CONSTRaining PARAMETERS OF WHITE-Dwarf BI-naries USING GRAVITATIONAL-Wave AND ELECTROMAGNETIC OBSERVATIONS

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(Dated: June 16, 2014)

ABSTRACT

The space-based gravitational wave (GW) detector, evolved Laser Interferometer Space Antenna (eLISA) is expected to observe millions of compact Galactic binaries that populate our Milky Way. GW measurements obtained from the eLISA detector are in many cases complimentary to possible electromagnetic (EM) data. In our previous papers, we have shown that the EM data can significantly enhance our knowledge of the astrophysically relevant GW parameters of the Galactic binaries, such as the amplitude and inclination. This is possible due to the presence of some strong correlations between GW parameters that are measurable by both EM and GW observations, for example the inclination and sky position. In this paper, we quantify the constraints in the physical parameters of the white-dwarf binaries, i.e. the individual masses, chirp mass and the distance to the source that can be obtained by combining the full set of EM measurements such as the inclination, radial velocities, distances and/or individual masses with the GW measurements. We find the following 2 − σ fractional uncertainties in the parameters of interest. The EM observations of distance constrains the the chirp mass to ∼ 15 − 25%, whereas EM data of a single-lined spectroscopic binary constrains the secondary mass and the distance with factors of 2 to ∼ 40%. The single-line spectroscopic data complemented with distance constrains the secondary mass to ∼ 25 − 30%. Finally EM data on double-lined spectroscopic binary constrains the distance to ∼ 30%. All of these constraints depend on the inclination and the signal strength of the binary systems. We also find that the EM information on distance and/or the radial velocity are the most useful in improving the estimate of the secondary mass, inclination and/or distance.

Subject headings: stars: binaries - gravitational waves, Galactic binaries - GW parameters, GW detectors - LISA

1. INTRODUCTION

Gravitational wave (GW) observations and electromagnetic (EM) observations can be used to study compact Galactic binaries independently and often these two ways provide different measurements of the same system. There are about ∼50 of these binaries that have been studied in the optical, UV, and X-ray wavelengths (e.g. Roelofs et al. 2010). This number is expected to grow by a factor of ∼100 when a space-based gravitational wave (GW) observatory like the recently eLISA1 will be in operation. This detector is expected to observe millions of compact Galactic binaries with periods shorter than about a few hours (Nelemans 2009; Amaro-Seoane et al. 2013), amongst other astrophysical sources. Of those millions of binaries we will be able to resolve several thousands. It has been shown (Shah et al. 2012) that for a non-eclipsing binary system (for example AM CVn), its EM measurement of the inclination, i can improve the error on the GW amplitude (A) significantly depending on the strength of the GW signal and the magnitude of the EM uncertainty in the inclination. A is a GW parameter which is given by a combination of the masses, orbital period and distance to the source:

\[ A = \frac{4(GM_c)^{5/3}}{c^4d} (\pi f)^{2/3}, \] (1)

where, d is the distance to the source, f is the source’s GW frequency (2/Porb), and M_c is the chirp mass defined as:

\[ M_c \equiv \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}. \] (2)

From the GW observations alone, one typically cannot measure the individual masses or the distance since they are degenerate via Eqs. 1, 2. In the rare cases that a precise orbital decay (\dot{f}), can be measured from GW data then the distance can be estimated (with the assumption that the frequency evolution is dictated by GW radiation only) by determining M_c from the measured f and \dot{f} via the equation (Peters & Mathews 1963):

\[ \dot{f} = \frac{96 \pi G^{5/3}}{5c^3} (\pi M_c)^{5/3} f^{11/3}. \] (3)

For the compact binaries that have been observed with the optical telescopes, a subset of which will also be detected by eLISA, their EM data often provide measurements of the orbital period P orb, the primary mass (m_1), sometimes the secondary mass (m_2), the distance (d) and the radial velocity amplitude (K_i). We use these measurements for a few binaries to show the quantitative

1 In preparation by ESA, expected launch in ∼2034
improvements in their GW and other physical parameters. Many of these binaries can/could still be found electromagnetically before or after eLISA discoveries.

We have previously shown that knowing sky positions from EM data can improve the GW uncertainties on $A$ and $\iota$ depending on the particular geometry and orientation of the binary systems (Shah et al. 2013). Thus, so far we have quantitatively studied the improvement factors in the uncertainties of the parameters that can be gained from prior knowledge of parameters which are common to both GW and EM observations, for example inclination, and sky position.

In this paper we go beyond constraining only those GW parameters which are also measured independently from the EM data. We explore various combinations of any possible EM observations and the GW measurements in constraining the useful parameters of the binaries that are astro-physically interesting, for example the individual masses. Because their GW signal is significantly affected we consider high-inclination (sometimes eclipsing) and (low inclination) binary systems. We review the GW data analysis methods in Sect. 2. In Sect. 3, we explore the information gained by combining EM measurements in different ways where the EM data can be the radial velocity of one of the binary components, $K_1$, $m_1$, $m_2$, $d$, and $P_{\text{orb}}$. Specifically, we classify various combinations into a number of scenarios in discussing the parameter constraints.

2. PARAMETER UNCERTAINTIES FROM GW OBSERVATIONS

For our analyses below, we consider one of the eLISA verification binaries J065133.33+284423.3 (J0651, hereafter; Brown et al. (2011)). We also consider a second (hypothetical) system with higher masses which we will refer to as “the high mass binary”. Their GW parameter values are listed in Table 1. Before looking at the EM data we briefly recap out GW data analysis method. We have used Fisher matrix studies (e.g. Cutler 1998) to extract the GW parameter uncertainties and correlations in the GW parameters that describe the compact binary sources. Our method and application of Fisher information matrix (FIM) for eLISA binaries together with their signal modeling and the noise from the detector and the Galactic foreground have been described in detail in Paper I. Most of the binaries will be monochromatic (e.g. $A$, frequency ($f$), polarization angle ($\psi$), initial GW phase ($\phi_0$), inclination ($\cos \iota$), ecliptic latitude ($\sin \beta$), and, ecliptic longitude ($\lambda$). From the GW signal of a binary and a Gaussian noise we can use FIM to estimate the parameter uncertainties. The inverse of the FIM is the variance-covariance matrix whose diagonal elements are the GW uncertainties and the off-diagonal elements are the correlations between the two parameters. We do the GW analyses of the above mentioned binaries for eLISA observations of two years. We note that Fisher-based method is a quick way of computing parameter uncertainties and their correlations in which these uncertainties are estimated locally at the true parameter values and therefore by definition the method cannot be used to sample the entire posterior distribution of the parameters. Additionally Fisher-based results hold in the limit of strong signals with a Gaussian noise (e.g. Vallisneri 2008; see also Appendix).
The strong increase in uncertainty trends for low inclination systems is due to the correlation between amplitude and inclination (\(\mu = 0.25 M_\odot\), \(m_2 = 0.55 M_\odot\), \(d = 1.0\ kpc\)). Clearly the high mass binary has larger S/N which gives smaller uncertainties in both of its parameters shown in open circles in the figure compared to that of J0651. Observe that inclination is a cyclic parameter and is bounded between 0° ≤ \(\iota\) ≤ 90° and yet we get very large uncertainties from Fisher matrix for lower inclinations systems shown in the right panel of in Figure [1]. This is due to the fact that Fisher matrix methods are based on the linearised signal approximation as a result of which it is not sensitive to the bounded parameters that describe the signal model [Vallisneri 2008]. In other words in FIM one computes the uncertainties in parameters based on variation of the signal with respect to the parameters at the true parameter values and the fact that far away from the true value the parameter has a bound is not taken into account by the FIM. When the uncertainty in a bounded parameter exceeds its physically allowed range, it means the quantity cannot be determined from GW data analysis. The dashed line in in Figure [2] indicates the value (at 90°) beyond which the uncertainties in \(\iota\) imply unphysical values for the inclination. Since the low inclination systems on the left-side of the plot are affected by this, corrections have to be applied to the corresponding (over-estimated) uncertainties in amplitude in the left panel by discarding the unphysical range in the inclination [Shah et al. 2013], Eq. C4. One way to correct these unphysical values is by taking a rectangular prior on the inclination. This in effect will cut off the posterior distribution in the parameters at the physical bounds described by the prior. Note that cutting off the error ellipses at lower inclinations in Figure [1] is reasonable because taking strict bounds far away form the real value about which Fisher uncertainties are computed will not change the shape and slope significantly. The cut off in the posterior distributions due to rectangular priors will skew the means of the parameter distributions away from the real value [Rodriguez et al. 2013, Eq. C4]. Furthermore we stress the fact that the Fisher matrix method is an estimate and cannot, follow the posterior in detail (see Appendix).

The normalized correlations between all the seven parameters for an eclipsing and non-eclipsing orientations of J0651 are listed in the variance-covariance matrices (VCM) in the Appendix. We will make use of these parameter uncertainties and their corresponding correlations when combining with various EM data in Sect. 3.

### 2.1. GW information only

We start by considering the case where we only have the GW data. From the GW observations, the astrophysical parameters of interest for a monochromatic source are its \(f\), \(A\) and \(\iota\). From the GW data analysis the frequency of the source will be very well determined, \(\sigma_f/f \sim 10^{-6}\) Hz (e.g. Paper I) for a \(10^{-13}\)Hz source, so we consider that \(f\) is essentially known with a fixed value. Given that most of the binaries that we will observe with eLISA will be binary white-dwarfs (WD) \([\text{Nelemans et al. 2001}]\), we can restrict their masses to \(m_1 \in [0.1, 1.4] M_\odot\). For simplicity we take uniform distributions for both masses in this range. This provides a distribution in the system’s chirp mass, which will provide an upper limit on the distance for the source. In Figure [3] we show these estimates in \(d\) with their 95 percentile (or 2−\(\sigma\) uncertainties) as a function of inclination for both J0651 (in black) and for the high mass binary with equal high mass components (in grey). The dashed line (in black) is the real value of the distance for both systems. The lower medians of distances at the lower inclinations for both systems are explained by the fact that at \(\iota = 5°\), the GW distribution of \(A\) has a long uniform tail. This is shown in Figure [4] where we compare the distributions of \(A\) for two inclinations: 5° (in black), and 90° (in grey) in the left panel. For a fixed distribution of \(M_c\), the corresponding distributions in \(d\) are shown in the right panel where the solid vertical lines are the distribution medians and dashed vertical line is the real value. We can see that \(A^{\text{90°}}_\text{median} > A^{\text{5°}}_\text{median}\) thus giving \(d^{\text{5°}}_\text{median} < d^{\text{90°}}_\text{median}\) via Eq. [4] for a fixed \(M_c\).

![Figure 3](image-url)

**Fig. 3.**—GW data only: 95 percentile in distance assuming finite mass for J0651 in black lines and the high mass binary in grey lines. The dashed line (in black) is the true value. For clarity the constraints for the high mass binary are shifted to the right. We do this for all the cases below.

TABLE 1

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A) ([10^{-22}])</td>
<td>1.67−6.71</td>
</tr>
<tr>
<td>(M_c) ([M_\odot])</td>
<td>0.32−0.70</td>
</tr>
<tr>
<td>(\phi_0) ([\text{rad}])</td>
<td>0.007</td>
</tr>
<tr>
<td>(f) ([\text{Hz}])</td>
<td>2.61−7.2</td>
</tr>
<tr>
<td>(\psi) ([\text{rad}])</td>
<td>0.72−0.101</td>
</tr>
<tr>
<td>(\sin \beta)</td>
<td>0.177−1.77</td>
</tr>
<tr>
<td>(S/N)</td>
<td>13°−50°</td>
</tr>
</tbody>
</table>

\(a\) for \(m_1 = 0.25 M_\odot, m_2 = 0.55 M_\odot, d = 1.0\ kpc\) \(b\) for \(m_1 = 0.8 M_\odot, m_2 = 0.8 M_\odot, d = 1.0\ kpc\)
fractional uncertainty compared to that of the $A$ and thus the relative error uncertainties in the chirp mass dominates those in the distance, which remain roughly constant for all inclinations.

3. COMBINING EM & GW OBSERVATIONS

In all the various scenarios we analyze below, we take the EM parameters with an uncertainty of 10% which is inspired by observational uncertainties of J0651. This binary is a well known EM source and also a guaranteed source for eLISA. J0651 is an eclipsing system and such an orientation of a nearby binary allows accurate EM measurements of it’s orbital parameters, and the masses (accuracies of $\sim 15\%$ (primary mass); $8\%$ (secondary mass)) from observing the spectra, radial velocities and eclipses of each star by the other (Brown et al. 2011). Furthermore its rate of change of orbital period has also been measured from follow-up high speed photometry from $\sim 1$ yr. worth of data to an accuracy of $\sim 30\%$ (Hermes et al. 2012), and this will improve in the course of time. In this section we classify specific (possible) scenarios where we could have one or more EM data on the white dwarf binary parameters. We explicitly state how much the knowledge of any of the various parameters that describe the physical properties of a binary system can be further improved if we can fold in various combinations of the existing EM and/or GW observations. We construct three specific scenarios below based on the typical knowledge from the EM observations:

- EM data on distance
- Single-line spectroscopic data (complemented with or without the distance measurement)
- Double-line spectroscopic data

In all the scenarios the GW information on amplitude, inclination and frequency from Sect. 2.1 are used.

3.1. Scenario 1: EM observation of the distance
Measuring distances accurately is made feasible by the \textit{Gaia} mission (de Bruijne \textit{et al.} 2012), a new astrometric satellite. \textit{Gaia} is expected to measure stellar parallaxes of millions of stars with arcsec accuracy depending on how bright a star is. For example at 1 kpc, J0651 ($g = 19.1$ mag) would have a $\sim 300$arcsec accuracy in the parallax measurement corresponding to a fractional accuracy in distance of $\sim 10\%$ (e.g. Bailer-Jones 2009). There is also some indication of the distance of the binary from its absolute magnitude. The uncertainties in $d$ from such measurements are also of the order of several percent or $10\%$ for the case of J0651 (Brown \textit{et al.} 2011). A sole EM measurement of the distance of a WD binary might be possible in cases where the system is identified as a WD binary but it is too faint to measure other parameters. For instance from the wide-field surveys it is often possible to identify WD from their colors (Verbeek \textit{et al.} 2013). Given the distance and the GW uncertainty in amplitude, we can trivially solve for the chirp mass, $M_c$ using Eq. (1). The resulting probability distribution functions (pdf) are computed by randomly drawing points from the given distributions and computing the parameter of interest for each draw. The 95% percentiles in the $M_c$ are shown in the left panel of Figure 5 for J0651 (in black) and the high mass binary (in grey) as a function of inclination. The dashed lines (in black for J0651 (in black) and the high mass binary (in grey) as shown in the left panel of Figure 5 for $\iota = 90^\circ$ (in thick black lines) compared to that of $\iota = 90^\circ$ (in thick black lines). For a fixed distribution in distance the corresponding distribution of $M_c$ is therefore overestimated for $\iota = 5^\circ$ shown in the right panel of Figure 5 compared to that of $\iota = 90^\circ$. The 95% percentiles of the chirp masses for both J0651 and the high mass binary are affected by these overestimated medians of the amplitudes at lower inclinations which cause significant offsets of the $M_c$ from their respective real values as can be seen in the left panel of Figure 5. Thus at lower inclinations where the medians in the amplitude are overestimates, the 95 percentiles in the chirp mass can be interpreted as upper limits of the chirp mass.

In order to calculate reliable constraints in $M_c$ at these small inclinations we have to do full (Bayesian) data analyses that takes into account the physical priors and gives us a better estimate of the expected posterior distributions in the desired parameters. The 95 percentile in $M_c$ for both systems decrease as a function of inclination as is expected from the propagation of uncertainty where $\sigma_{M_c} \propto \sigma_A$. Thus, knowing distance from EM observation gives us an estimate of the chirp mass where the constraints are tighter for the higher inclination (eclipsing) systems.

3.2. Scenario 2: EM observations of single-lined spectroscopic binary

Some measurements are unique to EM observations such as the radial velocity $K_1$, of one of the components ($m_1$) of the binary:

$$K_1 = \sin \iota \frac{m_2}{(m_1 + m_2)^{2/3}} \left( \frac{2\pi G}{P_{\text{orb}}} \right)^{1/3},$$

which can be used to measure inclination. We adopt the convention from the optical studies of the binary sources where the primary mass, $m_1$ is the brighter object and the dimmer secondary mass, $m_2$. Note that the inclination measurement from the GW data analysis, $\iota[\text{GW}]$ and from the radial velocity equation above i.e. $\iota[\text{RV}]$ are two independent measurements for the same system. We will show that these two are anti-correlated below in Sect. 3.2.3, yielding radial velocity measurements very useful.

3.2.1. Scenario 2a: EM data on $m_1$

Before looking at a real single-lined spectral binary we first consider the case that only the mass $m_1$ is known from the EM data. This is a viable scenario when determining $K_1$ is impossible and we may get an estimate of the primary mass from the photometry or the spectra. Assuming a double WD system, we take a uniform distribution for $m_2$, which together with the given $m_1$ constrains the $d$. The estimates of distance with their corresponding 95 percentiles as a function of inclination are shown in Figure 5 for both the J0651 (in black) and...
the high mass binary (in grey). The real value of distance is shown in the dashed (black) line. The offsets of the medians in the distance at low inclinations for both the binary systems can be explained in a similar way as in the previous sections, which is due to the overestimated medians of $A$ at lower inclinations as shown in the left panel of Figure 4. Additionally, the significant discrepancy between the median distance for J0651 vs. the high mass binary (at $i \geq 40^\circ$) is again due to the over-estimated value of the $M_e$ for J0651 assuming a uniform distribution $m_2$ distribution. This is shown in the right panel of Figure 4, where the vertical dashed lines are their corresponding true values of the $M_e$ and the vertical solid lines are the medians of the corresponding distributions. The simulated distribution of $M_e$ from an EM measurement of $m_1$ with a Gaussian width in its uncertainty together with an assumed uniform distribution in the unknown $m_2$ results in an overestimated median of the $M_e$ for J0651 compared to that of the high mass binary. This propagates in overestimating the median $d$ for J0651 at higher inclinations unlike for the high mass binary since its median $M_e$ is slightly underestimated. The flat priors on $m_2$ affect the constraints obtained in the $d$. The constraints in the distance from Figure 5 can be compared with those in Figure 6, where there was no EM information on any of the masses: the upper limits on $d$ for both J0651 and the high mass binary are constrained by a up to factor of $\sim 4$ better when $m_1$ is known for both binaries with 10% accuracy.

3.2.2. Scenario 2b: EM data on $m_1 \& d$

In this case we consider EM measurements of a single-lined spectroscopic binary where resolving one of the masses of the binary spectroscopically typically provides measurements on the primary mass and its radial velocity. We assume an uncertainty in radial velocity amplitude of 10 km/s found in the EM measurements (for e.g., Roelofs et al. 2006). Given $m_1$ and $K_1$ from the EM data and inclination from GW data $i(GW)$, we can numerically solve for $m_2$ via the $K_1$ formulation in Eq. 4. Assuming it is a WD, the $m_2$ is restricted to lie in $[0.1-1.4]M_\odot$. Then a fixed pair of [$A$, $i(GW)$] and the masses give us a distance. We calculate the resulting distributions in $m_2$ and the distance from the Gaussian distributions of $m_1$ and $K_1$ about their typical EM uncertainties and GW distributions in the inclination and amplitude. The 95 percentile in the secondary mass and the distance are shown in Figure 7 as a function of inclination for both J0651 (in black) and the high mass binary (in grey). Like in the scenarios discussed above, for the lowest inclinations, the over estimated FIM uncertainties of $A$ propagates into erroneous constraints on $m_2$, and $d$. Thus, at lower inclinations we have to use Bayesian methods to get their accurate GW uncertainties. Observe that the 95 percentile in the $m_2$ and the distance roughly similar and large from $5^\circ < i < 45^\circ$. This is again due to the influence of the GW distributions in $A$ at the lower inclinations, which have uniform distributions resulting into over-estimated medians (see Figure 1). However, for $i > 45^\circ$ the uncertainties for both $m_2$, and $d$ decrease with inclination and their medians stabilize at the true values. This is caused by the fact that at higher inclinations, the medians of GW amplitudes are close to the true values of the systems where the constraints on the GW parameters are also tighter with increasing inclination. Thus, the decreasing uncertainties in $i(GW)$ as a function of $i$ (see right panel of Figure 2) should result in the same behavior of $\sigma_{m_2}$ via Eq. 4. Since distance is computed using these $m_2$, the same behavior holds for the distance in the right panel.

3.2.3. Scenario 2c: EM data on $m_1$, $K_1 \& d$

Here the EM measurements of a single-lined spectroscopic binary in the previous subsection is complemented with a distance measurement from Gaia or from an estimate of the absolute magnitude. From the primary mass $m_1$, distance and the amplitude we immediately get the secondary mass, $m_2$. We will call this as the preliminary $m_2$ since this can be further improved by folding in the radial velocity measurement. As mentioned before the radial velocity measurement essentially provides an independent measurement of the inclination via Eq. 4. This can be seen in the following way: The GW parameters of the non-eclipsing J0651 are: $A_0$, $t_0 = 1.67 \times 10^{-22}$, $45^\circ$ whose VCM uncertainties are: $\sigma_A/A = 0.231$, and $\sigma_i = 0.75$ rad respectively. We also take a fixed radial velocity, $K_0$ corresponding to $m_1$, $m_2$ (listed in Table 1), and $t_0$. The 2-d Gaussian distribution from GW data with $1-\sigma$ uncertainties for these parameters is shown in the left panel of Figure 8. For each randomly selected pair of [$A$, $i(GW)$] and for a fixed $m_1$, and $d$, we can solve for the $m_2$ from Eq. 4. Using this $m_2$ for that fixed $m_1$ and $K_0$, we solve for $i(GW)$. For many points randomly picked in the [$A$, $i(GW)$] space the computed $i(RV)$ are compared with the corresponding $i(GW)$ in the right panel. The inclinations measured in two ways roughly anti-correlate. However we know that values of $i(RV)$ that are different from $i(GW)$ cannot be true. Thus, constraining the inclination of the system in a small area around $45^\circ$ along the diagonal line in the right panel also constrains $m_2$ and the amplitude. We make use of this in the case considered in this subsection. The preliminary $m_2$ and their 95 percentiles computed from EM data on $m_1$, $d$, and the GW data on $A$ as a function of inclination is shown in Figure 5, in grey lines in the left panel for J0651. The same for the high mass binary is also shown in the right panel in grey lines. From this $m_2$, given $m_1$, and $i(GW)$, the radial velocity, $K_{GW}$ is computed which is compared with the $K_1$ from the EM data. Since the EM measured $K_1$ is more precise, we keep the subset of those $K_{GW}$ and the respective $i(GW)$ weighted with a probability distribution function of the $K_1$ given by:

$$P_i = \frac{1}{\sqrt{2\pi} \sigma_{K_1}} \exp\left(-\frac{0.5 (K_{1,i(GW)} - K_1)^2}{\sigma_{K_1}}\right) dK_1.$$  

The final reduced 95 percentiles in $m_2$ are shown in black lines for J0651 in left panel, and the same is shown for the high mass binary in the right panel. Observe that the uncertainties in $m_2$ calculated in this way for lower inclinations is the similar to those at the higher inclinations. Thus, the advantage of folding in $K_1$ measurement is especially useful for lowering inclination systems with S/N
Fig. 7.— Scenario 2b: Constraints on the secondary mass and distance from combining single spectroscopic EM data: $m_1$, and $K_1$ with GW data on $A$, $\iota$ for J0651 (black) and the high mass binary (in grey).

Fig. 8.— Relation between inclination from GW observation vs. that from EM observations. Left: 2-d error ellipse from the GW data analysis in amplitude and $\cos \iota$ for J0651 with $\iota = 45^\circ$. Right: Relation between inclination from the left panel and inclination from Eq. [4].

$\sim 10$, where large GW uncertainties $A$ influence the constraints of the physical parameters in question. Furthermore, the constraints in $m_2$ can also be compared with the previous case in Figure [?], where we find that for the single-lined spectroscopic binary, knowing its distance to 10% significantly improves knowledge of the secondary mass at lower inclinations.

The key point in Scenarios 2b and 2c is that not all the [$A$, $\iota$ (GW)] pairs are consistent with the EM observations. Therefore both constraints on the GW data and other parameters also constrain the GW error ellipses. The 2-$\sigma$ uncertainties in the GW amplitude and GW inclination for these scenarios are shown in the Appendix.

3.3. Scenario 3: EM data on $m_1$, $K_1$, $m_2 \& K_2$

In this section we consider EM observations of a double-lined spectroscopic binary which translates to a set of measurements in the mass and radial velocity for each of the components: $m_1$, $K_1$, and $m_2$, $K_2$. Given the two masses and GW measurement on the amplitude we can immediately compute a preliminary distance. Additionally, we can also derive two sets of inclinations independently from the individual radial velocities and the masses, $\iota_{K_1}, \iota_{K_2}$ from Eq. [4]. These inclinations can be compared with the one measured from GW data, $\iota$ (GW). At lower inclinations, large uncertainties in $\iota$ (GW) essentially imply that those systems’ inclinations are undetermined and this also affects the amplitude due to the strong correlation between them. Thus, the independent estimates of $\iota_{K_1}, \iota_{K_2}$ from the EM data can be useful in constraining the GW amplitude. This reduced amplitude will further constrain the distance which is shown in the third panel in Figure [10]. In the figure both the observed $K_1, K_2$ are shown in the left panel in thick and thin black lines respectively. The inclination and the distance given the GW amplitude and both the masses are shown in grey line in the middle and right panels respectively. Observe that a 10% fractional error in each $K_1$ and $K_2$ translate into similar uncertainties of the distance and thus in the following figures we show the constraints from using $K_1$ data only. The constrained distances estimated in this way as a function of inclination is shown in
**Fig. 9.**—Scenario 2c: Same scenario as in Figure 7 with an additional EM measurement on the distance. Left: $2 - \sigma$ uncertainties for the secondary mass for J0651 where the grey colored lines are constraints from EM information on $m_1$, $d$ and GW $\alpha$. These reduce to the tighter constraints shown in black lines when EM data on $K_1$ is also used (see text). The dashed line in red is the real value of $m_2$. Right: Same for the high mass binary.

**Fig. 10.**—Scenario 3: Left: Distributions of the radial velocities from EM data for J0651 masses with the binary orientated at $i = 45^\circ$. Middle: Given EM data on $m_1, m_2$, and the corresponding $K_1, K_2$, the inclinations are calculate using Eq. 4 which is compared with inclination from GW data. Right: Constraints on the distance obtained solving Eq. 1 with: EM data on $m_1$, $K_1$, $m_2$, $K_2$ and GW data on $\alpha$, $i$.

Figure 11 for J0651 in left panel (also in black) and for the high mass binary in the right panel (in black). The grey lines in both the panels are the 95 percentiles in $d$ using only the masses from the EM data and the GW amplitude. Observe that at lower inclinations knowing masses and a radial velocity can improve the constraint in distances significantly. The uncertainties are smaller for J0651 at lowest inclinations because the relative 10% uncertainties in the $K_1$ have lower absolute uncertainties that propagate into the uncertainties of the distance.

Note that typically in practice EM data provides measurements of both the masses and only one of the radial velocities with $\sim 10\%$ precision. From the radial velocity formulation we have the relation: relation $m_1/m_2 = K_2/K_1$ which can be used to compute the remaining $K_i$. This provides a consistency check between EM and GW data. The EM data can be used to derive inclination measured from the radial velocities, $i[RV]$ which can be verified against the $i[GW]$ as shown in the middle panel in Figure 10.

**4. CONCLUSIONS**

We have quantified the possible constraints/improvements in the physical parameters of the white-dwarf (WD) binaries that are observable by the eLISA detector in the future when combined with the EM data. We do this for the binary parameters that are astrophysically interesting (masses and distance).

1. **GW data only**: Assuming a double white-dwarf system this scenario somewhat constrains the distance.

2. **Scenario 1**: GW data + distance $d$: This scenario constrains the chirp mass $M_c$.

3. **Scenario 2a**: GW data + primary mass $m_1$: This
Fig. 11.— Scenario 3: Same as in Figure 10 but for all inclinations for J0651 in left panel and the for the high mass binary in right panel. The constraints in black lines are from using $i[K_1]$ and the constraints in grey are from using $i[GW]$. The dashed line (in black) is the real values of the $d$.

Fig. 12.— Comparison of normalized CDFs in $M_c$, $m_2$, and $d$ for all the scenarios above for J0651 with $i = 25^\circ$. The vertical lines (in yellow) in all the panels are the true values of the parameters. The solid curves in grey are CDFs for the parameters when only GW data is available. Curves in dash-dotted lines are constraints for Scenario 1 (known distance $d$), dashed curves are for Scenario 2a (known primary mass $m_1$), solid curves are for Scenario 2b (known primary mass $m_1$ and radial velocity $K_1$), dotted curves are for Scenario 2c (known $m_1$, $K_1$ and $d$) and thin-dashed lines are for Scenario 3 (known both masses $m_1$, $m_2$ and radial velocities $K_1$, $K_2$).

Fig. 13.— Same as in Figure 12 for $i = 85^\circ$. 
scenario constrains the chirp mass and the distance.

4. *Scenario 2b*: GW data + single-lined spectroscopic binary i.e. \( m_1, K_1 \): This scenario constrains the secondary mass \( m_2 \) and the distance.

5. *Scenario 2c*: GW data + single-lined spectroscopic binary + \( d \): This scenario also constrains the secondary mass \( m_2 \).

6. *Scenario 3*: GW data + double-lined spectroscopic binary i.e. \( m_1, K_1 \) and \( m_2, K_2 \): This scenario constrains the distance.

All the \( 1 - \sigma \) EM accuracies are taken to be 10% of the real/measured values which is inspired by several EM observations. We compare below the constraints in the physical parameters of interest: secondary mass \( m_2 \), chirp mass \( M_c \) and the distance \( d \) as a function of the scenarios depending on the EM information available. Since the GW parameter uncertainties are significantly different for a low inclination (face-on) orientation than for a high inclination (edge-on) orientation, we do the comparison for a non-eclipsing J0651 with \( \iota = 25^\circ \) and an almost eclipsing J0651 with \( \iota = 85^\circ \) in Figures 12 and 13 respectively and conclude the following:

1. *Constraints on chirp mass, \( M_c \):* In the left panels of Figures 12 and 13 EM data on \( d \) constrains the 95 percentile of the system’s chirp mass (dash-dotted line) to \( 0.38^{+0.11}_{-0.09} M_\odot \) and \( 0.32^{+0.05}_{-0.03} M_\odot \) for face-on and eclipsing J0651 respectively. EM data on \( m_1 \) constrains the \( M_c \) (in thick-dashed line) to \( 0.36^{+0.13}_{-0.21} M_\odot \) which does not depend on the inclination. The normalized cumulative distributions (CDF) of the constraints on the distance are compared to that from GW data only which is shown in the grey line in both panels.

2. *Constraints on secondary mass, \( m_2 \):* In the middle panels of Figures 12 and 13 EM data on the \( m_1, K_1 \) constrain the 95 percentile of secondary mass, \( m_2 \) to \( 0.19^{+0.07}_{-0.07} M_\odot \) and \( 0.55^{+0.22}_{-0.13} M_\odot \) for face-on and eclipsing J0651 respectively (shown in solid lines). The set of data complemented with the distance further constrain the 95 percentile in \( m_2 \) with \( 0.55^{+0.18}_{-0.13} M_\odot \) and \( 0.55^{+0.16}_{-0.13} M_\odot \) for face-on and eclipsing J0651 respectively (shown in dotted lines). For comparison, the CDF of \( m_2 \) using only the GW data is shown in grey.

3. *Constraints on distance, \( d \):* In the right panels, of Figure 12 and 13 EM data on \( m_1 \) constrains the distance to \( 0.91^{+0.98}_{-0.98} \) kpc and \( 1.25^{+0.95}_{-0.95} \) kpc for face-on and eclipsing J0651 respectively (in thick-dashed lines). EM data on the \( m_1, K_1 \) constrain the 95 percentile in \( d \) with \( 0.32^{+0.17}_{-0.16} \) kpc and with \( 0.99^{+0.49}_{-0.35} \) kpc accuracy for face-on and eclipsing J0651 respectively (in solid lines). EM data on \( m_1, m_2, K_1 \) and \( K_2 \) constrain the 95 percentile in \( d \) to \( 0.96^{+0.29}_{-0.24} \) kpc and \( 1.01^{+0.29}_{-0.26} \) kpc for face-on and eclipsing J0651 respectively (in thin-dashed line). For comparison, the CDF of \( d \) using only the GW data and the assumption that the masses are WDs is shown in grey.

Thus, knowing distance and/or radial velocity of the primary component can significantly improve our knowledge of the binary system. These constraints change as a function of inclination of the binary that is shown in previous sections. In a forthcoming paper we will address the effect on these improvements by including the (possible) EM measurement of rate of change of the orbital period.

This work was supported by funding from FOM. We are very grateful to Michele Vallisneri for providing support with the *Synthetic LISA* and *Lisasolve* softwares.

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**APPENDIX**

VARIANCE-COVARIANCE MATRIXES OF J0651

We have listed the VCM matrices for the J0651 system with eclipsing and non-eclipsing configurations in our analysis. There are 7 parameters that describe them which are listed in the first row of the matrices below and for each binary, the values are listed in the row with \( \theta_i \). The diagonal elements are the absolute uncertainties in each the 7 parameters and the off-diagonal elements are the normalized correlations, i.e. \( c_{ii} = \sqrt{C_{ii}} \equiv \sigma_i \), \( c_{ij} = \frac{c_{ij}}{\sqrt{C_{ii}C_{jj}}} \). The strong correlations between parameters (i.e. whose magnitudes are \( \geq 0.9 \)) are marked in bold in the VCMs below. These correlations have been explained in Paper I.
VCM 1: Eclipsing J0651 ($i = 5^\circ$), S/N = 10.5.

\[
\begin{array}{cccccccc}
\theta_i & \mathcal{A} & \phi_0 & \cos i & f & \psi & \sin \beta & \lambda \\
\hline
\mathcal{A} & 0.08 \times 10^{-23} & - & 0.0 & 0.01 & 0.02 & 0.03 & -0.06 \\
\phi_0 & 0.907 & -0.01 & 0.01 & 0.01 & 0.11 & 0.08 & \\
\cos i & 0.707 & -0.01 & 0.01 & 0.07 & -0.33 & \\
f & 0.91 & 0.035 & -0.02 & 0.05 & \\
\psi & 2.982 \times 10^{-10} & -0.01 & -0.08 & -0.15 & \\
\sin \beta & 0.059 & 0.08 & \\
\lambda & 2.10 & 0.017 & \\
\end{array}
\]

VCM 2: Not-eclipsing J0651 ($i = 45^\circ$), S/N = 24.

\[
\begin{array}{cccccccc}
\theta_i & \mathcal{A} & \phi_0 & \cos i & f & \psi & \sin \beta & \lambda \\
\hline
\mathcal{A} & 3.86 \times 10^{-23} & 0.03 & -0.98 & -0.02 & 0.03 & -0.13 & 0.35 \\
\phi_0 & 0.739 & -0.03 & -0.19 & 0.16 & 0.15 & 0.10 & \\
\cos i & 0.19 & 0.02 & -0.01 & 0.13 & -0.36 & \\
f & 1.688 \times 10^{-9} & -0.98 & -0.06 & -0.21 & \\
\psi & 0.36 & 0.13 & 0.07 & \\
\sin \beta & 0.031 & -0.13 & \\
\lambda & 2.10 & 0.009 & \\
\end{array}
\]

CO NSTRAINTS IN $\mathcal{A}$ AND $i$ OF J0651

Figure 14 shows how the error ellipses of amplitude and inclination from GW observations reduce using EM observations for the different scenarios that we have described in Sects. 1 and 2. Knowing one of the masses (Scenario 2a) from the EM does not constrain the [\mathcal{A}, i] any more than the GW data alone. In other words the $m_2$ and $d$ are free parameters to satisfy the amplitude. The 95 percentiles in the amplitude are shown in grey in the figure which are the same as the case where we have GW data only. In fact these constraints in the amplitude decrease as a function of inclination as expected from the GW measurements (see Figure 2). Adding an EM measurement of the measured mass’s radial velocity (Scenario 2b) can constrain the [\mathcal{A}, i] slightly or significantly depending on inclination of the system which are shown in thick black lines. Finally complementing the mass and radial velocity of the brighter companion with the distance to the binary (Scenario 2c) significantly constrains the [\mathcal{A}, i] which is strongest for the lower inclinations as shown in the figure in thin black lines. Observe that EM information provide strongest improvements for low inclination systems where GW uncertainties in the amplitude and the inclination are very large.

THE DISTRIBUTION OF $\mathcal{A}$ AND $i$ AT LOWER INCLINATIONS

Here we show that while Fisher method gives an estimate on the parameter uncertainties and correlation between them without following the posterior in detail, it gives a reasonable estimate of the above quantities as long as the priors in the parameters are rectangular (i.e. not Gaussian) and are large enough to preserve the overall orientation of the posterior. We compute an estimate of the likelihood with a simple $\chi^2$ procedure on a 2D parameter distribution of $\mathcal{A}$, and $\cos i$, where the $\chi^2 = (1/(N-1))\sum_{i=0,\lambda}(h_0(t) - (h[i,j](t) + n(t))^2$, $h_0 = $ true signal, $h[i,j] = $ signal at a grid point and $n$ is a noise realisation, $N =$ total time samples. For an evenly placed parameters in a $10 \times 10$ grid, we take the average $\chi^2$ computed for 10 different noise realisations. Figure 15 shows the colored contours of 2D $\chi^2$ distribution for the case of $i = 65^\circ$ (in the left-panel) where the Fisher uncertainties are well within the physically allowed bounds. The over-plotted contour in thick solid line is $1 - \sigma$ uncertainty ellipse computed from Fisher matrix about the true values of $\mathcal{A}$ and $\cos i$ labelled with the white circle. This just shows that the $\chi^2$ distribution follows the shape and the slope of the Fisher distribution roughly, but not exactly as expected. The same is shown for $i = 10^\circ$ in the right-panel where the uncertainties hit the physical bounds and both the methods show a sharp cut-off at $\cos i = 1$. Here we see that again the Fisher uncertainties and correlation roughly follow that of the $\chi^2$, but with truncations at the boundaries. The deviation in the top-right is discussed in [Shah et al. (2012)]. It was argued that although the results of Fisher-based uncertainties imply that the $i = 5^\circ$ system is very similar to $i = 90^\circ$, this is unlikely because of the anti-correlation between $\mathcal{A}$ and $\cos i$ at high inclinations. At $i \gtrsim 45^\circ$ correlation between $\mathcal{A}$ and $\cos i$ decreases $i$.
and the high accuracy in the inclination itself actually suffices to distinguish the higher inclination systems. Thus we expect that that $\chi^2$ deviates from the Fisher estimate towards the top-right region of the Figure 15 in the right-panel.

Fig. 14.— 95 percentiles in the GW amplitude as a function of inclination for various sets of EM information as labeled. The real value for the amplitude is shown in black dashed-line.

Fig. 15.— Filled contour plot of the 2D $\chi^2$ averaged over 10 noise realisations for an evenly distributed grid of $A$ and $\cos \iota$ compared with the $1 - \sigma$ error ellipse (shown in thick solid line) from Fisher matrix $\iota = 65^\circ$ in the left-panel and $\iota = 10^\circ$ in the right-panel. The $\chi^2$ values are represented by the darker to lighter colors corresponding to lower to higher values in $\chi^2$. 