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Research

The spectral model of particle physics

The discovery of the Higgs particle at CERN in Geneva in 2012 formed the crown on the so-called Standard Model of particle physics. Despite its enormous phenomenological success, much of the underlying mathematics remains still to be understood. Walter van Suijlekom, Assistant Professor in mathematical physics at IMAPP, here lifts the curtain of what noncommutative geometry can already say about the Standard Model, offering an intriguing perspective of what space looks like at scales analysed by particle accelerators. Van Suijlekom's book ***Noncommutative Geometry and Particle Physics*** **has just appeared with Springer and gives an introduction to the subject. In this article, he starts his exposition with the famous mathematical question "Can one hear the shape of a drum?", and then moves to the noncommutative world, using not much more but matrix multiplication.**

This article was written on the occasion of the workshop 'Noncommutative Geometry and Particle Physics', organized at the Lorentz Center in Leiden in October 2013. See www.noncommutativegeometry.nl for more information on this workshop, and on the field in general.

Spectral geometry

Noncommutative geometry [11] can be considered as a generalization of *spectral geometry* to the quantum world. So, let us start with a brief tour through spectral geometry. One deals with the question how the geometric structure of a Riemannian manifold M — that is, a topological space that looks locally like Euclidean space — determines the spectrum of the Laplacian on M (cf. [10]). The inverse problem, how the manifold M is determined by the spectrum of the Laplacian leads to the famous question "Can one hear the shape of a drum?", as posed by Mark Kac in 1966 [16]. The answer to this question is "no", as is well known by now, e.g. through the construction of two isospectral

Figure 1 Two isospectral domains in $\mathbb{R}^2$ whose Laplacians have the same spectrum [14].

polygonal domains in $\mathbb{R}^2$ (two 'drums') (cf. Figure 1). Here the metaphoric sound of a Riemannian manifold is governed by the *Helmholtz equation* satisfied by the amplitude u of a wave on M,

$$\Delta_M u = k^2 u,$$

where Δ_M is the Laplacian and k is the wave number. This wave number can thus essentially be found by taking the 'square-root' of the Laplacian. More precisely, one searches for an operator that squares to Δ_M and analyses its spectrum of eigenvalues. It was Paul Dirac who found such a differential operator. Even though it does not always exist, it does so on Riemannian *spin* manifolds to which we will restrict. Let us consider some examples of Dirac operators for low-dimensional tori.

Dirac operators on the circle, 2-torus and 4-torus
We parametrize the circle $\mathbb{S}^1$ by an angle $t \in [0, 2\pi)$. The Dirac operator on the circle then reads

$$D_{\mathbb{S}^1} = -i\frac{d}{dt}.$$

The square $(D_{\mathbb{S}^1})^2 = -\frac{d^2}{dt^2}$ is indeed the Laplacian on the circle. Note that the eigenfunctions of $D_{\mathbb{S}^1}$ are the complex exponential functions

$$e^{int} = \cos nt + i \sin nt,$$

for any integer $n \in \mathbb{Z}$, with eigenvalue n. Hence, the spectrum of $D_{\mathbb{S}^1}$ is given by the set of integers $\mathbb{Z}$ and we arrive at the usual circular harmonics given by Fourier series.

Next, consider the two-dimensional torus $\mathbb{T}^2$. It can be parametrized by two angles $t_1, t_2 \in [0, 2\pi)$. The Laplacian then reads

$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$

At first sight it seems difficult to construct a differential operator that squares to $\Delta_{\mathbb{T}^2}$. In fact, squaring any linear combination of the two partial derivatives results in cross-terms:

$$\left(a\frac{\partial}{\partial t_1} + b\frac{\partial}{\partial t_2}\right)^2 = a^2\frac{\partial^2}{\partial t_1^2} + 2ab\frac{\partial^2}{\partial t_1 \partial t_2} + b^2\frac{\partial^2}{\partial t_2^2}$$

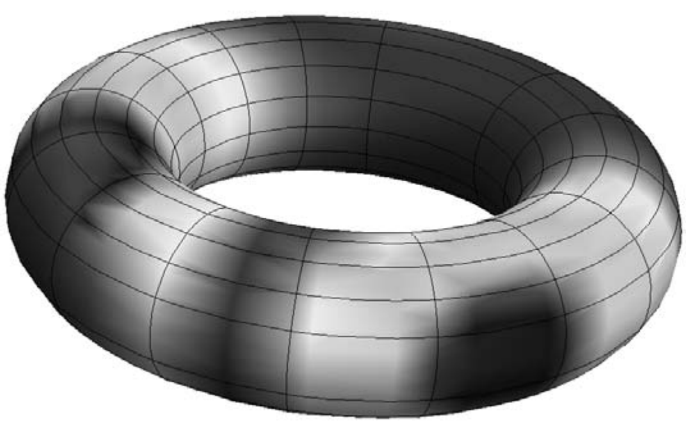

Figure 2 Wave function on $\mathbb{T}^2$ corresponding to n_1=2,n_2=4; grey levels correspond to the amplitude u of the wave.

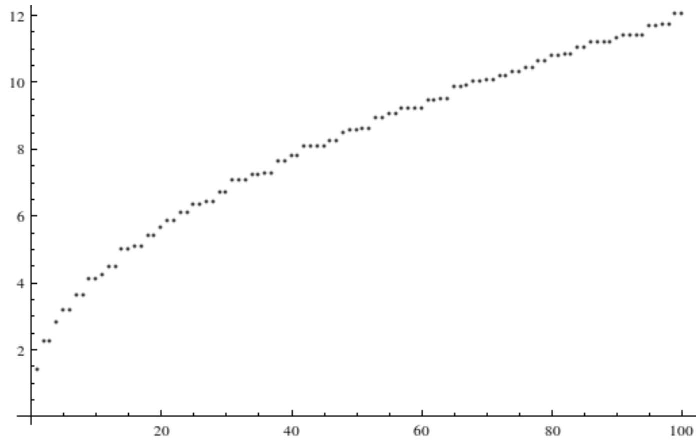

Figure 3 List of the eigenvalues of $D_{\mathbb{T}^2}$.

for any two complex numbers a and b. Of course, the demands $a^2 = b^2 = -1$ and $ab = 0$ cannot hold simultaneously.

This puzzle was solved by Dirac, who considered the possibility that a and b be complex *matrices*. Namely, if

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

then with $i^2 = -1$ we *do* have $a^2 = b^2 = -1$ and $ab + ba = 0$, as one can readily check.

Hence the Dirac operator on the torus is

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

which indeed satisfies $(D_{\mathbb{T}^2})^2 = \Delta_{\mathbb{T}^2}$. Since the eigenvalues of the Laplacian on the torus are given by $n_1^2 + n_2^2$ for integers n_1 and n_2, it follows that the spectrum of the Dirac operator $D_{\mathbb{T}^2}$ is

$$\left\{\sqrt{n_1^2 + n_2^2} : n_1, n_2 \in \mathbb{Z}\right\},$$

and is depicted in Figure 3. A typical eigenfunction of the Dirac operator on the torus is given in Figure 2.

Let us jump to four dimensions — of direct relevance to physics — and consider as a final example the Dirac operator on the 4-torus $\mathbb{T}^4$. We now have four angles t_1, t_2, t_3, t_4, and the Laplacian is

$$\Delta_{\mathbb{T}^4} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2} - \frac{\partial^2}{\partial t_3^2} - \frac{\partial^2}{\partial t_4^2}.$$

The same problem as above arises in the search for a differential operator that squares to $\Delta_{\mathbb{T}^4}$. Again, allowing for matrices solves the problem, but we need more as there are now four matrices that must square to -1 and mutually multiply to 0. Here, there is a beautiful appearance of Hamilton's *quaternions*. Recall that besides the complex i, the field of quaternions contains elements j and k that satisfy

$$i^2 = j^2 = k^2 = ijk = -1.$$

From this one can derive that $ij = -ji$, $ik = -ki$, et cetera. The Dirac

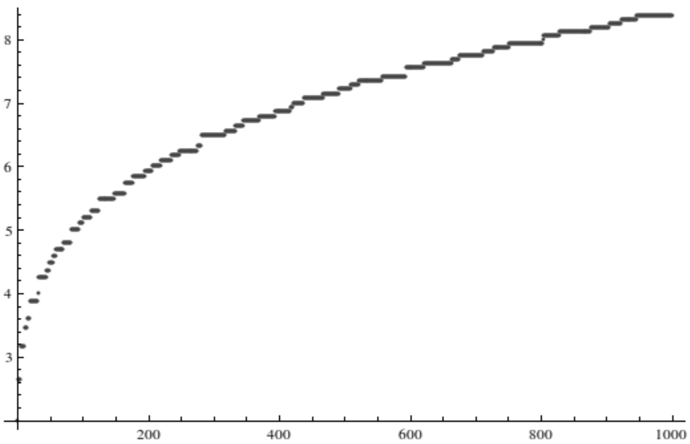

Figure 4 List of the eigenvalues of $D_{\mathbb{T}^4}$.

operator on $\mathbb{T}^4$ is conveniently written in terms of quaternions as

$$D_{\mathbb{T}^4} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} & 0 \end{pmatrix}, \quad (1)$$

A straightforward computation then shows that its square coincides with $\Delta_{\mathbb{T}^4}$. As a consequence, the spectrum of $D_{\mathbb{T}^4}$ is given by

$$\left\{ \sqrt{n_1^2 + n_2^2 + n_3^2 + n_4^2} : n_1, n_2, n_3, n_4 \in \mathbb{Z} \right\},$$

and is depicted in Figure 4.

Riemannian spin manifolds

More generally, a Dirac operator D_M on a Riemannian manifold (M, g) is a square-root (up to scalar terms) of the Laplacian on M. It exists when M is a Riemannian spin manifold, we refer to [2] for more details. What is important for us is that even though the eigenvalues of D_M do not completely determine M, certain information of it can be subtracted from the spectrum of D_M. A famous result is Weyl's asymptotic law, stating that the number $N_{D_M}(\Lambda)$ of eigenvalues smaller (in absolute value) than $\Lambda \geq 0$ is given asymptotically by

$$N_{D_M}(\Lambda) \sim \frac{\Omega_n \mathrm{Vol}(M)}{n(2\pi)^n} \Lambda^n,$$

in terms of the dimension n of M and Ω_n is the volume of the n-sphere. Hence, from the growth of the eigenvalues of D_M one can derive the dimension of M. For the tori in dimension two and four, this can already be seen from the parabolic shapes in Figures 3 and 4.

In the applications of noncommutative geometry to particle physics one interprets the above counting function $N_{D_M}(\Lambda)$ as a so-called *spectral action functional* [3–4] describing dynamics and interactions of the physical particles and fields. We will consider a smooth version of the counting function, to wit

$$\mathrm{Tr}\, f\left(\frac{D_M}{\Lambda}\right) = \sum_\lambda f\left(\frac{\lambda}{\Lambda}\right),$$

where f is a smooth version of a cutoff function, Tr is the trace, and the sum on the right-hand side is over all eigenvalues of D_M. For illustrational purposes, we will restrict in this article to the exponential cut-off function, that is to say, a Gaussian function (cf. Figure 5):

$$f(x) = e^{-x^2}. \quad (2)$$

The main reason for doing so is that $\mathrm{Tr}\, e^{-D_M^2/\Lambda^2}$ is the so-called heat kernel for the Laplacian D_M^2, whose asymptotics as $\Lambda \to \infty$ is well-known [2]. As a matter of fact, asymptotically we have

$$\mathrm{Tr}\, e^{-D_M^2/\Lambda^2} \sim \frac{\mathrm{Vol}(M)\Lambda^n}{(4\pi)^{n/2}}, \quad (3)$$

in concordance with Weyl's estimate above.

As should be clear by now, the spectrum of D_M does not capture all of the geometry of M. This can be improved by considering besides D_M also the space of smooth complex-valued functions on M, denoted by $C^\infty(M)$. For instance, the distance function on M can be written as

$$d(p,q) = \sup_{f \in C^\infty(M)} \left\{ |f(p) - f(q)| : \text{gradient } f \leq 1 \right\},$$

where the gradient of f can be controlled with the commutator $[D_M, f] = D_M f - f D_M$. For instance, on the circle we have $[D_{\mathbb{S}^1}, f] = -i\frac{df}{dt}$. The translation of distances between points via functions on that space is illustrated in Figure 6.

Finite noncommutative spaces

Let us consider finite spaces F, equipped with the discrete topology. That is, consider the space F consisting of N points:

$$1\bullet \quad 2\bullet \quad \cdots\cdots \quad N\bullet$$

The space $C^\infty(F)$ of smooth functions on such a finite space is simply given by $\mathbb{C}^N$: one complex number for each of the function values at the points of F. An element $f \in C^\infty(F)$ can be conveniently written as a diagonal matrix:

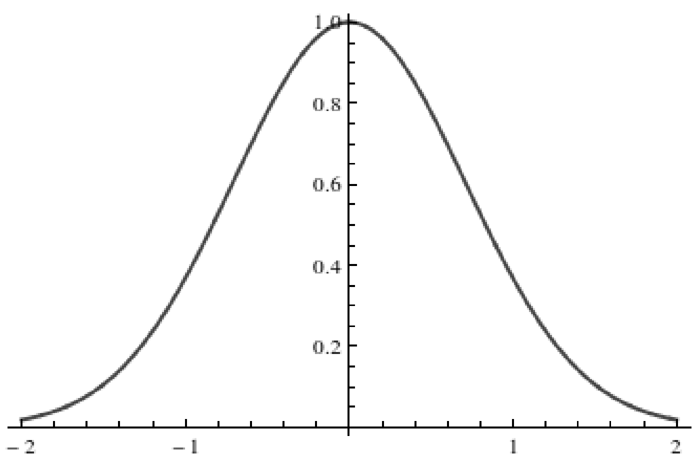

Figure 5 Smooth cutoff function given by equation 2.

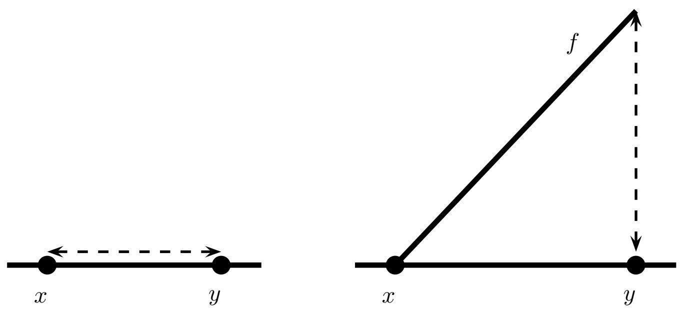

Figure 6 The distance between the points x and y can be translated to the distance between $f(x)$ and $f(y)$ for functions with gradient equal to 1.

$$f \rightsquigarrow \begin{pmatrix} f(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & f(N) \end{pmatrix},$$

and the matrix product corresponds to the pointwise product of functions: $fg(p) = f(p)g(p)$ for two functions f, g at any point p in F.

For such finite space there is an analogue of a Dirac operator, which in this finite case is an arbitrary hermitian matrix D_F. As before, a distance function on F can be defined as

$$d(p,q) = \sup_{f \in C^\infty(F)} \{|f(p) - f(q)| : \|[D_F, f]\| \leq 1\}, \tag{4}$$

where the 'gradient' $\|[D_F, f]\|$ is defined as the square root of the largest eigenvalue of the matrix $[D_F, f]^*[D_F, f]$. In fact, $d(p,q)$ is a generalized distance function on F as it can take the value ∞.

Example 1. Consider the space F consisting of two points:

$$F = {}_1\bullet \quad {}_2\bullet$$

Then, smooth functions are diagonal 2×2-matrices, so that

$$C^\infty(F) := \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{C} \right\},$$

where λ_1 is the function value at point 1, and λ_2 at point 2.

We can take as a 'finite Dirac operator' the hermitian matrix

$$D_F = \begin{pmatrix} 0 & \bar{c} \\ c & 0 \end{pmatrix}$$

for some constant $c \in \mathbb{C}$. The distance formula 4 then becomes

$$d(p,q) = \begin{cases} |c|^{-1}, & p \neq q, \\ 0, & p = q. \end{cases}$$

We conclude that the distance between 1 and 2 in F is dictated by the constant c that defines D_F.

The geometry of F gets much more interesting if we allow for a *noncommutative* structure at each point of F. That is, instead of diagonal matrices, we consider *block diagonal* matrices

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_N \end{pmatrix},$$

where the $a_1, a_2, \dots a_N$ are square matrices of size $n_1, n_2, \dots, n_N$, respectively, associated to the N points of F. Hence we will consider the vector space

$$V_F := M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C}), \tag{5}$$

where $M_n(\mathbb{C})$ stands for the space of $n \times n$-matrices with complex entries.

We will consider the vector space V_F of such block diagonal matrices as a replacement for functions on F. Since the matrix product is not commutative, we have enriched the perhaps not-so-interesting finite space F with a noncommutative structure.

As far as the finite Dirac operator is concerned, already in the commutative case this operator was given as a matrix, and its definition continues to make sense when considering block diagonal matrices in V_F. Thus, in order to describe a finite noncommutative space F we consider the pair given by the vector space V_F and a hermitian matrix D_F. Note that this is a purely linear-algebraic set of data, which explains the ease with which computations can be done in the context of particle physics.

Remark 2. For pedagogical purposes we carefully avoided the notion of an associative algebra, using only basic linear algebra concepts such as matrices and matrix multiplication. In order to connect to the usual terminology encountered in most texts on noncommutative geometry let us mention that the vector space V_F is an example of an associative $*$-algebra, with product given by matrix multiplication and $*$-structure given by hermitian conjugation.

Example 3. The two-point space can be given a noncommutative structure by considering the space V_F of 3×3 block diagonal matrices of the following form:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}, \tag{6}$$

with complex entries $\lambda, a_{11}, a_{12}, a_{21}$ and a_{22}. Hence, point 2 in F has a noncommutative structure given by 2×2 matrices.

A hermitian 3×3-matrix can then be chosen of the form

$$D_F = \begin{pmatrix} 0 & \bar{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

inspired by Example 1 and which turns out to be relevant for our physical applications later on. Of course, mathematically speaking any other choice of a hermitian matrix D_F is a valid one.

Perturbation semigroup

The approach we have sketched above to spectral (noncommutative) geometry is still static: the Dirac operator is fixed. We now make this more dynamical by perturbing the operator D_F by matrices in V_F, and D_M by functions on the manifold M. This naturally gives rise to the structure of a semigroup of perturbations [8]. We recall that in general a *semigroup* is defined as a set equipped with an associative multiplication.

Definition 4. Let V_F be the space defined in (5). We define the *perturbation semigroup of* V_F as the following subset in the tensor product $V_F \otimes V_F$:

$$\mathrm{Pert}(V_F) := \left\{ \sum_j A_j \otimes B_j \;\middle|\; \begin{array}{l} \sum_j A_j (B_j)^t = \mathbb{I} \\ \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j} \end{array} \right\},$$

where ^{t} denotes matrix transpose, $\mathbb{I}$ is the identity matrix in V_F, and $\overline{}$ denotes complex conjugation of the matrix entries.

The semigroup law in $\mathrm{Pert}(V_F)$ is given by the matrix product in $V_F \otimes V_F$, i.e. on Kronecker products $A \otimes B$, $A' \otimes B'$ the semigroup multiplication is

$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB').$$

The two conditions in the definition of $\mathrm{Pert}(A)$ are called the *normalization*, and *self-adjointness condition*.

Let us check that $\mathrm{Pert}(V_F)$ is indeed a semigroup. The normalization condition carries over to products,

$$\left(\sum_j A_j \otimes B_j \right) \left(\sum_k A'_k \otimes B'_k \right) = \sum_{j,k} (A_j A'_k) \otimes (B_j B'_k),$$

for which

$$\sum_{j,k} A_j A'_k (B_j B'_k)^t = \sum_{j,k} A_j A'_k (B'_k)^t (B_j)^t = \mathbb{I},$$

because matrix transpose reverses the order of the matrices. Similarly, one checks that the self-adjointness condition is respected when taking products of two elements in $\mathrm{Pert}(V_F)$.

Let us illustrate this rather abstract definition with some examples.

Example 5. Consider the two-point space with $V_F = \mathbb{C}^2$, i.e. the space of diagonal 2×2 matrices as considered in Example 1. Let e_{11}, e_{22} denote the standard basis of such diagonal matrices:

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we can write an arbitrary element of $\mathrm{Pert}(\mathbb{C}^2)$ in terms of this basis as

$$z_1 e_{11} \otimes e_{11} + z_2 e_{11} \otimes e_{22} + z_3 e_{22} \otimes e_{11} + z_4 e_{22} \otimes e_{22},$$

Photo: Bert Beelen

The founder of noncommutative geometry Alain Connes visiting the Radboud University Nijmegen (March 2014), here together with the author behind Foucault's pendulum in the Huygens building.

with complex coefficients $z_1, \ldots, z_4$. Since the matrix multiplication between e_{11} and e_{22} follows simple rules, the normalization condition becomes

$$z_1 = 1 = z_4.$$

Instead, the self-adjointness condition reads

$$z_2 = \overline{z_3}.$$

This leaves only one free complex parameter, say z_2, and we conclude that $\mathrm{Pert}(\mathbb{C}^2) \simeq \mathbb{C}$.

More generally one can show along the same lines that the perturbation semigroup $\mathrm{Pert}(\mathbb{C}^N)$ for the space of N points is given by $\mathbb{C}^{N(N-1)/2}$ with semigroup structure given by componentwise product.

Example 6. Let us consider a noncommutative example, to wit $V_F = M_2(\mathbb{C})$. We can identify $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ with $M_4(\mathbb{C})$ so that elements in $\mathrm{Pert}(M_2(\mathbb{C})$ are 4×4-matrices satisfying the normalization and self-adjointness condition. One can show that we have in a suitable basis:

$$\mathrm{Pert}(M_2(\mathbb{C})) = \left\{ \begin{pmatrix} 1 & v_1 & v_2 & iv_3 \\ 0 & x_1 & x_2 & ix_3 \\ 0 & x_4 & x_5 & ix_6 \\ 0 & ix_7 & ix_8 & x_9 \end{pmatrix} \;\middle|\; \begin{array}{l} v_1, v_2, v_3 \in \mathbb{R} \\ x_1, \ldots, x_9 \in \mathbb{R} \end{array} \right\}.$$

It is quite remarkable that the product of two such matrices is again of the same form, as it should be to form a semigroup. In fact, one can show that $\mathrm{Pert}(M_2(\mathbb{C}))$ is a semidirect product of semigroups,

$$\mathrm{Pert}(M_2(\mathbb{C})) \simeq \mathbb{R}^3 \rtimes S,$$

where S is the semigroup of 3×3 matrices of the form

$$\begin{pmatrix} x_1 & x_2 & ix_3 \\ x_4 & x_5 & ix_6 \\ ix_7 & ix_8 & x_9 \end{pmatrix},$$

where $x_1, \ldots, x_9$ are real numbers. More generally, one can identify a real vector space W and a semigroup S' such that

$$\mathrm{Pert}(M_N(\mathbb{C})) \simeq W \rtimes S'.$$

This is further worked out in the thesis [18] and in [19].

Example 7. Even though strictly speaking Definition 4 of the perturbation semigroup applies only to (noncommutative) finite topological spaces, let us see what we can say for the case of a smooth manifold M. The vector space V_F is replaced by the space of smooth complex-valued functions on M, denoted $C^\infty(M)$. Now, we can consider functions in the tensor product $C^\infty(M) \otimes C^\infty(M)$ as functions of two-variables. In other words, they are elements in $C^\infty(M \times M)$. The normalization and self-adjointness condition in $\mathrm{Pert}(C^\infty(M))$ translate accordingly and yield

$$\mathrm{Pert}(C^\infty(M)) = \left\{ f \in C^\infty(M \times M) \;\middle|\; \begin{array}{l} f(x,x) = 1 \\ f(x,y) = \overline{f(y,x)} \end{array} \right\},$$

where $x, y \in M$.

Let us then come back to the general set-up, with V_F as in equation (5) with block diagonal matrices of arbitrary (but fixed) size. As a first result we have:

Proposition 8. *Let $\mathcal{U}(V_F)$ be the unitary block diagonal matrices in V_F. This space forms a group which is a subgroup of the semigroup $\mathrm{Pert}(V_F)$.*

Proof. The space of unitary matrices in V_F forms a group with inverse of a unitary U given by U^*. If U is a unitary block diagonal matrix in V_F, then we claim that the Kronecker product $U \otimes \overline{U}$ defines an element in $\mathrm{Pert}(V_F)$. Indeed, the normalization condition is satisfied because of unitarity

$$U\overline{U}^t = UU^* = 1,$$

and $U \otimes \overline{U}$ trivially satisfies the self-adjointness condition. □

The significance of the perturbation semigroup becomes clear in its action on hermitian matrices. Indeed, an element $\sum_j A_j \otimes B_j \in \mathrm{Pert}(V_F)$ acts on a hermitian matrix D by matrix multiplication on the left and on the right as:

$$D \mapsto \sum_j A_j D B_j^t,$$

which is then considered as a perturbation of D. This action is compatible with the semigroup law, since

$$\sum_{j,k} (A_j A'_k) D (B_j B'_k)^t = \sum_j A_j \left(\sum_k A'_k D (B'_k)^t \right) (B_j)^t$$

and it respects hermiticity of D precisely because of the self-adjointness condition:

Alain Connes during the IMAPP Colloquium in Nijmegen (March 2014) presenting (part of) the Standard Model Lagrangian.

Photo: Bert Beelen

$$\left(\sum_j A_j D (B_j)^t \right)^* = \sum_j \overline{B_j} D (\overline{A_j})^t = \sum_j A_j D (B_j)^t.$$

The restriction of this action to the unitary group $\mathcal{U}(V_F)$ gives

$$D \mapsto UDU^*.$$

The crucial point is that conjugation by a unitary leaves the spectrum of D invariant. As such, the spectral action functional is an invariant under this action. In physics, this corresponds to *gauge invariance* and $\mathcal{U}(V_F)$ is recognized as the *gauge group*.

Let us conclude with a discussion on the action in the examples treated before.

Example 9. Let us consider the action of $\mathrm{Pert}(\mathbb{C}^2) \simeq \mathbb{C}$ (cf. Example 5) on the symmetric matrix

$$D_F = \begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix}.$$

One finds that $\phi \in \mathbb{C} \simeq \mathrm{Pert}(\mathbb{C}^2)$ acts as

$$D_F \mapsto \begin{pmatrix} 0 & c\phi \\ \overline{c\phi} & 0 \end{pmatrix}.$$

The group of unitary diagonal 2×2 matrices is $U(1) \times U(1)$ and an element (λ_1, λ_2) therein acts on the perturbed D_F, and consequently on ϕ as

$$\phi \mapsto \lambda_1 \overline{\lambda_2} \phi.$$

Example 10. Let us consider a noncommutative example, namely, the action of $\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$ on the operator D_F of Example 3. The perturbation semigroup behaves nicely with respect to direct sums and we find in this case that

$$\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C})) \simeq M_2(\mathbb{C}) \times \mathrm{Pert}(M_2(\mathbb{C})).$$

It turns out that only $M_2(\mathbb{C}) \in \mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$ acts non-trivially on the above D_F. If we label the entries of the first column of such a 2×2 matrix by ϕ_1 and ϕ_2 we arrive at

$$D_F \mapsto \begin{pmatrix} 0 & \overline{c\phi_1} & \overline{c\phi_2} \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}.$$

We will see later that the two fields ϕ_1 and ϕ_2 turn out to parametrize the famous *Higgs field* in physics.

The group of unitary block diagonal matrices is now $U(1) \times U(2)$ and an element (λ, u) therein acts as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \overline{\lambda} u \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{7}$$

Example 11. Let us end with a commutative but continuous example and consider a smooth manifold M. The action of $\mathrm{Pert}(C^\infty(M))$ (cf. Example 7 on the partial derivatives appearing in a Dirac operator D_M on a Riemannian spin manifold M is given by

$$\frac{\partial}{\partial x_\mu} \mapsto \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu} f(x, y)\bigg|_{y=x}, \qquad (\mu = 1, \ldots, n),$$

where $f \in C^\infty(M \times M)$ is such that $f(x, x) = 1$ and $\overline{f(x, y)} = f(y, x)$. In physics, one writes

$$A_\mu := \frac{\partial}{\partial y_\mu} f(x, y)\bigg|_{y=x},$$

which turns out to be the *electromagnetic potential* giving rise to the electromagnetic field that describes the photon. We refer e.g. to [15] for more details on the theory of electrodynamics.

A unitary element u in $C^\infty(M)$ acts by conjugation on the partial derivatives, or, which is the same, can be absorbed by the transformation

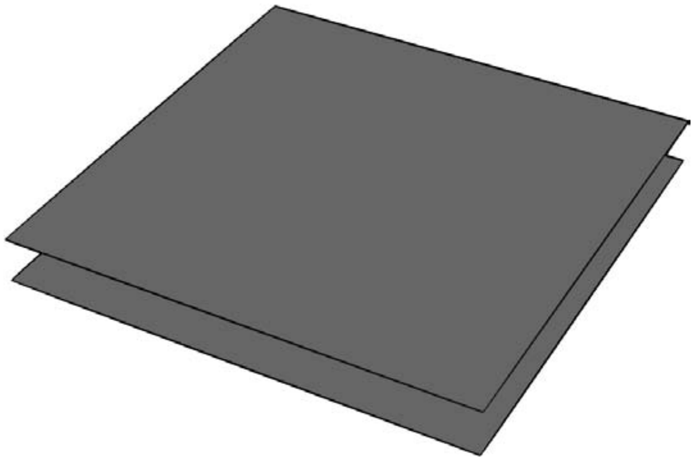

Figure 7 The product of M with the two-point space of Example 1 can be identified with the space consisting of two copies of M.

$$A_\mu \mapsto u A_\mu u^* + u \partial_\mu u^*,$$

which is the usual form of a gauge transformation in physics.

Applications to particle physics

We now combine a Riemannian spin manifold M with a finite noncommutative space F, considering the latter as an *internal space* at each point of M. In other words, we form the *direct product* $M \times F$ and consider matrix-valued maps from M to V_F as functions on this noncommutative space. Thus, if F describes a space of N points, possibly with some noncommutative structure at each point, the product $M \times F$ can be considered as a (noncommutative) space consisting of N copies of the manifold M (see Figure 7 for $N = 2$).

The next ingredient is the Dirac operator on $M \times F$ which is defined to be the product of D_M and D_F. More precisely, if M is four-dimensional we can write D_M as the following block matrix:

$$D_M = \begin{pmatrix} 0 & D_M^+ \\ D_M^- & 0 \end{pmatrix}.$$

This was indeed the case for the four-dimensional torus, where we had in equation (1):

$$D_M^{\pm} = \pm \frac{\partial}{\partial t_1} + i \frac{\partial}{\partial t_2} + j \frac{\partial}{\partial t_3} + k \frac{\partial}{\partial t_4}.$$

We combine this with the finite Dirac operator D_F by setting as a Dirac operator on the product $M \times F$:

$$D_{M \times F} = \begin{pmatrix} D_F & D_M^+ \\ D_M^- & -D_F \end{pmatrix}.$$

The crucial property of this specific form is that it squares to the sum of the two Laplacians on M and F:

$$D_{M \times F}^2 = D_M^2 + D_F^2,$$

which follows from a simple matrix calculation. This is very useful in the computation of the spectral action functional. Let us carry out this computation in the simple case that f is a Gaussian function as in (2).

Then, we can expand the exponential in powers of D_F:

$$\operatorname{Tr} e^{-D^2_{M\times F}/\Lambda^2} = \operatorname{Tr}\left(1 - \frac{D_F^2}{\Lambda^2} + \frac{D_F^4}{2\Lambda^4} - \cdots\right) e^{-D_M^2/\Lambda^2}. \tag{8}$$

If we use equation (3) in this expression and ignore terms proportional to Λ^{-1}, we arrive in dimension $n = 4$ at

$$\operatorname{Tr} e^{-D^2_{M\times F}/\Lambda^2} = \frac{\mathrm{Vol}(M)\Lambda^4}{(4\pi)^2} \operatorname{Tr}\left(1 - \frac{D_F^2}{\Lambda^2} + \frac{D_F^4}{2\Lambda^4}\right) + \mathcal{O}(\Lambda^{-1}).$$

As Λ is supposedly large, we will ignore the terms proportional to Λ^{-1}. Hence, up to overall constants, the spectral action functional yields a *potential* for D_F, i.e.

$$V(D_F) = \Lambda^4 - \Lambda^2 \operatorname{Tr} D_F^2 + \frac{1}{2}\operatorname{Tr} D_F^4. \tag{9}$$

This potential plays a crucial role in the Higgs spontaneous symmetry breaking mechanism, as we will now explain.

Noncommutative two-point space and the Higgs boson

Let us consider the space $M \times F$ where F is the two-point space introduced in Example 3. Then, the distance on the space $M \times F$ is the combination of the ordinary Riemannian distance on each copy of M, and the two copies are at distance $|c|^{-1}$ from each other.

If one includes the perturbations of D_F analysed in Example 10, then D_F becomes parametrized by the Higgs fields ϕ_1, ϕ_2, which may now vary over the points in M. The potential of equation (9) then becomes a potential for the complex field ϕ:

$$V(\phi) = \Lambda^4 - 2\Lambda^2(|\phi_1|^2 + |\phi_2|^2) + (|\phi_1|^2 + |\phi_2|^2)^2. \tag{10}$$

This is the famous 'mexican-hat' potential depicted in Figure 8. It is the starting point of the Higgs spontaneous symmetry breaking mechanism, as we will explain next.

First, note the circular symmetry in Figure 8, which in fact corresponds to the invariance of the potential under the $U(1) \times U(2)$-action of equation (7). However, in physics particles and fields tend to minimize potentials and it is already clear from the picture that any such minimum breaks this symmetry. This procedure is called *spontaneous symmetry breaking*. Essentially, a minimum of V sets ϕ_1 and ϕ_2 to certain fixed vacuum values, say v and 0 respectively. Accordingly, this freezes the distance between the two layers to be proportional to $|v|^{-1}$, as explained in Example 1. If one takes all constants and physical units properly into account, one derives from the recently measured mass of the Higgs boson (approximately 125.5 GeV) that the distance between the two layers in Figure 7 is of the order of $10^{-18}m$.

Noncommutative three-point space and a new particle?

We now consider the case that F is a three-point space, with the noncommutative structure dictated by the matrices

$$V_F = \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}).$$

That is to say, we consider matrices of the form

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix}$$

for complex numbers $\lambda_1, \lambda_2, a_{11}, a_{12}, a_{21}, a_{22}$.

We can make the following convenient choice of finite Dirac operator for this three-point space:

$$D_F := \begin{pmatrix} 0 & 0 & \overline{c} & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

Even though the matrix D_F contains mainly zeroes, the perturbations of it coming from the semigroup $\mathrm{Pert}(V_F)$ are rather non-trivial and give

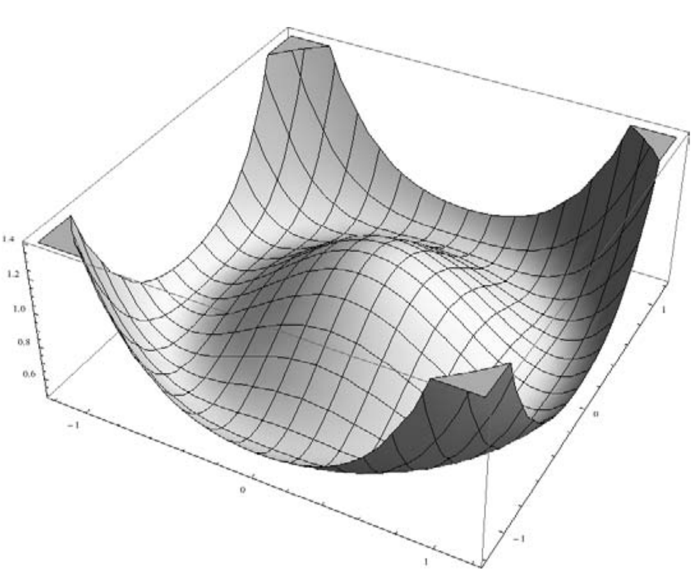

Figure 8 The 'mexican-hat' potential $V(\phi)$ of equation (10) in terms of $|\phi_1|$ and $|\phi_2|$.

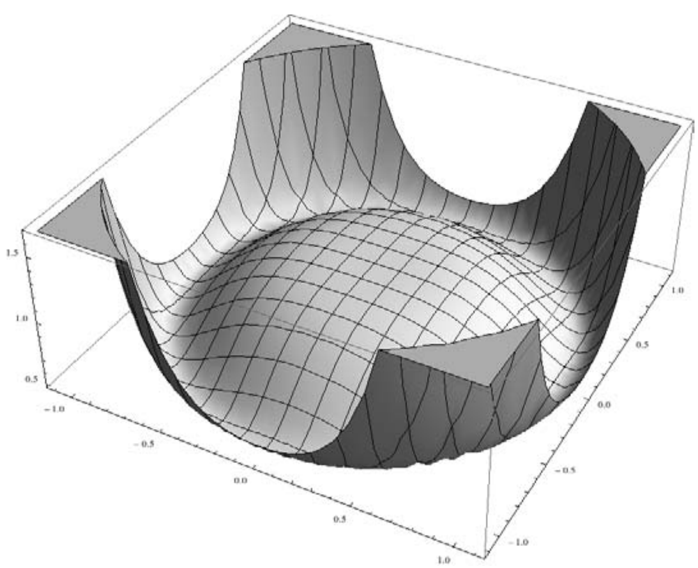

Figure 9 The 'bowler hat' potential $V(\sigma_1,\sigma_2)$ of equation (11) in terms of $|\sigma_1|$ and $|\sigma_2|$.

rise to two scalar fields σ_1 and σ_2. The potential derived in equation (9) becomes a potential for these fields, now of the form

$$V(\sigma_1,\sigma_2) = \Lambda^4 - 2\Lambda^2(|\sigma_1|^2 + |\sigma_2|^2)^2 + (|\sigma_1|^2 + |\sigma_2|^2)^4. \tag{11}$$

Note that this is a polynomial expression of order 8, as opposed to the order 4 encountered before for the Higgs field (cf. [8] for the full details on this example). The resulting 'bowler hat' potential is depicted in Figure 9.

Again, the potential $V(\sigma_1,\sigma_2)$ is invariant under the group of unitary matrices in V_F, which in this case is $U(1)\times U(1)\times U(2)$. If the fields (σ_1,σ_2) attain a minimum, this spontaneously breaks this symmetry. A similar discussion as before for the Higgs field also applies to the σ-field, freezing the two layers to be separated by an even smaller distance of $10^{-27}m$ (corresponding to the mass of the σ-particle to be of the order of 10^{12}GeV).

The Standard Model of particle physics

We now sketch how the above toy models extend and combine to give a noncommutative geometrical description of the Standard Model of particle physics. First, recall that the latter model is the result of decades of experimental and theoretical work in physics, explaining the dynamics and interactions of all existing elementary particles. Let us summarize the particle content (cf. Figure 10):

- *leptons:* electron (e), muon (μ), tauon (τ) and three neutrinos (ν_e,ν_μ,ν_τ).
- *quarks:* up (u), charm (c) and top (t), and down (d), strange (s) and bottom (b), all coming in three colours.
- *force carriers:* photon (electromagnetic force), Z and W-boson (weak nuclear force) and gluons (strong nuclear force).
- *Higgs boson:* giving mass to the Z and W-boson via the Higgs spontaneous symmetry breaking.

These particles are the building blocks of well-known particles such as the proton (built from two up quarks and one down quark), neutron (built from two down quarks and one up quark), pion, et cetera.

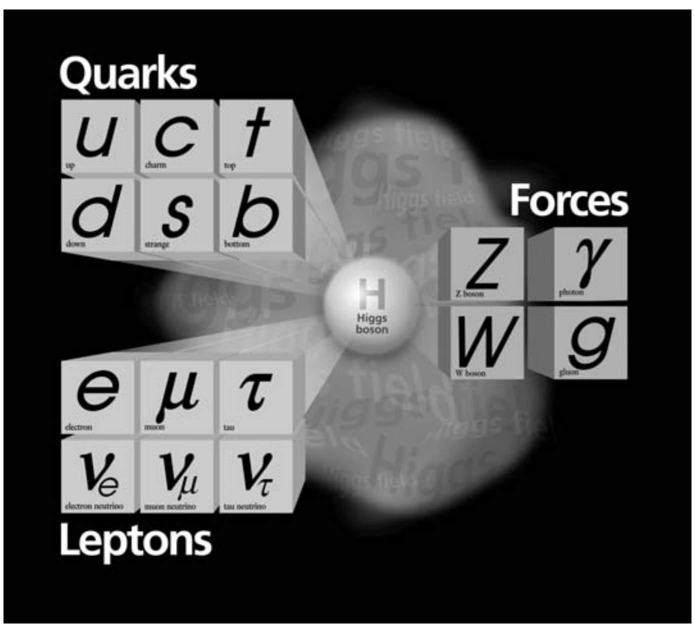

Figure 10 The particle content of the Standard Model.

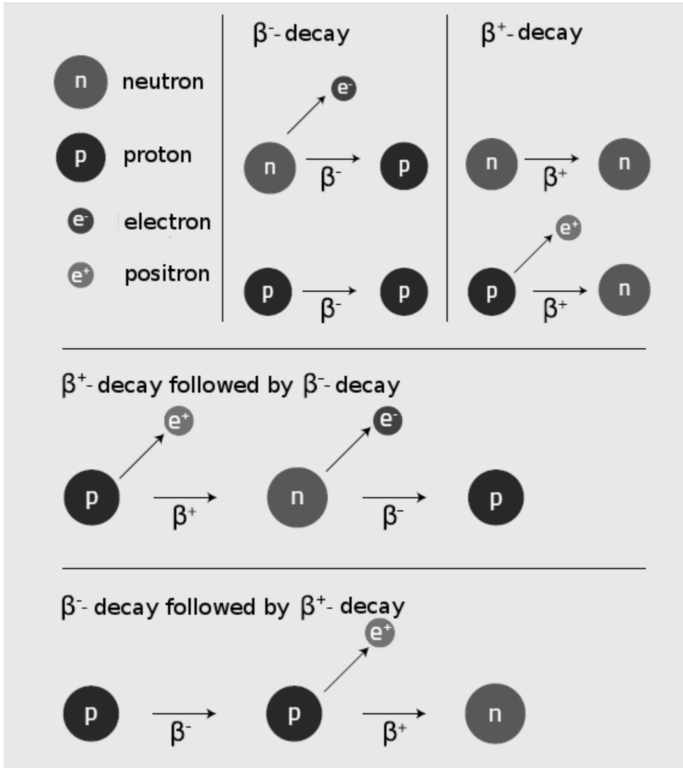

Illustration: Erick Vermeulen (*New Scientist*)

Figure 11 β-decay is a noncommutative physical process.

We will not describe the full dynamics and interactions of the Standard Model, as this can easily fill a textbook; we refer to [13] for a physicist's overview and to [1] for a mathematician-friendly introduction. Instead, we single out a typical decay process described by the Standard Model, and explain how it leads to a noncommutative structure.

We consider β^- and β^+-decay, which are two types of radioactive decay. The first, β^--decay, is the emission of an electron (and an electron-neutrino) by a neutron to form a proton (see Figure 11). This process is a weak interaction process, replacing a down quark in the neutron by an up quark to form a proton, at the same time emitting a W-boson. Subsequently, this W-boson decays into an electron and neutrino. Let us simplify this process by only considering what happens to neutron and proton:

$$\beta^- : n \mapsto p,$$
$$\beta^- : p \mapsto p.$$

The second line simply states that β^--decay is not concerned with decay of the proton, and leaves it as it is. Such a process calls for a representation by matrices: if we denote the basis vectors in $\mathbb{C}^2$ by p and n,

$$p = \begin{pmatrix}1\\0\end{pmatrix}, \qquad n = \begin{pmatrix}0\\1\end{pmatrix},$$

then we can represent

$$\beta^- = \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}.$$

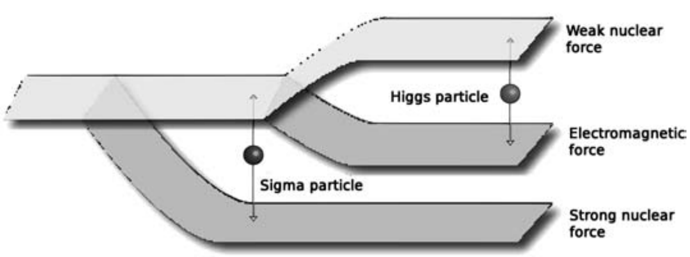

Illustration: Erick Vermeulen (*New Scientist*)

Figure 12 The block matrices in $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ allows to distinguish three layers in $M \times F$.

The 'reverse' process is β^+-decay, now a proton emits a positron (and neutrino) to form a neutron (see Figure 11). Again, in terms of the above basis vectors p and n we can represent β^+ as a two-by-two matrix:

$$\beta^+ = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

The crucial observation is that the two matrices representing β^- and β^+ do not commute:

$$\beta^-\beta^+ \neq \beta^+\beta^-,$$

as one can readily check.

It is exactly these two-by-two matrices that lead to the noncommutative structure encountered in Example 3. In fact, the block matrices in $\mathbb{C} \oplus M_2(\mathbb{C})$ are responsible for both the electromagnetic and the weak interaction. Moreover, the strong interaction between the quarks and their corresponding three colours is governed by three-by-three matrices, which can be described in complete analogy with the above description of β-decay. This results in the space of block matrices

$$V_F = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}).$$

In the full model [7] one further restricts $M_2(\mathbb{C})$ to the quaternions, but for the present illustrational purposes this point is irrelevant. By considering these matrices as functions on the noncommutative space F, we have essentially translated the noncommutativity of the above physical processes into geometry. When combined as $M \times F$ with a (space-time) manifold M, the direct summands in V_F each correspond to a copy of M, and form the domain of the electromagnetic, weak and strong interaction, respectively (see Figure 12).

As in the toy models described before, the dynamics on this finite noncommutative space F is governed by a symmetric matrix D_F. If we count the total number of leptons and quarks we come to a 96×96-dimensional matrix which is filled with masses for the leptons and quarks, and otherwise contains many zeroes. Now, the great advantage of the noncommutative approach is that from this geometrical data alone all bosons can now be derived, with a key role played by the perturbation semigroup of Definition 4. In fact, much as the photon was obtained in Example 11 by acting with $\mathrm{Pert}(C^\infty(M))$ on the Dirac operator D_M, extending this action to $\mathrm{Pert}(V_F)$ on that same D_M produces the photon, W and Z-boson, and gluons.

And what is more, the action of $\mathrm{Pert}(V_F)$ on D_F results in the Higgs boson, much as in the toy model discussed before. This is a great improvement over the usual formulation of the Standard Model, where the Higgs field is introduced by hand. Also a new, yet undiscovered σ particle appears, as in the toy model of the previous section. Moreover, the computation of the distances between the several layers of the previous two subsections translate *verbatim* to the Standard Model, yielding a picture (Figure 12) of three layers of space-time which are separated by a distance of $10^{-18}m$ and $10^{-27}m$, respectively.

Remark 12. The noncommutative description of the Standard Model was given in [7] (see also [12]). All mathematical details and nuances can be found therein. The σ-field was discovered in [6], but already tacitly present in [5]. A full mathematical description of it, including the description of the perturbation semigroup was given in [8–9]. We also refer to [20] and references therein.

Quantization of the theory on a lattice

In the previous sections we have sketched how the full Standard Model of particle physics can be derived from a noncommutative space, using not more than basic linear algebra. Even though this is quite an achievement, there is still the formidable problem to give a mathematically rigorous description of the quantization of the above system. At present, the derivation of the spectral action functional for the Standard Model, including e.g. the Higgs potential, is a mathematical derivation. However, the translation of it to realistic quantum particles and fields follows a more physics-style approach. It is clear that in order to have a proper understanding of the Standard Model of particle physics this aspect should be improved. It is the goal of my Vidi-research project to take a step in this direction.

We will analyse the quantization of gauge fields — such as the electromagnetic field — on a discrete space instead of in the continuum. That is, we replace M by a lattice, construct the quantum theory there, and then analyse the limit of small lattice spacing (Figure 13). The main challenge is to do this in a mathematically rigorous way, for which we intend to exploit the powerful functional analytical techniques coming from noncommutative geometry. One of the intriguing links with the above description of noncommutative finite spaces is that the replacement of M by a lattice is very similar to analysing the structure of the discrete spaces F using matrix algebra. In [17] we present a first exploration of this exciting interplay between noncommutative geometry, lattice gauge theory and quantization.

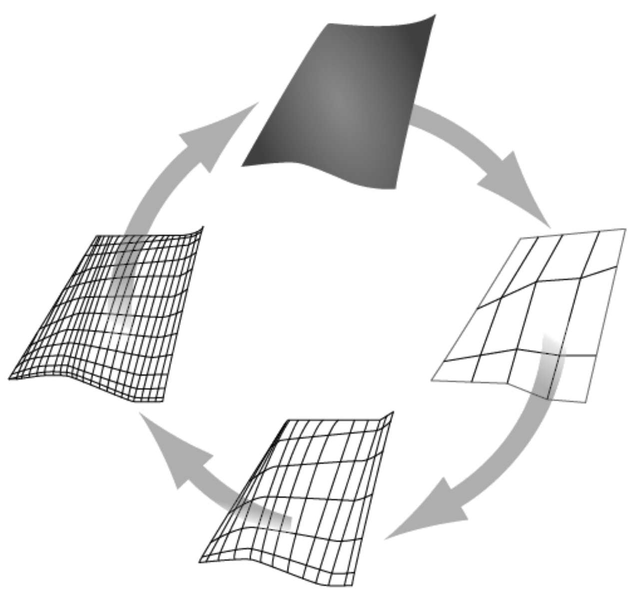

Illustration: Erick Vermeulen (*New Scientist*)

Figure 13 Quantum gauge fields are first described on a lattice, after which one takes the limit of small lattice spacing to analyse the quantum theory in the continuum.

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