

Renormalization Group Flow in CDT

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Abstract

We perform a first investigation of the coupling constant flow of the nonperturbative lattice model of four-dimensional quantum gravity given in terms of Causal Dynamical Triangulations (CDT). After explaining how standard concepts of lattice field theory can be adapted to the case of this background-independent theory, we define a notion of “lines of constant physics” in coupling constant space in terms of certain semiclassical properties of the dynamically generated quantum universe. Determining flow lines with the help of Monte Carlo simulations, we find that the second-order phase transition line present in this theory can be interpreted as a UV phase transition line if we allow for an anisotropic scaling of space and time.

1 Introduction

The formalism of Causal Dynamical Triangulations (CDT) provides a regularization of the putative theory of quantum gravity [1, 2]. Its underlying assumption is that the fundamental theory of quantum gravity can be understood purely in terms of quantum field-theoretical concepts. CDT quantum gravity shares this assumption with the asymptotic safety program, originally put forward by Weinberg [3], which was subsequently studied in a $(2 + \varepsilon)$ -dimensional expansion [4] and more recently with the help of the functional renormalization group equation [5]. Similarly, a key idea behind Hořava-Lifshitz gravity (HLG) [6] is to use ordinary quantum field theory to construct quantum gravity, but to circumvent the usual problem of non-renormalizability by explicitly breaking the four-dimensional diffeomorphism invariance of the continuum theory with the introduction of a preferred time foliation. In this setting one can naturally introduce terms with higher *spatial* derivatives in the action to render the theory renormalizable while keeping the theory unitary.

Their common field-theoretic basis, as well as coinciding results on the spectral dimension of spacetime on Planckian scales [7] and a similar phase structure of CDT and HLG [8, 9] make it natural to try to relate the three approaches – causal dynamical triangulations, asymptotic safety and Hořava-Lifshitz gravity – more directly.¹ Interesting examples of this include the formulation of a functional renormalization group equation for foliated spacetimes [14] and its application to projectable HLG at low energies [15], and an extension of CDT quantum gravity by the explicit addition of higher spatial derivative terms (albeit at this stage only in three spacetime dimensions [16]). Note that HLG does not appeal to an asymptotic safety scenario for the theory to make sense at high energies.

Although the distinguished notion of proper time of CDT looks superficially similar to the time foliation in HLG, its status is different because CDT does not possess any residual diffeomorphism invariance, which therefore cannot be broken either. The role of time in CDT was recently clarified further in a study in three dimensions, where it was verified explicitly that key results of CDT quantum gravity continue to hold in a version of the theory which does not possess preferred simplicial hypermanifolds that can be identified with surfaces of constant time [17]. This provides strong evidence that the notion of proper time that is naturally available in standard CDT is simply a convenient parameter to (partially) describe the spacetime geometry, and that its presence does not skew the results of the theory in an unwanted way. Of course, also this “non-foliated” version of CDT incorporates microscopic causality conditions, implying an asymmetry between

¹More distant relatives of CDT are group field theory [10] and so-called tensor models [11], which in specific limits can generate triangulations. These models are presumably more closely related to (Euclidean) Dynamical Triangulations [12, 13] than to CDT.

time and spatial directions that persists after Wick-rotating, just like in standard CDT. It is therefore conceivable that in part of the coupling constant space [8] the nonperturbative effective quantum action of CDT can be related to an anisotropic action of HLG-type, even though in the former no higher-order spatial derivative terms are added explicitly to the bare classical Einstein-Hilbert action. Let us also point out that the built-in unitarity of the CDT formalism – resulting from a well-defined transfer matrix [18] – is likely to affect the functional form of the dynamically generated quantum action, in a way we currently do not control explicitly.

In this article, we present a first attempt at establishing a concrete renormalization group flow in four-dimensional CDT quantum gravity (in the standard version and without higher-derivative terms in the bare action), assuming a straightforward identification of lattice proper distances with continuum proper distances. More specifically, with the help of computer simulations we determine trajectories of constant physics – interpreted in a specific way in terms of semi-classical observables we have at our disposal – in the coupling constant space spanned by the bare coupling constants of the lattice theory. Moving along these lines in the direction of smaller lattice spacing, we do not find evidence that they run into the second-order phase transition line, with the possible exception of the triple point of the phase diagram, where three transition lines meet. A slightly more general ansatz that allows for a relative scaling of time and space as the second-order transition is approached leads to a more interesting result, which can be interpreted as a proper UV limit. – In terms of procedure and first results, our investigation provides a reference frame and opens the door to a further systematic study of renormalization group flows in CDT and perhaps other nonperturbative lattice formulations of quantum gravity. This will involve more sophisticated arguments for an appropriate relative scaling of time and space near the phase transition, and hopefully a wider range of observables to provide alternative definitions of what it means to “keep physics constant”.

2 Causal Dynamical Triangulations

CDT is a theory of fluctuating geometries, which at the regularized level are represented by triangulated, piecewise flat spacetimes. It can be viewed as a lattice theory in the sense that the length assignments to the one-dimensional edges (links) of a given triangulation completely determine the piecewise flat geometry.² As already mentioned, a well-behaved causal structure is implemented on each Lorentzian triangulation with the help of a global time foliation that is

²Let us emphasize that these geometries are perfectly continuous (and not discrete, as is sometimes stated), despite the fact that curvature is distributed on them in a singular manner.

distinguished in terms of the simplicial structure. One sums over these geometries in the path integral, where the action is given by the Einstein-Hilbert action in Regge form, suitable for piecewise linear geometries (see the review [1] or the original articles [18] for further details). All triangulations can be obtained by suitably gluing together two types of building blocks, the so-called (4,1) and (3,2) four-simplices, leading (after Wick rotation) to a very simple form for the Euclidean Regge action S_E , namely,

$$S_E = -(\kappa_0 + 6\Delta)N_0 + \kappa_4(N_4^{(4,1)} + N_4^{(3,2)}) + \Delta(2N_4^{(4,1)} + N_4^{(3,2)}), \quad (1)$$

where $N_4^{(4,1)}$ and $N_4^{(3,2)}$ are the numbers of four-simplices of type (4,1) and (3,2) respectively, and N_0 is the number of vertices in the triangulation. The parameter κ_0 is proportional to a^2/G_0 where G_0 is the bare gravitational coupling constant and a denotes the length of (spatial) links. Similarly, κ_4 is proportional to the bare cosmological constant but will play no role here, since we will keep the number of four-simplices (almost) constant during the Monte Carlo simulations of the CDT lattice system.

The parameter Δ appearing in the action (1) requires a more detailed discussion. There are two types of edges that occur in the Lorentzian-signature triangulations before everything is Wick-rotated, spacelike links with squared length a^2 and timelike links with negative squared length $a_t^2 = -\alpha a^2$, where the parameter $\alpha > 0$ quantifies the relative magnitude of the two. We then perform a rotation to Euclidean signature by analytically continuing α in the lower-half complex plane from α to $-\alpha = \tilde{\alpha}$, so that

$$a_t^2 = -\alpha a^2 \quad \mapsto \quad a_t^2 = \tilde{\alpha} a^2, \quad \tilde{\alpha} > 0. \quad (2)$$

The original, Lorentzian Einstein-Hilbert action in Regge form depends on α and satisfies $iS_L(\alpha) = -S_E(-\alpha)$ when rotating from Lorentzian to Euclidean signature. The Euclidean action S_E is now a function of $\tilde{\alpha}$ (see [1] for details). It can be parametrized in the form (1), where Δ is now a function of $\tilde{\alpha}$, normalized such that the case of uniform edge lengths, $\tilde{\alpha} = 1$, corresponds to $\Delta = 0$.

At this stage Δ is not a coupling constant, but only a parameter in the action. Even for Δ different from zero the action continues to be the Euclidean Regge-Einstein-Hilbert action, merely reflecting the fact that some links are assigned a different length. However, in the effective quantum action Δ will appear as a coupling constant. The reason why this can happen is that the choice of coupling constants for which interesting fluctuating geometries are observed is far from the semiclassical region. In this nonperturbative region the measure used in the path integral becomes as important as the classical action, and Δ will effectively play the role of a coupling constant. We refer again to [1] for a detailed discussion, and examples of nongravitational lattice models where one encounters a similar

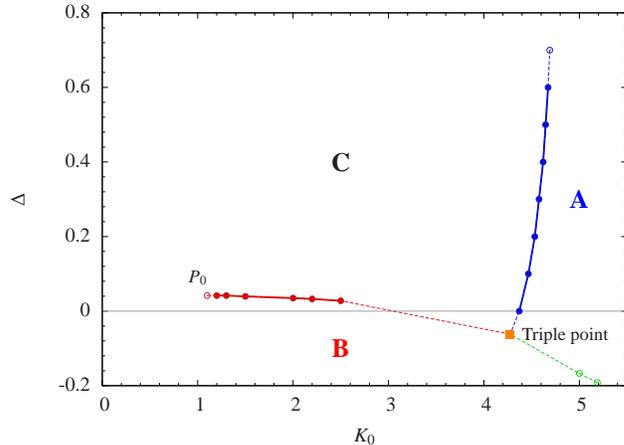


Figure 1: Phase diagram of CDT quantum gravity in four dimensions.

situation. In view of this, the coupling constant space of CDT quantum gravity is spanned by κ_0 and Δ .

For reference, we are showing in Fig. 1 the corresponding phase diagram, already reported in [19, 9]. It has three phases, denoted by A , B and C . Previous studies have shown that only phase C is interesting from the point of view of quantum gravity, in the sense that only there one seems to find quantum fluctuating geometries which are macroscopically four-dimensional. The properties of the quantum geometry in this phase have been studied in great detail [20, 19, 21, 22].

In the present work, we will follow standard lattice procedure by trying to trace the flow of the bare coupling constants inside phase C when we take the lattice spacing a to zero, while keeping physics constant. We know from [9] that the phase transition line separating phases B and C is of second order, while phases A and C are separated by a first-order transition. Our expectation is therefore that the flow lines will approach this second-order transition line when a goes to zero and continuum physics is kept constant.

3 Identifying paths of constant physics

For the purpose of illustration, consider a ϕ^4 -lattice scalar field theory with bare (dimensionless) mass term m_0 and bare dimensionless ϕ^4 -coupling constant λ_0 . Correspondingly, the effective action has a renormalized mass m_R and a renormalized coupling constant λ_R . Let us assume that λ_R is defined according to some specific prescription in terms of the four-point function. Similarly, assume that m_R is defined by some prescription related to the two-point function, for

example the exponential fall-off of the connected two-point function³. One can thus write $m_R a = 1/\xi$, where ξ is the correlation length measured in lattice units a . This relation specifies how one should scale the lattice spacing a to zero as a function of the correlation length ξ in order for m_R to stay constant. Once the actual value of m_R has been supplied from the outside, say by comparison with experiment, the value of $a(\xi)$ is fixed in physical units by measuring ξ .

In order to define a continuum limit where $a(\xi) \rightarrow 0$ while m_R is kept fixed one needs a divergent correlation length ξ , in other words, a phase transition point or phase transition line of second order in the (m_0, λ_0) -coupling constant space. The lattice ϕ^4 -theory has such a second-order phase transition line. Choosing specific initial values $m_0(0)$ and $\lambda_0(0)$ for the bare coupling constants, performing the functional lattice integral will determine the renormalized coupling $\lambda_R = \lambda_R(m_0(0), \lambda_0(0))$ corresponding to these values. The requirement that $\lambda_R(m_0, \lambda_0)$ stay constant when changing m_0 and λ_0 then defines a curve $(m_0(s), \lambda_0(s))$ in the plane spanned by the bare coupling constants, where s is an arbitrary curve parameter.

Along this curve the correlation length ξ will change. Assuming for simplicity that ξ is a monotonic function of s , one can parametrize the curve by ξ instead. Moving along the curve in the direction of increasing ξ will in general lead to the second-order phase transition line where ξ becomes infinite. At the same time, because of $a(\xi) = 1/(m_R \xi)$, the UV cut-off a will decrease. If the curve reaches the transition line at a point λ_0^* , this point will be a UV fixed point for the ϕ^4 -theory, corresponding to a renormalized mass m_R and a renormalized coupling constant λ_R , since approaching it one has $a(\xi) \rightarrow 0$. However, it can happen that a curve of constant λ_R does not reach the second-order phase transition line. If one cannot find a single curve of constant λ_R , for any starting point (m_0, λ_0) , which reaches such a critical point, one would conclude that the theory does not have a UV completion with a finite value of the renormalized coupling constant λ_R . For the four-dimensional scalar ϕ^4 -theory this turns out to be the case.

Assume for the sake of the argument that there *is* a UV fixed point λ_0^* somewhere on the second-order phase transition line⁴. The β -function then has a zero there, $\beta(\lambda_0^*) = 0$, since at fixed m_R and λ_R the coupling $\lambda_0(\xi)$ stops running for $\xi \rightarrow \infty$. Approaching the fixed point along such a trajectory, the behaviour of

³As usual in a lattice set-up, there is the question of lattice artifacts when defining m_R and λ_R , due to the finiteness of the lattice spacing and accompanying discretization effects. In the discussion below we ignore such technical issues because our focus will be on the essence of the renormalization group flow of the bare lattice coupling constants.

⁴Note that in formulas (3) and (5) below it is assumed that $\lambda_0^* \neq 0$. If $\lambda_0^* = 0$ the fixed point is Gaussian and the formulas have to be modified appropriately.

λ_0 near λ_0^* is described by

$$\lambda_0(\xi) = \lambda_0^* + \text{const.} \xi^{\beta'}, \quad \beta' = \left. \frac{d\beta}{d\lambda_0} \right|_{\lambda_0=\lambda_0^*}. \quad (3)$$

In the CDT quantum gravity theory it will be convenient to analyze the flow of bare coupling constants for fixed continuum physics under the additional assumption that the *physical* volume of spacetime is fixed and finite. With this in mind, one can reformulate the above coupling constant flow in ordinary lattice field theory in terms of so-called finite-size scaling. Consider the case of d dimensions and introduce a dimensionful physical d -volume V_d by $V_d := N_d a^d$, where N_d is the total number of d -dimensional elementary building blocks (hypercubes on a cubic lattice, simplices on a triangular lattice). We want to make sure that V_d can be viewed as constant along a trajectory of the kind described above, with m_R and λ_R kept fixed, in the continuum limit as $a(\xi) \rightarrow 0$. This can be achieved by keeping the ratio between the linear size $L = N_d^{1/d}$ of the lattice “universe” and the correlation length ξ fixed. In terms of the renormalized mass m_R and the lattice spacing $a(\xi)$ the ratio can also be written as

$$\frac{\xi^d}{N_d} = \frac{1}{(a(\xi)m_R)^d N_d} = \frac{1}{m_R^d V_d}. \quad (4)$$

Accordingly, moving along a trajectory of constant m_R and λ_R in the bare (m_0, λ_0) -coupling constant plane and changing N_d ensures that the quantum field theory in question has a finite continuum spacetime volume V_d . Furthermore, the equality (4) implies that the dependence on the correlation length ξ in (3) can be substituted by a dependence on the linear size $N_d^{1/d}$ in lattice units of the spacetime, leading to

$$\lambda_0(N_d) = \lambda_0^* + \text{const.} N_d^{\beta'/d}. \quad (5)$$

We noted above that the absence of a UV fixed point is signaled by the fact that no curve of constant λ_R reaches the phase transition line. In this case the correlation length ξ along curves will not go to infinity and the lattice spacing will not go to zero. Restated in terms of the discrete lattice volume it means that N_d will not go to infinity.

We have outlined in this section in some detail how to define and follow lines of constant physics in the ϕ^4 -lattice scalar field theory, because we want to apply the same technique to understand the UV behaviour of the lattice quantum gravity theory. Of course, it should be emphasized that the two theories differ in important ways. First, because ϕ^4 -theory is renormalizable in four dimensions, we know a priori that it suffices to study the flow in the bare couplings m_0 and λ_0 : if no UV fixed point is found along lines of constant λ_R in the (m_0, λ_0) -plane, it does not exist. On the other hand, gravity is not renormalizable, and restricting

the search for a UV fixed point to the two-dimensional coupling constant space spanned by (κ_0, Δ) – although suggestive because of the observed second-order transition line – may ultimately not be sufficient.

Second, while the meaning of lines of constant physics is relatively straightforward in ϕ^4 -theory, the same cannot be said about this concept in nonperturbative and background-independent quantum gravity, because any measure of length one uses is defined in terms of geometry, which is subject to the dynamics of the theory. As will become clear in the remainder of this paper, defining lines of constant physics in terms of suitable geometric observables needs considerable care and is at this stage much more tentative than in the case of scalar field theory.

4 Application to nonperturbative gravity

In the present application to quantum gravity, we will use the coupling constant flow in the form (5), staying at a constant spacetime volume $V_4 = N_4 a^4$ for the universe, where N_4 is the number of four-simplices⁵. How can we make sure that it is consistent to view V_4 as constant when we increase the lattice volume N_4 ? In the case of ordinary field theory we achieved this by using the physical correlation length as a fixed yardstick and requiring $m_R^d V_d$ to remain constant. Since in the CDT pure gravity model we do not have a similar simple correlation length at our disposal, we need to find another indicator of constant physics.

In phase C , at least somewhat away from the B - C phase boundary, the three-volume profile of the universe is to excellent approximation given by [21]

$$\langle N_3(i) \rangle_{N_4} = N_4 \frac{3}{4} \frac{1}{\omega N_4^{1/4}} \cos^3 \left(\frac{i}{\omega N_4^{1/4}} \right), \quad |i| \leq \frac{\pi}{2} \omega N_4^{1/4}, \quad (6)$$

and the variance of the spatial volume fluctuations $\delta N_3(i) := N_3(i) - \langle N_3(i) \rangle$ by

$$\langle (\delta N_3(i))^2 \rangle_{N_4} = \gamma^2 N_4 F \left(\frac{i}{\omega N_4^{1/4}} \right), \quad (7)$$

for a specific function F , whose details are not important for the discussion at hand. Both profiles are functions of the lattice time i . The number of spacelike three-simplices at fixed integer time i is denoted by $N_3(i)$, and the parameters ω and γ depend on the geometric properties of the triangular building blocks and the bare coupling constants κ_0 and Δ .

The profiles (6) and (7) represent finite-size scaling relations, and show in the first place that the time extension of the universe scales like $N_4^{1/4}$ and its spatial

⁵Strictly speaking, we are keeping the number $N_4^{(4,1)}$ of four-simplices of type $(4, 1)$ constant, see [21] for a discussion. The distinction is not important for our present analysis.

volume at a given time like $N_4^{3/4}$, as one would expect from a four-dimensional spacetime. This might seem like a triviality since we started out with four-dimensional building blocks, but in a set-up where no background geometry is put in by hand it is not: all our results are extrapolated to an infinite limit ($N_4 \rightarrow \infty$), and in this limit nonperturbative contributions from the summed-over path integral histories play an important role in bringing about the final outcome. To illustrate the point, no four-dimensional macroscopic scaling behaviour is found in phases A and B of the present model, although they are of course based on exactly the same (microscopically four-dimensional) building blocks. Similarly, one may in principle find deviations from such a scaling inside phase C when getting close to the second-order transition between phases B and C .

The data (6) and (7) extracted from the Monte Carlo simulations in phase C at fixed lattice volume N_4 allow us to interpret the ground state of geometry as a macroscopically four-dimensional quantum universe with a definite average volume profile and a definite behaviour of the average quantum fluctuations of the spatial volume around it. Moreover, making a specific identification of continuum proper time with lattice proper time (by fixing a relative constant for given values of the bare couplings), these properties are characteristic for a de Sitter universe [21].

Sufficiently far away from the phase boundaries of phase C the data summarized in relations (6) and (7) is compatible with the discretized action

$$S_{discr} = k_1 \sum_i \left(\frac{(N_3(i+1) - N_3(i))^2}{N_3(i)} + \tilde{k} N_3^{1/3}(i) \right), \quad (8)$$

which was reconstructed from measuring the correlation function of spatial three-volumes [21] and has the form

$$\langle \delta N_3(i) \delta N_3(i') \rangle_{N_4} = \gamma^2 N_4 F \left(\frac{i}{\omega N_4^{1/4}}, \frac{i'}{\omega N_4^{1/4}} \right), \quad (9)$$

where it is understood that the function F for identical arguments coincides with the function F on the right-hand side of eq. (7). For sufficiently large N_4 and to first approximation, the measured parameters k_1 and \tilde{k} in the reconstructed action (8) were shown to be independent of N_4 and the coefficient γ in (7) was shown to be related to k_1 by

$$\gamma \propto \frac{1}{\sqrt{k_1}}. \quad (10)$$

Phrased differently, for appropriate choice of the coupling \tilde{k} the classical solution to the discretized action (8), solved under the constraint of fixed N_4 , is well approximated by the observed distribution $\langle N_3(i) \rangle_{N_4}$ of (6). In addition, the

observed behaviour of the volume fluctuations, eqs. (7) and (9), is well described by expanding the action (8) to quadratic order around the average profile (6), thus leading to (10). Note finally that the coupling constant \tilde{k} is a function of ω if the distribution (6) is to represent the local minimum of S_{discr} for large, but fixed N_4 , namely

$$\tilde{k} = 9 \left(\frac{3}{4\omega^4} \right)^{2/3}. \quad (11)$$

A natural starting point for trying to relate the above results to continuum physics is to compare the effective action (8) for the spatial three-volume (constructed from numerical ‘‘observations’’) with a minisuperspace action for the scale factor of a homogeneous, isotropic universe with spatial slices of the same S^3 -topology. We can then ask which continuum minisuperspace actions can be matched to an emergent background like (6). The line element of (Euclidean) minisuperspace is

$$ds^2 = N^2(t)dt^2 + a^2(t)d\Omega_3^2, \quad (12)$$

where $a(t)$ is the scale factor, $N(t)$ the lapse function and $d\Omega_3^2$ the line element on the unit three-sphere, such that the spatial volume at time t is $V_3(t) = 2\pi^2 a^3(t)$.

As we have already argued in the introduction, Hořava-Lifshitz gravity provides a natural and potentially useful reference frame for nonperturbative properties of CDT quantum gravity. Also in our present analysis of the renormalization group flow we will use an extended class of reference metrics of type (12), including minisuperspace models of Hořava-Lifshitz type.

Recall that the quadratic part of the action of projectable⁶ HLG in four dimensions in terms of the three-metric $g_{ij}(x, t)$ and the extrinsic curvature $K_{ij}(x, t)$ reads

$$S_{cont} = \tilde{\kappa} \int dt d^3x N(t) \sqrt{g} (K_{ij}K^{ij} - \lambda K^2 + \tilde{\delta} {}^{(3)}R), \quad (13)$$

where $N(t)$ is the lapse function and ${}^{(3)}R$ is the intrinsic scalar curvature of the spatial three-geometry. For the parameter values $\lambda = 1$ and $\tilde{\delta} = -1$ one obtains the standard form of the Euclidean Einstein action, in which case one can identify $\tilde{\kappa} = 1/(16\pi G)$, where G is the gravitational coupling. The three terms in parentheses on the right-hand side of (13) are separately invariant under foliation-preserving diffeomorphisms, the invariance group of HLG.

Using the metric ansatz (12), with $a(t)$ re-expressed in terms of $V_3(t)$, the continuum HLG action (13) becomes

$$S_{cont} = \kappa \int dt N(t) \left(\frac{\dot{V}_3^2}{N^2 V_3} + \delta V_3^{1/3} \right), \quad \frac{\delta}{\tilde{\delta}} = \frac{18(2\pi^2)^{2/3}}{1 - 3\lambda}, \quad \frac{\kappa}{\tilde{\kappa}} = \frac{1 - 3\lambda}{3}. \quad (14)$$

⁶Because of the symmetry reduction to minisuperspace we are considering below, the difference between projectable and nonprojectable HLG will not play a role here.

Firstly, the equation of motion derived for $V_3(t)$ from the action (14), under the constraint that the total four-volume is V_4 , is solved by

$$V_3(\tau) = V_4 \frac{3}{4} \left(\frac{8\pi^2}{3\chi^3 V_4} \right)^{1/4} \cos^3 \left(\left(\frac{8\pi^2}{3\chi^3 V_4} \right)^{1/4} \tau \right), \quad N = \text{const.}, \quad (15)$$

which we have written in a form that facilitates comparison with the lattice expression (6). It is of course precisely the match of the lattice results with a classical \cos^3 -profile (15) that allows us to identify lattice time with a continuum time t , which is a *constant* multiple of continuum proper time τ ,

$$\tau = N t, \quad N = \text{const.} \quad (16)$$

The parameter χ in relation (15) is defined as

$$\chi^2 = \frac{9(2\pi^2)^{2/3}}{\delta}. \quad (17)$$

Computing the scale factor $a(t)$ corresponding to the volume profile (15) and substituting it into the line element (12) one obtains

$$ds^2 = d\tau^2 + R^2 \cos^2 \left(\frac{\tau}{\chi R} \right) d\Omega_3^2, \quad R = \left(\frac{3V_4}{8\pi^2 \chi} \right)^{1/4}. \quad (18)$$

Unless χ equals its general relativistic value $\chi = 1$ this describes a deformed four-sphere with time extension $\pi\chi R$ and spatial extension πR , R being the (maximal) radius of the spatial three-sphere⁷.

Next, comparing the continuum expressions (14)–(17) with the corresponding lattice expressions (6) and (8) and assuming $V_4 \propto N_4 a^4$, one is led to the identifications

$$\tau_i \propto \left(\frac{\chi^{3/4}}{\omega} \right) i \cdot a, \quad k_1 \propto \left(\frac{\omega}{\chi^{3/4}} \right)^2 a^2 \kappa, \quad \tilde{k} \propto \left(\frac{\chi^{3/4}}{\omega} \right)^{8/3} \delta. \quad (19)$$

We note that in the transition from lattice to continuum data only the ratio of ω and $\chi^{3/4}$ appears. The first relation in (19) reiterates our earlier assertion that the continuum proper time can be viewed as proportional to the integer lattice time multiplied by the lattice spacing, where the said ratio is now seen to enter.

Following the logic outlined at the beginning of this section, we would now like to define a path of constant continuum physics in the coupling constant space spanned by (κ_0, Δ) . In doing this, we want to keep the total four-volume

⁷The geometry of the deformed four-sphere is not smooth at $\tau = \pm\chi\pi R/2$. The intrinsic curvature is discontinuous but integrable at these points.

$V_4 \propto N_4 a^4$ fixed. This will enable us to take the lattice spacing $a \rightarrow 0$ by changing N_4 , a parameter we can control explicitly. Our definition of what constitutes “constant physics” will rely on the assumptions that (i) throughout phase C the behaviour of the three-volume is described adequately by the (semi-)classical continuum formulas derived above, and (ii) we can associate space- and time-like lattice units with continuum proper distances and proper times in a way that inside phase C is independent of κ_0 and Δ . More precisely, regarding this latter point it is sufficient to make the weaker assumption that the *ratio* of unit proper distance and unit proper time is a fixed number times the speed of light c throughout coupling constant space.⁸ This is equivalent to keeping fixed the ratio $\omega/\chi^{3/4}$ in relations (19).

Under these assumptions, keeping ω constant in the simulations implies a constant χ and thus a constant volume profile, giving us one criterion for constant, macroscopic physics. However, keeping $\omega(\kappa_0, \Delta)$ fixed is not sufficient to ensure that the emergent continuum universe is unchanged in the limit $N_4 \rightarrow \infty$. Denoting the typical size of volume fluctuations by $|\delta N_3(i)| := \langle N_3(i)N_3(i) \rangle_{N_4}^{1/2}$ and analogously for $|\delta V_3(\tau)|$, one has

$$\frac{|\delta V_3(\tau_i)|}{V_3(\tau_i)} = \frac{|\delta N_3(i)|}{\langle N_3(i) \rangle} \propto \frac{\gamma(\kappa_0, \Delta) \omega(\kappa_0, \Delta)}{N_4^{1/4}} \left(\propto \frac{\chi^{3/4}}{\sqrt{\kappa} V_4^{1/4}} \right), \quad (20)$$

where the result in parentheses follows from relations (10) and (19), and the scaling should be understood for fixed times τ . In view of the proportionality $\tau_i \propto i/N_4^{1/4}$ from (19) above, the discrete time label i used in $N_3(i)$ and $\delta N_3(i)$ should change proportional to $\tau N_4^{1/4}$ when changing N_4 . According to our assumptions the three-volume profile $V_3(\tau)$ and the fluctuation size $|\delta V_3(\tau)|$ are physical quantities, and the ratio $|\delta N_3(i)|/N_3(i)$ (with the interpretation of i just given) must therefore remain constant along any path of constant physics in the space of bare coupling constants.

First, note that staying at a given point (κ_0, Δ) while taking $N_4 \rightarrow \infty$ does *not* correspond to constant continuum physics. Rather, according to (20) it describes a situation where $V_3(\tau)$ (and V_4) go to infinity, and the fluctuations around this macroscopic universe become ever smaller relative to $V_3(\tau)$. Since we have already established that $\omega(\kappa_0, \Delta)$ must be kept fixed along a trajectory of constant physics, eq. (20) implies that as $N_4 \rightarrow \infty$ we must follow a path $(\kappa_0(N_4), \Delta(N_4))$ satisfying

$$\gamma(\kappa_0(N_4), \Delta(N_4)) \propto N_4^{1/4}, \quad \omega(\kappa_0(N_4), \Delta(N_4)) = \text{const.} \quad (21)$$

⁸Note that in the continuum expressions used above we have set $c = 1$, $\hbar = 1$ everywhere. Re-introducing them makes it explicit that the parameter χ defined in eq. (17) above has the dimension of an inverse velocity, and that the product $c \cdot \chi$ is therefore dimensionless.

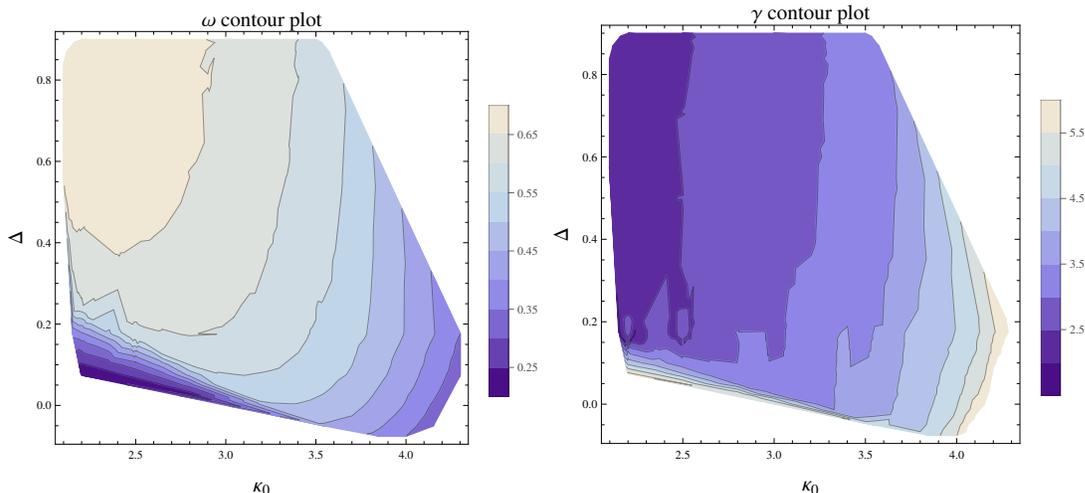


Figure 2: Contour plots in phase C of the parameters $\omega(\kappa_0, \Delta)$ (left) and $\gamma(\kappa_0, \Delta)$ (right), used to characterize trajectories of constant physics. They behave roughly oppositely, ω decreasing and γ increasing toward the bottom right. Their product $\omega \cdot \gamma$ changes only moderately, as illustrated by Fig. 3 below.

This pair of conditions can be regarded as the CDT equivalent of keeping V_4 constant in scalar field theory by insisting that the correlation length satisfies $\xi \propto N_d^{1/d}$, as discussed above. Furthermore, we read off from relation (20) that the conditions (21) are consistent with a physical situation where also the gravitational coupling constant κ is kept fixed. In the next section we will investigate whether it is possible to satisfy (21) in the limit as $N_4 \rightarrow \infty$.

5 Measuring indicators of constant physics

In phase C of the CDT phase diagram we have performed a systematic study measuring the distributions $N_3(i)$ for a fixed number N_4 of building blocks. By fitting, following the procedure outlined in [21], we can determine $\omega(\kappa_0, \Delta)$ and $\gamma(\kappa_0, \Delta)$ for given N_4 . Our analysis assumes that the values of $\omega(\kappa_0, \Delta)$ and $\gamma(\kappa_0, \Delta)$ will only change little with increasing N_4 . This assumption is well tested inside phase C , and for the fixed four-volume we have been using, namely, $N_4^{(4,1)} = 40.000$. Any significant changes in ω and γ must therefore be due to changes in the bare couplings κ_0 and Δ . A dense grid of measuring points in coupling constant space was used to collect the relevant data. Details of this computing-intensive process will be published elsewhere. The resulting contour plots for $\omega(\kappa_0, \Delta)$ and $\gamma(\kappa_0, \Delta)$ in the (κ_0, Δ) -plane (Fig. 2) can be interpreted directly in terms of constant physics: moving along any given line of constant ω on the left contour plot, we can read off from the right contour plot how γ changes along this line,

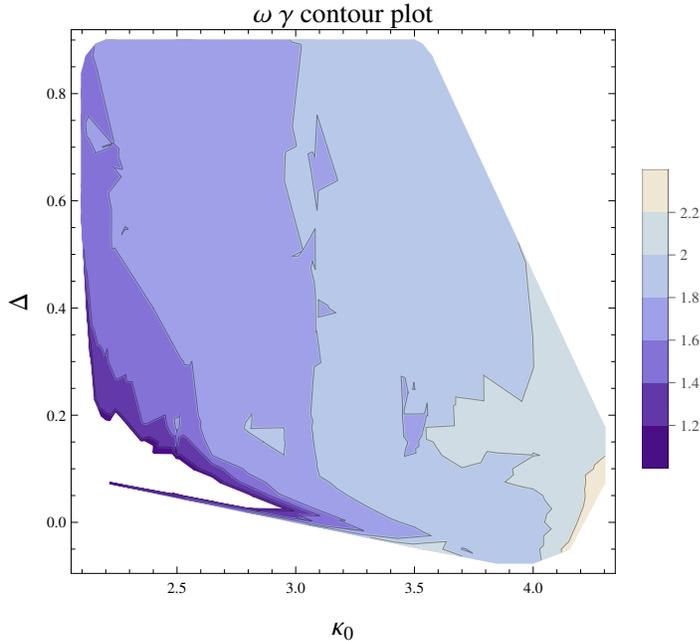


Figure 3: Contour plot of the product $\omega(\kappa_0, \Delta) \cdot \gamma(\kappa_0, \Delta)$ on coupling constant space.

and in particular whether it increases as desired for a UV limit.

Approaching the B - C phase boundary, which lies along the bottom of the two plots of Fig. 2, we observe that ω decreases significantly, while γ increases somewhat, as one would expect when approaching a second-order phase transition line. However, this increase does not appear to be large enough to result in an increase of the product $\omega(\kappa_0, \Delta) \cdot \gamma(\kappa_0, \Delta)$. According to the logic outlined above, this product should go to infinity in a UV limit where V_4 and κ stay constant while we take $a \rightarrow 0$. At least in the region where we can measure reliably, somewhat away from the transition line, the product $\omega \cdot \gamma$ changes little, as can be seen in Fig. 3. Close to the B - C phase transition line our results are not reliable. Autocorrelation times grow enormously, and the decrease in the parameter ω means that the universe becomes very short in the time direction, rendering the use of the effective action (8) questionable.

6 UV fixed point scenario

In the minisuperspace action (14) we have introduced a generalized inverse gravitational coupling constant κ , which has mass dimension 2 and incorporates a dependence on the HLG-parameter λ . We can introduce a corresponding dimensionless coupling $\hat{\kappa}$ via $\kappa(a) = \hat{\kappa}(a)/a^2$. Comparing with relations (19), we see

that $\hat{\kappa}(a) \propto (\omega/\chi^{3/4})^2 k_1(a)$. Of course, this identification is only meaningful as long as physics is well described by the effective actions (14) and (8). At least well inside phase C this is known to be the case.

For long-distance physics we expect κ to be a constant, implying that k_1 should behave like $k_1(a) \propto \kappa \cdot a^2 \propto \kappa (V_4/N_4)^{1/2}$. This implies the scaling behaviour $\gamma \propto 1/\sqrt{k_1} \propto N_4^{1/4}$, which we have already discussed earlier as a requirement of constant physics. However, this is not the behaviour one would in general expect to encounter at a UV fixed point. By definition a nonperturbative UV fixed point is one where the *dimensionless* coupling goes to a finite fixed value, $\hat{\kappa}(a) \rightarrow \hat{\kappa}^*$. Consequently, the analogue of the expansion (5) for the inverse gravitational coupling constant is given by

$$\hat{\kappa}(N_4) = \hat{\kappa}^* + \text{const. } N_4^{\beta'/4}, \quad \beta' < 0, \quad (22)$$

provided we are in the vicinity of the fixed point $\hat{\kappa}^*$ and move on a trajectory where V_4 is kept constant. According to relations (19) and (22) this implies a k_1 -behaviour of the form

$$k_1(N_4) \propto \left(\frac{\omega}{\chi^{3/4}} \right)^2 \hat{\kappa}^* \quad (\text{large } N_4). \quad (23)$$

Still assuming that our minisuperspace analysis provides a reliable frame of reference, this leads to

$$\frac{|\delta V_3(\tau_i)|}{V_3(\tau_i)} = \frac{|\delta N_3(i)|}{\langle N_3(i) \rangle} \propto \frac{\omega(\kappa_0(N_4), \Delta(N_4))}{\sqrt{k_1(N_4)} N_4^{1/4}} \propto \frac{\chi(\kappa_0(N_4), \Delta(N_4))^{3/4}}{\hat{\kappa}^* N_4^{1/4}} \quad (\text{large } N_4). \quad (24)$$

We conclude that this quotient cannot be kept constant in the neighbourhood of the UV fixed point and for constant χ , unless for some reason $\hat{\kappa}^* = 0$. One way to make a vanishing fixed-point value for k_1 appear natural is by explicitly invoking the HLG-parameter λ and discussing the UV fixed point in terms of the coupling constant $\tilde{\kappa}$, which appears in the continuum action (13). In terms of its dimensionless counterpart $\hat{\kappa}(a) := a^2 \tilde{\kappa}(a)$ one would make an ansatz

$$\hat{\kappa}(N_4) = \hat{\kappa}^* + \text{const. } N_4^{\beta'/4}, \quad \beta' < 0, \quad (25)$$

analogous to (22). However, because of $\kappa = (\frac{1}{3} - \lambda)\tilde{\kappa}$, in place of relation (23) one then obtains

$$k_1(N_4) \propto \left(\frac{1}{3} - \lambda \right) \left(\frac{\omega}{\chi^{3/4}} \right)^2 \hat{\kappa}^* \quad (\text{large } N_4). \quad (26)$$

This now leaves open the possibility of a vanishing k_1 at the ultraviolet fixed point, $k_1(N_4) \rightarrow 0$, *provided* one chooses to scale $\lambda \rightarrow 1/3$ at the same time.

Note that by doing so one gives up staying on a curve of constant physics, in the sense of keeping V_4 , V_3 , $|\delta V_3|$ and the shape of the emergent semiclassical minisuperspace geometry fixed. The reason is that according to

$$\chi^2 = \frac{1 - 3\lambda}{2\tilde{\delta}} \quad (27)$$

(c.f. eq. (14) and (17)) a change in λ implies a change in χ , the parameter describing the shape of the universe, unless we choose to scale $\tilde{\delta}$ precisely as $\tilde{\delta} \propto (1 - 3\lambda)$. According to our assumptions, ω then also changes. If $\tilde{\delta}$ stays constant or goes to zero slower than $(1 - 3\lambda)$, both $\chi \rightarrow 0$ and $\omega \rightarrow 0$ at the UV fixed point. Since we observe in our computer simulations that ω goes towards zero when we approach the B - C second-order phase transition line, the line appears as a candidate for UV fixed points in this particular scenario. Approaching it along some path where χ (and therefore ω) decreases but V_4 is kept fixed implies that $V_3 \propto (V_4/\chi)^{3/4}$ is no longer constant. Also the constancy criterion (20) for $|\delta V_3|/V_3$ can no longer be applied in a straightforward manner.

If on the other hand we choose to scale $\tilde{\delta}$ like $(1 - 3\lambda)$, we can maintain the concept of constant shape and three-volume V_3 for fixed V_4 . We are then back to the situation analyzed previously; γ has to grow proportional to $N_4^{1/4}$ along paths of constant ω , with the only difference that this now allows for a UV interpretation in terms of $\tilde{\kappa}$ rather than κ . However, as discussed in the previous section, there is little support for this growth from the data, at least in the region where we can measure reliably.

7 Discussion and conclusion

In this paper, we have presented the results of a first nonperturbative analysis of renormalization group flows in four-dimensional CDT quantum gravity. Since a second-order phase transition line has been found in this formulation of quantum gravity [9] – thus far a very rare occurrence in dynamical models of higher-dimensional geometry – how this line may be reached along suitably defined RG trajectories in phase space will give us important information about the theory’s ultraviolet regime. It will also allow us to make a closer comparison with continuum investigations of gravity in terms of functional renormalization group techniques and may provide an independent check on ultraviolet fixed point scenarios derived in this approach.

As explained in Secs. 3 and 4, we use conventional lattice methods to investigate the behaviour near the phase transition, adapted to the case of dynamical geometry, where we do not have a fixed background geometry to refer to and any physical yardstick for measuring distances has to be generated dynamically.

Taking a UV limit is achieved formally by sending the lattice spacing a to zero, but to make this into a physically meaningful prescription a has to be related to some *physical* length units. As illustrated by the scalar field example, this is usually done by referring to the correlation length. Alternatively, since in the case of gravity we currently do not have a suitable correlation length available, one may also refer to the total volume of the system, and re-express scaling relations near a fixed point in terms of this volume, as illustrated by eq. (5). This is the strategy we follow for gravity to make sure that we have a true, physical implementation of the ultraviolet limit. The difference with the scalar field case is that the macroscopic reference volume used is generated *dynamically*, and any possible dependence on the bare couplings should be considered carefully, because it can have an influence on how one defines ‘lines of constant physics’ on coupling constant space.

For the latter we have made the most direct ansatz available in CDT gravity, namely, to define constant physics in terms of the physical quantities characterizing the macroscopic universe that emerges as the ground state of the quantum dynamics. These are its total four-volume, its three-volume as a function of proper time and quantum fluctuations of the three-volume around its mean, the so-called volume profile. We have interpreted all of them physically in terms of a class of homogeneous and isotropic cosmological solutions of Hořava-Lifshitz type, and have assumed that this interpretation is valid throughout phase C , where we observe extended geometry. At the same time we have assumed that we can make an identification of lattice units in terms of continuum proper times and distances that likewise remains unchanged inside phase C . Conceptually, these are the most straightforward assumptions one can make, and it is important to understand what conclusions they lead to.

Concretely, we then defined lines of constant physics by keeping the shape parameter $\omega(\kappa_0, \Delta)$ constant, as well as the relative size of three-volume fluctuations, leading to the scaling requirement $\gamma(\kappa_0, \Delta) \propto N_4^{1/4}$ for the “fluctuation parameter” γ in the UV limit $N_4 \rightarrow \infty$. Analyzing the computer simulation data presented above we saw no concrete indication that the second-order B - C phase transition line is reached when flowing along any of the lines of constant physics. Instead, the lines of constant $\omega(\kappa_0, \Delta)$ run parallel to the B - C phase transition line if one starts in the vicinity of this line. Increasing $\gamma(\kappa_0, \Delta)$ along such a line brings one close to the triple point of the phase diagram. For the finite value of N_4 used here, curves of constant ω eventually turn away from the triple point and run parallel to the A - C transition line. However, this may well be a finite-volume effect, leaving open the possibility for flow lines to end up in the triple point. On the other hand, on the basis of the measurements made up to now the increase in γ when moving along a line of constant ω seems to be too slow to satisfy the criteria of constant physics for $N_4 \rightarrow \infty$.

However, as we described in Sec. 7, it *is* possible to view the B - C line as a second-order UV phase transition line for the HLG action (13), if we allow for a suitable scaling of “little lambda”, $\lambda \rightarrow 1/3$.⁹ In this interpretation an anisotropy between space and time develops as one moves along flow lines, corresponding to $\chi \rightarrow 0$ in (18).

It is clear that the next step in our investigation of renormalization group flows will be a more extended analysis of different UV scaling scenarios, where in particular our current assumption of “frozen” proper distance units throughout coupling constant space is relaxed, which will have consequences for how “constant physics” is defined. It would also allow us to consider a scenario where the shape of the emergent universes is interpreted in terms of round four-spheres, at least somewhat away from the phase transition, as we have done in previous work [19, 21], in contrast to the family of deformed spheres we have used here. It is clear that this can change the running of the renormalization group flows significantly, and improve on the results found in the present work, where we have adopted rather conservative assumptions about scaling and constant physics.

Using different notions of constant physics close to the phase transition is certainly well motivated by nonclassical features of quantum geometry already found on Planckian scales, like the anomalous behaviour of the spectral dimension [7], and by taking seriously anisotropic scaling scenarios à la Hořava in the UV, which we have already argued constitute a natural frame of reference for our investigation. There will be technical issues to deal with when investigating different scalings near the B - C transition line, including the fact that the time extension of the universe shrinks to only a few lattice spacings there, making any construction of an effective action imprecise. One obvious solution would be to increase the lattice size N_4 , but one also has to take into account the critical slowing-down near the B - C transition (as one would expect), which makes simulations there painfully slow. We are currently trying to circumvent this issue by using the so-called transfer matrix formalism [25], where a large time extension is not needed. Progress on this will be reported elsewhere.

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⁹The value $\lambda = 1/3$ is special in the sense that the Wheeler-DeWitt metric underlying the construction of the kinetic term in the action (13) becomes degenerate, and an extra constraint appears in the Hamiltonian analysis. HLG models setting $\lambda = 1/3$ from the outset have recently been considered in their own right [24].

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