Prodiscrete groups and Galois toposes

by Ieke Moerdijk

Mathematisch Instituut, Rijksuniversiteit Utrecht, 3508 TA Utrecht, the Netherlands

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A prodiscrete group is usually taken to be a topological group G isomorphic to the limit of a filtered projective system $\{G_i\}$ of discrete groups. Even if all the homomorphisms $G_i \to G_j$ in the system are surjections, G may turn out to be the trivial group (cf. 1.3 below), and this is the reason why the fundamental group of a scheme (or more generally, of a topos) has to be constructed as a *progroup*, i.e. a formal inverse system $\{G_i\}_i$ of discrete groups (e.g. Artin, Mazur (1969)). (Another well-known way out is to restrict one self to finite coverings only, and construct the fundamental group as a profinite topological group.)

The aim of this note is to point out that this kind of pathology does not occur if one computes the inverse limit in a category of generalized spaces, namely locales. In this sense, this is another illustration of the familiar fact that the category of locales, which contains the sober topological spaces as a full subcategory, is much better behaved than the category of topological spaces. Other striking examples of this phenomenon are, for instance, that the product of paracompact locales is again paracompact (Isbell (1972)), that the product of compact locales is compact, even in the absence of the axiom of choice (Johnstone (1981)), and that every connected and locally connected locale is path-connected (Moerdijk, Wraith (1986)). All these properties fail to hold for topological spaces.

In section 1, it will be shown that by actually taking the inverse limit, one obtains an equivalence between progroups with all homomorphisms surjective

("surjective progroups"), and prodiscrete localic groups (i.e. groupobjects in the category of locales).

There is still another kind of generalized space where one can faithfully represent such surjective progroups by actually taking the inverse limit, namely Grothendieck toposes. To any localic group one may associate the topos BG of right G-sets. To a progroup $\underset{\sim}{G} = \{G_i\}_i$ one may then associate the pointed Grothendieck topos $B\underset{\sim}{G} \underset{\mathrm{def}}{=} \varprojlim BG_i$, and this again gives an equivalence of categories:

$$\underline{\mathrm{Hom}}(\underset{\sim}{G}, \underset{\sim}{H}) \simeq \underline{\mathrm{Hom}}(B\underset{\sim}{G}, B\underset{\sim}{H})$$

for surjective progroups $\underset{\sim}{G}$ and $\underset{\sim}{H}$.

To prove this latter result, we will use two properties of the construction $G \longmapsto BG$ which are of general interest, namely

(A) Let $\mathscr{F} \xrightarrow{p} \mathscr{E}$ be a geometric morphism, and let G be an open localic group in $\mathscr{E}$. Then

$$B(\mathscr{E}, G) \times \mathscr{F} \simeq B(\mathscr{F}, p^{\#}G),$$

where $B(\mathscr{E}, G)$ denotes the topos of $\mathscr{E}$-objects with a continuous G-action, etc., see 2.3 below, and

(B) Let $\{G_i\}_i$ be a filtered inverse system of localic groups, and suppose each homomorphism $G_i \to G_j$ is an open surjection. Then

$$B(\varprojlim G_i) \simeq \varprojlim B(G_i),$$

see 2.4 below.

Standard Galois theory will provide a characterization of those toposes of the form BG, G a prodiscrete localic group, given in section 3.

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§ 1. PROGROUPS AS LOCALIC GROUPS

1.1. *Preliminaries on locales.* Our terminology concerning spaces, locales, etc. will be as in Johnstone (1982). So a *frame* is a complete Heyting algebra, and a morphism of frames is a function which preserves finite meets and arbitrary sups. *Locales* are the duals of frames; for a locale X, $\mathscr{O}(X)$ denotes the corresponding frame, the elements of which are called the opens of X. So a *map of locales*, or a *continuous map* $X \xrightarrow{f} Y$ is by definition a frame morphism $\mathscr{O}(Y) \xrightarrow{f^{-1}} \mathscr{O}(X)$. Every topological space T gives rise to a locale iT

defined by $\mathcal{O}(iT)=\mathcal{O}(T)$; i is a full embedding of sober topological spaces into locales.

We will assume that the reader is familiar with the basic facts about locales, such as e.g. presented in Joyal & Tierney (1984), chapters I–V (henceforth referred to as [JT] – note that Joyal and Tierney write space, resp. locale, for what we call locale, resp. frame!); see also Johnstone (1982), Isbell (1972).

Recall that a map $X \xrightarrow{f} Y$ is *open* if f^{-1} has a left adjoint $f_! : \mathcal{O}(X) \to \mathcal{O}(Y)$ satisfying a Frobenius identity: $f_!(U \wedge f^{-1}(V)) = f_!(U) \wedge V$. Open maps are stable under pullback as are open surjections; and for two maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, gf is open if f and g are, while g is open if gf is and f is a surjection [JT, ch. V].

A *point* of a locale X is a map $1 \xrightarrow{x} X$, where 1 is the terminal locale, i.e. $1=i$ of the one-point space. A neighbourhood of x is an open sublocale $U \subset X$ such that $1 \xrightarrow{x} X$ factors through U. A (*neighbourhood*) *base at* x is a cofinal system of neighbourhoods of x. The set $pt(X)$ of points of X is a topological space, making pt into a functor right adjoint to the functor i.

A locale X is *discrete* if $\mathcal{O}(X)$ is of the form $\mathscr{P}(S)$ = the powerset of S, for some set S. If X is discrete, then $\mathcal{O}(X) = \mathscr{P}(pt(X))$ where $pt(X)$ is taken as a set. The functor (sets)$\to$(locales) which associates to a set the corresponding discrete locale is left adjoint to the functor (locales)$\xrightarrow{pt}$(sets). A locale X is discrete iff the diagonal $X \to X \times X$ is open [JT, ch. V]. The product of two discrete locales is discrete.

Let $\mathbb{P}$ be a poset, and $p \in \mathbb{P}$. A sieve on p is a downwards closed subset of $\{q \mid q \leqq p\}$. A covering system on $\mathbb{P}$ is an assignment of a family Cov(p) of "covering sieves" to each p, such that (1) $\{q \mid q \leqq p\} \in \mathrm{Cov}(p)$ and (2) $S \in \mathrm{Cov}(p)$, $q \leqq p \Rightarrow \exists T \in \mathrm{Cov}(q)$, $T \subset \{r \in S \mid r \leqq q\}$. The set $\mathbb{F}$ of subsets $U \subseteq \mathbb{P}$ satisfying $p \leqq q \in U \Rightarrow p \in U$ and $\mathrm{Cov}(p) \subseteq U \Rightarrow p \in U$ form a frame. If X is the locale defined by $\mathcal{O}(X) = \mathbb{F}$, then $(\mathbb{P}, \mathrm{Cov}(-))$ is said to be a *presentation* of X. If $p \in \mathbb{P}$, the principal open $U_p = \bigcap \{U \in \mathbb{F} \mid p \in U\}$ is in $\mathbb{F}$, and if x is a point of X, $\{U_p \mid U_p$ is a neighbourhood of $x\}$ clearly forms a basis at x.

The following example is particularly relevant. Let $\{S_i\}_{i \in I}$ be a filtered inverse system of sets S_i and functions $f_{ij} : S_i \to S_j$ $(i \leqq j)$. Let $\mathbb{P}$ be the poset of pairs (s,i), $s \in S_i$, where $(s,i) \leqq (t,j)$ iff $i \leqq j$ and $f_{ij}(s) = t$. The covering sieves of an element (s,i) are the sieves of the form $\{(t,k) \mid k \leqq j,\ t \in S_k,\ (t,k) \leqq (s,i)\}$, for each given j. This defines a presentation of the locale $X = \varprojlim_I S_i$, where each S_i is considered as a discrete locale. So the points of X are precisely the sequences $x = (s_i)_{i \in I}$ such that $s_i \in S_i$ and $(s_i, i) \leqq (s_j, j)$ whenever $i \leqq j$. The opens of the form $U_{(s_i,i)}$ form a neighbourhood basis at x.

Colimits of locales are computed as limits of the corresponding frames (which, in turn, are computed as limits of the underlying sets). It is easy to check that if

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{p} Z$$

is a coequalizer of locales, and f and g are open, then so is p. Moreover, every open surjection of locales is a coequalizer of its kernel pair ([JT], p. 39).

1.2. *Localic groups.* A localic group is a group object in the category of locales, i.e. a locale G equipped with maps $G\times G \xrightarrow{m} G$, $G \xrightarrow{r} G$, $1 \xrightarrow{e} G$ satisfying the usual identities. Every group (in (sets)) gives rise to a (discrete) localic group; another example is that of a locally compact topological group T: $i(T)$ is then a localic group, because i preserves products of locally compact spaces (Isbell (1972), Johnstone (1982)). Localic groups are considered e.g. in Wraith (1981), Isbell et al. (to appear).

Let G be a localic group. If $H\subset G$ is a localic subgroup, we may define the quotient G/H as the coequalizer of locales

$$(1)\qquad R_H \rightrightarrows G \to G/H$$

where R_H is the equivalence relation defined by the pullback of locales

$$(2)\qquad \begin{tikzcd} R_H \arrow[r] \arrow[d] & G\times G \arrow[d, "m\circ(1\times r)"] \\ H \arrow[r] & G \end{tikzcd}$$

H is *normal* if the composition $G\times H \xrightarrow{\langle \pi_1,\pi_2,\pi_1\rangle} G\times H\times G \xrightarrow{\alpha} G$ $\alpha=m\circ$ $\circ(m\times r)$ factors through H. One cannot conclude that G/H is a localic group because $G/H\times G/H$ need not be a quotient of $G\times G$. However, if $H\subset G$ is open and normal, then G/H is a localic group, by the remarks on co-equalizers at the end of 1.1. This proves part of the following lemma.

LEMMA. Let $U\subset G$ *be an open localic subgroup. Then* G/U *is discrete; if* U *is moreover normal,* G/U *is a group making the quotient map* $G\to G/U$ *into a homomorphism of localic groups.*

PROOF. We only have to show that the diagonal $G/U\to G/U\times G/U$ is open (cf. 1.1). Consider the commutative diagram

$$\begin{tikzcd} R_U \arrow[r] \arrow[d] & G/U \arrow[d] \\ G\times G \arrow[r] & G/U\times G/U \end{tikzcd}$$

Since U is open, both maps $R_U \rightrightarrows G$ are open, and hence $G\to G/U$ is open (cf. 1.1). Since $R_U\to G/U$ is a surjection and $R_U\to G\times G\to G/U\times G/U$ are open, so is $G/U\to G/U\times G/U$.

1.3. *Progroups.* A progroup $\underset{\sim}{G}$ is a filtered inverse system of groups $\underset{\sim}{G}=\{G_i\}_{i\in I}$, where I is some filtered (small) indexing category, i.e. $\underset{\sim}{G}$ is a functor $I^{op}\rightarrow$(groups). Homomorphisms of progroups are defined by interpreting them as "formal" inverse limits: for progroups $\underset{\sim}{G}=\{G_i\}_{i\in I}$ and $\underset{\sim}{H}=\{H_j\}_{j\in J}$, we have

$$\mathrm{Hom}(\underset{\sim}{G},\underset{\sim}{H})=\varprojlim_j \varinjlim_i \mathrm{Hom}(G_i,H_j).$$

(See Appendix 2 of Artin, Mazur (1969) for some basic properties of progroups.) A progroup is called *surjective* if all the homomorphism $G_i\rightarrow G_j$ (for $j\rightarrow i$ in I) in the projective system are surjections.

Note that the indexing category I can be taken to be a filtered *poset* in this case.

Every progroup $\underset{\sim}{G}$ gives rise to a localic group $\varprojlim G_i$, simply by taking the inverse limit in the category of localic groups; this defines a functor

$$L:(\text{progroups})\rightarrow(\text{localic groups}).$$

It is a general property of inverse limits of locales that for a surjective progroup $\underset{\sim}{G}$, the projections $\varprojlim G_i\xrightarrow{\pi_j}G_j$ are all surjections ([JT, § IV.4], Moerdijk (1986), theorem 5.1). In particular, $\varprojlim G_i$ is a highly non-trivial locale.

The situation is quite different if one tries to take the inverse limit in the category of *topological groups*. In fact, it is possible for a surjective progroup $\underset{\sim}{G}=\{G_i\}_{i\in I}$ that $\varprojlim G_i=(0)$, as topological group.

One of the simplest examples occurring in the literature is the following one taken from Higman & Stone (1954).

EXAMPLE. Let ω_1 be the set of countable ordinals, let S_α be the set of strictly order preserving maps $[0,\alpha]\rightarrow\mathbb{R}$, and let $r_{\alpha\beta}:S_\beta\rightarrow S_\alpha$ be the restriction function, for $\alpha<\beta$. Each $r_{\alpha\beta}$ is surjective, but $\varprojlim_\alpha S_\alpha$ is the set of order-embeddings $[0,\omega_1)\rightarrowtail\mathbb{R}$, which is empty.

Now let V_α be the vector space (over $\mathbb{Q}$ say) with basis S_α, and let $l_{\alpha\beta}:V_\beta\twoheadrightarrow V_\alpha$ be the linear map induced by $r_{\alpha\beta}$. Suppose $v\in\varprojlim V_\alpha$ is non-zero; write $v=\{v_\alpha\}_\alpha$, where $v_\alpha=\sum_{s\in S_\alpha}q_s\cdot s$ for rationals q_s. Let $K_\alpha=\{s\in S_\alpha|q_s\neq 0\}$.

K_α is a finite subset of S_α, and $r_{\alpha\beta}(K_\beta)\supseteq K_\alpha$ whenever $\alpha<\beta$. So $\alpha<\beta$ implies that the cardinality of K_β is at least that of K_α. It follows that the cardinality of K_α must eventually be constant, say from α_0 onwards. So $r_{\alpha\beta}(K_\beta)=K_\alpha$ for $\alpha_0\leq\alpha<\beta$. But then $\varprojlim_{\alpha>\alpha_0} K_\alpha\neq\emptyset$ by Tychonov's theorem, and therefore $\varprojlim S_\alpha\neq\emptyset$, contradiction.

1.4. *Prodiscrete localic groups.* A localic group is prodiscrete if it is isomorphic to an inverse limit of discrete groups. Here are some equivalent descriptions.

PROPOSITION. *Let G be a localic group. The following assertions are equivalent:*

(1) *G is prodiscrete.*

(2) *G is isomorphic to the inverse limit of a projective system of discrete groups and surjections, indexed by a filtered poset.*

(3) *The open normal subgroups of G form a neighbourhood base at $1 \xrightarrow{e} G$.*

PROOF. (1)⇒(2). Let $G = \varprojlim_{i \in I} G_i$, where the G_i are discrete groups, and I is some small indexing category, and let $G \xrightarrow{\pi_i} G_i$ be the projection. Write $\tilde{G}_i = G/\mathrm{Ker}(\pi_i)$. Since $\mathrm{Ker}(\pi_i)$ is open, $G/\mathrm{Ker}(\pi_i)$ is a discrete group (cf. 1.2, lemma), and $G \xrightarrow{\pi_i} G_i$ factors as $G \xrightarrow{q_i} \tilde{G}_i \xrightarrow{u_i} G_i$. When $i \xrightarrow{\alpha} j$ is a map in I with corresponding homomorphism $G_j \xrightarrow{f_\alpha} G_i$, there is a unique factorization h_{ji} as in the diagram below, because $\mathrm{Ker}(\pi_j) \subset \mathrm{Ker}(\pi_i)$.

$$\begin{tikzcd} & G \arrow[dl, "q_j"'] \arrow[dr, "q_i"] & \\ G/\mathrm{Ker}(\pi_j) \arrow[rr, dashed, "h_{ji}"] \arrow[d, "u_j"'] & & G/\mathrm{Ker}(\pi_i) \arrow[d, "u_i"] \\ G_j \arrow[rr, "f_\alpha"] & & G_i \end{tikzcd}$$

h_{ji} is a surjection which does not depend on α, so we obtain a system $\{\tilde{G}_i\}$ indexed by a poset $\tilde{I}$ ($i \leq j$ in $\tilde{I}$ iff there is some $i \xrightarrow{\alpha} j$ in I). Let $\tilde{G} = \varprojlim \tilde{G}_i$, with projections $p_i : \tilde{G} \to \tilde{G}_i$. As remarked in 1.3, each p_i is a surjection of locales.

There are homomorphisms of localic groups

$$G \underset{b}{\overset{a}{\rightleftarrows}} \tilde{G}$$

defined by $\pi_j \circ b = u_j \circ p_j$, resp. $p_j \circ a = q_j$. One easily checks that $b \circ a$ and $a \circ b$ are both identities. So $G \cong \tilde{G}$.

(2)⇒(3) is obvious from the construction of inverse limits of discrete locales (cf. 1.1).

(3)⇒(1). Let $\{U_i\}_{i \in I}$ be the system of open normal subgroups (ordered by inclusion), and let $G_i = G/U_i$, with projection $G \xrightarrow{p_i} G_i$. This gives a homomorphism

$$p : G \to \varprojlim G_i$$

of localic groups. It is clear that p is dense (in the sense that for every $V \in \mathcal{O}(\varprojlim G_i)$, $p^{-1}(V) = 0$ implies that $V = 0$). Thus by a result of Isbell et al.

(to appear) – which says that the only dense localic subgroup of a localic group is the group itself – it suffices to show that p is the inclusion of a sublocale, i.e. that $p^{-1}: \mathcal{O}(\varprojlim G_i) \to \mathcal{O}(G)$ is onto. To this end, take $W \in \mathcal{O}(G)$. We claim that

$$(*) \qquad W = \bigvee \{p_i^{-1}(x) | i \in I, x \in G_i, p_i^{-1}(x) \leq W\},$$

which would complete the proof, since $p_i^{-1}(x) = p^{-1}(\pi_i^{-1}(x))$ where

$$\pi_i : \varprojlim G_j \to G_i$$

is the projection.

Only $\leq$ in (*) requires a proof. We will show that every continuous map $X \xrightarrow{a} G$ of locales which factors through W, also factors through

$$\bigvee \{p_i^{-1}(x) | i \in I, x \in G_i, p_i^{-1}(x) \leq W\}.$$

Indeed, such an $X \xrightarrow{a} G$ induces an isomorphism

$$\phi_a : X \times G \xrightarrow{\sim} X \times G,$$

$\phi_a = (\pi_1, m(r \times G)(a \times G))$ ("$\phi_a(x, g) = (x, a(x)^{-1} \cdot g)$"). Let $V = \phi_a(X \times W) \subset \subset X \times G$. Then $X \xrightarrow{(\mathrm{id}, e)} X \times G$ factors through V, since a factors through W. So by the construction of the product of locales, and since the U_i form a basis at $1 \xrightarrow{e} G$, there are an open cover $\{W_\alpha\}_{\alpha \in A}$ of X and $i_\alpha \in I$ such that $W_\alpha \times \times U_{i_\alpha} \leq V$ (all $\alpha \in A$). Since G_{i_α} is discrete, each W_α is covered by $\{W_\alpha^x\}_{x \in G_{i_\alpha}}$, where $W_\alpha^x = a^{-1} p_{i_\alpha}^{-1}(x) \wedge W_\alpha$. So $\{W_\alpha^x | \alpha \in A, x \in G_{i_\alpha}\}$ covers X. But the restriction of a to W_α^x factors through $p_{i_\alpha}^{-1}(x)$, and since $W_\alpha^x \times U_{i_\alpha} \leq V$ we have $W_\alpha^x \times p_{i_\alpha}^{-1}(x) = \phi_a^{-1}(W_\alpha^x \times U_{i_\alpha}) \leq X \times W$, from which it follows that $p_{i_\alpha}^{-1}(x) \leq W$ whenever $W_\alpha^x \neq 0$. This proves (*).

REMARK. Note that the characterization (3) of the preceding proposition fails if one replaces locales by topological spaces, even under the assumption that G is Hausdorff. For example, let $\mathbb{Z}$ be the additive group of integers, I a countably infinite set, and $\mathbb{Z}^I$ the direct product (with the product topology). Let $G = \{\alpha \in \mathbb{Z}^I | \alpha(i) \neq 0$ for only finitely many $i \in I\}$. G is a dense subgroup of $\mathbb{Z}^I$; G cannot be an inverse limit of discrete groups (in the category of topological groups), because any such is complete, and finite or uncountable.

1.5. *Homomorphisms of prodiscrete localic groups.* Let G, H be prodiscrete localic groups, and write $G = \varprojlim G_i$, $H = \varprojlim H_j$, where these are filtered inverse systems of discrete groups and surjections. It is easy to see that

$$\mathrm{Hom}(G, H) \simeq \varprojlim_j \varinjlim_i \mathrm{Hom}(G_i, H_j).$$

Indeed, the only non-immediate thing is to show that a continuous homomorphism $G \xrightarrow{f_j} H_j$ factors uniquely through some projection $G \xrightarrow{\pi_i} G_i$. Since H_j is discrete, $\mathrm{Ker}(f_j)$ is an open subgroup of G. But if $G = \varprojlim G_i$, then

$\{\mathrm{Ker}(\pi_i)\}_i$ is a neighbourhood base at $1 \xrightarrow{e} G$, so $\mathrm{Ker}(f_j) \supset \mathrm{Ker}(\pi_i)$ for some i, and hence f_j factors through some $G \to G/\mathrm{Ker}(\pi_i)$. Since $G \xrightarrow{\pi_i} G_i$ is an open surjection of locales, $G/\mathrm{Ker}(\pi_i) \cong G_i$, and given this i, the factorization $G_i \to H_j$ of f_j is unique.

1.6. THEOREM. *The functor* L : (progroups)$\to$(localic groups) *restricts to an equivalence of categories*

$$L : (\text{surjective progroups}) \xrightarrow{\sim} (\text{prodiscrete localic groups}).$$

PROOF. This restriction is essentially surjective by 1.4 and fully faithful by 1.5.

§ 2. PROGROUPS AS TOPOSES

2.1. *The topos of G-sets.* Let G be a localic group. A *G-set* is a set S with a right G-action $S \times G \xrightarrow{\cdot} S$: here S is considered as a discrete locale, and $\times$ denotes the product in the category of locales, so $S \times G \xrightarrow{\cdot} S$ is a continuous map of locales satisfying the usual condition for an action, namely that

$$\begin{tikzcd} S\times 1 \arrow[r, "S\times e"] \arrow[dr, "\pi_2"', "\sim"] & S\times G \arrow[d] \\ & S \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S\times G\times G \arrow[r, "S\times m"] \arrow[d, "\times G"] & S\times G \arrow[d] \\ S\times G \arrow[r] & S \end{tikzcd}$$

commute. A *map of G-sets* $(S, \cdot) \xrightarrow{f} (T, \cdot)$ is an action-preserving function $S \to T$ of sets, i.e.

$$\begin{tikzcd} S\times G \arrow[r, "f\times G"] \arrow[d] & T\times G \arrow[d] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

commutes. We will write BG for the category of G-sets.

The forgetful functor $BG \xrightarrow{U}$ Sets creates colimits and finite limits. Exponentials in BG are computed as follows: if S and T are G-Sets, T^S is the set of functions $S \to T$ such that for some open subgroup $U \subset G$, f is a map of U-sets for the restricted action:

$$\begin{tikzcd} S\times U \arrow[r, "f\times U"] \arrow[d] & T\times U \arrow[d] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

BG also has a subobject classifier, namely the two-point G-set $\{0,1\}$ with the trivial G-action. It is well known that BG is a Grothendieck topos, in fact an atomic Grothendieck topos (cf. Barr, Diaconescu (1980)), at least in the case where G is a topological group. This will also be immediate from the description of a site for BG, given in 2.3 below.

2.2. *Functorial properties.* A continuous homomorphism of localic groups $G \xrightarrow{\phi} H$ induces a geometric morphism $BG \xrightarrow{B\phi} BH$. The inverse image functor $(B\phi)^*$ sends an H-set $(S,\cdot)$ to the same set S with the induced G-action $S\times G \xrightarrow{S\times\phi} S\times H \longrightarrow S$. The underlying set functor U is the inverse image of a point $(\text{Sets}) \xrightarrow{p_G} BG$, and $p_G^* \circ (B\phi)^* \cong p_H^*$, so B is in fact a functor into *pointed* toposes. If ϕ is an open surjection, $B\phi^*$ has a left adjoint $(B\phi)_!$: for a G-set $S=(S,\cdot)$, $(B\phi)_!(S)$ is the coequalizer $S\underset{G}{\otimes}H$ of locales:

$$S\times G\times H \underset{v}{\overset{u}{\rightrightarrows}} S\times H \xrightarrow{p} S\underset{G}{\otimes}H$$

where $u = \cdot\times H$ and $v = S\times m\circ(\phi\times H)$, equipped with the obvious right G-action. $S\underset{G}{\otimes}H$ is indeed a discrete locale, since $S\times G\times H \xrightarrow{(u,v)} (S\times H)\times \times(S\times H)$ is open, and hence $S\underset{G}{\otimes}H$ has an open diagonal:

$$\begin{tikzcd}
S\times G\times H \arrow[r, "{(u,v)}"] \arrow[d, "pu=pv"] & (S\times H)\times(S\times H) \arrow[d, "p\times p"] \\
S\underset{G}{\otimes}H \arrow[r, "\delta"] & S\underset{G}{\otimes}H\times S\underset{G}{\otimes}H
\end{tikzcd}$$

Note that $(B\phi)_!\circ(B\phi)^*\cong\mathrm{id}$ when ϕ is an open surjection. In fact it easily follows that $B\phi$ is an atomic connected geometric morphism (cf. Barr, Diaconescu (1980)). A more general result is extensively discussed in Moerdijk (to appear).

2.3. *Stability.* Let $\mathscr{F} \xrightarrow{p} \mathscr{E}$ be a geometric morphism of Grothendieck toposes, let $\underline{\mathrm{Loc}}(\mathscr{E})$, $\underline{\mathrm{Loc}}(\mathscr{F})$ denote the categories of internal locales in $\mathscr{E}$ and in $\mathscr{F}$, respectively. The morphism p induces an adjoint pair of functors

$$\underline{\mathrm{Loc}}(\mathscr{F}) \underset{p_!}{\overset{p^\#}{\leftrightarrows}} \underline{\mathrm{Loc}}(\mathscr{E}),\ p_! \dashv p^\#.$$

$p_!$ is described by the formula

$$\mathscr{O}(p_!(Y)) \simeq p_*(\mathscr{O}(Y)).$$

For $X \in \underline{\mathrm{Loc}}(\mathscr{E})$, $p^{\#}(X)$ can be constructed as the pullback

$$\begin{tikzcd}
Sh_{\mathscr{F}}(p^{\#}X) \arrow[r] \arrow[d] & Sh_{\mathscr{E}}(X) \arrow[d] \\
\mathscr{F} \arrow[r] & \mathscr{E}
\end{tikzcd}$$

where $Sh_{\mathscr{E}}(X)$ denotes the topos of $\mathscr{E}$-sheaves on X, etc. Although $\mathscr{O}(p^{\#}(X))$ is usually much bigger than $p^{*}(\mathscr{O}(X))$, the latter poset at least defines a presentation of the locale $p^{\#}(X)$ in $\mathscr{F}$, see [JT], Moerdijk-Wraith (1986).

STABILITY LEMMA. *Let G be a localic group (in* Sets*), and let $\mathscr{F}$ be any Grothendieck topos. Then there is a canonical equivalence of toposes*

$$B(G) \times \mathscr{F} \simeq B(\mathscr{F}, \gamma^{\#} G),$$

where γ is the essentially unique geometric morphism $\mathscr{F} \to$ Sets.

REMARK. The same proof actually gives a more general equivalence

$$B(\mathscr{E}, G) \underset{\mathscr{E}}{\times} \mathscr{F} \simeq B(\mathscr{F}, p^{\#} G)$$

for any geometric morphism $\mathscr{F} \xrightarrow{p} \mathscr{E}$ and any localic group G in $\mathscr{E}$ such that $G \to 1$ is an open map of locales in $\mathscr{E}$.

PROOF. The result follows easily from the following description of a site $\mathbb{S}_G$ for BG: the objects of $\mathbb{S}_G$ are the sets G/U of "right cosets", where U is an open subgroup of G. G/U is indeed a set (i.e. discrete, cf. 1.2), and G acts on G/U in the obvious way. $\mathbb{S}_G$ is the full subcategory of BG whose objects are these G-sets of the form G/U. The maps $G/U \to G/V$ in $\mathbb{S}_G$ can be explicitly identified with those points $K \in G/V$ such that $U \subseteq p_V^{-1}(K)^{-1} \cdot p_V^{-1}(K) = m(r(p_V^{-1}(K)) \times p_V^{-1}(K))$, where $p_V : G \to G/V$ is the projection, and the topology of $\mathbb{S}_G$ is the atomic topology.

If $\mathscr{B}$ is a cofinal system of open subgroups (i.e. for every open subgroup U there is an open subgroup $V \in \mathscr{B}$ such that $V \subset U$) then the full subcategory of $\mathbb{S}_G$, whose objects are those of the form G/V where $V \in \mathscr{B}$, obviously still is a site for $\mathbb{S}_G$. But the open subgroups of $\gamma^{\#}(G)$ of the form $\gamma^{\#}(U)$ where U is an open subgroup of G form such a cofinal system for the open subgroups of $\gamma^{\#}(G)$. Moreover, $\gamma^{\#}$ preserves quotients of the form G/U, i.e. $\gamma^{*}(G/U) = \gamma^{\#}(G/U) \simeq \gamma^{\#}(G)/\gamma^{\#}(U)$, since $\gamma^{\#}$ preserves open surjections, and open surjections are coequalizers of their kernel pairs, as pointed out in 1.1. From these observations, the result easily follows.

2.4. *Inverse limits.* Let $G \xrightarrow{\phi} H$ be an open surjective map of localic groups. ϕ induces an adjoint pair of functors

$$\mathbb{S}_G \underset{T_\phi}{\overset{P_\phi}{\rightleftarrows}} \mathbb{S}_H, \; P_\phi \dashv T_\phi$$

defined by $P_\phi(G/U) = H/\phi(U)$, $T_\phi(H/V) = G/\phi^{-1}(U)$ (and a similarly obvious definition on maps) – in fact T_ϕ and P_ϕ are just the restrictions of $(B\phi)^*$ and $(B\phi)_!$, respectively.

Thus if

$$\cdots G_2 \xrightarrow{\phi_1} G_1 \xrightarrow{\phi_0} G_0$$

is an inverse sequence of localic groups, we obtain a system of sites and adjoint functors

$$\cdots \mathbb{S}_2 \underset{T_1}{\overset{P_1}{\rightleftarrows}} \mathbb{S}_1 \underset{T_0}{\overset{P_0}{\rightleftarrows}} \mathbb{S}_0$$

($\mathbb{S}_i = \mathbb{S}_{G_i}$, etc.), just as in Moerdijk (1986), § 3. Let $\mathbb{S}^\infty$ be the atomic site for $\varprojlim_n B(G_n)$ as constructed in loc. cit. (The objects of $\mathbb{S}^\infty$ are sequences $(S_n)_n$ where S_n is an object of $\mathbb{S}_n$ and $S_{n+1} = T_n(S_n)$ for $n \geqq$ some n_0.) Then $\mathbb{S}^\infty$ is precisely the site we obtain from the localic group $G^\infty = \varprojlim G_n$, when we take only those objects G^∞/U where U is of the form $p_n^{-1}(V)$ for some open subgroup $V \subset G_n$ ($G^\infty \xrightarrow{p_n} G_n$ is the projection). Open subgroups of this form obviously form a cofinal system $\mathscr{B}$, cf. 2.3. above. Thus we obtain an equivalence of toposes

$$\varprojlim B(G_n) \simeq B(\varprojlim G_n).$$

In fact, the following more general result holds:

INVERSE LIMIT THEOREM. *Let $\{G_i\}_i$ be a filtered inverse system of localic groups and continuous homomorphisms, which are all open surjections. Then there is a canonical equivalence of toposes*

$$\varprojlim B(G_i) \simeq B(\varprojlim G_i).$$

PROOF. For the case of an inverse *sequence*, the argument has just been sketched. For an arbitrary filtered inverse system, we may extend the base by an open surjection $\mathscr{E} \xrightarrow{p} \mathscr{S}$ such that in $\mathscr{E}$, the system contains a cofinal sequence (cf. Moerdijk (1986), lemma 5.3). Since $B(G_i) \times \mathscr{E} \simeq B(\mathscr{E}, p^\#(G_i))$ as proved in 2.3 above, the case of sequences now implies that $B(\mathscr{E}, \varprojlim p^\#(G_i)) \simeq$ $\simeq \varprojlim B(\mathscr{E}, p^\#(G_i))$. So again by 2.3, $B(\varprojlim G_i) \times \mathscr{E} \simeq \varprojlim B(G_i) \times \mathscr{E}$. The theorem now follows by the following lemma.

LEMMA. *Let*

$$\begin{tikzcd} \mathcal{G} \arrow[r, "q"] \arrow[d, "b"] & \mathcal{E} \arrow[d, "a"] \\ \mathcal{F} \arrow[r, "p"] & \mathcal{S} \end{tikzcd}$$

be a pullback square of toposes, and assume p is an open surjection. Then if b is an equivalence, so is a.

PROOF OF THE LEMMA. b is certainly connected, locally connected, hence so is a, by Moerdijk (1986), theorem 5.2 (v). In particular, a^* and b^* have left adjoints $a_!$ and $b_!$ such that the Beck-Chevalley condition $p^*a_! \cong b_!q^*$, or equivalently $a^*p_* \cong q_*b^*$, holds. Hence $a_!a^* \cong \mathrm{id}$ since a is connected locally connected, and moreover $q^*a^*a_! \cong b^*p^*a_! \cong b^*b_!q^* \cong q^*$ since b is an equivalence, and so $a^*a_! \cong \mathrm{id}$ since q^* is faithful, q being an open surjection.

2.5. COROLLARY. *Let G be a prodiscrete localic group, and let Sh(G) be the topos of sheaves on the underlying locale. Then the square*

$$\begin{tikzcd} Sh(G) \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & \text{Sets} \arrow[d, "p_G"] \arrow[dl, phantom, "\swarrow \mu_G"] \\ \text{Sets} \arrow[r, "p_G"'] & BG \end{tikzcd}$$

is a (pseudo-) pullback of toposes, where p_G is the canonical point of the topos BG given by $p_G^*(X, \cdot) = X$, and μ_G is the "generic" natural transformation whose component at $X \in BG$ is the map $X \times G \xrightarrow{(\cdot, \pi_2)} X \times G$.

PROOF. This is wellknown and easy to see in the case of a discrete group G. For prodiscrete groups the assertion follows by 2.4.

REMARK. There is an alternative proof of 2.5 which doesn't use 2.4. Moreover, the equivalence holds for a wider class of localic groups than just the prodiscrete ones (namely those groups which are étale complete when regarded as localic groupoid, cf. Moerdijk (to appear), § 7.4).

2.6. THEOREM. *Let G and H be localic groups, and assume that H is prodiscrete. The functor B induces an equivalence of categories*

$$\underline{\mathrm{Hom}}(G, H) \simeq \underline{\mathrm{Hom}}(BG, BH).$$

REMARK. Here on the right, $\mathrm{Hom}(BG, BH)$ denotes the category whose objects are pairs (f, α) where $f: BG \to BH$ is a geometric morphism and

$\alpha : fp_G \cong p_H$ is a 2-isomorphism; the morphisms in $\underline{\mathrm{Hom}}(BG, GH)$ from one such (f, α) to another (f', α') are just natural transformations $f^* \to f'^*$. On the left, $\underline{\mathrm{Hom}}(G, H)$ is the category whose objects are continuous homomorphisms $G \xrightarrow{\phi} H$, and whose morphisms $\phi \Rightarrow \psi$ are points $1 \xrightarrow{h} H$ such that $\psi = h^{-1}\phi h$.

PROOF. It is easy to see that $\underline{\mathrm{Hom}}(G, H) \to \underline{\mathrm{Hom}}(BG, BH)$ is full and faithful, if H is prodiscrete. To show that it is essentially surjective, take a map $(f, \alpha): BG \to BH$ of pointed toposes, i.e. $BG \xrightarrow{f} BH$ is a geometric morphism, and $\alpha : fp_G \Rightarrow p_H$ is a 2-isomorphism (a natural isomorphism $p_G^* f^* \to p_H^*$). Now consider the pseudo-pullback

$$(1) \qquad \begin{tikzcd} Sh(H) \arrow[r, "\gamma_H"] \arrow[d, "\gamma_H"'] & \mathrm{Sets} \arrow[d, "p_H"] \arrow[dl, phantom, "\Swarrow \mu_H"] \\ \mathrm{Sets} \arrow[r, "p_H"'] & BH \end{tikzcd}$$

of 2.5. There exists a similar commutative-up-to-isomorphism square for G, with a 2-isomorphism $\mu_G : p_G\gamma_G \Rightarrow p_G\gamma_G$ (but this need not be a pullback). Thus we obtain a 2-isomorphism from $p_H \circ \gamma_G$ to itself, namely

$$(\alpha \cdot \gamma_G) \circ (f \cdot \mu_G) \circ (\alpha^{-1} \cdot \gamma_G).$$

By the universal property of (1), there is a unique map $G \xrightarrow{\phi} H$ of locales such that $\mu_H \cdot \phi = (\alpha \cdot \gamma_G) \circ (f \cdot \mu_G) \circ (\alpha^{-1} \cdot \gamma_G)$. So $(\alpha \cdot \gamma_G) \circ (\mu_H \cdot \phi) = (\alpha \cdot \gamma_G) \circ (f \cdot \mu_G)$, which precisely says that α is in fact a natural isomorphism $f^* \to (B\phi)^*$. Moreover, it follows easily from uniqueness of ϕ that ϕ is in fact a homomorphism.

2.7. REMARK. In SGA 4 (p.319) Grothendieck et al. construct the "classifying topos" of an inverse system $\underline{G} = \{G_i\}_i$ of groups and *surjective* homomorphisms. It is not difficult to see that this topos coincides with the topos of continuous G-sets $B(\varprojlim G_i)$ considered in this paper, where $\varprojlim G_i$ is the inverse limit constructed as a *localic* group. (This was observed some time ago by M. Tierney.)

§ 3. GALOIS TOPOSES

We conclude this note by characterizing the toposes of the form BG, where G is a prodiscrete group. It will be apparent that the answer is essentially contained in Grothendieck's theory of Galois categories (cf. [SGA 1]). We first give the necessary definitions.

3.1. *Galois toposes, locally trivial coverings, connected locally connected toposes.* Recall that an atom A of a Grothendieck topos $\mathscr{E}$ is called a *normal atom*, or a *Galois object* of $\mathscr{E}$, if A is an $\mathrm{Aut}(A)$-torsor in $\mathscr{E}$, i.e. the canonical morphism $(\pi_1, \mathrm{ev}): A \times \gamma^*(\mathrm{Aut}(A)) \to A \times A$ is an $\mathscr{E}$-isomorphism. A *Galois* *topos* is a pointed connected atomic topos which is generated by its normal atoms.

An object X of a Grothendieck topos $\mathscr{E}$ is called *locally trivial* if there exists an epi $U \twoheadrightarrow 1$ and an isomorphism $U \times \gamma^*(S) \xrightarrow{\sim} U \times X$ over U, for some set S. X is called a locally trivial cover if moreover $X \to 1$ is epi. Finite limits and colimits of locally trivial objects are locally trivial; if $X \xrightarrow{f} Y$ is a morphism of locally trivial objects, then $\mathrm{im}(f) \subset Y$ is complemented subobject. So if Y is connected, f must be epi.

For connected, locally connected (= molecular) toposes, see [SGA 4], Barr, Paré (1980), or the Appendix of Moerdijk (1986).

3.2. THEOREM. *The following are equivalent, for a Grothendieck topos* $\mathscr{E}$:

(1) $\mathscr{E}$ *is a Galois topos.*

(2) $\mathscr{E}$ *is equivalent to BG for a prodiscrete localic group G.*

(3) $\mathscr{E}$ *is pointed, connected locally connected, and generated by its locally trivial coverings.*

(4) $\mathscr{E}$ *is pointed, connected locally connected, and every object of* $\mathscr{E}$ *is a sum of locally trivial coverings.*

PROOF. (1)$\Rightarrow$(4) is clear, since every Galois object of a connected topos is a locally trivial covering.

(4)$\Rightarrow$(3) obvious.

(3)$\Rightarrow$(1) is a standard argument from Galois theory: if X is a connected locally trivial covering of $\mathscr{E}$, say with $\alpha : U \times \gamma^*(S) \xrightarrow{\sim} U \times X$ over U, let $B = \mathrm{Iso}(\gamma^*(S), X)$, as a subobject of $X^{\gamma^*(S)}$ in $\mathscr{E}$. Then in particular, α gives a map $U \xrightarrow{\hat{\alpha}} B$, so $B \twoheadrightarrow 1$. There is an obvious map $\mathrm{Aut}(S) \to \mathrm{Aut}(B)$, and B is an $\mathrm{Aut}(S)$-torsor by the canonical map $B \times \gamma^*\mathrm{Aut}(S) \simeq B \times \mathrm{Aut}(S) \xrightarrow{\sim} B \times B$, so a fortiori $\gamma^*\mathrm{Aut}(B)$ acts transitively on B. Let $A \subset B$ be any component of B. A is locally trivial since B is, and $\mathrm{Aut}(A) = \{\alpha | A : \alpha \in \mathrm{Aut}(B),\ \alpha(A) \subset A\}$. So $\mathrm{Aut}(A)$ acts transitively on A, i.e. $A \times \gamma^*\mathrm{Aut}(A) \to A \times A$ is epi. It is also mono, since if

$$E \rightarrowtail A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} A$$

is an equalizer, E is locally constant, hence complemented in A, so $E = 0$ or $E = A$. Thus B is a sum of Galois objects, and B covers X by evaluation at any $s_0 \in S : B \twoheadrightarrow X$.

(2)$\Rightarrow$(1). If G is a prodiscrete group, then for every normal open subgroup U, G/U is a normal atom of BG, and these form a site, cf. 2.3.

(1)$\Rightarrow$(2). Let $\mathrm{Sets} \xrightarrow{p} \mathscr{E}$ be a point of the Galois topos $\mathscr{E}$, and let $\mathbb{A} \subset \mathscr{E}$ be the atomic site of Galois objects of $\mathscr{E}$. By the descent theorem ([JT], Moerdijk (1985)), $\mathscr{E} \simeq B(\mathrm{Aut}(p))$ where $\mathrm{Aut}(p)$ is the localic (!) group of automorphisms of p (a sublocale of the product $\prod_{A \in \mathbb{A}} \mathrm{Aut}\, p^*(A)$; note that $\mathrm{Aut}\, p^*(A)$ is not discrete if $p^*(A)$ is infinite). To see that $\mathrm{Aut}(p)$ is prodiscrete, let $\mathbb{D}$ be the diagram of the functor $p^*: \mathbb{A} \to \text{Sets}$ (the objects of $\mathbb{D}$ are pairs (x, A) with $x \in p^*(A)$, the morphisms $(x, A) \xrightarrow{f} (y, B)$ are $\mathscr{E}$-maps $A \longrightarrow B$ with $p^*(f)(x) = y$). Then $\text{Aut}(p) \simeq \varprojlim_{(x,A)} \text{Aut}(A)$ as localic groups, where each $\text{Aut}(A)$ is taken as a discrete group. (In "point-set notation", this isomorphism can be described by defining mutually inverse maps $\text{Aut}(p) \xrightarrow{\phi} \varprojlim_{(x,A)} \text{Aut}(A)$ and $\varprojlim_{(x,A)} \text{Aut}(A) \xrightarrow{\psi} \text{Aut}(p)$, by $\phi(\tau)_{(x,A)}: A \xrightarrow{\sim} A$ is the unique one with $p^*(\phi(\tau)_{(x,A)})(x) = \tau_A(x)$, and $\psi(\sigma)_A: p^*(A) \xrightarrow{\sim} p^*(A)$ is given by $\psi(\sigma)_A(x) = p^*(\sigma_{(x,A)})(x)$. This "point-set notation" can be justified using test spaces in the standard way.)

3.3. REMARK. To prove (1)$\Rightarrow$(2) of the preceding theorem, one may also show directly that the functor $F: \mathscr{E} \to B(\varprojlim_{(x,A)} \text{Aut}(A))$ is an equivalence, where $F(X) = (p^*(X), \mu_X)$ and the action $\mu_X: p^*(X) \times \varprojlim \text{Aut}(A) \to p^*(X)$ is defined as follows: write

$$p^*(X) \times \varprojlim \text{Aut}(A) = \coprod_{t \in p^*(X)} \text{Aut}(A),$$

and let $\mu_X(t, -): \varprojlim \text{Aut}(A) \to p^*(X)$ be the composite

$$\varprojlim \text{Aut}(A) \xrightarrow{\pi_{(x,A)}} \text{Aut}(A) \xrightarrow{u_x} p^*(A) \xrightarrow{p^*(\alpha)} p^*(X),$$

for any $A \xrightarrow{\alpha} X$ and $x \in p^*(A)$ with $p^*(\alpha)(x) = t$, where $u_x(\sigma) = p^*(\sigma)(x)$.

3.4. *Remark on the fundamental group.* Let $\mathscr{E}$ be a pointed connected locally connected topos. Let $\text{Cov}(\mathscr{E}) \subset \mathscr{E}$ be the full subcategory of $\mathscr{E}$ generated (as a topos) by the locally trivial covers of $\mathscr{E}$. Using Giraud's theorem, it is not hard to see that $\text{Cov}(\mathscr{E})$ is a pointed Grothendieck topos. By 3.2 above, $\text{Cov}(\mathscr{E})$ is of the form $B(G)$ for a prodiscrete localic group $\pi_1(\mathscr{E})$, which is unique up to isomorphism (Corollary 2.6). Of course, $\pi_1(\mathscr{E})$ is just the fundamental group of $\mathscr{E}$ defined as a progroup (cf. Artin, Mazur (1969)), modulo the equivalence 1.6. The map $\mathscr{E} \xrightarrow{\xi} B(\pi_1(\mathscr{E}))$ corresponding to the inclusion $\text{Cov}(\mathscr{E}) \subset \mathscr{E}$ is universal among maps of $\mathscr{E}$ into pointed toposes of the form $B(G)$, G prodiscrete; i.e. G induces an equivalence of categories

(1) $\quad \underline{\text{Hom}}(\mathscr{E}, BG) \simeq \underline{\text{Hom}}(B(\pi_1 \mathscr{E}), BG)$

(2) $\quad \simeq \underline{\text{Hom}}(\pi_1 \mathscr{E}, G)$,

(1)) simply because for a geometric morphism $\mathscr{E} \xrightarrow{f} BG$, f must necessarily map into $\text{Cov}(\mathscr{E}) \subset \mathscr{E}$, (2) is by 2.6). In particular, if G is discrete, BG classifies G-torsors, i.e. $\underline{\text{Hom}}(\mathscr{E}, BG)$ is equivalent to the category of G-torsors in $\mathscr{E}$, i.e. G-principal bundles over $\mathscr{E}$. Passing to isomorphism classes, we obtain the usual property of $\pi_1(\mathscr{E})$, by the fact that for G discrete, every geometric morphism is pointed, namely

$$\pi^1(\mathscr{E}; G) = [\pi_1(\mathscr{E}), G],$$

where $\pi^1(\mathscr{E}; G)$ denotes the set of isomorphism classes of G-torsors in $\mathscr{E}$, and $[\pi_1(\mathscr{E}), G]$ denotes the set of equivalence classes of continuous homomorphisms $\pi_1(\mathscr{E}) \to G$ (two such being equivalent if they differ by an inner automorphism of G), cf. Artin, Mazur (1969), § 10, and references cited there.

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