,j)$ as above such that $\phi_{i,j,*}[Z_J] = [Z_K]$. It is clear that there are no such choices unless $|K| = r$ or $|K| = r+1$. If $|K| = r$ then we can choose $j \notin K$ and $i \in \alpha_j^{-1}(K)$ arbitrarily; once these choices are made there is a unique $J \subset \{1,\ldots,m\}$ with $|J| = m$ such that $\phi_{i,j,*}[Z_J] = [Z_K]$. Note that the number of choices in this case is $(m+1-r)r$. Similarly, if $|K| = r+1$ we have to choose $j \in K$ and $i \notin \alpha_j^{-1}(K)$ and then there is again a unique choice for $J$ such that $\phi_{i,j,*}[Z_J] = [Z_K]$. In this case the number of choices is $(r+1)(m-r)$. □

**5.5.** *Lemma.* — *Notation and assumptions as in Theorem 5.1. If $\Gamma^n(Y,b) = 0$ in $\mathrm{A}(Y^n)$ then $\sum_{r=0}^{m+n} r^j(m+n-r)^j\cdot V_r^{(m+n)}$ lies in $\mathrm{Fil}^1\mathrm{A}(X^{m+n})$ for all $m \geqslant j \geqslant 0$.*

*Proof.* We use induction on $m$. For $m = 0$ the assumption that $\Gamma^n(Y,b) = 0$ means that $\bigl[\Delta_Y^{(n)}\bigr] \in \mathrm{Fil}^1\mathrm{A}(Y^n)$. Pulling back to $X^n$ and using that $a \sim \sigma(a)$ we find that $\sum_{r=0}^n V_r$ lies in $\mathrm{Fil}^1\mathrm{A}(X^n)$.

Assuming the assertion is true for some $m$, let us prove it for $m+1$. By Proposition 3.4, $\left[\Delta_Y^{(n)}\right] \in \mathrm{Fil}^1 \mathrm{A}(Y^n)$ implies that $\left[\Delta_Y^{(n+1)}\right] \in \mathrm{Fil}^1 \mathrm{A}(Y^{n+1})$. So the assertion for $j < m+1$ follows from the induction hypothesis, replacing $n$ with $n+1$.

It remains to consider the case $j = m+1$. Let

$$W = \Phi_*^{(m+n)} \left( \sum_{r=0}^{m+n} r^m (m+n-r)^m \cdot V_r^{(m+n)} \right).$$

By the induction assumption, $\sum_{r=0}^{m+n} r^m(m+n-r)^m \cdot V_r^{(m+n)}$ lies in $\mathrm{Fil}^1 \mathrm{A}(X^{m+n})$, and by the same argument as in 3.3, $\Phi_*\colon \mathrm{A}(X^n) \to \mathrm{A}(X^{n+1})$ respects the filtrations; hence, $W \in \mathrm{Fil}^1 \mathrm{A}(X^{m+n+1})$.

By Lemma 5.4, $W$ equals

$$\begin{aligned}
&\sum_{r=0}^{m+n} r^m(m+n-r)^m \cdot \Big( r(m+n+1-r) \cdot V_r^{(m+n+1)} + (r+1)(m+n-r) \cdot V_{r+1}^{(m+n+1)} \Big) \\
&= \sum_{s=0}^{m+n+1} \Big( s^{m+1}(m+n-s)^m(m+n+1-s) + (s-1)^m s(m+n+1-s)^{m+1} \Big) \cdot V_s^{(m+n+1)} \\
&= \sum_{s=0}^{m+n+1} s(m+n+1-s) \cdot \Big( s^m(m+n-s)^m + (s-1)^m(m+n+1-s)^m \Big) \cdot V_s^{(m+n+1)} .
\end{aligned}$$

Putting $x = s$ and $y = m+n+1-s$ we have

$$\begin{aligned}
s^m(m+n-s)^m + (s-1)^m(m+n+1-s)^m &= x^m(y-1)^m + (x-1)^m y^m \\
&= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \cdot (x^j + y^j) x^{m-j} y^{m-j} .
\end{aligned}$$

As $x + y = m+n+1$ is constant, we can rewrite this as $2x^m y^m + \sum_{j=0}^{m-1} c_j \cdot x^j y^j$ for some constants $c_0, \dots, c_{m-1}$. Hence

$$\begin{aligned}
W = 2 \cdot &\sum_{s=0}^{m+n+1} s^{m+1}(m+n+1-s)^{m+1} \cdot V_s^{(m+n+1)} \\
&+ \sum_{j=1}^{m} c_{j-1} \left( \sum_{s=0}^{m+n+1} s^j(m+n+1-s)^j \cdot V_s^{(m+n+1)} \right)
\end{aligned}.$$

As we have already shown that for $j < m+1$

$$\sum_{s=0}^{m+n+1} s^j(m+n+1-s)^j \cdot V_s^{(m+n+1)} \in \mathrm{Fil}^1 \mathrm{A}(X^{m+n+1}),$$

the same is true for the remaining term, i.e., for $j = m+1$. □

*Proof of Theorem* 5.1. Taking $m = n-1$ in Lemma 5.5 and using that $V_r^{(2n-1)} = V_{2n-1-r}^{(2n-1)}$, we find that

$$\sum_{r=0}^{n-1} \big( r(2n-1-r) \big)^j \cdot V_r^{(2n-1)} = 0 \quad \text{in } \mathrm{A}(X^{2n-1})/\mathrm{Fil}^1$$

for all $j \in \{0, 1, \ldots, n-1\}$. The $n \times n$ matrix

$$\Big(\big(r(2n-1-r)\big)^j\Big)_{r,j=0,\ldots,n-1}$$

is a Vandermonde matrix with distinct entries in the second column ($j = 1$). Therefore, $\big[\Delta_X^{(2n-1)}\big] = V_0^{(2n-1)} \in \mathrm{Fil}^1\mathrm{A}(X^{2n-1})$, which means that $\Gamma^{2n-1}(X, a) = 0$. $\square$

We refer to the forthcoming paper [15] of Claire Voisin for a generalization of this result to covers of arbitrary degree.

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