New methods to bound the critical probability in fractal percolation

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Abstract: We study the critical probability $p_c(M)$ in two-dimensional $M$-adic fractal percolation. To find lower bounds, we compare fractal percolation with site percolation. Fundamentally new is the construction of an computable increasing sequence that converges to $p_c(M)$. We prove that $p_c(2) > 0.881$ and $p_c(3) > 0.784$.

For the upper bounds, we introduce an iterative random process on a finite alphabet $A$, which is easier to analyze than the original process. We show that $p_c(2) < 0.993$, $p_c(3) < 0.940$ and $p_c(4) < 0.972$.

Keywords: fractal percolation, critical probability, upper and lower bounds

1 Introduction

Fractal percolation has been introduced by Mandelbrot in 1974 as a model for turbulence and is discussed in his book *The Fractal Geometry of Nature* [8]. Several equivalent formal definitions of this process can be found in the literature (see e.g. [3, 4, 6]). Here we only give an informal definition of the two-dimensional case. Let $K_0$ be the unit square and choose an integer $M \geq 2$ and a parameter $p \in [0, 1]$. To obtain $K_1$, divide $K_0$ into $M^2$ equal subsquares, each of which survives with probability $p$ and is discarded with probability $1 - p$, independently of all other subsquares. Now do the same procedure in all surviving squares, in order to obtain $K_2$. Iterating this process gives a decreasing sequence of sets $(K_n)_{n \in \mathbb{N}}$, see Figure 1. Let $K = \bigcap_{n \in \mathbb{N}} K_n$ be the limit set.

It was shown in 1988 by Chayes, Chayes and Durrett [3] that there exists a non-trivial critical value $p_c(M)$ such that a.s. the largest connected component in $K$ is a point for $p < p_c(M)$ and with positive probability there is a connected component intersecting opposite sides of the unit square for $p \geq p_c(M)$.

For all $M \geq 2$, the value of $p_c(M)$ is unknown. Several attempts have been
made to find bounds for $p_c(M)$. It is easy to see that $K$ is empty a.s. if $p \leq 1/M^2$, which implies $p_c(M) > 1/M^2$. The argument in [3] is already a bit smarter: any left-right crossing has to cross the line $\{1/M\} \times [0, 1]$ somewhere. A crossing of this line in $K_n$ means that there is a pair of adjacent squares on opposite sides of this line. Such pairs form a branching process with mean offspring $p^2M$ and consequently $p_c(M) > 1/\sqrt{M}$. For the case $M = 2$ this was sharpened by White in 2001 to $p_c(2) \geq 0.810$, who used a set that dominates $K$ and has a simpler structure to study.

Sharp upper bounds are harder to obtain. The first idea to get rigorous upper bounds for $M \geq 2$ was given by Chayes, Chayes and Durrett [3], but (in their own words) these bounds are ridiculously close to 1. For $M = 3$, they show that $p_c(3) < 0.9999$ (although in fact one can prove that $p_c(3) < 0.993$ with their method), which was improved by Dekking and Meester [5] to $p_c(3) < 0.991$. Chayes et al only treat $M = 3$, but they point out that the same idea works for any $M \geq 3$. The case $M = 2$ can be treated by comparing with $M = 4$. As is noted by van der Wal [10], a coupling argument gives $p_c(2) \leq 1 - (1 - \sqrt{p_c(4)})^4$. Following this approach gives $p_c(4) < 0.998$ and $p_c(2) < 1 - 10^{-12}$.

In this paper we present ideas to find significantly sharper lower and upper bounds. To find lower bounds, we compare fractal percolation with site percolation. In particular, we prove the following result (we will present a precise definition of $\pi_n(p,M)$ in section 2.2):

**Theorem 1** Let $M$ be fixed. Define $\pi_n(p) = \mathbb{P}(\text{two sides are connected in } K_n(p))$. If $\pi_n(p) < p_c^{\text{site}}$ for some $n$, then $p < p_c(M)$.

This theorem leads to the construction of a increasing computable sequence $(p_n^c(M))_{n=0}^\infty$ of lower bounds for $p_c(M)$. However, these computations are quite demanding: to find $p_n^c(M)$, one needs to consider all possible realizations of $K_n$. In section 3 we develop methods to bound $(p_n^c(M))$ from below by classifying realizations of $K_n$ into some different types, where the set of types $\mathcal{A}$ does not depend on $n$. The fractal percolation iteration process now induces an iterative random process on $\mathcal{A}$, which is easier to analyze than the original process. Specifically, the recursive structure allows us to investigate the limit for large $n$. Similar ideas are discussed for the upper bounds, but here we do not need the coupling with site percolation. For the
cases $M = 2, 3$ and $4$, we use these insights to give computer aided proofs
for the following bounds:

**Theorem 2** The following bounds hold for $p_c(M)$, $M = 2, 3, 4$:

1. $p_c(2) > 0.881$ and $p_c(3) > 0.784$;
2. $p_c(2) < 0.993$, $p_c(3) < 0.940$ and $p_c(4) < 0.972$.

## 2 Lower bounds for $p_c(M)$

In this section we develop methods to calculate lower bounds for the critical value of two-dimensional fractal percolation. First we briefly introduce site percolation and then we prove a coupling with fractal percolation that allows us to find lower bounds for $p_c(M)$. In particular, we construct an increasing sequence of lower bounds and we prove that this sequence converges to $p_c(M)$. At the end of this section we show how to use these ideas to obtain numerical results.

### 2.1 Site percolation

Consider the infinite two-dimensional square lattice in which each vertex is open with probability $p$ and closed otherwise. In this model the percolation probability $\zeta(p)$ is defined as the probability that the origin belongs to an infinite open cluster. The critical probability is given by

$$p_{site} := \inf \{ p : \zeta(p) > 0 \}.$$

It has been shown by van den Berg and Ermakov [1] that $p_{site} > 0.556$. The following classical property (see e.g. [7] and the references therein) will be used to couple site percolation to fractal percolation.

**Property 1** Take a box of $n \times n$ vertices. Suppose $p < p_{site}$. Then the probability that there is an open cluster intersecting opposite sides of the box converges to 0 as $n \to \infty$.

### 2.2 Coupling site percolation and fractal percolation

In the fractal percolation model one usually adopts the following definitions: a set in the unit square is said to percolate if it contains a connected component intersecting both the left side and the right side of the square. Let

$$\theta_n(p, M) = \mathbb{P}(K_n(p, M) \text{ percolates}), \quad \theta(p, M) = \mathbb{P}(K(p, M) \text{ percolates}).$$
The critical probability is defined as
\[ p_c(M) := \inf \{ p : \theta(p, M) > 0 \}. \]

We will often suppress some of the dependence on \( M \) and \( p \). It is well known (see \cite{9}) that
\[
\lim_{n \to \infty} \theta_n(p) = \theta(p) = \mathbb{P}(\bigcap_{n=0}^{\infty} \{ K_n(p) \text{ percolates} \}).
\]

To obtain a proper coupling, we will slightly modify the above definitions. For example, the set \([0, 1/2]^2 \cup [1/2, 1]^2\) percolates (see Figure 2). We would like to ignore such diagonal connections in fractal percolation, since in site percolation diagonal connections do not exist. Therefore we redefine percolation as follows:

**Definition 1** We say \( K_n \) percolates if it contains \( M^n \)-adic squares \( S_1, \ldots, S_k \) such that
- \( S_1 \) intersects the left side of the square \( \{0\} \times [0, 1] \).
- \( S_k \) intersects the right side of the square \( \{1\} \times [0, 1] \).
- \( S_i \) shares a full edge with \( S_{i+1} \) for \( 1 \leq i \leq k-1 \).

A diagonal connection in \( K_n \) can only be present in \( K_{n+1} \) if both subsquares in the corners survive. This means that the connection is preserved with probability at most \( p^2 \). As a consequence, if \( K_n \) does not percolate in the sense of Definition 1, then there will be no percolation in the limit since all (countably many) diagonal connections break down almost surely. It follows that this modification of the definition does not change the limiting percolation probability. From now on also connections, crossings and connected components in \( K_n \) are similarly redefined.

**Definition 2** Denote the four sides of the unit square by \( B_1, \ldots, B_4 \). If \( K_n \) contains \( M^n \)-adic squares \( S_1, \ldots, S_k \) such that
- \( S_1 \) intersects \( B_i \) and \( S_k \) intersects \( B_j \) for some \( i \neq j \),
- \( S_i \) shares a full edge with \( S_{i+1} \) for \( 1 \leq i \leq k-1 \),
then we say that two sides are connected in \( K_n \).

**Proof of Theorem** First define delayed fractal percolation: \( F_{m,n} \) is constructed in the same way as \( K_{m+n} \), the only difference being that we do not discard any squares in the first \( m \) construction steps. So we first divide the unit square into \( M^m \times M^m \) subsquares and only then we start the fractal percolation process in each of these squares. Delayed fractal percolation
stochastically dominates fractal percolation. Then we have the following inequalities (illustrated for $M = m = 2$ and $n = 3$ in Figure 3):

$$\theta_{m+n}(p, M) \leq P(F_{m,n}(p, M) \text{ percolates}) \leq \theta_1(\pi_n(p), M^m). \quad (1)$$

The first inequality follows from the fact that $F_{m,n}$ is stochastically larger than $K_{m+n}$. The second inequality can be explained as follows. Suppose we have two types of squares: realizations of $K_n(p, M)$ in which two sides are connected (type 1) and realizations in which no sides are connected (type 2). Suppose we tile a larger square with $M^{2m}$ independent realizations of $K_n(p, M)$. So the probability on a type 1 square is $\pi_n(p)$. This larger square is a (scaled) realization of $F_{m,n}(p, M)$. Now replace all type 1 squares by a full square and discard all type 2 squares. This gives a realization of $K_1(\pi(p), M^m)$. Moreover, we claim that this replacement procedure can not destroy percolation.

To prove this claim, suppose we have a left-right crossing in $F_{m,n}(p, M)$. This crossing successively traverses $n$th level squares $S_1, \ldots, S_k$, where $S_i$ shares an edge with $S_{i+1}$ for $1 \leq i \leq k-1$. If $S_i$ is of type 1, then $S_i$ will be replaced by a full square. If $S_i$ is of type 2, then the crossing enters and leaves $S_i$ at the same side of $S_{i'}$, i.e. $S_{i-1} = S_{i+1}$ (note that $S_1$ and $S_k$ can not be of type 2). We conclude that if we remove the type 2 squares from $S_1, \ldots, S_k$, then still each of these squares shares an edge with its predecessor and successor. From this observation the claim follows.

A first level fractal percolation set can be seen as site percolation in a finite box. Suppose $\pi_n(p) < p_{\text{site}}^{\text{site}}$ for some $n$ and let the box size $M^m$ tend to $\infty$. By Property 1 we arrive at

$$\lim_{m \to \infty} \theta_1(\pi_n(p), M^m) = 0. \quad (2)$$

Therefore, if $\pi_n(p) < p_{\text{site}}^{\text{site}}$, by (1) we find that

$$\lim_{m \to \infty} \theta_{m+n}(p, M) = 0,$$

which is equivalent to $\theta(p, M) = 0$ and hence $p < p_c(M)$. \hfill $\Box$
2.3 A convergent sequence of lower bounds for \( p_c(M) \)

In this section we define a sequence of lower bounds for \( p_c(M) \). We prove that this sequence converges to the \( p_c(M) \). Let

\[
p^n_c(M) = \sup \{ p : \pi_n(p, M) < p_{\text{site}} \}.
\]  

Note that \( \pi_n(p^n_c) = p_{\text{site}} \) for all \( n \), since \( \pi_n(p) \) is continuous in \( p \). Since \( \pi_n(p) \) is strictly decreasing in \( n \), it follows that \( \pi_{n+1}(p^n_c) < p_{\text{site}} \) for all \( n \) and hence by Theorem \[\text{[1]}\] indeed \( p^n_c(M) < p_c(M) \) for all \( n \). The strict monotonicity in \( n \) also implies that \( p^n_c(M) \) is increasing. The obvious question now is whether \( (p^n_c(M))^{\infty}_{n=0} \) converges to \( p_c(M) \). We will show that this is indeed the case. First we need that \( \pi_n(p) \) goes to zero if the fractal percolation is subcritical. Basically this is known, since \( K \) is almost surely disconnected when \( p < p_c(M) \) (see [2, 3]).

Lemma 1 If \( p < p_c(M) \), then \( \lim_{n \to \infty} \pi_n(p, M) = 0 \).

Proof Suppose \( p < p_c(M) \), so \( \theta(p) = 0 \). Note that a.s. there is an \( n \) such that in \( K_n \) the two squares in the top left and bottom left corner are discarded already. Conditioned on this event, a connection in the limiting set from the left side to any other side can only occur if it horizontally crosses the vertical strip \( S \) consisting of the squares \([0, M^{-n}] \times [jM^{-n}, (j + 1)M^{-1}] \) for \( j = 1, \ldots, M^n - 2 \). From selfsimilarity and subcriticality it follows that in the limit each of these squares has zero probability to contain a component connecting opposite sides (horizontally and vertically). So in \( K \) a horizontal crossing of \( S \) can only occur if it crosses a block of two vertically adjacent squares. But as Dekking and Meester showed (Lemma 5.1 in [5]), \( \theta(p) = 0 \) implies that such a block crossing has zero probability as well. It follows that connections from the left side to any other side have zero probability to occur. If this holds for the left side, then it holds for all sides, and so \( \lim_{n \to \infty} \pi_n(p) = 0 \).

The previous lemma makes it easy to prove convergence:

Proposition 1 The sequence \( (p^n_c(M))^{\infty}_{n=0} \) converges to \( p_c(M) \) if \( n \to \infty \).

Proof Let \( \varepsilon > 0 \) and suppose \( p = p_c(M) - \varepsilon \). Then by Lemma \[\text{[1]}\] there exists an \( N \) such that \( \pi_n(p) < p_{\text{site}} \) for all \( n \geq N \). Therefore, \( p^n_c(M) > p \) for all \( n \geq N \).}

The theory developed so far gives in principle an algorithmic way to calculate an increasing and converging sequence of lower bounds for \( p_c(M) \): compute the polynomial \( \pi_n(p) \) and solve \( \pi_n(p) = p_{\text{site}} \). Since \( p_{\text{site}} \) is not known exactly, we replace it in our calculations by the lower bound of van den Berg and Ermakov. This leads to lower bounds \( \tilde{p}^n_c \) for \( p_c \) that are smaller.
than $p_c^n$, but they still converge to $p_c$ (the proof of Proposition 1 still works).

**Example 1** For $M = 2$ and $M = 3$ we find

\[
\begin{align*}
\pi_1(p, 2) &= 1 - (1 - p)^4, \\
\pi_1(p, 3) &= 1 + (1 - p)^4(p^5 + 4p^4(1 - p) + 6p^3(1 - p)^2 - 1).
\end{align*}
\]

Solving $\pi_1(p, 2) = 0.556$ and $\pi_1(p, 3) = 0.556$ gives $\tilde{p}_c(2) \approx 0.183$ and $\tilde{p}_c(3) \approx 0.178$. Therefore, $p_c(2) > 0.183$ en $p_c(3) > 0.178$. ■

To find sharper bounds than in the above example, we should take larger values of $n$. However, for large $n$, the functions $\pi_n(p)$ are very complicated polynomials, and it is not clear how to find them in reasonable time. In the next section we will discuss a way to avoid this problem.

### 3 Classifying realizations

The number of possible realizations of $K_n$ and their complexity rapidly increases as $n$ grows. In this section we introduce a way to reduce the complexity without losing too much essential information on the connectivity structure in $K_n$. Basically, we will divide the boundary of the unit square in some segments and the presence or absence of connections between these segments will determine the type of a realization of $K_n$. We take a finite set of symbols (also called letters), each representing a type, that does not depend on $n$. This set will be called the alphabet $\mathcal{A}$. We will analyze the probabilities that $K_n$ is of a certain type. These ideas can be used to obtain both lower and upper bounds for $p_c(M)$.

Let $\mathcal{X}_n$ be the set of all possible realizations of $K_n$. For each $n$ we will define a map $\mathcal{C}_n : \mathcal{X}_n \rightarrow \mathcal{A}$. The sequence of maps $\mathcal{C} = (\mathcal{C}_n)_{n=0}^{\infty}$ will be called a classification. In Section 4 we will give a detailed description of the alphabet and classification. For now we only state that $\mathcal{A}$ will be a partially ordered set, having a unique minimum and maximum, with the property that

\[
\mathcal{C}_n(K_n) = \begin{cases} 
\min(\mathcal{A}) & \text{if } K_n = \emptyset, \\
\max(\mathcal{A}) & \text{if } K_n = [0, 1]^2.
\end{cases}
\]

The letters from $\mathcal{A}$ can be used to create words. For our purposes we only need two-dimensional square words of size $M \times M$, denoted by

\[
w = (w_{i,j})_{0 \leq i, j \leq M-1} = \begin{array}{cccc}
w_{0,M-1} & \cdots & w_{M-1,M-1} \\
\vdots & \ddots & \vdots \\
w_{0,0} & \cdots & w_{M-1,0}
\end{array},
\]

where $w_{i,j} \in \mathcal{A}$. The set of all such words will be denoted by $\mathcal{A}^{M \times M}$. Since $K_n$ can be obtained by tiling $[0, 1]^2$ by scaled realizations of $K_{n-1}$, there is a
natural way to associate realizations of $K_n$ to words in $A^{M \times M}$. First define the tiles of $K_n$ as follows
\[
K_n(i, j) := K_n \cap \left( \left( \frac{i}{M}, \frac{i+1}{M} \right) \times \left( \frac{j}{M}, \frac{j+1}{M} \right) \right), \quad 0 \leq i, j \leq M - 1.
\]
Now rescale and translate them into the unit square:
\[
\hat{K}_n(i, j) := M \left( K_n(i, j) - \left( \frac{i}{M}, \frac{j}{M} \right) \right), \quad 0 \leq i, j \leq M - 1.
\]
Then each $\hat{K}_n(i, j)$ either is the empty set, or it can be seen as a realization of $K_{n-1}$. Now we can map $K_n$ to a word in $A^{M \times M}$, provided $C_{n-1}$ is known. Define $W_n : K_n \to A^{M \times M}$ by
\[
W_n(K_n)_{i,j} = C_{n-1}(\hat{K}_n(i, j)).
\]
(5)
So far we discussed two maps on $K_n$: one that maps realizations to $M \times M$ words (the map $W_n$) and one that maps realizations to single letters (the map $C_n$). If the word $W_n(K_n)$ completely determines $C_n(K_n)$, we say the classification is regular:

**Definition 3** Let $C = (C_n)_{n=0}^{\infty}$ be a classification. If there exists a map $\phi : A^{M \times M} \to A$ such that
\[
C_n = \phi \circ W_n, \quad n \geq 1,
\]
then we say $C$ is a regular classification and $\phi$ is called the word code of $C$.

Note that a regular classification is uniquely defined by its word code: (4) defines $C_0$ and (5) together with the regularity defines $C_n$ if $C_{n-1}$ is known.

**Example 2** Let $M = 2$ and take the alphabet $A = \{\square, \blacksquare\}$, where $\min A = \square$ and $\max A = \blacksquare$. For $w \in A^{2 \times 2}$ define
\[
\phi(w) = \begin{cases} 
\square & \text{if } w = \square \square, \\
\blacksquare & \text{otherwise.}
\end{cases}
\]
Let $C = (C_n)_{n=0}^{\infty}$ be regular with word code $\phi$. Then $C_0$ is determined by (4):
\[
C_0(K_0(p)) = C_0([0, 1]^2) = \max A = \blacksquare.
\]
For $n \geq 1$, $C_n = \phi \circ W_n$. For instance, if $K_1 = [\frac{1}{2}, 1]^2$, then
\[
C_1(K_1) = \phi(W_1([\frac{1}{2}, 1]^2)) = \phi \left( \begin{array}{c} C_0(\hat{K}_1(1, 2)) \\ C_0(\hat{K}_1(2, 2)) \end{array} \right) = \phi \left( \begin{array}{c} C_0(\emptyset) \\ C_0(0, 1]^2) \end{array} \right) = \phi \left( \begin{array}{c} \square \\ \blacksquare \square \square \end{array} \right) = \blacksquare.
\]
We now want to analyze the probabilities $P(C(K_n(p)) = a)$, where $a \in \mathcal{A}$. Suppose $\mathcal{C}$ is a regular classification with word code $\phi$. Let

$$\mathcal{P}_\mathcal{A} = \left\{ x \in [0,1]^{||\mathcal{A}||} : ||x||_1 = 1 \right\}$$

be the set of all probability vectors on $\mathcal{A}$. For $x \in \mathcal{P}_\mathcal{A}$ and $a \in \mathcal{A}$, we denote the probability that $x$ assigns to $a$ by $x_a$. Take $x \in \mathcal{P}_\mathcal{A}$, and suppose we construct an $M \times M$ word $w$ in which all letters are chosen independently according to $x$. Define $F_\mathcal{C}(x) \in \mathcal{P}_\mathcal{A}$ by

$$(F_\mathcal{C}(x))_a = P_x(\phi(w) = a), \quad a \in \mathcal{A}.$$ 

The function $F_\mathcal{C} : \mathcal{P}_\mathcal{A} \to \mathcal{P}_\mathcal{A}$ will be the key to calculate the probabilities $P(C(K_n(p)) = a)$ in an iterative way, as is shown in the next lemma. Define $\tau^n(p) \in \mathcal{P}_\mathcal{A}$ by

$$\tau^n_a(p) = P(\mathcal{C}(K_n(p)) = a).$$

Let $\tau^\Box$ and $\tau^\blacksquare$ be the vectors that assign full probability to $\min(\mathcal{A})$ and $\max(\mathcal{A})$ respectively:

$$\tau^\Box_a = \begin{cases} 1, & a = \min(\mathcal{A}) \\ 0, & \text{otherwise.} \end{cases} \quad \tau^\blacksquare_a = \begin{cases} 1, & a = \max(\mathcal{A}) \\ 0, & \text{otherwise.} \end{cases}$$

For $S \subseteq \mathcal{A}$, we write

$$\tau^n_S(p) = P(\mathcal{C}(K_n(p)) \in S) = \sum_{a \in S} \tau^n_a(p).$$

**Lemma 2** If the classification $\mathcal{C}$ is regular, then

$$\tau^{n+1}(p) = F_\mathcal{C}(p\tau^n(p) + (1-p)\tau^\Box) \quad \text{with initial condition} \quad \tau^0(p) = \tau^\blacksquare.$$ 

**Proof** The letters in $\mathcal{W}_{n+1}(K_{n+1}(p))$ are independent. With probability $p$ a letter corresponds to a scaled realization of $K_n(p)$, with probability $1-p$ it corresponds to an empty square. So each letter occurs according to the probability vector $p\tau^n(p) + (1-p)\tau^\Box$. Together with the definition of $F_\mathcal{C}$, this gives the recursion. The initial condition follows from (4). \qed

This recursion formula is essentially a generalization of the recursion given in [5].

### 3.1 Strategy for lower bounds

Recall that our main obstacle in finding sharp bounds for $p_c$ is that $\pi_n(p)$ is hard to compute. The recursion of Lemma 2 gives a tool to dominate $\pi_n(p)$ by something that is easier to compute. The strategy to find lower bounds for $p_c$ is as follows. Define an alphabet $\mathcal{A}$ with subset $\mathcal{A}_\pi$ and
a regular classification \( \mathcal{C} \) (by choosing a word code) in such a way that \( \pi_n(p) \leq \mathbb{P}(\mathcal{C}(K_n(p))) \in \mathcal{A}_n \pi \) for all \( n \). Now take some \( n \) and search for the largest \( p \) for which the latter probability is smaller than 0.556. Then it follows that \( \pi_n(p) < p_{c,\text{site}} \) and therefore \( p < p_c \) by Theorem 1. We will give an example (which only gives a very moderate bound) to illustrate this procedure.

**Example 3** A lower bound for \( M = 2 \) Let \( \mathcal{C} \) be the classification of Example 2. By induction it follows that \( \mathcal{C}_n(K_n) = \square \) if \( K_n \) is nonempty. So, if we choose \( \mathcal{A}_x = \{ \square \} \), then definitely \( \pi_n(p) \leq \mathbb{P}(\mathcal{C}(K_n(p))) \in \mathcal{A}_n \pi = \tau_n \pi(p) \).

From the definition of \( \phi \) it follows that in this case

\[
(F_\phi(x)) \square = \mathbb{P}_x(\phi(w) = \square) = 1 - \mathbb{P}_x \left( w = \begin{array}{l}
\square \\
\square
\end{array} \right) = 1 - (x \square)^4 = 1 - (1 - x \square)^4.
\]

Note that \((p \tau^n(p) + (1 - p)\tau^n \phi) \square = p \tau^n \pi(p)\) and apply Lemma 2.

\[
\tau_{n+1}^\square(p) = (F_p(p \tau^n(p) + (1 - p)\tau^n \phi)(\square)) = 1 - (1 - p \tau_n \pi(p))^4, \quad \text{with} \quad \tau_0^\square(p) = 1.
\]

Writing \( G_p(x) := \tau_1^n(px) = 1 - (1 - px)^4 \) leads to

\[
\tau_{n+1}^\square(p) = G_p(\tau_n \pi(p)) = G_{p}^{n+1}(1).
\]

The function \( G_p(x) \) is increasing on \([0, 1]\) and \( G_p(1) \leq 1\), so \( \tau_n \square(p) \) decreases to the largest fixed point of \( G_p \). For \( p = 0.33 \) the fixed point is still below 0.556 and we find \( \pi_{50}(p) \leq \tau_{50}^\square(p) \approx 0.554 < 0.556 \leq p_{c,\text{site}} \). Consequently \( p_c(2) > 0.33 \) by Theorem 1.

### 3.2 Strategy for upper bounds

Our recipe to find upper bounds for \( p_c(M) \) is a bit more involved. We start by defining a partial ordering on the set of probability vectors \( \mathcal{P}_{\mathcal{A}_n} \). A set \( S \subseteq \mathcal{A} \) will be called increasing if \( a \in S \) implies \( b \in S \) for all \( b \geq a \). For \( x, y \in \mathcal{P}_{\mathcal{A}_n} \), we now write \( x \geq y \) if

\[
\sum_{a \in S} x_a \geq \sum_{a \in S} y_a, \text{ for all increasing } S \subseteq \mathcal{A}_n.
\]

We say the function \( F_\phi \) is increasing if \( F_\phi(x) \geq F_\phi(y) \) for \( x \geq y \).

**Lemma 3** Let \( \mathcal{C} \) be a regular classification for which \( F_\phi \) is increasing. Then \((\tau^n(p))_{n=0}^\infty\) is decreasing and \( \tau^n(p) := \lim_{n \to \infty} \tau^n(p) \) exists.

**Proof** Let \( S \) be any nonempty increasing subset of \( \mathcal{A}_n \). Then \( \max(\mathcal{A}_n) \in S \), and since \( \tau_{\max(\mathcal{A}_n)}^\square = 1 \) we have

\[
\sum_{a \in S} \tau_{a}^\square = 1 \geq \sum_{a \in S} x_a, \text{ for all } x \in \mathcal{P}_{\mathcal{A}_n}.
\]
Since \( \tau^0(p) = \tau^{\mathcal{B}} \), it follows that
\[
\tau^0(p) \geq x, \text{ for all } x \in \mathcal{P}_{\mathcal{A}},
\]
and in particular \( \tau^0(p) \geq \tau^1(p) \). Now we use induction: suppose \( \tau^n(p) \geq \tau^{n+1}(p) \). Then \( \tau^{n+1}(p) = F_\mathcal{E}(p\tau^n(p) + (1 - p)\tau^\mathcal{D}) \geq F_\mathcal{E}(p\tau^{n+1}(p) + (1 - p)\tau^\mathcal{D}) = \tau^{n+2}(p) \) since \( F_\mathcal{E} \) is increasing. So \( (\tau^n(p))_{n=1}^\infty \) is decreasing.

To prove existence of \( \lim_{n \to \infty} \tau^n(p) \), we will show that \( \lim_{n \to \infty} \tau^n_a(p) \) exists for all \( a \in \mathcal{A} \). Let \( S_a = \{ b \in \mathcal{A} : b \succeq a \} \). Then \( S_a \) and \( S_a \setminus \{ a \} \) are both increasing sets. Since \( (\tau^n(p))_{n=0}^\infty \) is decreasing, \( (\tau^n_{S_a}(p)) \) and \( (\tau^n_{S_a \setminus \{ a \}}(p)) \) are decreasing real-valued sequences, bounded from below by 0. Therefore, their limits exist and also
\[
\lim_{n \to \infty} \tau^n_a(p) = \lim_{n \to \infty} \left( \tau^n_{S_a}(p) - \tau^n_{S_a \setminus \{ a \}}(p) \right)
\]
exists. Since \( \mathcal{A} \) is finite, these limiting probabilities uniquely determine \( \tau^\infty(p) \). \( \square \)

To find upper bounds for \( p_c(M) \), we want to bound \( \theta(p) \) away from 0. We will construct an alphabet \( \mathcal{A} \) with an increasing subset \( \mathcal{A}_0 \subseteq \mathcal{A} \) and a regular classification \( \mathcal{C} \) for which \( F_\mathcal{E} \) is increasing. We will do this in such a way that \( \tau^\mu_\mathcal{C}(p) := \mathbb{P}(\mathcal{C}(K_n(p)) \in \mathcal{A}_0) \leq \theta_n(p) \) for all \( n \). If we can prove that \( \tau^\infty(p) > 0 \), then it follows that \( \theta(p) > 0 \) and hence \( p_c < p \). Finding \( \tau^\infty(p) \) exactly might be not so easy, but the following lemma gives the key to find a lower bound for it.

**Lemma 4** Let \( \mathcal{C} \) be a regular classification for which \( F_\mathcal{E} \) is increasing.

1. If \( F_\mathcal{E}(px + (1 - p)\tau^\mathcal{D}) \geq x \) for some \( x \in \mathcal{P}_{\mathcal{A}} \) and \( p \in (0, 1] \), then
   \[
   \tau^\infty(p) \geq x.
   \]
   2. If in addition \( \sum_{a \in \mathcal{A}_0} x_a > 0 \) and \( \theta_n(p) \geq \tau^\mu_n(p) \) for all \( n \), then
   \[
   p > p_c.
   \]

**Proof** We show by induction that \( \tau^n(p) \geq x \) for all \( n \in \mathbb{N} \). First note that \( \tau^0(p) = \tau^{\mathcal{B}} \geq x \). Now suppose \( \tau^n(p) \geq x \) for some \( n \). Then by Lemma 2
\[
\tau^{n+1}(p) = F_\mathcal{E}(p\tau^n(p) + (1 - p)\tau^\mathcal{D}) \geq F_\mathcal{E}(px + (1 - p)\tau^\mathcal{D}) \geq x.
\]
Hence indeed \( \tau^n(p) \geq x \) for all \( n \in \mathbb{N} \) and consequently \( \tau^\infty(p) \geq x \).

For the second statement, observe that
\[
\theta(p) = \lim_{n \to \infty} \theta_n(p) \geq \lim_{n \to \infty} \tau^\mu_n(p) = \tau^\infty(p) \geq \sum_{a \in \mathcal{A}_0} x_a > 0,
\]
where we used that $\mathcal{A}_\mu$ is an increasing set. Consequently $p_c < p$. □

The crucial question now is if for given $\mathcal{A}$, $\mathcal{A}_\mu$ and $\mathcal{C}$ there exists $x \in \mathcal{P}_{\mathcal{A}}$ satisfying all requirements and how we can find it. Here we give a guideline to find numerical results. First approximate the fixed point $\tau^\infty(p)$ by iterating the recursion of Lemma 2. If $p$ is too small, then $\tau^\infty(p) = 0$ and no suitable $x$ will exist. If $p$ is large enough, then $\tau^\infty(p) > 0$. In the latter case, if $x$ is an approximation for $\tau^\infty(p)$, we have

$$\sum_{a \in \mathcal{A}_\mu} x_a > 0 \quad \text{and} \quad F_C(px + (1 - p)\tau^\infty) \approx x.$$ 

Now apply the following trick: let $\varepsilon$ be a small positive number, then

$$\sum_{a \in \mathcal{A}_\mu} x_a > 0 \quad \text{and} \quad F_C((p + \varepsilon)x + (1 - (p + \varepsilon))\tau^\infty) \geq x.$$ 

Now we have an $x$ that fits into the conditions of Lemma 4, and therefore $p_c < p + \varepsilon$. Before we give an example of the procedure to find upper bounds for $p_c(M)$, we will first construct suitable alphabets in the next section.

4 Construction of the alphabet and word codes

Let $E = \{e_1, \ldots, e_n\}$ be a collection of closed line segments whose union is the boundary of $[0, 1]^2$. Assume that they do not overlap, i.e. the intersection of two segments is at most a single point. We number them clockwise starting from $(0, 0)$. We will define the alphabet $\mathcal{A}_E$ by means of non-crossing equivalence relations on $E$.

**Definition 4** Let $A = \{a_1, \ldots, a_n\}$ be an ordered set. A non-crossing equivalence relation on $A$ is a set $R \subseteq A \times A$ with the following properties:

1. $(a_i, a_i) \in R, \quad i = 1, \ldots, n$,
2. $(a_i, a_j) \in R \Leftrightarrow (a_j, a_i) \in R, \quad i, j = 1, \ldots, n$,
3. If $(a_i, a_j), (a_j, a_k) \in R$, then $(a_i, a_k) \in R, \quad i, j, k = 1, \ldots, n$,
4. If $i < j < k < l$ and $(a_i, a_k), (a_j, a_l) \in R$, then $(a_i, a_j) \in R$.

If $(a_i, a_j) \in R$, we say $a_i$ and $a_j$ are equivalent and write $a_i \sim_R a_j$, or simply $a_i \sim a_j$. The set $[a_i] = \{a_j \in A : a_i \sim a_j\}$ is called the equivalence class of $a_i$.

The first three properties are the usual reflexivity, symmetry and transitivity. First consider the simplest case,

$$E = \{e_1, e_2, e_3, e_4\} = \{ \{0\} \times [0, 1], [0, 1] \times \{1\}, \{1\} \times [0, 1], [0, 1] \times \{0\} \}.$$
An equivalence relation on $E$ can be graphically represented as a square with some connections (the equivalences) between the boundaries. For example, the symbol $\Box$ represents the equivalence relation
\[
\{(e_1, e_1), (e_1, e_4), (e_2, e_2), (e_2, e_3), (e_3, e_3), (e_4, e_1), (e_4, e_4)\},
\]
which has equivalence classes $\{e_1, e_4\}$ and $\{e_2, e_3\}$. Doing this for all non-crossing equivalence relations on $E$ gives the following set of symbols:
\[
\mathcal{A}_E = \{\Box, \Box, \Box, \Box, \Box, \Box, \Box, \Box, \Box, \Box, \Box, \Box\}.
\]
It will be convenient to identify these symbols with the corresponding equivalence relations. Now we can easily define a partial order on $\mathcal{A}_E$: we write $a \preceq b$ if $a$ is contained in $b$. The alphabet has a unique minimum and maximum, $\min(\mathcal{A}_E) = \Box$ and $\max(\mathcal{A}_E) = \Box$.

In a way similar to the identification of letters with equivalence relations on $E$, we will identify $M \times M$ words with equivalence relations on $E_{M \times M} = \bigcup_{0 \leq i, j \leq M-1} E + (i, j)$. Suppose $w \in \mathcal{A}_{E_{M \times M}}$. Each letter $w_{i,j}$ from this word is an equivalence relation on $E$, and therefore $w_{i,j} + (i, j) \times (i, j)$ is an equivalence relation on $E + (i, j)$. Henceforth,
\[
R_w := \bigcup_{0 \leq i, j \leq M-1} w_{i,j} + (i, j) \times (i, j)
\]
is a binary relation on $E_{M \times M}$. In general $R_w$ is not an equivalence relation. It can be easily checked that $R_w$ is both reflexive and symmetric. However, $R_w$ might fail to be transitive. For example, if $w_{0,0} = w_{0,1} = \Box$ then
\[
([0, 1] \times \{0\}, [0, 1] \times \{1\}) \in R_w \text{ and } ([0, 1] \times \{1\}, [0, 1] \times \{2\}) \in R_w,
\]
but $([0, 1] \times \{0\}, [0, 1] \times \{2\}) \notin R_w$. Let $\bar{R}_w$ be the transitive closure of $R_w$ (that is, $\bar{R}_w$ is the smallest equivalence relation on $E_{M \times M}$ that contains $R_w$). Figure 4, right panel, shows a $3 \times 3$ word over $\mathcal{A}_E$. The sides of the dashed squares are elements of the set $E_{M \times M}$ and two elements are equivalent with respect to $\bar{R}_w$ if there is a solid connection between them.

The theory above does not change essentially if we choose $E$ to be a larger set of line segments that all have equal length. A special case arises if all segments have length $M^{-n}$. In this case $E = \{e_1, \ldots, e_{4M^n}\}$ with $e_i = \{0\} \times [(i - 1)M^{-n}, iM^{-n}]$ for $1 \leq i \leq M^n$ et cetera. We will denote the corresponding alphabet by $\mathcal{A}_{E,M^n}$. As a final remark we note that the size of $\mathcal{A}_E$ is given by the Catalan number $\frac{1}{|E| + 1} \binom{2|E|}{|E|}$.
Figure 4: Left: A realization of $K_2$ for $M = 3$. Right: A $3 \times 3$ word $w$ over $A_{3,0}$. If we use the word code $\Psi_{3,0}$, then this word corresponds to the realization on the left.

4.1 Weak and strong connectivity

In this section we will define our word codes. Suppose $w$ is an $M \times M$ word over the alphabet $A_{M,n}$, so $w \in A_{M,n}^{M \times M}$. Let $E$ be the corresponding segment set. We define the boundary segment set of $E$ by

$$\partial E_{M \times M} := \{ e \in E_{M \times M} : e \cap \partial [0, M]^2 = e \}.$$ 

The set $\partial E_{M \times M}$ contains $4M^n+1$ segments and we number them clockwise starting from $(0,0)$: $e_1^w, \ldots, e_{4M^n+1}^w$. Now we partition $\partial E_{M \times M}$ into the sub-sets

$$E_i^w = \{ e_{M(i-1)+j}^w : 1 \leq j \leq M \}, \quad 1 \leq i \leq 4M^n. \quad (7)$$

Let $\sim$ denote equivalence with respect to $R_w$. If $e \sim f$ for some $e \in E_i^w$ and $f \in E_j^w$, we say $E_i^w$ and $E_j^w$ are connected. If $E_{i_1}^w, \ldots, E_{i_m}^w$ form a chain of pairwise connected sets, we say $E_{i_1}^w$ and $E_{i_m}^w$ are weakly connected. If there exists an equivalence class $C$ such that $|C \cap E_i^w| > M/2$ and $|C \cap E_j^w| > M/2$ or if $i = j$, we say $E_i^w$ and $E_j^w$ are strongly connected. Observe that weak and strong connectivity are non-crossing equivalence relations on $\{E_1^w, \ldots, E_{4M^n}^w\}$.

These notions provide tools to define word codes: a word determines a non-crossing equivalence relation on $\{E_1^w, \ldots, E_{4M^n}^w\}$ that will be mapped in the obvious way to a non-crossing equivalence relation on $E = \{e_1, \ldots, e_{4M^n}\}$, which is just a letter in $A_{M,n}$. In this way, define word codes

$$\Phi_{M,n} : A_{M,n}^{M \times M} \rightarrow A_{M,n} \quad \text{and} \quad \Psi_{M,n} : A_{M,n}^{M \times M} \rightarrow A_{M,n}.$$
based on weak and strong connectivity respectively. If $\overline{R}_{w_1} \subseteq \overline{R}_{w_2}$, then

$$\Phi_{M,n}(w_1) \leq \Phi_{M,n}(w_2) \quad \text{and} \quad \Psi_{M,n}(w_1) \leq \Psi_{M,n}(w_2).$$

Consequently, if $\mathcal{C}$ is a regular classification defined by one of these word codes, then $F_\mathcal{C}$ is increasing.

**Example 4** Consider the word $w$ over $\mathcal{A}_{3,0}$ as shown in Figure 4. In this case $E_{M \times M}$ contains 12 boundary segments, $e_1^w, \ldots, e_{12}^w$. Take partition sets $E_1^w, \ldots, E_4^w$ as in (7). For instance,

$$E_3^w = \{e_7^w, e_8^w, e_9^w\} = \{(3) \times [2, 3], (3) \times [1, 2], (3) \times [0, 1]\}$$

contains the three segments at the right side. Then $E_1^w, E_2^w$ and $E_4^w$ are all pairwise connected and $E_3^w$ is connected to $E_4^w$. Consequently $E_3^w$ is weakly connected to $E_i^w$ for all $i$ and $j$. Therefore $\Phi_{3,0}(w) = \mathbb{1}$.

Since $e_1^w, e_2^w, e_5^w$ and $e_6^w$ are in the same equivalence class, $E_1^w$ and $E_2^w$ are strongly connected. There are no other $i$ and $j$, $i \neq j$ for which $E_i^w$ and $E_j^w$ are strongly connected. Therefore $\Psi_{3,0}(w) = \mathbb{0}$.

The idea behind the definitions of $\Phi_{M,n}$ and $\Psi_{M,n}$ is that they guarantee the following key properties:

**Property 2** Define a classification $\mathcal{C}$ by the word code $\Phi_{M,k}$. Then the following implication holds: if $e, f \in E$ are connected in $K_n$, then $e \sim f$ in $\mathcal{C}_n(K_n)$.

**Proof** For $n = 0$, we have $\mathcal{C}_0(K_0) = \max(\mathcal{A}_{M,k})$, which means that all segments are equivalent. So the statement holds for $n = 0$.

Now suppose the statement is true for some $n$, and take a realization of $K_{n+1}$. Let $w = \mathcal{C}_{n+1}(K_{n+1})$. In each of the tiles $K_{n+1}(i, j)$, the induction hypothesis applies. So if $e, f \in E_{M \times M}$ are connected in (a scaled, translated version of) the tile $K_{n+1}(i, j)$, then the corresponding segments in the letter $w_{i,j}$ are equivalent. If $e_a, e_b \in E$ are connected in $K_{n+1}$, then this connection successively traverses some tiles. Hence, there exists segments $s_1, \ldots, s_m \in E_{M \times M}$ such that $s_1 \in E_a^w, s_m \in E_b^w$ and for which $s_i \sim u, s_{i+1}, i = 1, \ldots, m - 1$ by the induction hypothesis. Therefore $s_1 \sim u, s_m$ and $E_a^w$ is weakly connected to $E_b^w$. Consequently $e_a \sim e_b$ in $\Phi_{M,k}(w) = \mathcal{C}_{n+1}(K_{n+1})$.

**Property 3** Define a classification $\mathcal{C}$ by the word code $\Psi_{M,k}$. Then the reversed implication holds: if $e \sim f$ in $\mathcal{C}_n(K_n)$, then $e$ and $f$ are connected in $K_n$.

Before proving this property, we introduce some terminology. Let $C_0 = I$ be an interval and fix integers $M$ and $k > M/2$. Construct $C_1$ by subdividing $I$ into $M$ subintervals of equal length and let $k$ of them survive. Repeat this process in each of the surviving subintervals. The sets in the resulting
sequence $C_0, C_1, \ldots$ will be called $M$-adic fractal majority subsets of $I$. If $A$ and $B$ are $M$-adic fractal majority subsets of $I$, then $A \cap B$ contains an interval. If $e, f \in E$ and there is a connected component in $K_n(M)$ containing $M$-adic fractal majority subsets of both $e$ and $f$, we say there is a fractal majority connection between $e$ and $f$.

**Proof of Property** [3] We will actually prove a stronger statement. Since $K_0 = [0, 1]^2$ all boundary segments are connected to each other in $K_0$ by a fractal majority connection. Induction hypothesis: if $e \sim f$ in $C_n(K_n)$, then $e$ and $f$ are connected in $K_n$ by a fractal majority connection.

Suppose $e_a, e_b \in E$ and there is a connected component in $K_{n+1}(M)$ containing $M$-adic fractal majority subsets of both $e_a$ and $e_b$. We say there is a fractal majority connection between $e_a$ and $e_b$.

Consider the alphabet $A_{M,k}$ and let $E$ be the corresponding segment set. Partition $E$ into four sets, each corresponding to one of the sides of the unit square:

$$E_i = \{ e_{1+i[E]/4}, \ldots, e_{(1+i)[E]/4} \}, \quad i = 0, 1, 2, 3.$$ 

and define

$$A_\pi = \{ a \in A_{M,k} : \exists e \in E_i, f \in E_j, i \neq j, \text{ such that } e \sim_a f \},$$

$$A_\mu = \{ a \in A_{M,k} : \exists e \in E_1, f \in E_3, \text{ such that } e \sim_a f \}. \quad (8)$$

The following lemma shows that the alphabets and word codes as defined in this section are suitable for our purposes, see the discussion in Section [3].

**Lemma 5** Take the alphabet $A = A_{M,k}$ and define $A_\pi$ and $A_\mu$ as in (8).

1. Define a classification $\mathcal{C}$ by the word code $\Phi_{M,k}$. Then

$$\tau_\pi^n(p) := \mathbb{P}(\mathcal{C}(K_n(p)) \in A_\pi) \geq \pi_n(p), \quad \text{for all } n.$$ 

2. Define a classification $\mathcal{C}$ by the word code $\Psi_{M,k}$. Then

$$\tau_\mu^n(p) := \mathbb{P}(\mathcal{C}(K_n(p)) \in A_\mu) \leq \theta_n(p), \quad \text{for all } n.$$ 

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These statements follow from Property 2 and 3 respectively.

Now we are ready to give an example illustrating how to find upper bounds. We will keep the example as simple as possible, so that it can be checked by hand. Therefore our alphabet will contain only two letters and we will use a simplification of the word code $\Psi_{3,0}$. Nevertheless, it leads to a bound that already improves upon the best bound known so far.

Example 5 (An upper bound for $M = 3$) Let $A = \{\min(A_{3,0}), \max(A_{3,0})\} = \{\square, \blacklozenge\}$ and let $E$ be the boundary segment set corresponding to $A_{3,0}$. Partition $\partial E_{M \times M}$ into four sets as in (7). Define a regular classification $C$ by the word code 

$$
\phi(w) = \begin{cases} 
\square & \text{if } E_i^w \text{ and } E_j^w \text{ strongly connected for all } i, j, \\
\blacklozenge & \text{otherwise.}
\end{cases}
$$

This classification is increasing. Define $A_\mu = \{\blacklozenge\}$, which is an increasing subset of $A$. Since $\phi(w) \subseteq \Psi_{3,0}(w)$ for all $w \in A_{M \times M}$, the second statement of Lemma 5 also applies to this classification. For $x \in \mathcal{P}_\mathcal{A}$ given by $\mu = p^{\square} + (1 - p)^{\blacklozenge}$, write $x = (p, 1 - p)$. Counting all words for which $E_1^w, \ldots, E_4^w$ are strongly connected gives

$$
(F_C(x))_{\square} = \mathbb{P}_x(\phi(w) = \square) = p^9 + 9p^8(1 - p) + 20p^7(1 - p)^2.
$$

Now choose $p = 0.984$ and $y = (0.9720, 0.028)$. Then $(F_C(y))_{\square} \approx 0.9721 > y_{\square}$ and hence $F_C(y) \geq y$. Since $\sum_{a \in A_\mu} y_a = y_{\square} > 0$ all conditions for Lemma 4 are fulfilled. Hence $p_c(3) < 0.984$.

4.2 Monotonicity and convergence

So far we developed some tools to find bounds for $p_c(M)$. One would expect that taking larger alphabets results in sharper bounds, since we can approximate the connectivity structure in $K_n$ more accurately. In this section we show that this is indeed the case and that the lower bounds even convergence to $p_c(M)$ if the alphabet size goes to infinity.

For the word code $\Phi_{M,k}$ over $\mathcal{A}_{M,k}$, define the corresponding classification and let $\tau^n_\mu(p)$ be defined as before. Then define a critical value as follows:

$$
p_c(\Phi_{M,k}) := \sup \{ p : \tau^n_\mu(p) < p_c^{site} \}.
$$

Let $\mathcal{A} = \mathcal{A}_{M,k}$ and define $C$ by the word code $\Psi_{M,k}$. Let $\tau^n_\mu(p)$ be as before. Also here we define a critical value:

$$
p_c(\Psi_{M,k}) := \inf \{ p : \tau^n_\mu(p) > 0 \}.
$$

Now we have the following proposition:
Proposition 2 The sequence \( (p_c(\Phi_{M,k}))_{k=0}^\infty \) is increasing and \( (p_c(\Psi_{M,k}))_{k=0}^\infty \) is decreasing. Moreover,
\[
\lim_{k \to \infty} p_c(\Phi_{M,k}) = p_c(M).
\]

Proof Denote the segment set corresponding to the alphabet \( \mathcal{A} \) by \( E_{M,k} = \{ e_1, \ldots, e_{4M^k} \} \). Let \( \partial E_{M,k} \) be the set of boundary segments of \( M \times M \) words over \( \mathcal{A}_{M,k} \). Partition \( \partial E_{M,k} \times M \) into subsets \( E_1, \ldots, E_{4M^k} \) as in [7]. Each of these partition sets contains \( M \) segments, and the union of all \( 4M^{k+1} \) segments is equal to \( \partial[0, M]^2 \). There is a one-to-one correspondence between these segments and the elements of \( E_{M,k+1} \); if \( e \in E_{M,k+1} \), then \( Me \in \partial E_{M,k} \). Define
\[
C_i = \{ e \in E_{M,k+1} : Me \in E_i \}, \quad i = 1, \ldots, 4M^k.
\]

For \( e_i \in E_{M,k} \), we have \( e_i = \bigcup_{e \in C_i} e \). The elements of \( C_i \) will be called the children of their parent \( e_i \). Let \( \mathcal{A}_k^\partial \) and \( \mathcal{A}_k^\mu \) be defined according to (9).

Define regular classifications \( \mathcal{C}_k \) and \( \mathcal{C}_k^{k+1} \) by the word codes \( \Phi_{M,k} \) and \( \Phi_{M,k+1} \). Denote the corresponding probability vectors by \( k_\tau \) and \( k_\tau^{k+1} \). By induction on \( n \) it follows that if two segments are equivalent in \( \mathcal{C}_n^{k+1}(K_n) \), then their parents are equivalent in \( \mathcal{C}_n^{k}(K_n) \). Therefore, if \( \mathcal{C}_n^{k+1}(K_n) \in \mathcal{A}_k^\tau \) then \( \mathcal{C}_n^{k}(K_n) \in \mathcal{A}_n^\tau \). Consequently, \( k_\tau^{n}(p) \geq k_\tau^{n+1}(p) \) and so
\[
p_c(\Phi(M, k+1)) \geq p_c(\Phi(M, k)).
\]

Now define \( \mathcal{C}_k \) and \( \mathcal{C}_k^{k+1} \) by the word codes \( \Psi_{M,k} \) and \( \Psi_{M,k+1} \). By induction on \( n \); if \( e_i, e_j \in E_{M,k} \) are equivalent in \( \mathcal{C}_n^{k}(K_n) \), then the sets of children \( C_i \) and \( C_j \) are strongly connected in \( \mathcal{C}_n^{k+1}(K_n) \). So, if \( \mathcal{C}_n^{k}(K_n) \in \mathcal{A}_\mu^k \) then \( \mathcal{C}_n^{k+1}(K_n) \in \mathcal{A}_\mu^{k+1} \). Henceforth, \( k_\mu^{n}(p) \leq k_\mu^{n+1}(p) \), so
\[
p_c(\Psi(M, k+1)) \leq p_c(\Psi(M, k)).
\]

Having shown the monotonicity of the two sequences, we now turn to the convergence of \( p_c(\Psi(M, k)) \). Take the alphabet \( \mathcal{A}_{M,k} \) and define \( \mathcal{C}_k \) by the word code \( \Phi_{M,k} \). A realization of \( K_n \) consists of \( M^n \times M^n \) squares, and letters in \( \mathcal{A}_{M,k} \) have \( M_k \) boundary segments at each side. This means that for \( n \leq k \) the classification describes the connectivity structure exactly: two segments in \( \mathcal{C}_k(K_n) \) are equivalent if and only if they are connected in \( K_n \). So \( \pi_n(p) = k\pi_n(p) \) if \( n \leq k \). This implies that for \( n \leq k \) we can rewrite (5):
\[
p_c^n(M) = \sup \left\{ p : k\pi_n(p) < p_{\text{site}}^c \right\}
\leq \sup \left\{ p : k\pi^\infty(p) < p_{\text{site}}^c \right\} = p_c(\Phi_{M,k}) \leq p_c(M),
\]
where we used that \( k\pi(p) \) decreases in \( n \) by Lemma 3. Proposition 1 states that \( p_c^n(M) \) converges to \( p_c(M) \), so we conclude that \( \lim_{k \to \infty} p_c(\Phi_{M,k}) = p_c(M) \). \( \square \)
5 Numerical results

In this section we present our numerical results. The recursion of Lemma 2 is the main tool to perform the calculations. Our implementation in Matlab (everything available from the author on request) gives the following results:

**Proposition 3** Take the alphabet $\mathcal{A}_{M,k}$ and define a classification $\mathcal{C}$ by the word code $\Phi_{M,k}$. Let $n = 1000$ and define $\tau^n_\pi(p)$ as before. Then
- For $M = 2$ and $k = 0$, we have $\tau^n_\pi(0.785) < p_c^{site}$.
- For $M = 2$ and $k = 1$, we have $\tau^n_\pi(0.859) < p_c^{site}$.
- For $M = 3$ and $k = 0$, we have $\tau^n_\pi(0.715) < p_c^{site}$.

**Corollary 1** $p_c(2) > 0.859$ and $p_c(3) > 0.715$.

**Proof** This follows from Lemma 5 and Theorem 1. □

Figure 5 illustrates for the case $M = 2$ and $k = 0$ how $\tau^n_\pi(p)$ behaves as a function of $n$ for some values of $p$. The values of $\tau^n_\pi(p)$ were calculated by iterating the recursion of Lemma 2.

![Figure 5: Plot of $\tau^n_\pi(p)$ for $p = 0.7 + 0.01k$ where $k = 0, \ldots, 9$ as functions of $n$. Especially note the difference between $p = 0.78$ and $p = 0.79$.](image)

For larger values of $k$ the computations were too complicated to perform in a reasonable computation time. For example, the segment set $E_{2,2}$ contains 16 segments and therefore the alphabet $\mathcal{A}_{2,2}$ already contains $\frac{1}{17}(\frac{32}{16}) =
A

One can improve this even a bit more by taking the alphabet \( \tau \) word code that is based on weak connectivity. For this classification we find analogous to our previous approach, we define a classification by choosing the word code that is based on weak connectivity. For this classification we find \( \tau_n^{50} (0.876) < p_c^{\text{site}} \), which implies \( p_c (2) > 0.876 \).

One can improve this even a bit more by taking the alphabet \( \mathcal{A}_{2,2} \) and defining a word code \( \Phi_{2,2} \) that is a bit simpler than \( \Phi_{2,2} \) as follows. If at least one of the letters in a \( 2 \times 2 \) word \( w \) equals \( \min (\mathcal{A}_{2,2}) \), then \( \Phi_{2,2} (w) = \Phi_{2,2} (w) \). Otherwise, define \( \Phi_{2,2} (w) \) by first mapping each of the four letters to \( \mathcal{A}_{2,3/2} \) and then mapping the new word to \( \mathcal{A}_{2,2} \), in both steps using weak connectivity. This simplifies the required calculations a lot, and leads to \( \tau_n^{200} (0.881) < p_c^{\text{site}} \). Since \( \Phi_{2,2} (w) \supseteq \Phi_{2,2} (w) \), we have \( \tau_n (p) \geq \tau_n (p) \geq \pi_n (p) \) for all \( n \). We conclude that \( p_c (2) > 0.881 \).

For \( M = 3 \) we improved the lower bound of Corollary by using segments of length \( 1/3 \) at the left and the right side of \( [0,1]^2 \) and segments of length \( 1 \) at the bottom and the top side. Denote the resulting alphabet by \( \mathcal{A}_{3,1/2} \) and choose the word code based on weak connectivity. This gives \( \tau_n^{100} (0.784) < p_c^{\text{site}} \), whence \( p_c (3) > 0.784 \).

These calculations have been checked by Arthur Bik, a mathematics student at Delft University of Technology. He independently implemented the algorithms and reproduced all results, except the bound \( p_c (2) > 0.881 \). This was due to the fact that his program was not fast enough to perform the calculations in a reasonable time. Concluding, the best lower bounds we found are

**Theorem 3** \( p_c (2) > 0.881 \) and \( p_c (3) > 0.784 \).

Now let us turn to the upper bounds. The strategy described in Section 3 leads to the following results:

**Proposition 4** Take the alphabet \( \mathcal{A}_{M,k} \) and define a classification \( \Psi_{M,k} \) by the word code \( \Psi_{M,k} \). Let \( n = 1000 \). The conditions \( F_n (px + (1-p)\Sigma \phi) \geq x \) and \( \sum_{a \in \mathcal{A}_n} x_a > 0 \) hold if \( x \) and \( p \) are chosen as follows:

- For \( M = 3 \) and \( k = 0 \), choose \( p = 0.958 \) and \( x = 7.579 \).
- For \( M = 4 \) and \( k = 0 \), choose \( p = 0.972 \) and \( x = 7.579 \).

**Corollary 2** \( p_c (3) < 0.958 \) and \( p_c (4) < 0.972 \).

**Proof** This follows from Lemma 4 and Lemma 5.
For $M = 3$, the result can be sharpened by using the alphabet $\mathcal{A}_{3,1/2}$. The classification is again defined by strong connectivity. In that case the choice $p = 0.940$ and $x = \tau^{1000}(0.9399)$ satisfies all conditions, so $p_c(3) < 0.940$. The algorithm for $M = 4$ can be slightly adapted to find a bound for $M = 2$. Each realization of $K_n$ for $M = 4$ can be seen as a realization of $K_{2n}$ for $M = 2$. Therefore, we can still use the word code $\Psi_{4,0}$. The only thing that changes is the way the probabilities are computed. Given $\tau^n(p)$, the letters in the $4 \times 4$ word $w = \Psi_n(K_{n+2})$ occur according to the following rule: The word $w$ consists of four $2 \times 2$ blocks. In each of these blocks either all letters are equal to $\min(\mathcal{A})$ (with probability $1-p$) or they are independent of each other chosen according to $p\tau^n(p) + (1-p)\tau^0$ (with probability $p$). Basically we collapse two construction steps of $K_n$ into one step. These ingredients determine the recursion. Performing the calculations we find that the conditions are satisfied for $p = 0.993$ and $x = \tau^{1000}(0.9929)$, henceforth $p_c(2) < 0.993$.

Also for the upper bounds Arthur Bik checked our results. He independently reproduced our bounds, except for the bound $p_c(3) < 0.940$ (for similar reasons as before). Summarizing, our best upper bounds are

**Theorem 4** $p_c(2) < 0.993$, $p_c(3) < 0.940$ and $p_c(4) < 0.972$.

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**References**


