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The Solution

The latest designs

One of the oldest problems in combinatorics — dating back to mid-nineteenth century work of, independently, Plücker, Kirkman and Steiner — asserts the existence of what are referred to as 'Steiner systems' (subject essentially to only some simple divisibility conditions). These highly symmetric structures are foundational to the field of combinatorial design theory. Until this year, this existence conjecture has remained largely open. In January 2014, Peter Keevash announced a proof of the conjecture, which also confirms a more general conjecture for designs. In this article Ross Kang describes the conjecture and the solution of Keevash.

The Reverend Thomas P. Kirkman, a Church of England clergyman, in 1850 posed the following innocent puzzle in *The Lady's and Gentleman's Diary*, a recreational mathematics magazine of the time:

> *"Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast."*

This became known as *Kirkman's schoolgirl problem*. Its popularity partly led to the systematic study of discrete structures called Steiner systems.

These structures and their relatives (defined precisely below) are foundational to the field of combinatorial design theory. They are collections of finite sets that have specified regularity and incidence properties. Although their properties can be quite useful, they impose particular algebraic, arithmetic or geometric restrictions. Indeed, a very difficult challenge is to prove the existence of such systems, let alone construct, count or classify them. Questions of this nature can be traced as far back as 1835, with a brief remark implicitly about the existence of Steiner triple systems made in a book of J. Plücker.

A *design* $B(n,q,r,\lambda)$ with parameters (n,q,r,λ) is a collection B of q-element subsets of an n-element set X with the property that every r-element subset of X belongs to exactly λ members of B. These are so named for their relevance to experimental design in statistics, especially when $r = 2$, but they also have other applications in several areas of mathematics and computer science. A *Steiner system* $S(n,q,r)$ with parameters (n,q,r) is just a design with parameters $(n,q,r,1)$. The schoolgirl problem asked for a particularly nice type of $S(15,3,2)$. The well-known *Fano plane* in Figure 1 is an $S(7,3,2)$ and the finite projective planes form an important class of Steiner systems.

Observe that some simple divisibility conditions among the parameters must hold in order for $B(n,q,r,\lambda)$ to exist. In particular, $\binom{q-i}{r-i}$ must divide $\lambda\binom{n-i}{r-i}$ for each $i \in \{0,1,\ldots,r-1\}$, as seen by considering the members of B that contain any fixed i-element subset of X. A natural and long-standing conjecture, referred to in literature as the 'Existence Conjecture', is that for fixed q, r, λ these divisibility conditions are not only necessary but also sufficient for the existence of designs, apart from a finite number of exceptional n.

Until this year, only the cases of the Existence Conjecture with $r = 2$ had been settled, by a constructive proof due to Richard M. Wilson (Caltech), a seminal work carried out in the seventies during Wilson's doctorate at Ohio State. Another breakthrough was by Luc Teirlinck (Auburn) in 1987: for each r, he constructed infinitely many 'non-trivial' designs, that is, $B(n,q,r,\lambda)$ with λ independent of n. No Steiner system with $r \geq 6$ was known to exist.

In January at Oberwolfach, Peter Keevash announced the proof of the Existence Conjecture in its entirety, and he posted a manuscript online soon after ('The existence of designs', arxiv:1401:3665, 56 pp.). In March, Dutch mathematicians had the opportunity to hear him discuss this remarkable result at a discrete mathematics seminar in Eindhoven. Keevash is a 35-year-old British mathematician who works mainly in probabilistic and extremal combinatorics. He

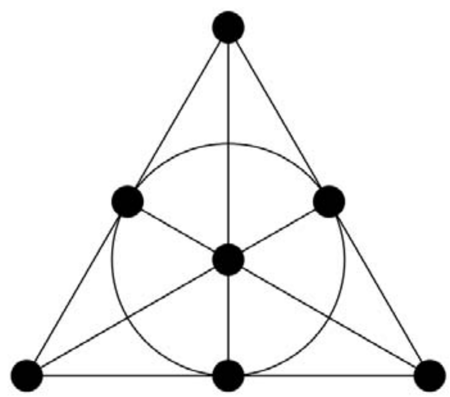

Figure 1 Fano plane.

Peter Keevash during a hiking break from a workshop in Barbados

obtained his bachelor's in Cambridge and his doctorate in Princeton, and was afterwards at Caltech (where he began this work), Birmingham and Queen Mary University of London, before taking up a professorship in Oxford last autumn.

The proof is probabilistic in nature, repeatedly using the following basic principle: having set up a probability space over a set of combinatorial objects, to find *just one* with some desired property it suffices to prove that the probability of a random object having that property is *positive*. Called the *probabilistic method*, this principle appears tautological, but has proven powerful and versatile. It was pioneered by the prolific, collaborative, itinerant Hungarian mathematician Paul Erdős (1913–1996). For interest, the probabilistic method is central to new national master's courses (Mastermath 'Advanced Combinatorics' and 'Probabilistic and Extremal Combinatorics') taught by Tobias Müller (Utrecht) and the author.

An early hint that the probabilistic method could be helpful in the hunt for designs was when Vojtěch Rödl (Emory) in 1985 established an approximate form of the Existence Conjecture for Steiner systems. In doing so, he confirmed a two-decade-old conjecture of Erdős and H. Hanani, and at the same time introduced a technique which has since had wide use in discrete mathematics. (in August Rödl was a main plenary speaker at the ICM in Seoul.) This technique, evocatively named the *Rödl nibble*, is an iterated form of the probabilistic method. It builds up an object gradually over a sequence of steps, each of which is a single application of the 'ordinary' probabilistic method, so that at each step the partial object so far built retains some structural properties that allow the sequence to continue (at least for a time).

In fact, the nibble is integral to Keevash's proof of the Existence Conjecture. Rödl's original approach was to repeatedly, randomly select a set of disjoint q-element subsets to add to the system and then forbid any of the intersected q-element subsets from being selected later. When done skillfully, this procedure can be run until nearly all the elements of X have been covered, yielding an asymptotically optimal Steiner system; however, this approach eventually breaks down as the structure of the remaining eligible q-element subsets deteriorates. The ingenuity of Keevash's approach, which circumvents this difficulty, is that he first constructs (partly by probabilistic means) a special algebraically-defined template — this template induces a partial Steiner system and also admits a flexible set of possible local modifications. The utility of the template is that, after a careful nibble procedure is run to produce another partial Steiner system that is almost complementary, it is possible to use some admissable local modifications of the template to reconcile the difference, so the two partial Steiner systems can be merged into one full one. This is a grossly simplified sketch, and, besides the fact that we discussed only Steiner systems, there are for instance inherent difficulties of handling sparse combinatorial structures, and moreover the proof relies on establishing a stronger result for quasirandom simplicial complexes by a top-level induction on r. The full proof is extremely delicate and it is a phenomenal achievement.

Although the guarantees on n have not yet been carefully optimised, they are not of tower type, as seen say in applications of Szemerédi's regularity lemma. The proof thus constitutes a randomised algorithm that could conceivably be implemented to produce concrete examples, say, for $r = 6$, but this has yet to be carried out.

Keevash's work is a cadence to a fundamental open problem, but certainly there are many further questions to pursue. Just one first step might be to consider if related existence questions for other symmetric rigid combinatorial structures could be resolved with similar techniques. It is well worth mentioning recent work of Kuperberg, Lovett and Peled (2012), in which a different probabilistic approach, a special local central limit theorem for certain lattice random walks, was used to attack several similar questions, where for example they had significantly improved upon Teirlinck's result.

Chatting at drinks following the talk in Eindhoven, Aart Blokhuis (Eindhoven) reminded us of another long-standing and major conjecture in the area, one that is very unlikely to be confirmed with probabilistic methods: projective planes only exist for prime power orders.