LOCAL MAPS OF TOPOSES

P. T. JOHNSTONE and I. MOERDIJK

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Introduction

The notion of local (Grothendieck) topos was introduced by Grothendieck and Verdier in SGA 4 [11, VI, 8.4]; a Grothendieck topos $\mathcal{E}$ is said to be local if the global sections functor $\mathcal{E} \to \mathcal{F}$ has a right adjoint as well as its usual left adjoint, i.e. if it is the inverse image part of a geometric morphism $\mathcal{F} \to \mathcal{E}$, as well as the direct image of the unique morphism $\mathcal{E} \to \mathcal{F}$. The example which the authors of [11] particularly had in mind was the topos of sheaves on a space (such as the Zariski spectrum of a local ring) containing a point whose only neighbourhood is the whole space. (Such points have been called focal by Freyd [9]; the similarity between ‘focal’ and ‘local’ is deliberate.) They also showed that, for a locally coherent topos $\mathcal{E}$, there is a process of ‘localization’ which mirrors the passage from the Zariski spectrum of a ring to the spectrum of one of its localizations.

Since then, the notion of local topos has not attracted much attention, apart from the occasional passing reference in papers about other things. We believe that the time is now ripe, if not overdue, for a more detailed investigation of the notion, for a number of reasons which we shall now describe briefly.

In the first place, the property of being local, like many of the properties of toposes studied in [11], is not so much a property of the topos $\mathcal{E}$ itself as of the geometric morphism $\mathcal{E} \to \mathcal{F}$; thus there is scope for studying toposes which are local over some base topos other than the classical topos of sets, or (to change our terminology slightly) studying local maps between toposes. (Note: throughout this paper we shall restrict ourselves to maps of toposes (i.e. geometric morphisms) which are bounded in the sense of [13, Definition 4.43]; we make this restriction in order to be assured of the existence of all the pullbacks which we shall wish to consider, but many of our results are not crucially dependent on it. To emphasize the unimportance of the restriction, we shall denote the 2-category of toposes and bounded geometric morphisms simply by $\mathcal{S}op$. ) Thus, in addition to the properties of local toposes studied in [11], there is scope for studying properties of local maps (stability under composition, pullback, etc.) which cannot even be conveniently formulated in terms of the Grothendieck–Verdier definition.

Secondly, the restriction to the locally coherent case in the construction of localizations in [11] seems to have been a matter of necessity rather than desirability (despite the remarks on p. 321 of [11]): Grothendieck and Verdier simply did not have techniques available for handling filtered inverse limits in the non-locally-coherent case. With the more powerful techniques available now, we shall show (in § 3 of this paper) that the notion of localization makes perfectly good sense for arbitrary (bounded) pointed toposes over a base, and we shall also

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provide some evidence (concerning the notion of 'the germ of a manifold at a point') to show that it is of interest in cases other than those considered by Grothendieck and Verdier.

Thirdly, Lawvere [23] has recently argued cogently that one should distinguish, in the study of Grothendieck toposes, between the 'generalized spaces' which conform to the traditional view of what a Grothendieck topos is (see [19]) and those 'gros toposes' which are themselves categories of generalized spaces. (The same point has been made, in a rather more diffuse way, by Grothendieck in [10].) Of course, the original 'gros topos' (whose definition was suggested by J. Giraud) was studied in SGA 4 [11, IV, 4.10]; as yet it is not entirely clear what one should take as the axiomatic definition of a 'gros topos', but it is at least clear that such a topos should be local over \( \mathcal{S} \). Therefore, while our results in this paper do not add anything directly to our understanding of gros toposes, we hope that they will be of some use in the study of this concept.

A few words are in order about the techniques used to prove the results in this paper. An earlier version of the paper [26], written by the second author, contained most of the results in the present version, but the proofs were often different; they relied on the characterizations of local toposes and local maps in terms of sites, which appear as 1.7 and 1.8 in the present paper. In the present version, in keeping with the view (mentioned earlier) that one should avoid invoking the boundedness of geometric morphisms any more than is strictly necessary, we have avoided the use of sites as far as possible in the proofs of general results (though we have retained the two characterizations just mentioned, since they are invaluable for the computation of specific examples), and have instead chosen to rely largely on the powerful 2-categorical techniques which are now available in \( \mathcal{Top} \); in particular, we make extensive use of an intrinsic 2-categorical characterization of local toposes (see 1.5 below), which was not emphasized in the earlier version.

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1. Local maps of toposes

We shall work throughout in the context of toposes defined and bounded over some base topos \( \mathcal{S} \), which we shall treat as if it were 'the' category of sets. Our fundamental definition is taken from [11, VI, 8.4.6]:

1.1. Definition. A topos \( \mathcal{E} \) is local (over \( \mathcal{S} \)) if the global sections functor \( y_*: \mathcal{E} \to \mathcal{S} \) is the inverse image of a map of \( \mathcal{S} \)-toposes.

We shall be interested particularly in the relativized version of this definition: we shall say that a map of toposes \( f: \mathcal{S} \to \mathcal{E} \) is local if \( \mathcal{S} \) is a local \( \mathcal{E} \)-topos, i.e. if
the direct image functor $f_* : \mathcal{F} \to \mathcal{G}$ has an $\mathcal{E}$-indexed right adjoint. Let us recall what this means: for every object $E$ of $\mathcal{E}$, $f$ induces a geometric morphism $f/E : \mathcal{F}/f^* E \to \mathcal{G}/E$, whose inverse image is simply given by

$$(f/E)^*(X \xrightarrow{\alpha} E) = (f^* X \xrightarrow{f^* \alpha} f^* E),$$

whilst its direct image is defined by pullback: given $(Y \xrightarrow{\beta} f^* E)$, its image under $(f/E)_*$ is the left-hand arrow in

$$\begin{array}{c}
\bullet \\
E \xrightarrow{\eta_E} f_* f^* E
\end{array} \xrightarrow{f_* \beta} f_* Y$$

where $\eta$ is the unit of $(f^* - f_*)$. Now $f$ is local if, for every $E$, $(f/E)_*$ has a right adjoint $(f/E)^+$, and these right adjoints commute with pulling back; i.e. for every $u : E \to E'$, the square

$$\begin{array}{ccc}
\mathcal{G}/E & \xrightarrow{(f/E)^+} & \mathcal{F}/f^* E \\
\uparrow{u^*} & & \uparrow{(f^* u)^*} \\
\mathcal{G}/E' & \xrightarrow{(f/E')^+} & \mathcal{F}/f^* E'
\end{array}$$

commutes (up to canonical isomorphism).

1.2. EXAMPLES. (a) Let $X$ be a topological space, and suppose there is a point $x$ of $X$ whose only neighbourhood is the whole space. Then the stalk functor $F \mapsto F_x$ for sheaves on $X$ is easily seen to coincide with the global sections functor $F \mapsto F(X)$; so $\text{Sh}(X)$ is a local topos. This situation occurs, for example, when $X$ is the spectrum of a local ring, and $x$ is the point corresponding to its maximal ideal. (It was this example, of course, which gave rise to the name ‘local topos’.) Further examples of spaces with this property will be found in [1].

(b) The notion of ‘gros topos’ does not seem to admit a precise definition as yet; but it is a well-established fact that gros toposes are generally local. (Indeed, Lawvere [23] has suggested that this property should be taken as part of the definition of a gros topos.) For example, let $\mathcal{F}$ be the topos of sheaves for the open cover topology on the category $\text{Top}$ of topological spaces (or, if you insist, a suitable small full subcategory thereof); then, in addition to its left adjoint sending sets to (functors represented by) discrete spaces, the global sections functor $\mathcal{T} \to \mathcal{F}$ also has a right adjoint defined by means of indiscrete spaces. More generally, for any space $X$ there is a local geometric morphism $\mathcal{T}/X \to \text{Sh}(X)$ (cf. [11, IV, 4.10]). We shall investigate this and similar examples in 1.9 below.

(c) Let $C$ be a small category (by which we mean an internal category in $\mathcal{F}$). If $C$ has a terminal object $t$, then the direct image functor $\lim_{C} : \mathcal{F}^\text{op} \to \mathcal{F}$ may be identified with the functor ‘evaluate at $t$’, which is an inverse image functor; so $\mathcal{F}^\text{op}$ is a local topos. (We shall encounter a generalization of this result from small categories to sites in 1.7 below.) In particular, the classifying topos of an essentially algebraic theory in the sense of Freyd [8] (that is, a lim-theory in the sense of Coste [3]) is local.
(d) Generalizing the last sentence of the previous paragraph, let \( T \) be a geometric theory which has a term model—that is, a model each of whose elements is the interpretation of some closed term in the language of \( T \), two such elements being equal if and only if the equality of the corresponding terms is provable in \( T \). (For an essentially algebraic \( T \), the term model is the free \( T \)-model on no generators.) Then it will follow from 1.5 below that the classifying topos of \( T \) is local.

(e) Any topos \( \mathcal{E} \) can be embedded as an open subtopos of a local topos by Artin glueing [11, IV, 9.5]: let \( \gamma_*: \mathcal{E} \to \mathcal{F} \) be the global sections functor, and form the comma category \( \mathcal{E}' \) whose objects are triples \((E, S, \alpha)\) with \( E \in \text{ob } \mathcal{E}, S \in \text{ob } \mathcal{F} \) and \( \alpha: S \to \gamma_*E \). It is well known (cf. [31]) that this category is a topos; it is local because its global sections functor coincides with the inverse image of the closed inclusion \( \mathcal{F} \to \mathcal{E} \), that is, the functor \((E, S, \alpha) \mapsto S\).

The above are all examples of toposes which are local over the base topos \( \mathcal{F} \).

To recognize when a morphism \( f: \mathcal{E} \to \mathcal{F} \) of \( \mathcal{F} \)-toposes is local, it will be useful to have a criterion which does not require us to consider the \( \mathcal{E} \)-indexing of the right adjoint to \( f_* \) (it is tacitly assumed throughout that all the functors we consider are \( \mathcal{F} \)-indexed). On the way to this, we require a lemma which ought to be a well-known triviality, but which we cannot remember hearing or seeing before.

1.3. LEMMA. Let \( L: \mathcal{C} \to \mathcal{D}, R: \mathcal{D} \to \mathcal{C} \) be functors with \( L \perp R \). If there exists some natural isomorphism between \( RL \) and the identity functor on \( \mathcal{C} \), then the unit of the adjunction is an isomorphism.

**Proof.** We may transport the monad structure on \( RL \), arising from the adjunction, along the given isomorphism to obtain a monad structure \((\eta, \mu)\) on \( \text{id}_\mathcal{C} \). Now one of the monad identities tells us that \( \mu \eta \) is the identity natural transformation on \( \text{id}_\mathcal{C} \); but by the naturality of either \( \eta \) or \( \mu \) we must have \( \mu \eta = \eta \mu \). So \( \eta \) and \( \mu \) are inverse isomorphisms; the result follows by transporting back along the given isomorphism \( RL \to \text{id}_\mathcal{C} \).

Recall that a geometric morphism \( f: \mathcal{F} \to \mathcal{E} \) is said to be connected if \( f^* \) is full and faithful (cf. [18, 1.12]). By a well-known lemma on adjunctions, this is equivalent to the unit map \( \text{id}_\mathcal{E} \to f_* f^* \) being an isomorphism; by the lemma just proved, it is equivalent to the existence of some isomorphism \( \text{id}_\mathcal{E} \to f_* f^* \). We thus have

1.4. PROPOSITION. The morphism \( f: \mathcal{F} \to \mathcal{E} \) is local if and only if \( f \) is connected and \( f_* \) has a right adjoint.

**Proof.** If \( f_* \) has a right adjoint, then it is certainly the inverse image of a morphism of \( \mathcal{F} \)-toposes \((c: \mathcal{E} \to \mathcal{F}, \text{ say})\), since it preserves finite limits. The condition for \( c \) to be a morphism of \( \mathcal{E} \)-toposes is that there should be an isomorphism \( fc \equiv \text{id}_\mathcal{E} \), which by the remarks above is equivalent to \( f \) being connected.

Let \( f: \mathcal{F} \to \mathcal{E} \) be a local map of \( \mathcal{F} \)-toposes. The morphism \( \mathcal{E} \to \mathcal{F} \), which was christened \( c \) in the proof of the last proposition, is called the centre of \( f \) (cf. [11, VI, 8.4.6]). If \( p: \mathcal{E} \to \mathcal{F} \) is any other morphism of \( \mathcal{E} \)-toposes, then we have a
natural transformation

\[ c^* \cong p^* f^* c^* = p^* f^* f_* \overset{p^* \varepsilon}{\longrightarrow} p^* , \]

where \( \varepsilon \) is the counit of \((f^* \dashv f_*)\); it is not hard to see that this makes \( c \) into an initial object in the category \( \mathcal{L}\mathcal{O}\mathcal{P}/\mathcal{E}(\mathcal{E}, \mathcal{F}) \) of sections of \( f \). The possession of an initial section does not, however, characterize local maps, even those with codomain \( \mathcal{S} \) (as may be seen by taking \( \mathcal{F} \) to be the coproduct of \( \mathcal{S} \) and an \( \mathcal{S} \)-topos with no points). But if we ‘stabilize’ the condition under change of base, we do get a characterization of local toposes:

1.5. Theorem. Let \((\mathcal{S} \overset{\gamma}{\rightarrow} \mathcal{S})\) be an \( \mathcal{S} \)-topos, \( c \) a point of \( \mathcal{S} \). Then the following are equivalent:

(i) \( \mathcal{S} \) is local over \( \mathcal{S} \), and \( c \) is its centre;

(ii) \( c \) is ‘universally initial’ among points of \( \mathcal{S} \); that is, for any \( \mathcal{S} \)-topos \((\mathcal{G} \overset{\delta}{\rightarrow} \mathcal{S})\), the composite \( c\delta \) is initial in the category \( \mathcal{L}\mathcal{O}\mathcal{P}/\mathcal{I}(\mathcal{G}, \mathcal{F}) \) of ‘\( \mathcal{G} \)-valued points’ of \( \mathcal{S} \).

Proof. (i) \( \Rightarrow \) (ii). Let \( p: \mathcal{G} \rightarrow \mathcal{F} \) be any \( \mathcal{G} \)-valued point of \( \mathcal{F} \). Clearly, natural transformations \((c\delta)^* = \delta^* c^* \rightarrow p^* \) correspond bijectively to natural transformations \( \gamma_* = c^* \rightarrow \delta_* p^* \). But \( \gamma_* \) may be identified with the representable functor \( \text{hom}_{\mathcal{F}}(1_{\mathcal{G}}, -) \), so by the Yoneda lemma these correspond to elements of \( \delta_* p^*(1_{\mathcal{G}}) \cong \delta_*(1_{\mathcal{G}}) \equiv 1_{\mathcal{G}} \). Since \( 1_{\mathcal{G}} \) has a unique element, we deduce that \( c\delta \) is initial in \( \mathcal{L}\mathcal{O}\mathcal{P}/\mathcal{I}(\mathcal{G}, \mathcal{F}) \).

(ii) \( \Rightarrow \) (i). Consider the case where \( \mathcal{F} = \mathcal{G} \), \( \delta = \gamma \); taking \( p \) to be the identity geometric morphism, we get a natural transformation \((c\gamma)^* = \gamma^* c^* \rightarrow \text{id}_{\mathcal{G}} \), or equivalently \( c^* \rightarrow \gamma_* \). But since \( \gamma_* \) is representable and \( c^*(1_{\mathcal{G}}) \equiv 1_{\mathcal{G}} \), we also have a natural transformation \( \gamma_* \rightarrow c^* \). The composite \( \gamma_* \rightarrow c^* \rightarrow \gamma_* \) is the identity, since it corresponds to an endomorphism of the representing object \( 1_{\mathcal{G}} \); and the composite \( c^* \rightarrow \gamma_* \rightarrow c^* \) is the identity, since it is an endomorphism of the initial object of \( \mathcal{L}\mathcal{O}\mathcal{P}/\mathcal{I}(\mathcal{G}, \mathcal{F}) \). Thus we have shown that \( c^* \cong \gamma_* \).

Remark. So far as we are aware, the first explicit appearance of condition (ii) of Theorem 1.5 was in Proposition 5.9 of [15], where it was verified (by ad hoc methods) for the particular local topos considered in that paper.

As a consequence of Theorem 1.5, we may prove the result stated in Example 1.2(d): for if \( M \) is a term model of a theory \( T \), then it is immediate from the definition that \( M \) (or rather \( \delta^* M \)) is initial in the category of \( \mathcal{G} \)-valued models of \( T \), for any \( \mathcal{G} \). The converse result is true ‘modulo uncertainty of language’: if \( M \) is a universally initial model of \( T \), then it becomes a term model in some enrichment of the language of \( T \).

As another consequence of 1.5, we have

1.6. Corollary. Let \( f: Y \rightarrow X \) be a continuous map of sober topological spaces. Then the induced morphism \( f: \text{Sh}(Y) \rightarrow \text{Sh}(X) \) is local if and only if there exists a continuous section \( c: X \rightarrow Y \) of \( f \) with \( cf(y) \leq y \) for all \( y \in Y \) (equivalently, such that \( c(x) \) is the least element of the fibre \( f^{-1}(x) \), for each \( x \in X \)).
Proof. The necessity of the condition follows from 1.5 on taking \( \mathcal{S} = \text{Sh}(X) \), \( \mathcal{F} = \text{Sh}(Y) \) and \( \mathcal{G} = \text{Set} \) (recalling that, for sober \( X \) and \( Y \), the category of geometric morphisms \( \text{Sh}(X) \to \text{Sh}(Y) \) is equivalent to the poset of continuous maps \( X \to Y \) with the (pointwise) specialization ordering—see [13, 7.24]). The sufficiency does not follow immediately from 1.5 as stated: all we can conclude from the condition given here is that condition (ii) of 1.5 holds whenever the topos \( \mathcal{G} \) is spatial. But in the proof of \( (ii) \Rightarrow (i) \) in 1.5, we used only the particular cases \( \mathcal{G} = \mathcal{F} \) and \( \mathcal{G} = \mathcal{S} \) of (ii), both of which are covered by what we know.

It follows from 1.6 that the only maps between Hausdorff spaces which induce local maps of sheaf toposes are homeomorphisms. However, there are non-trivial examples of maps satisfying the condition of 1.6: for example, let \( A \) be a normal distributive lattice, let \( X \) be the space of maximal ideals of \( A \), and let \( Y \) be its space of prime ideals. Then (cf. [17, II, 3.6]) the inclusion \( X \to Y \) admits a continuous retraction \( f : Y \to X \), which sends each prime ideal to the unique maximal ideal containing it; since the specialization ordering on \( Y \) is the opposite of the inclusion ordering on prime ideals, it follows that \( f \) satisfies the condition of 1.6.

Next, we turn to a characterization of local toposes in terms of sites, as was promised in 1.2(c). Let \( \mathcal{F} \) be an \( \mathcal{S} \)-topos, and let \( \mathcal{D} \) be a site (in \( \mathcal{S} \)) for \( \mathcal{F} \) whose underlying category \( \mathcal{D} \) has a terminal object \( t \). We say \( \mathcal{D} \) is local if (it is internally valid in \( \mathcal{S} \) that) whenever \( (d_i \to t)_{i \in I} \) is a cover in \( \mathcal{D} \), there exists \( i \in I \) such that \( d_i \to t \) has a section. Clearly, this condition holds if the topology on \( D \) is trivial.

1.7. PROPOSITION. For an \( \mathcal{S} \)-topos \( \mathcal{F} \), the following are equivalent:

(i) \( \mathcal{F} \) is local over \( \mathcal{S} \);

(ii) every subcanonical site for \( \mathcal{F} \) in \( \mathcal{S} \) (with a terminal object) is local;

(iii) there exists a local site for \( \mathcal{F} \) in \( \mathcal{S} \).

Proof. (i) \( \Rightarrow \) (ii). Let \( \mathcal{D} \) be a subcanonical site for \( \mathcal{F} \) with a terminal object; we identify its underlying category \( \mathcal{D} \) with a full subcategory of \( \mathcal{S} \), and its terminal object with \( 1_\mathcal{S} \). Since \( \mathcal{F} \) is local, the global section functor \( \text{hom}_\mathcal{S}(1_\mathcal{S}, -) \) preserves colimits and, in particular, epimorphic families; so if \( (d_i \to t)_{i \in I} \) is a cover in \( \mathcal{D} \), then \( (\text{hom}(1, d_i) \to 1)_{i \in I} \) is epimorphic in \( \mathcal{S} \), that is, some \( \text{hom}(1, d_i) \) must be inhabited.

(ii) \( \Rightarrow \) (iii). This is trivial.

(iii) \( \Rightarrow \) (i). Let \( \mathcal{D} \) be a local site for \( \mathcal{F} \). Then the global section functor \( \mathcal{F} \to \mathcal{S} \) may be identified with the functor ‘evaluate at \( t \)’, where \( t \) is the terminal object of \( \mathcal{D} \), and its right adjoint is given by right Kan extension along \( t : 1 \to D \). It is easy to verify that this latter functor takes values in the category of sheaves on \( \mathcal{D} \); explicitly, its value at \( S \in \text{ob} \mathcal{S} \) is the presheaf

\[
d \mapsto S^{\text{hom}(t,d)}
\]

which is a sheaf since \( \text{hom}_\mathcal{D}(t, -) \) sends covers to epimorphic families.

Once again, this proposition has an interpretation in terms of theories: if \( \mathcal{D} \) is the syntactic site obtained from a coherent theory \( \mathbb{T} \), then \( \mathcal{D} \) is local precisely when \( \mathbb{T} \) has the disjunction and existence properties. (We can generalize
'coherent' to 'geometric' here, provided we interpret the disjunction property as applying to arbitrary \( \mathcal{S} \)-indexed disjunctions.) Thus we obtain as a byproduct the result that (modulo uncertainty of language) a geometric theory has the disjunction and existence properties precisely when it has a term model.

Proposition 1.7 has a 'relative' form, which it will be convenient to make explicit. For this purpose, we recall the notion of continuous fibration which was introduced in [25]: if \( \mathcal{C} \) and \( \mathcal{D} \) are sites, a continuous fibration from \( \mathcal{D} \) to \( \mathcal{C} \) is a pair of functors

\[
\begin{array}{c}
\mathcal{D} \\
\downarrow P \\
\downarrow T \\
\mathcal{C}
\end{array}
\]

such that \( T \) is flat, continuous and a right adjoint right inverse to \( P \), and such that for every \( d \in \text{ob} \mathcal{D} \) the functor \( P/d : D/d \rightarrow C/Pd \) has a right adjoint right inverse \( T_d \). Such a fibration induces a geometric morphism \( f : \text{Sh}(\mathcal{D}) \rightarrow \text{Sh}(\mathcal{C}) \), where \( f_* \) is the functor 'compose with \( T \)' and \( f^* \) is 'compose with \( P \) and then sheafify'. In [25] it was shown that, given a morphism \( f : \mathcal{F} \rightarrow \mathcal{G} \) of bounded \( \mathcal{S} \)-toposes and a site \( \mathcal{C} \) for \( \mathcal{S} \), one can find a site \( \mathcal{D} \) for \( \mathcal{F} \) such that \( f \) is induced by a continuous fibration from \( \mathcal{D} \) to \( \mathcal{C} \). (Specifically, \( \mathcal{D} \) is obtained by 'externalizing' an internal site for \( \mathcal{F} \rightarrow \mathcal{G} \) in \( \mathcal{S} \) whose underlying category has a terminal object.)

Let \( T : \mathcal{C} \rightarrow \mathcal{D} \) be any functor between (the underlying categories of) sites in \( \mathcal{S} \). We say that \( T \) has the covering lifting property if any cover of the form \( (d_i \rightarrow Tc_i)_{i \in I} \) in \( \mathcal{D} \) has a refinement of the form \( (Tc_i \rightarrow Tc)_{i \in I} \), where \( (c_i \rightarrow c)_{i \in I} \) is a cover in \( \mathcal{C} \).

1.8. THEOREM. Let \( f : \mathcal{F} \rightarrow \mathcal{G} \) be a morphism of bounded \( \mathcal{S} \)-toposes, and let \( \mathcal{C} \) be a site of definition for \( \mathcal{G} \) over \( \mathcal{S} \). Then the following are equivalent:

(i) \( f \) is local;
(ii) there is a site \( \mathcal{D} \) for \( \mathcal{F} \) such that \( f \) is induced by a continuous fibration \( (P, T) \) where \( T \) has the covering lifting property;
(iii) there is a site \( \mathcal{D} \) for \( \mathcal{F} \) such that \( f \) is induced by a flat continuous functor \( T : \mathcal{C} \rightarrow \mathcal{D} \) which is full and faithful and has the covering lifting property.

Proof. (i) \( \Rightarrow \) (ii). Let \( \mathcal{B} \) be a local site for \( \mathcal{F} \rightarrow \mathcal{G} \), and let \( \mathcal{D} = \mathcal{C} \times \mathcal{B} \) be the semidirect product as defined in [25]. We recall from [25] that the objects of \( \mathcal{D} \) are pairs \( (c, b) \), where \( c \in \text{ob} \mathcal{C} \) and \( b \) is an object of \( B(c) \), and that the functor \( T \) is defined by \( Tc = (c, t) \), where \( t \) is the terminal object of \( B(c) \). Now suppose we have a covering family \( ((c_i, b_i) \rightarrow (c, t))_{i \in I} \) in \( \mathcal{D} \); then since \( \mathcal{B} \) is local, we have

\[
c \models (\exists i \in I)(b_i \rightarrow t \text{ has a section}),
\]

which means that there exists a cover \( (c_j \rightarrow c)_{j \in J} \) in \( \mathcal{C} \), and, for each \( j \in J \), an index \( i(j) \in I \) such that the restriction to \( c_j \) of \( b_{i(j)} \rightarrow t \) admits a section. The family \( (Tc_j \rightarrow Tc)_{j \in J} \) is thus a refinement of \( ((c_i, b_i) \rightarrow Tc)_{i \in I} \), and we have verified that \( T \) has the covering lifting property.

(ii) \( \Rightarrow \) (iii). This is trivial.

(iii) \( \Rightarrow \) (i). Given \( T \) as in (iii), we can describe \( f_* \) as the functor 'compose with \( T \)' (and \( f^* \) as the left Kan extension of \( C \rightarrow D \rightarrow \text{Sh}(\mathcal{D}) \) along the Yoneda embedding \( C \rightarrow \text{Sh}(\mathcal{C}) \)). As in the last part of the proof of 1.7, the obvious
candidate for the right adjoint $f^+$ of $f_*$ is the functor defined at the presheaf level by right Kan extension along $T$, that is,

$$f^+(X)(d) = \text{hom}(D(T-, d), X),$$

and we have to verify that this functor sends sheaves on $C$ to sheaves on $D$. But this follows from the covering lifting property: given a cover $(\beta_i; d_i \to d)_{i \in I}$ in $D$ and a compatible family of natural transformations $\tau_i: D(T-, d_i) \to X$, we have to construct $\tau: D(T-, d) \to X$ such that for each $i \in I$ the diagram

$$\begin{array}{ccc}
D(T-, d_i) & \xrightarrow{\tau_i} & X \\
\downarrow D(T-, \beta_i) & & \downarrow \tau \\
D(T-, d) & \xrightarrow{\tau} & X
\end{array}$$

commutes. Given $\gamma: Tc \to d$, let $(\alpha_j; c_j \to c)_{j \in J}$ be a cover in $C$ whose image under $T$ refines the pullback of $(\beta_i)_{i \in I}$ along $\gamma$, that is, such that for each $j \in J$ there exist $i(j) \in I$ and a commutative diagram

$$\begin{array}{ccc}
Tc_j & \xrightarrow{T\alpha_j} & Tc \\
\downarrow \delta_j & & \downarrow \gamma \\
d_{i(j)} & \xrightarrow{\beta_{i(j)}} & d
\end{array}$$

Then we have an element $\tau_{i(j)}(\delta_j)$ of $X(c_j)$, which may be checked to be independent of the choice of $i(j)$ and of $\delta_j$, and to be part of a compatible family of elements of the $X(c_j)$ ($j \in J$), so determining a unique element of $X(c)$ which we define to be $\tau(\gamma)$. The verification that $\tau$ is natural, and the remaining details of the proof, are straightforward.

We have thus shown that $f_*$ has a right adjoint $f^+$; and since $f^+$ is just (the restriction to sheaf subcategories of) the right Kan extension along the full and faithful functor $T$, it is itself full and faithful. Hence the left adjoint $f^*$ of $f_*$ must also be full and faithful, that is, $f$ is connected. So by Proposition 1.4, $f$ is local.

**Remark.** If we are willing to vary our site of definition for $\mathcal{E}$, we may add a fourth equivalent condition to those of Theorem 1.8:

(iv) $f$ is induced by a continuous fibration $(P, T)$ such that $T$ has a right adjoint which preserves covers.

Clearly, if $T$ has such a right adjoint $R$, then $T$ has the covering lifting property: for if $(d_i \to Tc_i)_{i \in I}$ is a cover in $D$, then $(Rd_i \to RTc_i \equiv c)_{i \in I}$ is a cover in $C$ (note that $RT \equiv \text{id}_C$, since $T$ is full and faithful) whose image under $T$ refines the given cover. In the converse direction, the construction of $R$ given (ii) of Theorem 1.8 requires certain completeness assumptions on $C$, and so we may have to enlarge our originally chosen site for $\mathcal{E}$; we omit the details, since we shall not need to use condition (iv) hereafter.

1.9. **Examples.** (a) Let $\mathcal{F} = \text{Sh}(\mathcal{T}op)$ be the gros topos of topological spaces, as in 1.2(b), and let $X$ be a space. If $\mathcal{O}(X)$ denotes the poset of open subsets of $X$ (with its canonical topology), then the inclusion functor $\mathcal{O}(X) \to \mathcal{T}op/X$ satisfies condition (iii) of 1.8; so the canonical geometric morphism $\mathcal{F}/yX \to \text{Sh}(X)$ is
local. (Here $y$ denotes the Yoneda embedding $\mathcal{C}^{op} \to \mathcal{I} \). Compare [11, IV, 4.10].

(b) Similarly, let $\mathcal{Z} = \mathcal{Sh}(\mathcal{Z})$ be the Zariski topos over a field $k$ (that is, let $\mathcal{Z}$ be the dual of the category of finitely-presented $k$-algebras, with the topology defined by surjective families of Zariski-open inclusions). To an object $A = k[x_1, \ldots, x_n]/I$ of $\mathcal{Z}$, we may associate the algebraic variety

$$V_A = \{ \alpha \in K^n \mid f(\alpha) = 0 \text{ for all } f \in I \},$$

where $K$ is a suitable algebraically closed field containing $k$. Then to a basic Zariski-open subset

$$D(f) = \{ \alpha \in V_A \mid f(\alpha) \neq 0 \}$$

of $V_A$ we may associate the Zariski-open inclusion which is the dual of the ring homomorphism $A \to A[f^{-1}]$; this defines a functor

$$\mathcal{B}(V_A) \to Z/A$$

(where $\mathcal{B}(V_A)$ is the poset of basic open subsets of $V_A$), which satisfies condition (iii) of 1.8, and so we have a local morphism

$$\mathcal{Z}/yA \to \mathcal{Sh}(V_A).$$

(c) Again, let $\mathcal{G} = \mathcal{Sh}(\mathcal{G})$ be the topos introduced as a model of Synthetic Differential Geometry by Dubuc [4] (see also [27, 28]). Recall that the objects of (the category underlying) the site $\mathcal{G}$ are the duals $\tilde{A}$ of $\mathcal{C}^\infty$-rings of the form $A = C^\infty(\mathbb{R}^n)/I$, where $I$ is a germ-determined ideal; with such a $\mathcal{C}^\infty$-ring we may associate the subset

$$Z(I) = \{ \alpha \in \mathbb{R}^n \mid f(\alpha) = 0 \text{ for all } f \in I \}$$

of $\mathbb{R}^n$. Arguments similar to those of (b) then show that we have a local morphism $\mathcal{G}/y\tilde{A} \to \mathcal{Sh}(Z(I))$.

The examples in 1.9 are all instances of a general phenomenon, which may provide a clue to what the general definition of a 'gros topos' ought to be. Let $\mathcal{C}$ be a category with finite limits, and let $L$ be a class of morphisms of $\mathcal{C}$ such that

(i) $L$ contains isomorphisms, and is closed under composition, and

(ii) $L$ is stable under pullback along arbitrary morphisms of $\mathcal{C}$.

We define a topology on $\mathcal{C}$ by saying that $(c_i \to c)_{i \in I}$ is a cover if each $c_i \to c$ is in $L$ and the family of set maps $(\Gamma c_i \to \Gamma c)_{i \in I}$ is surjective, where $\Gamma = \mathcal{C}(1, -)$ is the global sections functor. (Note that this automatically makes $\mathcal{C}$ a local site. This situation has been studied by Penon [30] and Dubuc [5], and is closely related to that considered by Grothendieck and Verdier in [11, IV, 4.10.6].) For any object $c$ of $\mathcal{C}$, we define a topology on $\Gamma c$ by specifying that the basic open subsets are the images of maps $\Gamma c' \to \Gamma c$, where $(c' \to c) \in L$; note that if $c \to c'$ is any morphism of $\mathcal{C}$, then the induced map $\Gamma c \to \Gamma c'$ is continuous. Then we have

1.10. PROPOSITION. Let $\mathcal{C}$ and $L$ be as above, and let $\mathcal{E}$ denote the topos of sheaves on $\mathcal{C}$. Then, for each object $c$ of $\mathcal{C}$, there is a local morphism $\mathcal{E}/yc \to \mathcal{Sh}(\Gamma c)$, which is natural in $c$; that is, for each $c \to c'$ in $\mathcal{C}$, the diagram

$$\mathcal{E}/yc \to \mathcal{E}/yc' \to \mathcal{Sh}(\Gamma c) \to \mathcal{Sh}(\Gamma c')$$

commutes up to isomorphism.
Proof. Following Dubuc [5], we define a functor $T: \mathcal{O}(\Gamma c) \to \mathcal{E}/\mathcal{Y}c$ which satisfies condition (iii) of 1.8 (for the canonical topology on $\mathcal{E}/\mathcal{Y}c$): if $U$ is an open subset of $\Gamma c$, we define $T(U) \to \mathcal{Y}c$ to be the union of the images of all maps $\mathcal{Y}c' \to \mathcal{Y}c$ induced by morphisms $c' \to c$ in $L$ such that $\Gamma c' \to \Gamma c$ factors through $U$. It is straightforward to verify that $T$ is flat, continuous and a full embedding; we shall show that it has the covering lifting property.

Let $(X_i \to T(U))_{i \in I}$ be an epimorphic family in $\mathcal{E}$. If $(\alpha: 1 \to c) \in U$, then, since $U$ is open, there exists $\beta: c' \to c$ in $L$ such that $\alpha$ factors as $\beta \alpha'$, and such that the image of $\Gamma \beta$ is contained in $U$ (so that $\gamma \beta$ factors through $T(U) \to \mathcal{Y}c$). Regarding this factorization as an element of $T(U)(c')$, we deduce that there is a cover $(\gamma_j: c_j \to c', j \in J)$ such that, for each $j$, the restriction of this element along $\gamma_j$ is in the image of some $X_i(c_j) \to T(U)(c_j)$. But since $(\gamma_j)_{j \in J}$ is a cover, $\alpha'$ must factor through some $\gamma_j$. Hence the open sets of the form $\text{Im}(\beta \gamma)$ (where $\beta$ as well as $\gamma$ is allowed to vary) form a cover of $U$; and the image of this cover under $T$ refines the given epimorphic family.

Finally, if $\alpha: c \to c'$ is any morphism of $C$, it is easy to verify that the diagram

$$
\begin{array}{ccc}
\mathcal{O}(\Gamma c') & \xrightarrow{\mathcal{O}(\Gamma \alpha)} & \mathcal{O}(\Gamma c) \\
\downarrow T' & & \downarrow T \\
\mathcal{E}/\mathcal{Y}c' & \xrightarrow{(\gamma \alpha)^*} & \mathcal{E}/\mathcal{Y}c
\end{array}
$$

commutes up to isomorphism, from which the naturality statement follows.

2. Stability properties of local maps

The class of local geometric morphisms is stable under many familiar constructions on $\mathfrak{X}\mathbf{op}$. We start with composition, which is fairly trivial.

2.1. Proposition. (i) Any equivalence is local.
(ii) A composite of local maps is local.
(iii) In a commutative triangle

$$
\begin{array}{ccc}
g & \xrightarrow{g} & \mathcal{F} \\
h & \searrow & \mathcal{E} \\
& & f
\end{array}
$$

of geometric morphisms, if $h$ is local and $g$ is connected, then $f$ is local. Moreover, if $c$ is the centre of $h$, then $gc$ is the centre of $f$.

Proof. (i) and (ii) are obvious. For (iii), we use Proposition 1.4; clearly, $f$ is connected if $g$ and $h$ are. Moreover, for objects $E, F$ of $\mathcal{E}$, $\mathcal{F}$ respectively, we have natural bijections

$$
\begin{array}{c}
E \to f_* F \\
f^* E \to F
\end{array}
$$

$$
\begin{array}{c}
h^* E \cong g^* f^* E \to g^* F \\
E \to h_* g^* F = c^* g^* F
\end{array}
$$

so $f_* = c^* g^*$ has a right adjoint $g_* c_*$. 
However, if we weaken the hypothesis ‘\( g \) is connected’ in 2.1(iii) to ‘\( g \) is surjective’, then the conclusion may fail, even if we add the further hypothesis that \( g \) is open (or even atomic).

2.2. Example. Let \( C \) and \( D \) be the finite categories

\[
\begin{array}{ccc}
C & \xrightarrow{\alpha} & D \\
\downarrow{\beta} & & \downarrow{\gamma} \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
C' & \xrightarrow{\alpha'} & D' \\
\downarrow{\beta'} & & \downarrow{\gamma'} \\
\end{array}
\]

respectively, and let \( g: D \to C \) be the obvious quotient map. Then \( g \) is surjective on objects and a discrete fibration, so the induced map \( \mathcal{S}^{D\text{op}} \to \mathcal{S}^{C\text{op}} \) is surjective and atomic. Now \( D \) has a terminal object, so \( \mathcal{S}^{D\text{op}} \) is local over \( \mathcal{S} \) by Example 1.2(c). But \( \mathcal{S}^{C\text{op}} \) is not local over \( \mathcal{S} \): the global sections functor \( \lim_{\mathcal{S}} \) sends a presheaf \( P \) to the equalizer of \( P(\alpha) \) and \( P(\beta) \), and so fails to preserve the epimorphism \( \text{hom}_{\mathcal{S}}(\mathcal{S}, d) \to 1 \).

Likewise, 2.1(iii) does not have a ‘dual’: in a commutative triangle as in 2.1(iii), even if we assume that both \( f \) and \( h \) are local, and further that \( g \) has a right inverse, there is still no reason why \( g \) should be local.

2.3. Example. Let \( \mathcal{E} = \mathcal{S} \), and let \( \mathcal{F} \) and \( \mathcal{G} \) be the classifying toposes (over \( \mathcal{S} \)) for the theories of abelian groups and of rings (not necessarily with 1), respectively. Then \( \mathcal{F} \) and \( \mathcal{G} \) are both local over \( \mathcal{S} \), by Example 1.2(d). Let \( g: \mathcal{G} \to \mathcal{F} \) correspond to the underlying additive group of the generic ring, and \( q: \mathcal{F} \to \mathcal{G} \) to the ring obtained by equipping the generic abelian group with the zero multiplication. Clearly, \( gq = \text{id}_{\mathcal{G}} \) (and \( g \) and \( q \) preserve the centres of \( \mathcal{F} \) and \( \mathcal{G} \); but \( q \) is not left adjoint to \( g \), and in fact \( g \) is not even connected. \( g \) does have a left adjoint, corresponding to the free ring generated by the generic abelian group, but the unit of the adjunction is not an isomorphism.)

Next, we consider pullbacks.

2.4. Theorem. Suppose that

\[
\begin{array}{ccc}
\mathcal{F}' & \xrightarrow{b} & \mathcal{F} \\
\downarrow{f'} & & \downarrow{f} \\
\mathcal{E}' & \xrightarrow{a} & \mathcal{E} \\
\end{array}
\]

is a pullback square with \( f \) local. Then \( f' \) is local. Moreover, if \( c \) is the centre of \( f \), then the centre of \( f' \) is the morphism \( c': \mathcal{E}' \to \mathcal{F}' \) induced by \( \text{id}: \mathcal{E}' \to \mathcal{E} \) and \( ca: \mathcal{E} \to \mathcal{F} \).

Proof. For this we use Theorem 1.5. Since pullbacks in \( \mathcal{X}\text{op} \) are \( \text{Cat} \)-enriched, for any \( \mathcal{E}' \)-topos \( (\mathcal{G} \xrightarrow{g} \mathcal{E}') \) we have an equivalence of categories

\[
\mathcal{X}\text{op}/\mathcal{E}'(\mathcal{G}, \mathcal{F}') = \mathcal{X}\text{op}/\mathcal{E}(\mathcal{G}, \mathcal{F})
\]

induced by composition with \( b \). In particular, \( c'g \) is initial in the former if and only if \( bc'g = cag \) is initial in the latter.
2.5. REMARK. Note in passing that it follows from the last sentence of the statement of 2.4 that the Beck condition \( a^*f_* \equiv f^*_b \) holds for any pullback square as in 2.4.

2.6. THEOREM. In the pullback square of 2.4, suppose that \( a \) is an open surjection and \( f' \) is local. Then \( f \) is local.

Proof. We make heavy use of the main result of [21] (see also [24]) that open surjections are effective descent morphisms. Form the diagram

\[
\begin{array}{ccc}
\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}' & \xrightarrow{q_0} & \mathcal{F}' \\
\downarrow f'' & & \downarrow f' \\
\mathcal{G}' \times_{\mathcal{G}} \mathcal{G}' & \xrightarrow{p_0} & \mathcal{G}'
\end{array}
\]

where the \( p_i \) and \( q_i \) are projections, and \( f'' \) is the pullback of \( f \) along \( ap_0 \equiv ap_1 \) (and hence the pullback of \( f' \) along either \( p_0 \) or \( p_1 \)). Then it follows from 2.4 that \( f'' \) is local, and that its centre \( c'' \) is the pullback of the centre \( c' \) of \( f' \) along either \( q_0 \) or \( q_1 \). Thus we have natural isomorphisms

\[
bc'p_0 \equiv bq_0c'' \equiv bq_1c'' \equiv bc'p_1,
\]

and the composite of these is easily seen to satisfy the coherence conditions required to give a factorization of \( c'^*b^* \) through the category of objects of \( \mathcal{G}' \) equipped with descent data for \( a \); that is, we have a functor \( c^* \colon \mathcal{F} \to \mathcal{G} \) such that \( a^*c^* \equiv c'^*b^* \). Since \( \mathcal{G} \), \( \mathcal{F} \) are comonadic over \( \mathcal{S}' \), \( \mathcal{F}' \) respectively, we may now apply a standard adjoint-lifting result [12] to obtain a right adjoint \( c^* \) for \( c^* \); and the natural isomorphisms \( f'^*b^* \equiv c'^*b^* \) and \( c'^*f'^*a^* \equiv a^* \) may be 'lifted' to isomorphisms \( f_* \equiv c^* \) and \( c^*f_* \equiv \text{id}_{\mathcal{G}} \), since they commute with the appropriate descent data.

The techniques used in the above proof are closely related to those used in [25] to establish a similar descent theorem for stably connected morphisms. The next result (or at least its Corollary) is also reminiscent of results proved in [25].

2.7. PROPOSITION. Let \( (\mathcal{E}_i)_{i \in I} \) be (the vertices of) a diagram of \( \mathcal{S}' \)-toposes (indexed by a small category \( I \)) for which the Cat-enriched limit \( \mathcal{E} = \lim_I \mathcal{E}_i \) exists in \( \mathcal{X}_\mathcal{Op}/\mathcal{S}' \). If each \( \mathcal{E}_i \) is local over \( \mathcal{S}' \), and the transition maps \( \mathcal{E}_i \to \mathcal{E}_j \) in the diagram are all connected, then \( \mathcal{E} \) is local over \( \mathcal{S}' \).

Proof. Let \( c_i \) be the centre of \( \mathcal{E}_i \). By 2.1(iii), the transition maps in the diagram preserve the \( c_i \) (up to isomorphism), so the latter define a point \( c \) of the limit. The fact that \( c \) is universally initial among points of \( \mathcal{E} \) follows immediately from the fact that each \( c_i \) is universally initial.

2.8. COROLLARY. Let \( (\mathcal{E}_i)_{i \in I} \) be a filtered inverse system of \( \mathcal{S}' \)-toposes, with inverse limit \( \mathcal{E} = \lim_{\leftarrow} \mathcal{E}_i \). If the transition maps \( \mathcal{E}_i \to \mathcal{E}_j \) of the system are all local, then the projections \( \mathcal{E} \to \mathcal{E}_i \) are also local.

Proof. By working in \( \mathcal{X}_\mathcal{Op}/\mathcal{E}_i \) with the diagram indexed by \( I/i \) and noting that
this has the same limit as the original \( I \)-indexed diagram), we may reduce this to the case considered in 2.7.

Next, we consider retracts:

2.9. PROPOSITION. A retract of a local morphism is local; i.e., if

\[
\begin{array}{ccc}
\mathcal{B} & \overset{j}{\to} & \mathcal{F} \\
\downarrow f & & \downarrow g \\
\mathcal{A} & \overset{i}{\to} & \mathcal{E} \\
\end{array}
\]

is a commutative diagram with \( ri = \text{id}_\mathcal{A} \) and \( sj = \text{id}_\mathcal{B} \), and \( g \) is local, then \( f \) is local.

Proof. By Theorem 2.4, the pullback of \( g \) along \( i \) is local, and \( f \) is a retract of it in \( \mathcal{X}_{\text{op}}/\mathcal{A} \). So we may reduce to the case when \( \mathcal{A} = \mathcal{E} \) and \( i \) and \( r \) are identity morphisms; now, for any \( \mathcal{E} \)-topos \( \mathcal{G} \), \( \mathcal{X}_{\text{op}}/\mathcal{G}(\mathcal{G}, \mathcal{B}) \) is a retract of \( \mathcal{X}_{\text{op}}/\mathcal{G}(\mathcal{G}, \mathcal{F}) \).

In general, if \( \mathcal{C} \) and \( \mathcal{D} \) are categories and \( \mathcal{C} \to \mathcal{D} \to \mathcal{C} \) are functors making \( \mathcal{C} \) a retract of \( \mathcal{D} \), \( R \) need not preserve the initial object 0 of \( \mathcal{D} \) (if it has one); but if we write \( \alpha \) for the unique morphism 0 \( \to PI0 \) in \( \mathcal{D} \), and \( \beta \) for the composite

\[
R \alpha \to R IR0 \equiv R0
\]

in \( \mathcal{C} \), then \( \beta \) is easily shown to be idempotent, and if it splits then its image is an initial object of \( \mathcal{C} \) (cf. [14, Lemma 1.5]). Now idempotents do split in categories of the form \( \mathcal{X}_{\text{op}}/\mathcal{G}(\mathcal{G}, \mathcal{B}) \), since they have filtered (\( \mathcal{C} \)-indexed) colimits (cf. [13, Corollary 7.14]); so this category has an initial object for every \( \mathcal{G} \). Moreover, the construction of this initial object is 'natural in \( \mathcal{G} \)' in a suitable sense, so we deduce that \( f \) has a universally initial section.

2.10. COROLLARY. An \( \mathcal{I} \)-topos \( \mathcal{E} \) is local if and only if it is a retract of \( \mathcal{E} \) (cf. Example 1.2(e)).

Proof. One direction is immediate from 1.2(e) and Proposition 2.9. Conversely, suppose \( \mathcal{E} \) is local. Regarding \( \mathcal{E} \) as the lax colimit of the diagram \( \mathcal{E} \to \mathcal{F} \) in \( \mathcal{X}_{\text{op}} \) (cf. [31]), we see that geometric morphisms \( \mathcal{E} \to \mathcal{F} \) (for any \( \mathcal{F} \)) correspond to diagrams of the form

\[
\begin{array}{ccc}
\mathcal{E} & \overset{\gamma}{\to} & \mathcal{F} \\
\downarrow \alpha & & \downarrow \beta \\
\mathcal{G} & & \mathcal{G}
\end{array}
\]

But we have such a diagram with \( \mathcal{F} = \mathcal{E} \), namely

\[
\begin{array}{ccc}
\mathcal{E} & \overset{\text{id}}{\to} & \mathcal{E} \\
\downarrow \gamma & & \downarrow \gamma \\
\mathcal{G} & \overset{\text{id}}{\to} & \mathcal{G}
\end{array}
\]

where \( \gamma \) is the centre of \( \mathcal{E} \); so we have a morphism \( r: \mathcal{E} \to \mathcal{E} \), which is clearly a one-sided inverse for the inclusion \( \mathcal{E} \to \mathcal{E} \).
Remark. In the proof of 2.10, we have apparently used even less of the assertion ‘$c$ has a universally initial point’ than we did in proving the implication (ii)$\Rightarrow$(i) of 1.5: merely the fact that there exists a natural transformation $c\gamma\to\text{id}_g$. From this fact alone, we cannot deduce that $c^*\equiv\gamma_*$, but merely that $\gamma_*$ is a retract of $c^*$; however, since idempotents split in $\text{Top}/\mathcal{P}(\mathcal{S},\mathcal{E})$, this is sufficient to prove that $\gamma$ is local.

Another important stability property of local morphisms concerns the hyperconnected-localic factorization (cf. [16]). It is, however, not true in full generality that local morphisms are stable under this factorization: that is, given a commutative diagram

$$
\begin{array}{ccc}
G' & \rightarrow & G \\
\downarrow h & & \downarrow f \\
G & \rightarrow & G
\end{array}
$$

(*)&

in which $f$ and $h$ are local and the rows are hyperconnected-localic factorizations, we cannot conclude that $g$ is local. For example, let $E$ be the object classifier (over $\mathcal{S}$) and $F$ the classifier for the theory of inhabited objects; then $E$ is local over $\mathcal{S}$ by 1.2(d) or (e), but $F$ is not, since the category of non-empty sets has no initial object. However, $F$ is hyperconnected over $\mathcal{S}$, since it may be represented as the topos of presheaves on a strongly connected category (the dual of the category of non-empty finite sets), and the morphism $F\rightarrow E$ which classifies the generic inhabited object is localic (cf. [19]). So we have a diagram of the form (*):

$$
\begin{array}{ccc}
F & \rightarrow & E \\
\downarrow \text{id} & & \downarrow \text{id} \\
F & \rightarrow & F
\end{array}
$$

in which the two outer vertical arrows are local, but the middle one is not.

The best that we can do in this direction concerns the special case of (*) in which the morphism $f$ is an equivalence:

2.11. Proposition. Let $f: \mathcal{F}\rightarrow\mathcal{E}$ be a local map of $\mathcal{S}$-toposes, and let $E\rightarrow E'\rightarrow\mathcal{F}$, $F\rightarrow F'\rightarrow F$ be hyperconnected-localic factorizations. Then the morphism $F'\rightarrow E'$ induced by $f$ is local.

Proof. The functors $f^*$ and $f_*$ both preserve finite limits, and so restrict to functors between the posets of subobjects of 1 in $E$ and $F$. If we take these posets, with their canonical topologies, as sites of definition for $E'$ and $F'$, then it is easy to see that the restrictions of $f^*$ and $f_*$ satisfy the conditions of 1.8(ii): the fact that they form a continuous fibration is immediate, and $f^*$ has the covering lifting property because $f_*$ preserves covers. So the result follows from Theorem 1.8.

We conclude this section with a result which enables us to describe such things as the object of (Dedekind) real numbers and the internal Baire space in gros toposes of the kind considered in Proposition 1.10. Many particular cases of this result have been considered before [6, Theorem; 27, Theorem 5.1; 5, Theorem
2.12], but the following seems to be the most natural and general formulation of the idea.

First, some terminology. Recall that a locale \( X \) is said to be *totally unordered* or a \( T_U \)-locale [17, III, 1.5] if, for any parallel pair of locale maps

\[
Y \xrightarrow{f} X, \quad g \leq f
\]

\( f \leq g \) (in the specialization order) implies that \( f = g \). (This notion has been called '\( T_U \)' by other writers [6, 27]; but see [29] for reasons why this name should be avoided.) We shall introduce the term *grouplike topos* for an \( \mathcal{S} \)-topos \( \mathcal{E} \) such that, for any \( \mathcal{S} \)-topos \( \mathcal{S} \), \( \mathcal{S}\text{op}/\mathcal{S}(\mathcal{E}, \mathcal{S}) \) is a groupoid. Recalling that the localic reflection \( \mathcal{S} \rightarrow \text{Sh}(\text{Sub}_\mathcal{S}(1)) \) induces, for any localic \( \mathcal{S} \)-topos \( \mathcal{E} \), an equivalence

\[
\mathcal{S}\text{op}/\mathcal{S}(\mathcal{E}, \mathcal{S}) \cong \mathcal{S}\text{op}/\mathcal{S}(\text{Sh}(\text{Sub}_\mathcal{S}(1)), \mathcal{S}) \cong \text{Loc}(\text{Sub}_\mathcal{S}(1), \text{Sub}_\mathcal{S}(1)),
\]

we see that the topos of sheaves on a \( T_U \)-locale is grouplike; examples of non-localic grouplike toposes include the topos of \( G \)-sets for any group \( G \) (or more generally the topos of presheaves on a groupoid).

2.12. **Lemma.** Let \( f: \mathcal{F} \rightarrow \mathcal{E} \) be a local morphism of \( \mathcal{S} \)-toposes, and let \( \mathcal{E} \) be a grouplike \( \mathcal{S} \)-topos. Then composition with \( f \) induces an equivalence of categories

\[
\mathcal{S}\text{op}/\mathcal{S}(\mathcal{E}, \mathcal{S}) \rightarrow \mathcal{S}\text{op}/\mathcal{S}(\mathcal{F}, \mathcal{S}).
\]

**Proof.** The centre \( c \) of \( f \) is left adjoint to \( f \) in the 2-category \( \mathcal{S}\text{op} \), so the functor 'compose with \( c \)' is right adjoint to 'compose with \( f \)' But any adjunction between groupoids is an equivalence.

We further recall (cf. [22]) that the *internalization* of a locale \( X \) in an \( \mathcal{S} \)-topos \( \mathcal{E} = \text{Sh}(\mathcal{C}) \) is the sheaf \( X_{\mathcal{E}} \) defined by

\[
X_{\mathcal{E}}(c) = [\mathcal{S}\text{op}/\mathcal{S}(\mathcal{E}/yc, \text{Sh}(X))],
\]

where the square brackets denote 'set of isomorphism classes of objects of . . .'. For example, if \( X \) is the locale of 'formal' real numbers, we obtain in this way the object of Dedekind real numbers in \( \mathcal{E} \) [7]. Combining Lemma 2.12 with Proposition 1.10, we immediately obtain:

2.13. **Corollary.** Let \( \mathcal{E} = \text{Sh}(\mathcal{C}) \) be as in 1.10, and let \( X \) be any \( T_U \)-locale. Then the internalization of \( X \) in \( \mathcal{E} \) is isomorphic to the sheaf whose value at \( c \) is the set of continuous maps \( \Gamma c \rightarrow X \).

For a non-grouplike topos \( \mathcal{E} \), a local map \( \mathcal{F} \rightarrow \mathcal{E} \) will not in general give rise to an equivalence \( \mathcal{S}\text{op}/\mathcal{S}(\mathcal{E}, \mathcal{S}) \rightarrow \mathcal{S}\text{op}/\mathcal{S}(\mathcal{F}, \mathcal{S}) \); but it will still induce a bijection between the connected components of these two categories. For example, taking \( \mathcal{G} \) to be an Eilenberg–Mac Lane topos in the sense of Wraith [32], we recover the result (already well known to Grothendieck and Verdier [11, IV, 4.10.5(c)], and of course easy to prove by other methods) that a gros topos has the same cohomology as the corresponding petit topos.
3. **Localizations of toposes**

The localization of a topos \( \mathcal{E} \) at a point \( p \) was introduced by Grothendieck and Verdier [11, VI, 8.4.2], who described it as a (filtered) inverse limit. We shall give a more conceptual description of it, which will enable us to deduce its basic properties rather more simply, and then show that our description is equivalent to that of Grothendieck and Verdier.

Informally, the localization \( \text{Loc}_p(\mathcal{E}) \) of an \( \mathcal{S} \)-topos \( \mathcal{E} \) at a point \( p \) should be the universal solution to the problem of 'converting' \( \mathcal{E} \) into a local \( \mathcal{S} \)-topos in such a way that \( p \) becomes its centre. Now the universal solution to the problem of modifying a category \( \mathcal{C} \) so that a given object \( c \) becomes initial is clearly the co-slice category \( c/\mathcal{C} \); so, in view of 1.5, this means that we are seeking a local topos \( \text{Loc}_p(\mathcal{E}) \) such that, for any \( \mathcal{S} \)-topos \( (\mathcal{F}, \mathcal{G}) \), we have an equivalence

\[
\mathcal{X}p/S(\mathcal{F}, \text{Loc}_p(\mathcal{E})) \simeq p\delta/(\mathcal{X}p/S(\mathcal{F}, \mathcal{E}))
\]

which is natural in \( \mathcal{F} \). Clearly, if such a topos exists, it is unique up to canonical equivalence.

To construct \( \text{Loc}_p(\mathcal{E}) \), we begin by recalling that the bicategory \( \mathcal{X}p/S \) of bounded \( \mathcal{S} \)-toposes admits tensors with the category 2, that is, for any \( \mathcal{S} \)-topos \( \mathcal{G} \) there is a universal diagram of the form

\[
\begin{array}{ccc}
2 \uparrow \mathcal{G} & \xrightarrow{\beta_0} & \mathcal{G} \\
\downarrow \theta & & \\
\beta_1 & & \\
\end{array}
\]

in \( \mathcal{X}p/S \). (If \( \mathcal{G} \) is the classifying topos for a geometric theory \( T \), \( 2 \uparrow \mathcal{G} \) classifies the theory whose models are homomorphisms of \( T \)-models. In addition \( 2 \uparrow \mathcal{G} \) may be described as the exponential \( \mathcal{G}^T \), where \( T \) is the Sierpiński topos over \( \mathcal{S} \); but its construction is a precursor rather than a consequence of the general theory of exponentiable toposes, as may be seen from [20].) The diagram

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{id} & \mathcal{G} \\
\downarrow \text{id} & & \\
\mathcal{G} & \xleftarrow{\text{id}} & \\
\end{array}
\]

induces a geometric morphism \( \Delta: \mathcal{G} \to 2 \uparrow \mathcal{G} \), which is clearly a one-sided inverse for both \( \beta_0 \) and \( \beta_1 \); and there are natural transformations

\[
\Delta \beta_0 \to \text{id}_{2 \uparrow \mathcal{G}} \to \Delta \beta_1
\]

which make \( \Delta \) left adjoint to \( \beta_0 \) and right adjoint to \( \beta_1 \) in \( \mathcal{X}p/S \). In particular, \( (\beta_0: 2 \uparrow \mathcal{G} \to \mathcal{G}) \) is a local \( \mathcal{G} \)-topos, with centre \( \Delta \).

3.1. **Definition.** Let \( \mathcal{G} \) be an \( \mathcal{S} \)-topos, \( p \) a point of \( \mathcal{G} \). We define the localization of \( \mathcal{G} \) at \( p \) to be the pullback

\[
\begin{array}{ccc}
\text{Loc}_p(\mathcal{G}) & \to & 2 \uparrow \mathcal{G} \\
\downarrow & & \downarrow \beta_0 \\
\mathcal{S} & \to & \mathcal{G} \\
\end{array}
\]

By 2.4, \( \text{Loc}_p(\mathcal{G}) \) is a local \( \mathcal{S} \)-topos. Now let \( (\delta: \mathcal{F} \to \mathcal{S}) \) be any \( \mathcal{S} \)-topos; then
morphisms $\mathcal{F} \to \text{Loc}_p(\mathcal{E})$ over $\mathcal{S}$ correspond to morphisms $f: \mathcal{F} \to 2 \sqcap \mathcal{E}$ such that $\partial_0 f \equiv p \delta$, and hence to diagrams of the form

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\partial_1} & \mathcal{E} \\
\delta & \uparrow & \\
\mathcal{S} & \xrightarrow{f} & p
\end{array}
$$

(*)

in $\mathsf{Top}/\mathcal{S}$; that is, we have our desired equivalence

$$
\mathsf{Top}/\mathcal{S}(\mathcal{F}, \text{Loc}_p(\mathcal{E})) = p \delta/\mathsf{Top}/\mathcal{S}(\mathcal{F}, \mathcal{E})).
$$

There are various alternative ways of expressing this equivalence. For example, we have already observed (in the proof of 2.10) that diagrams of the form (*) correspond to morphisms $\mathcal{F} \to \mathcal{E}$ over $\mathcal{S}$ (where $\mathcal{F}$ is defined as in 1.2(e)) which carry the closed point of $\mathcal{F}$ into $p$; so we have

3.2. LEMMA. Let $\mathbf{P} \mathbf{Top}/\mathcal{S}$ denote the 2-category of (bounded) $\mathcal{S}$-toposes equipped with a specified point, and geometric morphisms and natural transformations preserving the specified points (up to isomorphism). Then the assignment $(\mathcal{F}, p) \mapsto \text{Loc}_p(\mathcal{E})$ is a functor $\mathbf{P} \mathbf{Top}/\mathcal{S} \to \mathsf{Top}/\mathcal{S}$, right adjoint to the functor which sends $\mathcal{F}$ to $\mathcal{F}$ together with its closed point.

Another, more 'relaxed', way of making pointed $\mathcal{S}$-toposes into a 2-category is to allow the 1-cells $(\mathcal{F}, q) \to (\mathcal{E}, p)$ to be pairs $(f, \alpha)$ where $f: \mathcal{F} \to \mathcal{E}$ is a geometric morphism over $\mathcal{S}$ and $\alpha: p \to fq$ is a (not necessarily invertible) natural transformation. We shall denote the 2-category so obtained by $\mathbf{P} \mathbf{Top}/\mathcal{S}$. If $\mathbf{L} \mathbf{Top}/\mathcal{S}$ denotes the full sub-2-category of $\mathsf{Top}/\mathcal{S}$ consisting of local $\mathcal{S}$-toposes, then it follows at once from 1.5 that the functor which equips a local $\mathcal{S}$-topos with its centre defines a full embedding $\mathbf{L} \mathbf{Top}/\mathcal{S} \to \mathbf{P} \mathbf{Top}/\mathcal{S}$.

3.3. PROPOSITION. The assignment $(\mathcal{F}, p) \mapsto \text{Loc}_p(\mathcal{E})$ is a functor $\mathbf{P} \mathbf{Top}/\mathcal{S} \to \mathbf{L} \mathbf{Top}/\mathcal{S}$, right adjoint to the full embedding defined above.

Proof. First, we indicate how $(\mathcal{F}, p) \mapsto \text{Loc}_p(\mathcal{E})$ becomes functorial on $\mathbf{P} \mathbf{Top}/\mathcal{S}$. Let $(f, \alpha): (\mathcal{F}, q) \to (\mathcal{E}, p)$ be a morphism of $\mathbf{P} \mathbf{Top}/\mathcal{S}$; then we can form the diagram

$$
\begin{array}{ccc}
\text{Loc}_q(\mathcal{F}) & \xrightarrow{\partial_1} & \mathcal{F} & \xrightarrow{f} & \mathcal{E} \\
\partial_0 & \uparrow \theta & \uparrow \alpha & \downarrow q & \downarrow p \\
\mathcal{S} & \xrightarrow{id} & \mathcal{S}
\end{array}
$$

which on 'pasting' yields a diagram of the required form to induce a morphism $\text{Loc}_q(\mathcal{F}) \to \text{Loc}_p(\mathcal{E})$ over $\mathcal{S}$. The definition of the functor on 2-cells of $\mathbf{P} \mathbf{Top}/\mathcal{S}$, and the verification that it is indeed functorial, are straightforward.

The adjunction is established in a similar manner. Let $\mathcal{F}$ be a local $\mathcal{S}$-topos, with centre $c$, and suppose we are given $(f, \alpha): (\mathcal{F}, c) \to (\mathcal{E}, p)$ in $\mathbf{P} \mathbf{Top}/\mathcal{S}$. 
Then we obtain a morphism \( \mathcal{F} \to \text{Loc}_p(\mathcal{E}) \) from the diagram

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\text{id}} & \mathcal{F} \\
\uparrow & & \uparrow f \\
\mathcal{F} & \xrightarrow{c} & \mathcal{E}
\end{array}
\]

Conversely, given \( g: \mathcal{F} \to \text{Loc}_p(\mathcal{E}) \), the diagram

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{g} & \text{Loc}_p(\mathcal{E}) \\
\uparrow c & & \downarrow 2 \neg \neg \mathcal{E} \\
\mathcal{G} & \xrightarrow{\text{id}} & \mathcal{G}
\end{array}
\]

defines a morphism \((\mathcal{F}, c) \to (\mathcal{E}, p)\) in \( \mathfrak{P}\text{Top}/\mathcal{S} \). (Note: in the above diagrams, any cell without a 2-arrow in it is supposed to commute up to a (canonical) 2-isomorphism whose name has been suppressed in the interests of clarity.) Once again, the verification of the remaining details is straightforward.

The next result is an immediate consequence of Proposition 3.3, though it could almost as easily have been derived directly from the definition of \( \text{Loc}_p(\mathcal{E}) \).

3.4. COROLLARY. For a pointed \( \mathcal{S} \)-topos \((\mathcal{E}, p)\), the following are equivalent:

(i) \( \mathcal{E} \) is local over \( \mathcal{S} \), and \( p \) is its centre;

(ii) the composite

\[
\text{Loc}_p(\mathcal{E}) \xrightarrow{2 \neg \neg} \mathcal{E}
\]

(i.e. the counit of the adjunction of Proposition 3.3) is an equivalence.

Note in passing that, for any \((\mathcal{E}, p)\), the counit map \( \text{Loc}_p(\mathcal{E}) \to \mathcal{E} \) is actually a morphism of \( \mathfrak{P}\text{Top}/\mathcal{S} \), and not just of \( \mathfrak{P}\text{Top}/\mathcal{S} \)—this follows from the definition and the second part of the statement of Theorem 2.4.

Various stability properties of localization also follow directly from the definition. For example,

3.5. PROPOSITION. Localization is stable under change of base: that is, if we are given a pullback square

\[
\begin{array}{ccc}
\mathcal{G}' & \xrightarrow{\gamma} & \mathcal{E} \\
\mathcal{G} & \xrightarrow{f} & \mathcal{G}
\end{array}
\]

and a section \( p \) of \( \gamma \), then (writing \( p' \) for the induced section of \( \gamma' \)) there is a pullback square

\[
\begin{array}{ccc}
\text{Loc}_p(\mathcal{E}') & \xrightarrow{f} & \text{Loc}_p(\mathcal{E}) \\
\mathcal{G}' & \xrightarrow{f} & \mathcal{G}
\end{array}
\]

Proof. Let \( \delta: \mathcal{T} \to \mathcal{S}' \) be any \( \mathcal{S}' \)-topos. By definition, morphisms \( \mathcal{T} \to \)}
Loc$_p(\mathcal{E}')$ over $\mathcal{F}'$ correspond to diagrams of the form

\[
\begin{array}{ccc}
\mathcal{F} & \rightarrow & \mathcal{E}' \\
\downarrow \delta & & \downarrow \rho' \\
\mathcal{S} & \rightarrow & \mathcal{E}
\end{array}
\]

in $\text{Loc}/\mathcal{F}'$; but since $\mathcal{E}'$ is a (Cat-enriched) pullback, these correspond to diagrams of the form

\[
\begin{array}{ccc}
\mathcal{F} & \rightarrow & \mathcal{E} \\
\downarrow f \delta & & \downarrow \rho \\
\mathcal{S} & \rightarrow & \mathcal{E}
\end{array}
\]

in $\text{Loc}/\mathcal{S}$, and hence to morphisms $\mathcal{F} \rightarrow \text{Loc}_p(\mathcal{E})$ over $\mathcal{S}$.

The stability of localization under composition of geometric morphisms is more complicated, but not intrinsically any more difficult, than stability under pullback. Suppose we are given a composable pair $\mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{S}$ of geometric morphisms, plus sections $p$, $q$ for $\gamma$, $\delta$ respectively. Then there are five, formally different, ways of localizing $\mathcal{F}$ over $\mathcal{S}$:

(i) we may form the localization $\text{Loc}_{p}(\mathcal{F})$ in $\text{Loc}/\mathcal{S}$;
(ii) we may form $\text{Loc}_{q}(\mathcal{F})$ in $\text{Loc}/\mathcal{E}$, and then form $\text{Loc}_{q}(\text{Loc}_{p}(\mathcal{F}))$ in $\text{Loc}/\mathcal{S}$, where $\bar{q}$ is the centre of $\text{Loc}_{q}(\mathcal{F})$;
(iii) we may form the pullback

\[
\begin{array}{ccc}
\mathcal{F}' & \rightarrow & \mathcal{F} \\
\downarrow \delta' & & \downarrow \delta \\
\text{Loc}_{p}(\mathcal{E}) & \rightarrow & \mathcal{E}
\end{array}
\]

where $\epsilon$ is the counit of the adjunction of 3.3, and then form $\text{Loc}_{q-p}(\mathcal{F}')$, where $\bar{p}$ is the centre of $\text{Loc}_{p}(\mathcal{E})$ and $q'$ is the section of $\delta'$ obtained on pullback from $q$;
(iv) we may form the pullback $\mathcal{F}'$ as in (iii), and then form $\text{Loc}_{q}(\mathcal{F}')$ in $\text{Loc}/\text{Loc}_{p}(\mathcal{E})$;
(v) we may form the pullback

\[
\begin{array}{ccc}
\mathcal{F}'' & \rightarrow & \text{Loc}_{q}(\mathcal{F}) \\
\downarrow \delta'' & & \downarrow \delta \\
\text{Loc}_{p}(\mathcal{E}) & \rightarrow & \mathcal{E}
\end{array}
\]

By 2.1(ii) and 2.4, each of the above constructions yields a local $\mathcal{S}$-topos equipped with a $p\text{Loc}/\mathcal{S}$ morphism into $(\mathcal{F}, qp)$.

We might have hoped that these five constructions would yield equivalent toposes, but this hope is soon dispelled by Example 2.3: for if $\mathcal{E}$ and $\mathcal{F}$ are both local over $\mathcal{S}$ (with centres $p$ and $qp$ respectively) but $\delta$ is not local, then constructions (i) and (iii) will (up to equivalence) simply produce $\mathcal{F}$ itself,
whereas the other three will produce $\text{Loc}_q(\mathcal{F})$. However, this is all that can go wrong:

3.6. **Proposition.** With the notation developed above, the toposes $\text{Loc}_{q^p}(\mathcal{F})$ and $\text{Loc}_{q'^p}(\mathcal{F}')$ are equivalent (as toposes over $\mathcal{F}$); and so are the toposes $\text{Loc}_{q^p}(\text{Loc}_q(\mathcal{F})), \text{Loc}_q(\mathcal{F}')$ and $\mathcal{F}''$.

**Proof.** Note first that the equivalence of $\text{Loc}_q(\mathcal{F}')$ and $\mathcal{F}''$ follows directly from Proposition 3.5. Next, observe that both $\delta$ and $q$ are morphisms of $\mathcal{X}\text{Top}/\mathcal{S}$ (between $(\mathcal{F}, qp)$ and $(\mathcal{E}, p)$); so by functoriality of localization we obtain morphisms

$$\text{Loc}_p(\mathcal{E}) \xrightarrow{\bar{q}} \text{Loc}_{q^p}(\mathcal{F}) \xrightarrow{\delta} \text{Loc}_p(\mathcal{E})$$

with $\bar{q} \equiv \text{id}$. Now by the definition of $\delta$ we have $e\delta = \delta f$ (where $f: \text{Loc}_{q^p}(\mathcal{F}) \to \mathcal{F}$ is the counit map), so we obtain a map $\theta: \text{Loc}_{q^p}(\mathcal{F}) \to \mathcal{F}'$ with $\delta' \theta \equiv \delta$ and $e'\theta \equiv f$. Moreover, we have

$$\delta' \theta \bar{q} \equiv \delta \bar{q} \equiv \text{id} \equiv \delta' q' \quad \text{and} \quad e' \theta \bar{q} \equiv f \bar{q} \equiv q e \equiv e' q',$$

so that $\theta \bar{q} \equiv q'$; that is, $\theta$ is a morphism of $\mathcal{X}\text{Top}/\text{Loc}_p(\mathcal{E})$.

We wish to show that $\theta$ is actually a morphism in $\mathcal{X}\text{Top}/\mathcal{S}$ from $(\text{Loc}_{q^p}(\mathcal{F}), \bar{q} \bar{p})$ to $(\mathcal{F}', q' \bar{p})$ (where $\bar{q} \bar{p}$ is the centre of $\text{Loc}_{q^p}(\mathcal{F})$), that is, that $\theta \bar{q} \equiv q' \bar{p}$. By what we have just done, it is sufficient to show that $\bar{q} \bar{p} \equiv q \bar{p}$; but since

$$f \bar{q} \bar{p} \equiv q e \bar{p} \equiv q \bar{p} \equiv f \bar{q},$$

we see that they are factorizations through $f$ of isomorphic morphisms $(\mathcal{F}, \text{id}) \to (\mathcal{F}, qp)$ in $\mathcal{X}\text{Top}/\mathcal{S}$, and so must be isomorphic.

Now, by 3.3, $\theta$ induces a morphism $\varphi: \text{Loc}_{q^p}(\mathcal{F}) \to \text{Loc}_{q'^p}(\mathcal{F}')$ in $\mathcal{X}\text{Top}/\mathcal{S}$; but we also have a morphism $\psi: \text{Loc}_{q^p}(\mathcal{F}') \to \text{Loc}_{q^p}(\mathcal{F})$, obtained by applying the localization functor to $e'$: $(\mathcal{F}', q' \bar{p}) \to (\mathcal{F}, q \bar{p})$ in $\mathcal{X}\text{Top}/\mathcal{S}$. The rest of the verification that $\varphi$ and $\psi$ are inverse to each other, and that they define an equivalence of $\mathcal{S}$-toposes, is straightforward.

Finally, if we go through the above argument starting from $\text{Loc}_q(\mathcal{F})$ instead of $\mathcal{F}$, we obtain an equivalence $\text{Loc}_{q^p}(\text{Loc}_q(\mathcal{F})) \equiv \text{Loc}_{q'^p}(\mathcal{F}'')$, where $q''$ is the section of $\delta''$ obtained on pullback from $\bar{q}$. But since $\mathcal{F}''$ is already local over $\mathcal{S}$, the latter topos is equivalent to $\mathcal{F}''$.

We now turn to the relation between our definition of $\text{Loc}_p(\mathcal{E})$ and that given by Grothendieck and Verdier [11, VI, 8.4.2]. Recall that if $p$ is a point of an $\mathcal{S}$-topos $\mathcal{E}$, a *neighbourhood* of $p$ is defined to be a pair $(X, x)$ where $X$ is an object of $\mathcal{E}$ and $x \in p^*(X)$. The neighbourhoods of $p$ form a category in an obvious way; we denote it by $\text{Nbd}(p)$. Grothendieck and Verdier define $\text{Loc}_p(\mathcal{E})$ to be the inverse limit

$$\lim_{(X, x) \in \text{Nbd}(p)} \mathcal{E}/X.$$ 

(Anyone who is worried about taking limits over large categories may replace $\text{Nbd}(p)$ by a small subcategory consisting of those $(X, x)$ for which $X$ lies in some given generating family for $\mathcal{E}$.)
To identify our $\text{Loc}_p(\mathcal{E})$ with the above inverse limit, we first construct a cone over the diagram with vertex $\text{Loc}_p(\mathcal{E})$. Recall that, by a special case of Diaconescu's theorem [13, 4.37(i)], a geometric morphism $\text{Loc}_p(\mathcal{E}) \rightarrow \mathcal{E}/X$ corresponds to a geometric morphism $e: \text{Loc}_p(\mathcal{E}) \rightarrow \mathcal{E}$ together with a global section of $e^*(X)$ in $\text{Loc}_p(\mathcal{E})$. But we have such a morphism $e$, namely the counit of the adjunction of 3.3; and since $e$ is actually a morphism of $\mathcal{P}_{\text{op}}/\mathcal{S}$, as we observed earlier, we have isomorphisms

$$p^*(X) \cong \hat{p}^*e^*(X) \cong \hat{\gamma}_*e^*(X),$$

where $\hat{p}$ is the centre of $\text{Loc}_p(\mathcal{E})$, and $\hat{\gamma}: \text{Loc}_p(\mathcal{E}) \rightarrow \mathcal{S}$. So, given an object $(X, x)$ of $\text{Nbd}(p)$, we obtain a global section of $e^*(X)$ and hence a morphism $e_{X,x}: \text{Loc}_p(\mathcal{E}) \rightarrow \mathcal{E}/X$ lying over $e$; and since the isomorphisms (*) are natural in $X$, it is easy to see that the $e_{X,x}$ form a cone over the diagram considered earlier.

3.7. THEOREM. $\text{Loc}_p(\mathcal{E}) = \lim_{\rightarrow(\mathcal{E}/X)} e_{\mathcal{E}/X}$.

Proof. Let $(\mathcal{E}, \mathcal{S})$ be an $\mathcal{S}$-topos, and $(f_{X,x}: \mathcal{F} \rightarrow \mathcal{E}/X)_{(X,x) \in \text{Nbd}(p)}$ a cone with vertex $\mathcal{F}$. Reversing the argument just given, we can regard the cone as being specified by a morphism $f: \mathcal{F} \rightarrow \mathcal{E}$ over $\mathcal{S}$ (viz. $f = f_{1,*}$, where $*$ is the unique element of $p^*(1)$) together with, for each $(X, x)$, a global section $\varphi_{X,x}$ of $f^*(X)$, subject to compatibility conditions between the $\varphi_{X,x}$. But the latter say precisely that the assignment $x \mapsto \varphi_{X,x}$, regarded as a family of functions $\varphi_{X,x}: p^*(X) \rightarrow \delta_*f^*(X)$, is a natural transformation $p^* \rightarrow \delta_*f^*$, or equivalently $\delta^*p^* \rightarrow f^*$; so we have a diagram of the form required to induce a geometric morphism $\mathcal{F} \rightarrow \text{Loc}_p(\mathcal{E})$. It is straightforward to verify that this construction yields an equivalence between $\mathcal{F}_{\text{op}}/\mathcal{F}, \text{Loc}_p(\mathcal{E})$ and the category of cones with vertex $\mathcal{F}$.

REMARK. In [11] Grothendieck and Verdier were able to prove that $\text{Loc}_p(\mathcal{E})$ is local only in the case where $\mathcal{E}$ is locally coherent over $\mathcal{S}$. Since we have identified our notion of localization with theirs, we see that this restriction was unnecessary. (In fact one of the authors has shown that it is possible to proceed directly from the Grothendieck–Verdier definition of $\text{Loc}_p(\mathcal{E})$, and a site of definition for $\mathcal{E}$, to construct a site of definition for $\text{Loc}_p(\mathcal{E})$, and to show that the latter is a local site.)

Grothendieck and Verdier justify their neglect of the non-locally-coherent case by the observation that it seems unlikely to provide any examples of interest, given the fact that if $X$ is a Hausdorff space then the localization of $\text{Sh}(X)$ at any of its points is equivalent to $\mathcal{S}$. However, the implied dichotomy between ‘locally coherent’ and ‘Hausdorff’ does not appear to be justified. The correct context for the result about all localizations being trivial would seem to be the following, which includes a good many examples of locally coherent toposes (for example, the topos of $G$-sets for any group $G$) as well as the topos of sheaves on any $T_U$-locale.

3.8. PROPOSITION. A topos $\mathcal{E}$ is grouplike if and only if $\mathcal{G}_0: 2\mid \mathcal{G} \rightarrow \mathcal{E}$ is an equivalence. In particular, any localization of a grouplike topos is equivalent to $\mathcal{S}$. 
Proof. The first sentence is a translation of the fact that a category $\mathcal{C}$ is a groupoid if and only if the domain map $\mathcal{C}^2 \to \mathcal{C}$ is an equivalence. The second follows immediately from it and Definition 3.1.

One may also ask what can be said about those $\mathcal{S}$-toposes whose localizations are all localic over $\mathcal{S}$. Clearly, they include all localic $\mathcal{S}$-toposes; for if $\mathcal{E}$ is localic (i.e. classifies a propositional theory) over $\mathcal{S}$, then $2 \lhd \mathcal{E}$ is also localic over $\mathcal{S}$, and hence the domain map $2 \lhd \mathcal{E} \to \mathcal{E}$ is localic. But we also have

3.9. Lemma. Let $\mathcal{E}$ be an $\mathcal{S}$-topos, $X$ an object of $\mathcal{E}$. Then the canonical diagram

$$
\begin{array}{ccc}
2 \lhd (\mathcal{E}/X) & \xrightarrow{\partial_0} & \mathcal{E}/X \\
\downarrow & & \downarrow \\
2 \lhd \mathcal{E} & \xrightarrow{\partial_0} & \mathcal{E}
\end{array}
$$

is a pullback.

Proof. This is easily seen by considering the geometric theories classified by the four toposes involved. Suppose $\mathcal{E}$ classifies a theory $\mathcal{T}$; then we can think of the object $X$ as a 'geometric construct' in the theory $\mathcal{T}$, that is, an object constructible in a functorial way from an arbitrary $\mathcal{T}$-model, and $\mathcal{E}/X$ classifies the theory whose models are $\mathcal{T}$-models equipped with a distinguished element of this construct. But if $M_1$ and $M_2$ are models equipped with such distinguished elements, and $f: M_1 \to M_2$ is a homomorphism preserving the distinguished elements, then the distinguished element in $M_2$ is uniquely determined by that in $M_1$, together with $f$. This says that, for any $\mathcal{S}$-topos $\mathcal{F}$, giving a morphism $\mathcal{F} \to 2 \lhd (\mathcal{E}/X)$ is equivalent to giving a pair of morphisms $\mathcal{F} \to 2 \lhd \mathcal{E}$, $\mathcal{F} \to \mathcal{E}/X$ which compose to give the same morphism $\mathcal{F} \to \mathcal{E}$.

3.10. Corollary. If $\mathcal{E}$ is an étendue, then the domain map $2 \lhd \mathcal{E} \to \mathcal{E}$ is localic. Hence any localization of $\mathcal{E}$ is localic over $\mathcal{S}$.

Proof. If $\mathcal{F}$ is localic over $\mathcal{S}$ (equivalently, classifies a propositional geometric theory) then $2 \lhd \mathcal{F}$ is clearly also localic over $\mathcal{S}$, and hence over any $\mathcal{S}$-topos into which it maps; in particular, the domain map $2 \lhd \mathcal{F} \to \mathcal{F}$ is localic. Now if $\mathcal{E}$ is an étendue, then there is an object $X$ with global support in $\mathcal{E}$ such that $\mathcal{E}/X$ is localic over $\mathcal{S}$; hence by Lemma 3.9 the pullback of $\partial_0: 2 \lhd \mathcal{E} \to \mathcal{E}$ along $\mathcal{E}/X \to \mathcal{E}$ is localic. But the property of being localic descends along open surjections (since the hyperconnected-localic factorization is stable under pullback), and so $\partial_0: 2 \lhd \mathcal{E} \to \mathcal{E}$ is localic. The second sentence of Corollary 3.10 follows immediately from the first, since localic morphisms are stable under pullback.

In contrast to Proposition 3.8, we do not know whether the implication in the first sentence of Corollary 3.10 is reversible; but it seems not improbable.

We conclude the paper by discussing an example of a context (in general, a non-locally-coherent one) in which the notion of localization seems likely to be useful. Specifically, we shall consider the idea of the 'germ' of a smooth manifold
At a point \( p \). Clearly, given \( M \) and \( p \), it would be useful to have a 'space' representing the notion of the germ at \( p \) of a smooth map (and the study of Synthetic Differential Geometry was largely motivated by the desire to have such 'spaces' available). If we wish this 'space' to be a topos, it is no use trying \( \text{Loc}_p(\text{Sh}(M)) \), since by 3.8 this contains no information about \( M \) or \( p \). Another possibility which suggests itself is to take the filterpower \( \text{Sh}(M)/\mathcal{N}_p \), where \( \mathcal{N}_p \) is the filter of open neighbourhoods of \( p \) in \( M \) (regarded as a filter of subobjects of 1 in \( \text{Sh}(M) \)); this is always a topos, but never a Grothendieck topos unless \( p \) is an isolated point [2].

Perhaps a better way of representing a manifold \( M \) by a topos is the topos \( \mathcal{M}/yM \), where \( \mathcal{M} \) is the topos of sheaves for the open cover topology on the category \( \text{Mf} \) of smooth manifolds. Then \( \mathcal{M} \) is a local \( \mathcal{S} \)-topos (and indeed satisfies the conditions of Proposition 1.10, apart from the existence of arbitrary finite limits in the category \( \text{Mf} \); but pullbacks along open inclusions always exist, so that the proof of 1.10 still applies to give a local morphism \( \mathcal{M}/yM \rightarrow \text{Sh}(M) \) for each manifold \( M \)). Moreover, the assignment \( M \mapsto \mathcal{M}/yM \) is a full embedding \( \mathcal{Mf} \rightarrow \mathcal{M}/yM \), since \( y \) is a full embedding \( \mathcal{Mf} \rightarrow \mathcal{M} \). In particular, a point \( p \) of \( M \) may be regarded as a section \( p: \mathcal{M} \rightarrow \mathcal{M}/yM \) of the canonical map \( \mathcal{M}/yM \rightarrow \mathcal{M} \); composing this with the centre \( c \) of \( \mathcal{M} \), we obtain a point \( pc \) of \( \mathcal{M}/yM \).

If we take the category \( \mathcal{Mf}/M \), with the open cover topology, as a site of definition for \( \mathcal{M}/yM \), then the point \( pc \) corresponds to the flat continuous functor on \( \mathcal{Mf}/M \) which sends a manifold \( (N \rightarrow M) \) over \( M \) to (the underlying set of) the fibre \( \pi^{-1}(p) \). Clearly, we cannot expect this functor to tell us anything about the local nature of \( M \) at \( p \); and indeed if we localize \( \mathcal{M}/yM \) at \( pc \) we simply recover the topos \( \mathcal{M} \), since from Lemma 3.9 we obtain

\[
\text{Loc}_{pc}(\mathcal{M}/yM) = \text{Loc}_c(\mathcal{M}) = \mathcal{M}.
\]

However, we may define another point \( \tilde{p} \) of \( \mathcal{M}/yM \), corresponding to the flat continuous functor on \( \mathcal{Mf}/M \) which sends \( (N \rightarrow M) \) to the set of germs at \( p \) of (smooth) sections of \( \pi \). Of course, evaluation at \( p \) defines a natural transformation from this functor to the previous one, and hence a 2-cell \( \tilde{p} \Rightarrow pc \) in \( \mathcal{S}\text{op} \).

We write \( \mathcal{M}_p \) for the localization of \( \mathcal{M}/yM \) at the point \( \tilde{p} \). (Note that 3.9 again allows us to regard \( \mathcal{M}_p \) as a localization of \( \mathcal{M} \) itself, at the point which is the composite

\[
\mathcal{S} \rightarrow \mathcal{M}/yM \rightarrow \mathcal{M}.
\]

We claim that the topos \( \mathcal{M}_p \) is a suitable candidate for 'the germ of the manifold \( M \) at \( p \). In support of this claim, we prove

3.11. Proposition. For any manifold \( N \), the category \( \mathcal{S}\text{op}/\mathcal{M}(\mathcal{M}_p, \mathcal{M}/yN) \) is equivalent to the (discrete) set of germs at \( p \) of smooth maps \( M \rightarrow N \).

Proof. We use Theorem 3.7, together with the results of [25], to construct an explicit site of definition for \( \mathcal{M}_p \), starting from the site \( \mathcal{Mf}/M \) for \( \mathcal{M}/yM \). First we note that the full subcategory of \( \text{Nbd}(\tilde{p}) \) whose objects are of the form \((U \rightarrow M, x)\), with \( U \) an open subset of \( M \) containing \( p \), \( i \) the inclusion map, and
x the germ at p of the identity map on U, is co-initial in Nbd(p); so we may replace the inverse limit in the statement of 3.7 by an inverse limit over this subcategory. It is now not difficult to see, using the techniques of [25], that a site for \( \mathcal{M}_p \) may be obtained by taking the category whose objects are maps \((f: N \to M)\) in \( \mathcal{M}_f \), and whose morphisms

\[
(f_1: N_1 \to M) \to (f_2: N_2 \to M)
\]

are equivalence classes of smooth maps \( f_i^{-1}(U) \to N_2 \) over \( M \), where \( U \) is an open neighbourhood of \( p \) in \( M \) (the equivalence relation being the obvious one which identifies such a map with its restriction to \( f_i^{-1}(V) \) for any smaller neighbourhood \( V \) of \( p \)); we omit the explicit description of the topology on this site, since we shall not need to know anything about it beyond the fact that it is subcanonical.

Now let \( \theta: \mathcal{M}_p \to \mathcal{M} \) be the canonical geometric morphism. By the special case of Diaconescu's theorem mentioned earlier, we know that \( \mathcal{X} \text{op}/\mathcal{M}(\mathcal{M}_p, \mathcal{M}/yN) \) is equivalent to the discrete category of global sections of \( \theta^*(yN) \) in \( \mathcal{M}_p \). But we have

\[
\theta^*(yN) \equiv y(\pi: M \times N \to M),
\]

where \( \pi \) denotes the first product projection; and the morphisms in the category just described from the terminal object \((id: M \to M)\) to \((\pi: M \times N \to M)\) are easily seen to correspond to germs at \( p \) of smooth maps \( M \to N \). So the result is proved.

References