

On an independence result in the theory of lawless sequences

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The open data axiom LS3 for lawless sequences is actually an infinite list of axiom schemata: for each n we have

$$\text{LS3}(n): A(\alpha_1, \dots, \alpha_n) \wedge \bigwedge_{i < j \leq n} \alpha_i \neq \alpha_j \rightarrow \exists u_1 \ni \alpha_1 \dots \exists u_n \ni \alpha_n \\ \forall \beta_1 \in u_1 \dots \forall \beta_n \in u_n (\bigwedge_{i < j \leq n} \beta_i \neq \beta_j \rightarrow A(\beta_1, \dots, \beta_n));$$

here α_i, β_i range over lawless sequences, and the u_i range over finite sequences; ' $\alpha \in u$ ' stands for ' α has initial segment u '. In [D] it was shown that LS3(1) does not imply LS3(2) by using Cohen generic sequences. In [DL], this method was used to show that LS3(2) does not imply LS3(3).

The aim of this note is to give simple proofs of these facts, by using the models described in [HM]. Our method also shows that LS3(3) does not imply LS3(4), but we have not been able to prove a similar independence result for larger n . For $n \geq 4$ a different approach seems necessary for showing $\text{LS3}(n) \not\rightarrow \text{LS3}(n+1)$.

We observe here that the models described below all satisfy the axioms LS1 (decidable equality) and LS2 (density), and that the models which show that $\text{LS3}(n) \not\rightarrow \text{LS3}(n+1)$, $n = 1, 2, 3$, also yield the corresponding result $\text{LS4}(n) \not\rightarrow \text{LS4}(n+1)$ for the continuity axiom LS4. Thus we obtain

THEOREM. *For $n = 1, 2, 3$, there exists a model satisfying LS1, LS2, LS3(n), LS4(n), but neither satisfying LS3($n+1$), nor LS4($n+1$).*

This note is far from selfcontained. All unexplained notation is as in [HM], and we assume familiarity with the LS-model described in section 5.2. of [HM]. As in [DL], it will be notationally more convenient to consider only 0–1-sequences.

THE FIRST MODEL

Let M be the monoid of finite sequences of 0's and 1's, with the operation given by

$$u/v = v \text{ with the initial segment replaced by } u,$$

that is,

$$(u/v)_n = \begin{cases} u(n), & \text{if } n < \text{lth}(u) \\ v(n), & \text{if } \text{lth}(u) \leq n < \text{lth}(v) \end{cases}$$

M may be regarded as a submonoid of the monoid $Cts(C, C)$ of continuous functions from Cantor Space to itself, by identifying u with the function $x \rightarrow u/x$ in $Cts(C, C)$. In sheaves over this monoid M equipped with the open cover topology, the internal exponent $2^{\mathbb{N}}$ appears as $Cts(C, C)$ (with restrictions given by right composition), and it was shown in [HM] (section 2.3) that if we interpret the domain of lawless sequences as the subsheaf of $Cts(C, C)$ generated by M , open data and continuity in a single parameter (LS3(1), LS4(1)) hold.

On the other hand, open data and continuity in two lawless parameters cannot hold, as follows easily from the observation that in this model

$$\Vdash \forall \alpha, \beta \exists n \forall m \geq n \alpha(m) = \beta(m).$$

To see this, choose two elements u and v of M , and let n be the maximum of $\text{lth}(u)$ and $\text{lth}(v)$. Then if $w \in M$ and $m \geq n$, we find for any $x \in C$

$$u/w/x(m) = v/w/x(m) = \begin{cases} w(m), & m < \text{lth}(w) \\ x(m), & \text{otherwise.} \end{cases}$$

So $\Vdash u/w(m) = v/w(m)$. Thus $\Vdash \forall m \geq n \ u(m) = v(m)$. This proves

PROPOSITION 1. *There is a model for lawless sequences satisfying (decidable equality, density, and) LS3(1), LS4(1), but not LS3(2), LS4(2).*

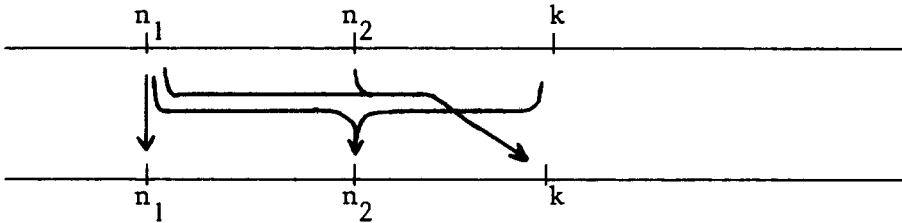
THE SECOND MODEL

We will now describe a model for LS3(2), LS4(2), in which LS3(3), LS4(3) do not hold. As in [HM] (section 5.2), we define a space T and a group G of automorphisms of T . Let $(v_n)_n$ be an enumeration of $2^{<\mathbb{N}}$ in which each sequence occurs infinitely many times. For $u \in 2^{<\mathbb{N}}$, let V_u denote the canonical basic open subset of C (Cantor space) determined by u . Now let $T = \prod_n V_{v_n}$. (Observe that T is homeomorphic to C .)

We consider three types of homeomorphisms from T to itself. The first two types (1) and (2) are defined as in [HM], section 5.2 (but with Baire space replaced by Cantor space). In addition, we have a third type

- (3) for each triple n_1, n_2, k of distinct natural numbers with $v_{n_1} + v_{n_2} = v_k$ (+ denotes pointwise addition modulo 2), a homeomorphism $h[n_1, n_2, k]$ defined by

$$\begin{aligned} h(x)_k &= x_{n_1} + x_{n_2} \\ h(x)_{n_2} &= x_k + x_{n_1} \\ h(x)_m &= x_m \text{ for all other } m. \end{aligned}$$



Thus, we have

$$\pi_k \circ h = \pi_{n_1} + \pi_{n_2}, \pi_{n_2} \circ h = \pi_{n_1} + \pi_k, \text{ and } \pi_m \circ h = \pi_m \text{ for } m \in \mathbb{N} \setminus \{k, n_2\}.$$

Now let G be the subgroup of the group of automorphisms of T generated by all homeomorphisms of types (1), (2), (3).

Our interpretation will be the standard interpretation in sheaves over T with a G -action (as in [HM], section 5.2.). The sort L of lawless 0 – 1-sequences will be interpreted by the subsheaf (of internal Cantor space) generated by all functions $T \rightarrow C$ which are of the form $\pi_{n_1} + \dots + \pi_{n_p}$, for a set of p distinct natural numbers $\{n_1, \dots, n_p\}$.

Note that in this model the axioms of decidable equality and density for L are satisfied. We will now show that LS3(2) is also satisfied in this model. For this, we need two lemmas.

LEMMA 2. *If $\{n_1, \dots, n_p\}$ and $\{m_1, \dots, m_q\}$ are two distinct sets of natural numbers, listed without repetitions, then there are numbers i and j , $i \neq j$, and a homeomorphism $h \in G$ such that*

$$\pi_i \circ h = \pi_{n_1} + \dots + \pi_{n_p}, \pi_j \circ h = \pi_{m_1} + \dots + \pi_{m_q}.$$

PROOF. If $p = 1$, we can find a composition h of homeomorphisms of type (3) which leave π_{n_1} unchanged and add $\pi_{m_1}, \dots, \pi_{m_q}$; i.e. for some $k \in \mathbb{N}$,

$$\begin{aligned} \pi_{n_1} \circ h &= \pi_{n_1}, \pi_k \circ h = \pi_{m_1} + \dots + \pi_{m_q} \\ (\text{and } \pi_l \circ h &= \pi_l, \text{ all } l \in \mathbb{N} \setminus \{k, m_1, \dots, m_q\}). \end{aligned}$$

If $p \neq 1$, first find an h which reduces $\pi_{m_1} + \dots + \pi_{m_q}$ to a single projection, i.e.

$$\pi_k \circ h = \pi_{m_1} + \dots + \pi_{m_q}.$$

Then apply the case $p = 1$ to the pair of (distinct!) sets $\{k\}$, $\{l_1, \dots, l_r\}$, where l_1, \dots, l_r are such that $(\pi_{l_1} + \dots + \pi_{l_r}) \circ h = \pi_{n_1} + \dots + \pi_{n_p}$. \square

LEMMA 3. Let $A(\alpha_1, \dots, \alpha_p)$ be a formula with variables $\alpha_1, \dots, \alpha_p$ of sort L , and all other variables lawlike. Let $U = \pi_0^{-1}(V_{u_0}) \cap \dots \cap \pi_k^{-1}(V_{u_k})$ be a basic open in T with $U \subseteq [A(\pi_{n_1}, \dots, \pi_{n_p})]$, where n_1, \dots, n_p are p mutually distinct natural numbers, and $k \geq n_p$. Then for each p -tuple mutually distinct numbers m_1, \dots, m_p ,

$$\pi_{m_1}^{-1}(V_{k_{n_1}}) \cap \dots \cap \pi_{m_p}^{-1}(V_{k_{n_p}}) \subseteq [A(\pi_{m_1}, \dots, \pi_{m_p})].$$

PROOF. As lemma 5.2.4. in [HM]. \square

COROLLARY 4. In the model described above, LS3(2) holds, but LS3(3) does not hold.

PROOF. It is clear that LS3(3) cannot hold, since we can find three lawless sequences α, β, γ in the model such that $\Vdash \alpha + \beta = \gamma$. LS3(2) does hold, i.e.

$$\begin{aligned} & \Vdash \forall \alpha_1, \alpha_2 (\alpha_1 \neq \alpha_2 \wedge A(\alpha_1, \alpha_2) \rightarrow \exists u_1 \ni \alpha_1 \exists u_2 \ni \alpha_2 \\ & \quad \forall \beta_1 \in u_1 \forall \beta_2 \in u_2 (\beta_1 \neq \beta_2 \rightarrow A(\beta_1, \beta_2))). \end{aligned}$$

To see this, choose two distinct sections α_1, α_2 of the sheaf L . We may assume that α_1, α_2 are global sections of the form $\alpha_1 = \pi_i, \alpha_2 = \pi_j, i \neq j$, since such sections generate (by lemma 2). Now suppose $x \in [A(\pi_i, \pi_j)]$. By lemma 3 we can find finite sequences $u_1 \ni x_i, u_2 \ni x_j$ such that

$$(1) \quad \pi_i^{-1}(V_{u_1}) \cap \pi_j^{-1}(V_{u_2}) \subseteq [A(\pi_i, \pi_j)],$$

i.e.

$$[\pi_i \in u_1 \wedge \pi_j \in u_2 \rightarrow A(\pi_i, \pi_j)] = T.$$

But then it holds that

$$[\forall \beta_1 \in u_1 \forall \beta_2 \in u_2 (\beta_1 \neq \beta_2 \rightarrow A(\beta_1, \beta_2))] = T.$$

For if $\{n_1, \dots, n_p\}$ and $\{m_1, \dots, m_q\}$ are distinct, we can find i', j' ($i' \neq j'$) and an $h \in G$ such that

$$(2) \quad \pi_{i'} \circ h = \pi_{n_1} + \dots + \pi_{n_p}, \pi_{j'} \circ h = \pi_{m_1} + \dots + \pi_{m_q}$$

(lemma 2), and by lemma 3, (1) implies that

$$(3) \quad [\pi_{i'} \in u_1 \wedge \pi_{j'} \in u_2 \rightarrow A(\pi_{i'}, \pi_{j'})] = T.$$

Hence also

$$\begin{aligned} & [\pi_{n_1} + \dots + \pi_{n_p} \in u_1 \wedge \pi_{m_1} + \dots + \pi_{m_q} \in u_2 \rightarrow A(\pi_{n_1} + \dots \\ & \quad \dots + \pi_{n_p}, \pi_{m_1} + \dots + \pi_{m_q})] \\ & = [\pi_{i'} \circ h \in u_1 (= u_1 \circ h) \wedge \pi_{j'} \circ h \in u_2 \rightarrow A(\pi_{i'} \circ h, \pi_{j'} \circ h)] \\ & = h^{-1} [\pi_{i'} \in u_1 \wedge \pi_{j'} \in u_2 \rightarrow A(\pi_{i'}, \pi_{j'})] = T. \quad \square \end{aligned}$$

Summarizing, we have first shown that ‘singleton projections’ generate (lemma

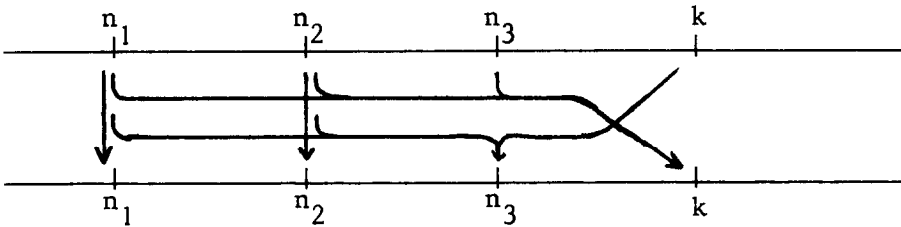
2). This enabled us to prove LS3(2) just as the full LS3 is proved in [HM] (lemma 3, corollary 4). In a similar way, we can show that LS4(2) holds in this model. LS4(3), however, cannot hold, since for three lawless sequences $\alpha_1, \alpha_2, \alpha_3$, it cannot be decided on the basis of initial segments whether $\alpha_3 = \alpha_1 + \alpha_2$, or not. This shows that we have obtained the following

PROPOSITION 5. *There is a model for lawless sequences in which LS3(2) and LS4(2) hold, but LS3(3) and LS4(3) do not hold.*

THE THIRD MODEL

A slight modification of the model just described suffices to obtain a model for LS3(3), LS4(3) which is not a model for LS3(4), LS4(4). The space T remains the same, but the definition of the group G is different. Besides the homeomorphisms of types (1) and (2) from [HM], we now take as a third type all homeomorphisms of the form $h[n_1, n_2, n_3, k]$, where n_1, n_2, n_3, k are distinct natural numbers such that $v_k = v_{n_1} + v_{n_2} + v_{n_3}$. $h = h[n_1, n_2, n_3, k]$ is defined by

$$\begin{aligned} h(x)_k &= x_{n_1} + x_{n_2} + x_{n_3} \\ h(x)_{n_3} &= x_{n_1} + x_{n_2} + x_k \\ h(x)_m &= x_m \text{ for all other } m. \end{aligned}$$



The interpretation is again the standard interpretation in sheaves over T with G -action, but now L is the sheaf generated by the global elements $\pi_{n_1} + \dots + \pi_{n_p}$, for $\{n_1, \dots, n_p\}$, a set of p distinct natural numbers, and p is odd. Observe that this sheaf is closed under the action of G (i.e. right composition with elements of G preserves ‘oddness’).

In this model, LS3(4) does not hold, since we can find four distinct lawless sequences $\alpha, \beta, \gamma, \delta$ such that $\Vdash \alpha + \beta + \gamma = \delta$. Similarly, LS4(4) does not hold, since we cannot continuously decide whether $\alpha + \beta + \gamma = \delta$ or not. LS3(3) and LS4(3), however, do hold. This is proved as for LS3(2) and LS4(2) in the second model described above, but one has to be slightly more careful now in showing that ‘singleton-projections’ generate for triples, i.e.

LEMMA 6. *Let $\{n_1, \dots, n_p\}, \{m_1, \dots, m_q\}, \{l_1, \dots, l_r\}$ be distinct sets of natural numbers, listed without repetitions, and with p, q, r odd. Then there exists a homeomorphism $h \in G$ and (distinct) coordinates i, j, k such that*

$$\pi_i \circ h = \pi_{n_1} + \dots + \pi_{n_p}, \quad \pi_j \circ h = \pi_{m_1} + \dots + \pi_{m_q}, \quad \pi_k \circ h = \pi_{l_1} + \dots + \pi_{l_r}.$$

SKETCH OF PROOF. As in lemma 2, the general case is easily reduced to the case $p = 1$. Thus, we have three distinct sets

$$\{n\}, \{m_1, \dots, m_q\}, \{l_1, \dots, l_r\}.$$

And again, not bothering about the coordinates l_1, \dots, l_r , but keeping n invariant, we may as well assume that $q = 1$; i.e. we find a composition h of homeomorphisms of type (3) such that

$$\begin{aligned} \pi_n \circ h &= \pi_n \\ \pi_m \circ h &= \pi_{m_1} + \dots + \pi_{m_q} \text{ for some } m, \\ \text{while } \pi_{l_1} + \dots + \pi_{l_r} &= (\pi_{s_1} + \dots + \pi_{s_{r'}}) \circ h, \\ &\text{for some coordinates } s_1, \dots, s_{r'}, \text{ where } r' \text{ is still odd.} \end{aligned}$$

Thus, we have three distinct sets of the form

$$\{n\}, \{m\}, \{s_1, \dots, s_{r'}\}.$$

If $r' = 1$, we are done. Otherwise, $r' \geq 3$, so the third set contains an element which is distinct both from m and from n . But in this case it is not difficult to see that we can find a composition h' of homeomorphisms of type (3) which reduces $\pi_{s_1} + \dots + \pi_{s_{r'}}$ to a single projection, but leaves the coordinates n and m invariant. \square

We have now obtained the following proposition:

PROPOSITION 7. *There is a model for lawless sequences in which LS3(3) and LS4(3) hold, but LS3(4) and LS4(4) do not hold.*

THE PROBLEM WITH MORE PARAMETERS

It is perhaps useful to indicate why this approach does not work in the case of more parameters. To obtain a model for LS3(4) $\not\sim$ LS3(5) in a similar way, one is inclined to put sums of four lawless sequences in the sheaf L (to falsify LS3(5)), and to add homeomorphisms to G which reduce such sums to single projections. However, if one puts such sums in L , one has to do so homogeneously (in order to obtain open data in four parameters $\alpha_1, \dots, \alpha_4$ for the formula $\exists \delta (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \delta)$). But then one finds four-tuples of lawless sequences projected from the sets $\{n_1, n_2, n_3, n_4\}$, $\{n_3, n_4, n_5, n_6\}$, $\{n_5, n_6, n_7, n_8\}$, $\{n_1, n_2, n_7, n_8\}$, for example, i.e.

$$\exists \alpha_1 \exists \alpha_2 \exists \alpha_3 \exists \alpha_4 (\bigwedge_{1 \leq i < j \leq 4} \alpha_i \neq \alpha_j \wedge \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0)$$

will hold in the model. This clearly contradicts LS3(4).

The reason why the proof of lemma 6 above fails in this case lies in the fact that the analog of 'preservation of oddness' for the elements of G does not hold: if one adds sums of four tuples, one may find a sum $\pi_{n_1} + \pi_{n_2} + \pi_{n_3} + \pi_{n_4}$, where $\pi_{n_4} \circ h$ is the sum of, say, $\pi_{n_1}, \pi_{n_2}, \pi_{n_5}, \pi_{n_6}$, for some $h \in G$, which leaves the coordinates n_1, n_2, n_3 unchanged. Hence one has also added sums of three coordinates, since $(\pi_{n_1} + \pi_{n_2} + \pi_{n_3} + \pi_{n_4}) \circ h = \pi_{n_1} + \pi_{n_2} + \pi_{n_3} + \pi_{n_1} + \pi_{n_2} + \pi_{n_5} +$

$+ \pi_{n_6} = \pi_{n_3} + \pi_{n_5} + \pi_{n_6}$. Therefore, in trying to prove the analog of lemma 6 one may, after having reduced three out of the four sets to singletons, end up with four sets which look like

$$\{n\}, \{m\}, \{k\}, \{n, m, k\}.$$

In fact, such a situation *must* occur if one starts with the four sets $\{n_1, n_2, n_3, n_4\}$, $\{n_3, n_4, n_5, n_6\}$, $\{n_5, n_6, n_7, n_8\}$, $\{n_1, n_2, n_7, n_8\}$ considered above, since the relation $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$ must be preserved by the action of G on the elements of the sheaf L .

This suggests that for the case of more parameters, a totally different approach is needed.

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