Chapter 12
Phenomenology of the Noncommutative Standard Model

In Theorems 11.10 and 11.11, we have derived the full Lagrangian for the Standard Model from the almost-commutative manifold $M \times F_{SM}$. The coefficients in this Lagrangian are given in terms of:

- the value $f(0)$ and the moments f_2 and f_4 of the function f in the spectral action;
- the cut-off scale Λ in the spectral action;
- the vacuum expectation value v of the Higgs field;
- the coefficients a, b, c, d, e of (11.3.2) that are determined by the mass matrices in the finite Dirac operator D_F.

One can find several relations among these coefficients in the Lagrangian, which we shall derive in the following section. Inspired by the relation ${g_3}^2 = {g_2}^2 = \frac{5}{3}{g_1}^2$ obtained from (11.3.4), we will assume that these relations hold at the unification scale. Subsequently, we use the renormalization group equations to obtain predictions for the Standard Model at 'lower' (i.e. particle accelerator) energies.

12.1 Mass Relations

12.1.1 Fermion Masses

Recall from (11.4.3) that we defined the mass matrices m_x of the fermions by rewriting the matrices Y_x in the finite Dirac operator D_F. Inserting the formula (11.4.3) for Y_x into the expression for a given by (11.3.2), we obtain

$$a = \frac{af(0)}{\pi^2 v^2} \mathrm{Tr}\big(m_\nu^* m_\nu + m_e^* m_e + 3m_u^* m_u + 3m_d^* m_d\big),$$

which yields

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$$\mathrm{Tr}\big(m_\nu^* m_\nu + m_e^* m_e + 3m_u^* m_u + 3m_d^* m_d\big) = \frac{\pi^2 v^2}{f(0)}.$$

From (11.3.13) we know that the mass of the W-boson is given by $M_W = \frac{1}{2} v g_2$. Using the normalization (11.3.4), expressing g_2 in terms of $f(0)$, we can then write

$$f(0) = \frac{\pi^2 v^2}{8{M_W}^2}. \tag{12.1.1}$$

Inserting this into the expression above, we obtain a relation between the fermion mass matrices m_x and the W-boson mass M_W, viz.

$$\mathrm{Tr}\big(m_\nu^* m_\nu + m_e^* m_e + 3m_u^* m_u + 3m_d^* m_d\big) = 2{g_2}^2 v^2 = 8{M_W}^2. \tag{12.1.2}$$

If we assume that the mass of the top quark is much larger than all other fermion masses, we may neglect the other fermion masses. In that case, the above relation would yield the constraint

$$m_{\text{top}} \lesssim \sqrt{\frac{8}{3}} M_W. \tag{12.1.3}$$

12.1.2 The Higgs Mass

We obtain a mass m_h for the Higgs boson h by writing the term proportional to h^2 in (11.3.10) in the form

$$\frac{b\pi^2}{2a^2 f(0)} 4v^2 h^2 = \frac{1}{2}{m_h}^2 h^2.$$

Thus, the Higgs mass is given by

$$m_h = \frac{2\pi\sqrt{b}v}{a\sqrt{f(0)}}. \tag{12.1.4}$$

Inserting (12.1.1) into this expression for the Higgs mass, we see that M_W and m_h are related by

$${m_h}^2 = 32\frac{b}{a^2}{M_W}^2.$$

Next, we introduce the quartic Higgs coupling constant λ by writing

$$\frac{b\pi^2}{2a^2 f(0)} h^4 =: \frac{1}{24} \lambda h^4.$$

From (11.3.4) we then find

$$\lambda = 24 \frac{b}{a^2} {g_2}^2, \tag{12.1.5}$$

so that the (tree-level) Higgs mass can be expressed in terms of the mass M_W of the W-boson, the coupling constant g_2 and the quartic Higgs coupling λ as

$$m_h{}^2 = \frac{4\lambda M_W{}^2}{3{g_2}^2}. \tag{12.1.6}$$

12.1.3 The Seesaw Mechanism

Let us consider the mass terms for the neutrinos. The matrix D_F described in Sect. 11.1 provides the Dirac masses as well as the Majorana masses of the fermions. After a rescaling as in (11.4.3), the mass matrix restricted to the subspace of H_F with basis $\{\nu_L, \nu_R, \overline{\nu_L}, \overline{\nu_R}\}$ is given by

$$\begin{pmatrix} 0 & m_\nu^* & m_R^* & 0 \\ m_\nu & 0 & 0 & 0 \\ m_R & 0 & 0 & \bar{m}_\nu^* \\ 0 & 0 & \bar{m}_\nu & 0 \end{pmatrix}.$$

Suppose we consider only one generation, so that m_ν and m_R are just scalars. The eigenvalues of the above mass matrix are then given by

$$\pm \frac{1}{2} m_R \pm \frac{1}{2} \sqrt{m_R{}^2 + 4m_\nu{}^2}.$$

If we assume that $m_\nu \ll m_R$, then these eigenvalues are approximated by $\pm m_R$ and $\pm \frac{m_\nu{}^2}{m_R}$. This means that there is a heavy neutrino, for which the Dirac mass m_ν may be neglected, so that its mass is given by the Majorana mass m_R. However, there is also a light neutrino, for which the Dirac and Majorana terms conspire to yield a mass $\frac{m_\nu{}^2}{m_R}$, which is in fact much smaller than the Dirac mass m_ν. This is called the *seesaw mechanism*. Thus, even though the observed masses for these neutrinos may be very small, they might still have large Dirac masses (or Yukawa couplings).

From (12.1.2) we obtained a relation between the masses of the top quark and the W-boson by neglecting all other fermion masses. However, because of the seesaw mechanism it might be that one of the neutrinos has a Dirac mass of the same order of magnitude as the top quark. In that case, it would not be justified to neglect all other fermion masses, but instead we need to correct for such massive neutrinos.

Let us introduce a new parameter ρ (typically taken to be of order 1) for the ratio between the Dirac mass m_ν for the tau-neutrino and the mass m_{top} of the top quark at unification scale, so we write $m_\nu = \rho m_{\text{top}}$. Instead of (12.1.3), we then obtain the restriction

$$m_{\text{top}} \lesssim \sqrt{\frac{8}{3+\rho^2}} M_W. \tag{12.1.7}$$

12.2 Renormalization Group Flow

In this section we evaluate the renormalization group equations (RGEs) for the Standard Model from ordinary energies up to the unification scale. For the validity of these RGEs we need to assume the existence of a 'big desert' up to the grand unification scale. This means that one assumes that:

- there exist no new particles (besides the known Standard Model particles) with a mass below the unification scale;
- perturbative quantum field theory remains valid throughout the big desert.

Furthermore, we also ignore any gravitational contributions to the renormalization group flow.

12.2.1 Coupling Constants

In (11.3.1) we introduced the coupling constants for the gauge fields, and we obtained the relation ${g_3}^2 = {g_2}^2 = \frac{5}{3}{g_1}^2$. This is precisely the relation between the coupling constants at (grand) unification, common to grand unified theories (GUT). Thus, it would be natural to assume that our model is defined at the scale Λ_{GUT}. However, it turns out that there is no scale at which the relation ${g_3}^2 = {g_2}^2 = \frac{5}{3}{g_1}^2$ holds exactly, as we show below.

The renormalization group β-functions of the (minimal) standard model read

$$\frac{dg_i}{dt} = -\frac{1}{16\pi^2} b_i g_i^3; \qquad b = \left(-\frac{41}{6}, \frac{19}{6}, 7\right),$$

where $t = \log \mu$. At first order, these equations are uncoupled from all other parameters of the Standard Model, and the solutions for the running coupling constants $g_i(\mu)$ at the energy scale μ are easily seen to satisfy

$$g_i(\mu)^{-2} = g_i(M_Z)^{-2} + \frac{b_i}{8\pi^2} \log \frac{\mu}{M_Z}, \tag{12.2.1}$$

where M_Z is the experimental mass of the Z-boson:

$$M_Z = 91.1876 \pm 0.0021 \text{ GeV}.$$

For later convenience, we also recall that the experimental mass of the W-boson is

$$M_W = 80.399 \pm 0.023 \text{ GeV}. \tag{12.2.2}$$

The experimental values of the coupling constants at the energy scale M_Z are known too, and are given by

$$\begin{aligned} g_1(M_Z) &= 0.3575 \pm 0.0001, \\ g_2(M_Z) &= 0.6519 \pm 0.0002, \\ g_3(M_Z) &= 1.220 \pm 0.004. \end{aligned}$$

Using these experimental values, we obtain the running of the coupling constants in Fig. 12.1. As can be seen in this figure, the running coupling constants do not meet at any single point, and hence they do not determine a unique unification scale Λ_{GUT}. In other words, the relation ${g_3}^2 = {g_2}^2 = \frac{5}{3}{g_1}^2$ cannot hold exactly at any energy scale, unless we drop the big desert hypothesis. Nevertheless, in the remainder of this section we assume that this relation holds at least approximately and we will come back to this point in the next section. We consider the range for Λ_{GUT} determined by the triangle of the running coupling constants in Fig. 12.1. The scale Λ_{12} at the intersection of $\sqrt{\frac{5}{3}}g_1$ and g_2 determines the lowest value for Λ_{GUT}, given by

$$\Lambda_{12} = M_Z \exp\left(\frac{8\pi^2(\frac{3}{5}g_1(M_Z)^{-2} - g_2(M_Z)^{-2})}{b_2 - \frac{3}{5}b_1}\right) = 1.03 \times 10^{13} \text{ GeV}. \tag{12.2.3}$$

The highest value Λ_{23} is given by the solution of $g_2 = g_3$, which yields

$$\Lambda_{23} = M_Z \exp\left(\frac{8\pi^2(g_3(M_Z)^{-2} - g_2(M_Z)^{-2})}{b_2 - b_3}\right) = 9.92 \times 10^{16} \text{ GeV}. \tag{12.2.4}$$

We assume that the Lagrangian we have derived from the almost-commutative manifold $M \times F_{SM}$ is valid at some scale Λ_{GUT}, which we take to be between Λ_{12} and Λ_{23}. All relations obtained in Fig. 12.1 are assumed to hold approximately at this scale, and all predictions that will follow from these relations are therefore also only approximate.

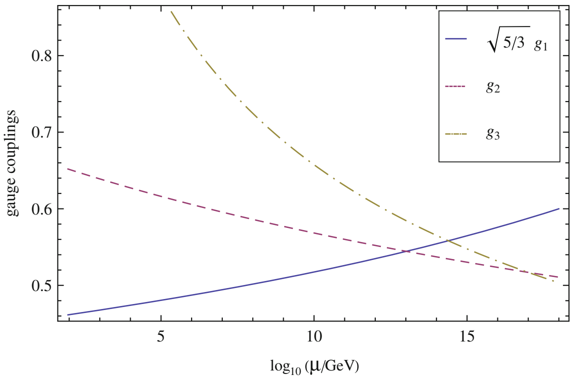


Fig. 12.1 The running of the gauge coupling constants

12.2.2 Renormalization Group Equations

The running of the neutrino masses has been studied in a general setting for non-degenerate seesaw scales. In what follows we consider the case where only the tau-neutrino has a large Dirac mass m_ν, which cannot be neglected with respect to the mass of the top-quark. In the remainder of this section we calculate the running of the Yukawa couplings for the top-quark and the tau-neutrino, as well as the running of the quartic Higgs coupling. Let us write y_{top} and y_ν for the Yukawa couplings of the top quark and the tau-neutrino, defined by

$$m_{\text{top}} = \frac{1}{2}\sqrt{2}y_{\text{top}}v, \qquad m_\nu = \frac{1}{2}\sqrt{2}y_\nu v, \tag{12.2.5}$$

where v is the vacuum expectation value of the Higgs field.

Let m_R be the Majorana mass for the right-handed tau-neutrino. By the Appelquist–Carazzone decoupling theorem we can distinguish two energy domains: $E > m_R$ and $E < m_R$. We again neglect all fermion masses except for the top quark and the tau neutrino. For high energies $E > m_R$, the renormalization group equations are given by

$$\begin{aligned}
\frac{dy_{\text{top}}}{dt} &= \frac{1}{16\pi^2}\left(\frac{9}{2}y_{\text{top}}^2 + y_\nu^2 - \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2\right)y_{\text{top}},\\
\frac{dy_\nu}{dt} &= \frac{1}{16\pi^2}\left(3y_{\text{top}}^2 + \frac{5}{2}y_\nu^2 - \frac{3}{4}g_1^2 - \frac{9}{4}g_2^2\right)y_\nu,\\
\frac{d\lambda}{dt} &= \frac{1}{16\pi^2}\Bigg(4\lambda^2 - (3{g_1}^2 + 9{g_2}^2)\lambda + \frac{9}{4}({g_1}^4 + 2{g_1}^2{g_2}^2 + 3{g_2}^4)\\
&\qquad + 4(3y_{\text{top}}^2 + {y_\nu}^2)\lambda - 12(3y_{\text{top}}^4 + {y_\nu}^4)\Bigg).
\end{aligned} \tag{12.2.6}$$

Below the threshold $E = m_R$, the Yukawa coupling of the tau-neutrino drops out of the RG equations and is replaced by an effective coupling

$$\kappa = 2\frac{{y_\nu}^2}{m_R},$$

which provides an effective mass $m_l = \frac{1}{4}\kappa v^2$ for the light tau-neutrino. The renormalization group equations of y_{top} and λ for $E < m_R$ are then given by

$$\begin{aligned}\frac{dy_{\text{top}}}{dt} &= \frac{1}{16\pi^2}\left(\frac{9}{2}y_{\text{top}}^2 - \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2\right)y_{\text{top}},\\ \frac{d\lambda}{dt} &= \frac{1}{16\pi^2}\Bigg(4\lambda^2 - (3{g_1}^2 + 9{g_2}^2)\lambda + \frac{9}{4}({g_1}^4 + 2{g_1}^2{g_2}^2 + 3{g_2}^4)\\ &\qquad + 12y_{\text{top}}^2\lambda - 36y_{\text{top}}^4\Bigg).\end{aligned} \tag{12.2.7}$$

Finally, the equation for y_ν is replaced by an equation for the effective coupling κ given by

$$\frac{d\kappa}{dt} = \frac{1}{16\pi^2}\left(6y_{\text{top}}^2 - 3{g_2}^2 + \frac{\lambda}{6}\right)\kappa. \tag{12.2.8}$$

12.2.3 Running Masses

The numerical solutions to the coupled differential equations of (12.2.6), (12.2.7) and (12.2.8) for y_{top}, y_ν and λ depend on the choice of three input parameters:

- the scale Λ_{GUT} at which our model is defined;
- the ratio ρ between the masses m_ν and m_{top};
- the Majorana mass m_R that produces the threshold in the renormalization group flow.

The scale Λ_{GUT} is taken to be either $\Lambda_{12} = 1.03 \times 10^{13}$ GeV or $\Lambda_{23} = 9.92 \times 10^{16}$ GeV, as given by (12.2.3) and (12.2.4), respectively. We now determine the numerical solution to (12.2.6), (12.2.7) and (12.2.8) for a range of values for ρ and m_R. First, we need to start with the initial conditions of the running parameters at the scale Λ_{GUT}. Inserting the top-quark mass $m_{\text{top}} = \frac{1}{2}\sqrt{2}y_{\text{top}}v$, the tau-neutrino mass $m_\nu = \rho m_{\text{top}}$, and the W-boson mass $M_W = \frac{1}{2}g_2 v$ into (12.1.7), we obtain the constraints

$$y_{\text{top}}(\Lambda_{GUT}) \lesssim \frac{2}{\sqrt{3+\rho^2}}g_2(\Lambda_{GUT}), \qquad y_\nu(\Lambda_{GUT}) \lesssim \frac{2\rho}{\sqrt{3+\rho^2}}g_2(\Lambda_{GUT}),$$

where (12.2.1) yields the values $g_2(\Lambda_{12}) = 0.5444$ and $g_2(\Lambda_{23}) = 0.5170$.

Furthermore, from (12.1.5) we obtain an expression for the quartic coupling λ at Λ_{GUT}. Approximating the coefficients a and b from (11.3.2) by $a \approx (3+\rho^2)m_{\text{top}}^2$ and $b \approx (3+\rho^4)m_{\text{top}}^4$, we obtain the boundary condition

$$\lambda(\Lambda_{GUT}) \approx 24\frac{3+\rho^4}{(3+\rho^2)^2}g_2(\Lambda_{GUT})^2.$$

Using these boundary conditions, we can now numerically solve the RG equations of (12.2.6) from Λ_{GUT} down to m_R, which provides us with values for $y_{\text{top}}(m_R)$, $y_\nu(m_R)$ and $\lambda(m_R)$. At this point, the Yukawa coupling y_ν is replaced by the effective coupling κ with boundary condition

$$\kappa(m_R) = 2\frac{y_\nu(m_R)^2}{m_R}.$$

Next, we numerically solve the RG equations of (12.2.7) and (12.2.8) down to M_Z to obtain the values for y_{top}, κ and λ at ‘low’ energy scales.

The running mass of the top quark at these energies is given by (12.2.5). We find the running Higgs mass by inserting λ into (12.1.6). We shall evaluate these running masses at their own energy scale. For instance, our predicted mass for the Higgs boson is the solution for μ of the equation $\mu = \sqrt{\lambda(\mu)/3}v$, in which we ignore the running of the vacuum expectation value v.

The effective mass of the light neutrino is determined by the effective coupling κ, and we choose to evaluate this mass at scale M_Z. Thus, we calculate the masses by

$$\begin{aligned} m_{\text{top}}(m_{\text{top}}) &= \frac{1}{2}\sqrt{2}y_{\text{top}}(m_{\text{top}})v, \\ m_l(M_Z) &= \frac{1}{4}\kappa(M_Z)v^2, \\ m_h(m_h) &= \sqrt{\frac{\lambda(m_h)}{3}}v, \end{aligned}$$

where, from the W-boson mass (12.2.2) we can insert the value $v = 246.66 \pm 0.15$. The results of this procedure for m_{top}, m_l and m_h are given in Table 12.1. In this table, we have chosen the range of values for ρ and m_R such that the mass of the top-quark and the light tau-neutrino are in agreement with their experimental values

$$m_{\text{top}} = 172.0 \pm 0.9 \pm 1.3 \text{ GeV}, \qquad m_l \leq 2 \text{ eV}.$$

For comparison, we have also included the simple case where we ignore the Yukawa coupling of the tau-neutrino (by setting $\rho = 0$), in which case there is no threshold at the Majorana mass scale either. As an example, we have plotted the running of λ, y_{top}, y_ν and κ for the values of $\Lambda_{GUT} = \Lambda_{23} = 9.92 \times 10^{16}$ GeV, $\rho = 1.2$, and $m_R = 3 \times 10^{12}$ GeV in Figs. 12.2, 12.3, 12.4 and 12.5.

Table 12.1 Numerical results for the masses m_{top} of the top-quark, m_l of the light tau-neutrino, and m_h of the Higgs boson, as a function of Λ_{GUT}, ρ, and m_R

Λ_{GUT} (10^{13} GeV)	1.03	1.03	1.03	1.03	1.03	1.03	1.03
ρ	0	0.90	0.90	1.00	1.00	1.10	1.10
m_R (10^{13} GeV)	–	0.25	1.03	0.30	1.03	0.35	1.03
m_{top} (GeV)	183.2	173.9	174.1	171.9	172.1	169.9	170.1
m_l (eV)	0	2.084	0.5037	2.076	0.6030	2.080	0.7058
m_h (GeV)	188.3	175.5	175.7	173.4	173.7	171.5	171.8
Λ_{GUT} (10^{16} GeV)	9.92	9.92	9.92	9.92	9.92		
ρ	0	1.10	1.10	1.20	1.20		
m_R (10^{13} GeV)	–	0.30	2.0	0.35	9900		
m_{top} (GeV)	186.0	173.9	174.2	171.9	173.5		
m_l (eV)	0	1.939	0.2917	1.897	6.889×10^{-5}		
m_h (GeV)	188.1	171.3	171.6	169.1	171.2		
Λ_{GUT} (10^{16} GeV)	9.92	9.92	9.92	9.92			
ρ	1.30	1.30	1.35	1.35			
m_R (10^{13} GeV)	0.40	9900	100	9900			
m_{top} (GeV)	169.9	171.6	169.8	170.6			
m_l (eV)	1.866	7.818×10^{-5}	8.056×10^{-3}	8.286×10^{-5}			
m_h (GeV)	167.1	169.3	167.4	168.4			

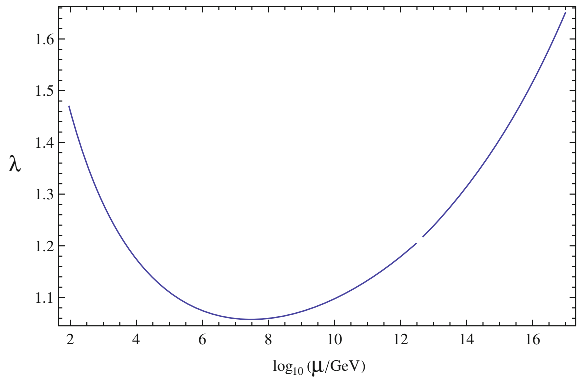


Fig. 12.2 The running of the quartic Higgs coupling λ for $\Lambda_{GUT} = 9.92 \times 10^{16}$ GeV, $\rho = 1.2$, and $m_R = 3 \times 10^{12}$ GeV

For the allowed range of values for ρ and m_R that yield plausible results for m_{top} and m_l, we see that the mass m_h of the Higgs boson takes its value within the range

$$167 \text{ GeV} \leq m_h \leq 176 \text{ GeV}.$$

The errors in this prediction, which result from the initial conditions (other than m_{top} and m_l) taken from experiment, as well as from ignoring higher-loop corrections to the RGEs, are smaller than this range of possible values for the Higgs mass, and therefore we may ignore these errors.

Fig. 12.3 The running of the top-quark Yukawa coupling y_{top} for $\Lambda_{GUT} = 9.92 \times 10^{16}$ GeV, $\rho = 1.2$, and $m_R = 3 \times 10^{12}$ GeV

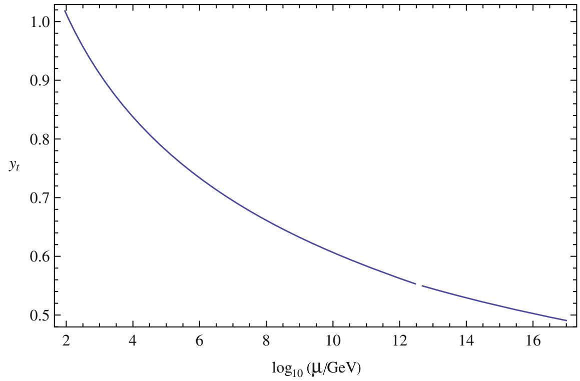


Fig. 12.4 The running of the tau-neutrino Yukawa coupling y_ν for $\Lambda_{GUT} = 9.92 \times 10^{16}$ GeV, $\rho = 1.2$, and $m_R = 3 \times 10^{12}$ GeV

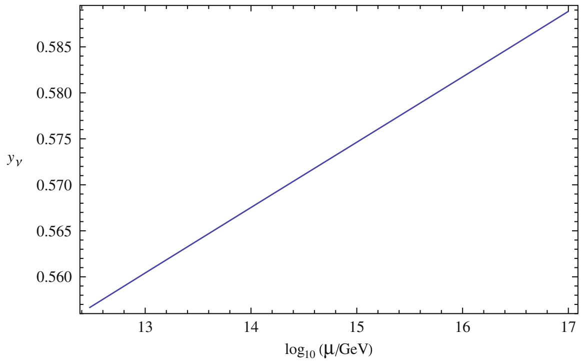
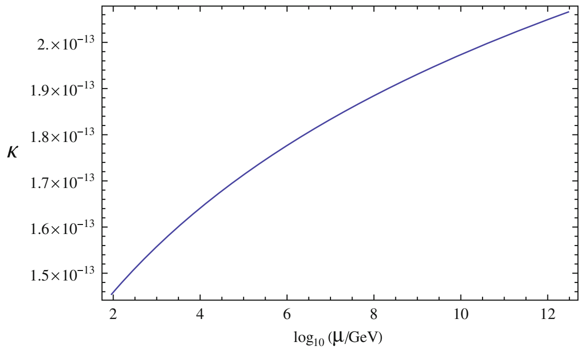


Fig. 12.5 The running of the effective coupling κ for $\Lambda_{GUT} = 9.92 \times 10^{16}$ GeV, $\rho = 1.2$, and $m_R = 3 \times 10^{12}$ GeV



12.3 Higgs Mass: Comparison to Experimental Results

It is time to confront the above predicted range of values with the discovery of a Higgs boson with a mass $m_h \simeq 125.5$ GeV at the ATLAS and CMS experiments at the Large Hadron Collider at CERN. At first sight, this experimentally measured value seems to be at odds with the above prediction and seems to falsify the description of the Standard Model as an almost-commutative manifold. However, let us consider more closely the (main) hypotheses on which the above prediction is based, discussing them one-by-one.

The almost-commutative manifold $M \times F_{SM}$: An essential input in the above derivation is the replacement of the background manifold M by a noncommutative space $M \times F_{SM}$. We motivated the structure of F_{SM} by deriving it from a list of finite irreducible geometries, along the way imposing several mathematical constraints (cf. Sect. 11.1). The strength of this approach was that it allowed for a *derivation* of all the particles and symmetries of the Standard Model from purely geometrical data. Moreover, the spectral action resulted in the Lagrangian of the Standard Model, including Higgs mechanism.

The incompatibility of the prediction of the Higgs mass with experiment might be resolved by considering almost-commutative manifolds that go beyond the Standard Model by dropping some of the aforementioned mathematical constraints; we will discuss a recently proposed possibility in the next Section.

Ultimately, one should also consider noncommutative manifolds that are not the product of M with a finite space F (see Note 7 on page 228).

The spectral action: The bosonic Lagrangian was derived from the asymptotic expansion of the spectral action $\mathrm{Tr} f(D/\Lambda)$.

Adopting Wilson's viewpoint on the renormalization group equation this Lagrangian was considered the bare Lagrangian at the cutoff scale Λ. The renormalization group equations then dictate the running of the renormalized, physical parameters.

Alternatively, one can consider the spectral action for $M \times F_{SM}$ in a perturbative expansion in the fields, as in Sect. 7.2.2, leading to unexpected and an intriguing behaviour for the propagation of particles at energies larger than the cutoff Λ (see Note 9 on page 228).

Yet another alternative is to consider Λ as a regularization parameter, allowing for an interpretation of the asymptotic expansion of $\mathrm{Tr} f(D/\Lambda)$ as a higher-derivative gauge theory. It turns out that conditions can be formulated on the Krajewski diagram for F that guarantee the (super)renormalizability of the asymptotic expansion of the spectral action for the corresponding almost-commutative manifold $M \times F$ (see Note 9 on page 228).

Big desert: In our RGE-analysis of the couplings and masses we have assumed the big desert up to the GUT-scale: no more elementary particles than those present in the Standard Model exist at higher energies (and up to the GUT-scale). This is a good working hypothesis, but is unlikely to be true. The main reason for this is the mismatch of the running coupling constants at the GUT-scale (Fig. 12.1). This indicates that new physics is expected to appear before this scale. As already suggested, it might very well be that this new physics can be described by considering almost-commutative manifolds that go beyond the Standard Model. We will discuss such a possibility in the next Section.

Renormalization group equations: We exploited renormalization group techniques to run couplings and masses down from the GUT-scale to ordinary energies. The renormalization group equations were derived in a perturbative approach to quantum field theory, which was supposed to be valid at all scales. Moreover, we have adopted the one-loop beta-functions, something which can definitely be improved. Even though this might lead to more accurate predictions, it is not expected to resolve the incompatibility between the predicted range for m_h and the experimentally measured value.

Gravitational effects: In our analysis we have discarded all possible gravitational effects on the running of the couplings constants. It might very well be that gravitational correction terms alter the predicted values to a more realistic value.

12.4 Noncommutative Geometry Beyond the Standard Model

Let us then drop some of the above hypotheses, and demonstrate how a small correction of the space $M \times F_{SM}$ gives an intriguing possibility to go beyond the Standard Model, solving at the same time a problem with the stability of the Higgs vacuum given the measured low mass m_h.

Namely, in the definition of the finite Dirac operator D_F of Eq. 11.1.2, we can replace Y_R by $Y_R\sigma$, where σ is a real scalar field on M. Strictly speaking, this brings us out of the class of almost-commutative manifolds $M \times F$, since part of D_F now varies over M. Nevertheless, it fits perfectly into the more general class of topologically non-trivial almost-commutative geometries. In fact, it is enough to consider the trivial fiber bundle $M \times H_F$, for which an endomorphism $D_F(x) \in \mathrm{End}(H_F)$ is allowed to depend smoothly on $x \in M$.

The scalar field σ can also be seen as the relic of a spontaneous symmetry breaking mechanism, similar to the Higgs field h in the electroweak sector of the Standard Model. Starting point is the almost-commutative manifold $M \times F_{PS}$ based on the algebra $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ with which we started Chap. 11. The gauge group corresponding to F_{PS} is $SU(2) \times SU(2) \times SU(4)$ and the corresponding model is called **Pati–Salam unification**. It turns out that the spectral action for $M \times F_{PS}$ yields a spontaneous symmetry breaking mechanism that *dynamically* selects the algebra $A_F \subset M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ of Proposition 11.1.

Let us then replace Y_R by $Y_R\sigma$ and analyze the additional terms in the spectral action. In Proposition 11.9 we insert a σ for every Y_R that appears, to arrive at

$$\begin{aligned}\mathcal{L}'_H(g_{\mu\nu}, \Lambda_\mu, Q_\mu, H, \sigma) := & \frac{bf(0)}{2\pi^2}|H|^4 - \frac{2af_2\Lambda^2}{\pi^2}|H|^2 + \frac{ef(0)}{\pi^2}\sigma^2|H|^2 \\ & - \frac{cf_2\Lambda^2}{\pi^2}\sigma^2 + \frac{df(0)}{4\pi^2}\sigma^4 + \frac{af(0)}{2\pi^2}|D_\mu H|^2 \\ & + \frac{1}{4\pi^2}f(0)c(\partial_\mu\sigma)^2,\end{aligned}$$

where we ignored the coupling to the scalar curvature.

As before, we exploit the approximation that m_{top}, m_ν and m_R are the dominant mass terms. Moreover, as before we write $m_\nu = \rho m_{top}$. That is, the expressions for a, b, c, d and e in (11.3.2) now become

$$\begin{aligned} a &\approx m_{top}^2(\rho^2 + 3), \\ b &\approx m_{top}^4(\rho^4 + 3), \\ c &\approx m_R^2, \\ d &\approx m_R^4, \\ e &\approx \rho^2 m_R^2 m_{top}^2. \end{aligned}$$

In a unitary gauge, where $H = \begin{pmatrix} h \\ 0 \end{pmatrix}$, we arrive at the following potential:

$$\mathcal{L}_{pot}(h, \sigma) = \frac{1}{24}\lambda_h h^4 + \frac{1}{2}\lambda_{h\sigma} h^2 \sigma^2 + \frac{1}{4}\lambda_\sigma \sigma^4 - \frac{4g_2^2}{\pi^2} f_2 \Lambda^2 (h^2 + \sigma^2),$$

where we have defined coupling constants

$$\lambda_h = 24 \frac{\rho^4 + 3}{(\rho^2 + 3)^2} g_2^2, \qquad \lambda_{h\sigma} = \frac{8\rho^2}{\rho^2 + 3} g_2^2, \qquad \lambda_\sigma = 8 g_2^2. \tag{12.4.1}$$

This potential can be minimized, and if we replace h by $v + h$ and σ by $w + \sigma$, respectively, expanding around a minimum for the terms quadratic in the fields, we obtain:

$$\begin{aligned} \mathcal{L}_{pot}(v + h, w + \sigma)|_{\text{quadratic}} &= \frac{1}{6} v^2 \lambda_h v^2 + 2vw\lambda_{h\sigma}\sigma h + w^2 \lambda_\sigma \sigma^2 \\ &= \frac{1}{2} \begin{pmatrix} h & \sigma \end{pmatrix} M^2 \begin{pmatrix} h \\ \sigma \end{pmatrix}, \end{aligned}$$

where we have defined the mass matrix M by

$$M^2 = 2 \begin{pmatrix} \frac{1}{6}\lambda_h v^2 & \lambda_{h\sigma} vw \\ \lambda_{h\sigma} vw & \lambda_\sigma w^2 \end{pmatrix}.$$

This mass matrix can be easily diagonalized, and if we make the natural assumption that w is of the order of m_R, while v is of the order of M_W, so that $v \ll w$, we find that the two eigenvalues are

$$m_+^2 \sim 2\lambda_\sigma w^2 + 2\frac{\lambda_{h\sigma}^2}{\lambda_\sigma} v^2,$$
$$m_-^2 \sim 2\lambda_h v^2 \left(\frac{1}{6} - \frac{\lambda_{h\sigma}^2}{\lambda_h \lambda_\sigma}\right).$$

We can now determine the value of these two masses by running the scalar coupling constants λ_h, $\lambda_{h\sigma}$ and λ_σ down to ordinary energy scalar. The renormalization group equations for these couplings are given by

$$\frac{d\lambda_h}{dt} = \frac{1}{16\pi^2}\Big(4\lambda_h^2 + 12\lambda_{h\sigma}^2 - (3{g_1}^2 + 9{g_2}^2)\lambda_h + \frac{9}{4}({g_1}^4 + 2{g_1}^2{g_2}^2 + 3{g_2}^4)$$
$$+ 4(3y_{\text{top}}^2 + {y_\nu}^2)\lambda_h - 12(3y_{\text{top}}^4 + {y_\nu}^4)\Big),$$
$$\frac{d\lambda_{h\sigma}}{dt} = \frac{1}{16\pi^2}\Big(8\lambda_{h\sigma}^2 + 6\lambda_{h\sigma}\lambda_\sigma + 2\lambda_{h\sigma}\lambda_h$$
$$- \frac{3}{2}\left(g_1^2 + 3g_2^2\right)\lambda_{h\sigma} + 2(3y_{\text{top}}^2 + {y_\nu}^2)\lambda_{h\sigma}\Big),$$
$$\frac{d\lambda_\sigma}{dt} = \frac{1}{16\pi^2}\Big(8\lambda_{h\sigma}^2 + 18\lambda_\sigma^2\Big).$$

As before, at lower energy the coupling y_ν drops out of the RG equations and is replaced by an effective coupling.

At one-loop, the other couplings obey the renormalization group equations of the Standard Model, that is, they satisfy (12.2.6) and (12.2.7). As before, we can solve these differential equations, with boundary conditions at Λ_{GUT} given for the scalar couplings by (12.4.1). The result varies with the chosen value for Λ_{GUT} and the parameter ρ. The mass of σ is essentially given by the largest eigenvalue m_+ which is of the order 10^{12} GeV for all values of Λ_{GUT} and the parameter ρ. The allowed mass range for the Higgs, i.e. for m_-, is depicted in Fig. 12.6. The expected value $m_h = 125.5$ GeV is therefore compatible with the above noncommutative model. Furthermore, this calculation implies that there is a relation (given by the red line in the Figure) between the ratio m_ν/m_{top} and the unification scale Λ_{GUT}.

We conclude that with noncommutative geometry we can proceed beyond the Standard Model, enlarging the field content of the Standard Model by a real scalar field with a mass of the order of 10^{12} GeV. At the time of writing of this book (Spring 2014), this is completely compatible with experiment and also guarantees stability of the Higgs vacuum at higher energy scales. Of course, the final word is to experiment in the years to come. What we can say at this point is that noncommutative geometry provides a fascinating dialogue between abstract mathematics and concrete measurements in experimental high-energy physics.

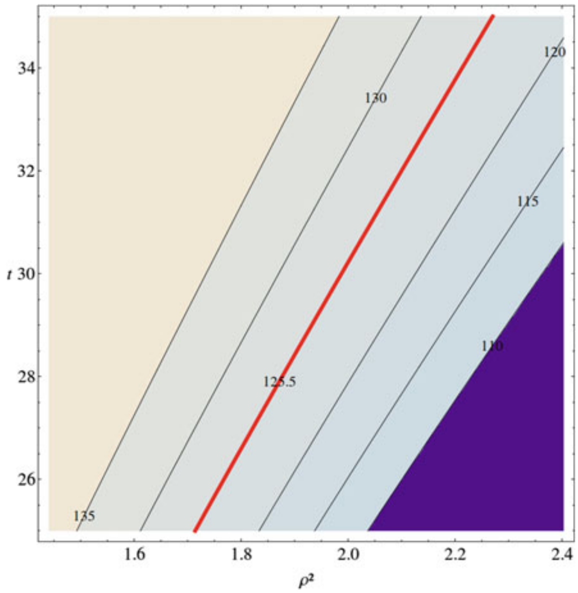


Fig. 12.6 A contour plot of the Higgs mass m_h as a function of ρ^2 and $t = \log(\Lambda_{GUT}/M_Z)$. The red line corresponds to $m_h = 125.5$ GeV

Notes

1. In the first part of this Chapter, we mainly follow [1, Sect. 5] (see also [2, Chap. 1, Sect. 17]). In Sect. 12.2 we have also incorporated the running of the neutrino masses as in [3] (see also [4]).

Section 12.1 Mass Relations

2. Further details on the see-saw mechanism can be found in e.g. [5].

Section 12.2 Renormalization Group Flow

3. The renormalization group β-functions of the (minimal) standard model are taken from [6–8] and [9]. We simplify the expressions by ignoring the 2-loop contributions, and instead consider only the 1-loop approximation. The renormalization group β-functions are [6, Eq. (B.2)] or [9, Eq. (A.1)]).
4. The experimental masses of the Z and W-boson and the top quark, as well as the experimental values of the coupling constants at the energy scale M_Z are found in [10].
5. In arriving at (12.2.6) we have followed the approach of [3] where two energy domains are considered: $E > m_R$ and $E < m_R$. The Appelquist–Carazzone decoupling theorem is found in [11]. For the renormalization group equations, we refer to [7, Eq. (B.4)], [12, Eq. (14) and (15)] and [8, Eq. (B.3)].

Section 12.3 Higgs Mass: Comparison to Experimental Results

6. The discovery of the Higgs boson at the ATLAS and CMS experiments is published in [13, 14].

7. The spectral action has also been computed for spectral triples that are not the product of M with a finite space F, and which are further off the 'commutative shore'. These include the noncommutative torus [15], the Moyal plane [16, 17], the quantum group $SU_q(2)$ [18] and the Podleś sphere S_q^2 [19].
8. The generalization of noncommutative geometry to *non-associative geometry* is analyzed in [20, 21].
9. The bosonic Lagrangian derived from the spectral action was interpreted in [22] à la Wilson [23] as the bare Lagrangian at the cutoff scale Λ. A perturbative expansion of the full spectral action was obtained in [24–26], leading to unexpected and an intriguing behaviour for the propagation of particles at energies larger than the cutoff Λ. Alternatively, the interpretation of Λ as a regularization parameter has been worked out in [27–30], including the derivation of renormalizability conditions on the Krajewski diagrams.
10. Other searches beyond the Standard Model with noncommutative geometry include [31–35], adopting a slightly different approach to almost-commutative manifolds as we do (cf. Note 3 on page 118). The intersection between supersymmetry and almost-commutative manifolds is analyzed in [36–40].
11. A possible approach to incorporate gravitational effects in the running of the coupling constants is discussed in [41].

Section 12.4 Noncommutative Geometry Beyond the Standard Model

12. For stability bounds on the Higgs mass, we refer to [42].
13. The small correction to the space $M \times F_{SM}$ was realized in [43] (and already tacitly present in [44]) and we here confirm their conclusions. The class of topologically non-trivial almost-commutative geometries has been worked out in [45–47]. The spontaneous symmetry breaking of the noncommutative description of the Pati–Salam model [48] was analyzed in [49, 50], after generalizing inner fluctuations to real spectral triples that do not necessarily satisfy the first-order condition (4.3.1).
14. In [51] an alternative approach is considered, taking the 'grand' algebra $M_4(\mathbb{H}) \oplus M_8(\mathbb{C})$ from the list of [52], but where now the condition of bounded commutators of D with the algebra is not satisfied.
15. The renormalization group equations for the couplings $\lambda_h, \lambda_{h\sigma}, \lambda_\sigma$ have been derived in [53].

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