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Every étendue comes from a local equivalence relation

Anders Kock
Mathematics Institute, Aarhus University, Ny Munkegade, DK-8000 Aarhus, Denmark

Ieke Moerdijk
Mathematics Institute, University of Utrecht, Budapestlaan 6, 3508 TA Utrecht, The Netherlands

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Abstract

We first prove that, under suitable connectedness assumptions, the equivariant sheaves for a local equivalence relation on a space (or a locale) form an étendue topos. Our main result is that conversely, every étendue can be obtained in this way.

Introduction

An étendue is a topos \( \mathcal{T} \) for which an object \( U \in \mathcal{T} \) exists such that \( U \to 1 \) is epi and the slice topos \( \mathcal{T}/U \) is localic, that is, \( \mathcal{T}/U \) is equivalent to the category of sheaves on a locale. These étendue topoi were introduced by Grothendieck and Verdier [1, p. 478 ff.] in the context of foliations and local equivalence relations. It was suggested that for a suitable local equivalence relation \( r \) on a topological space, the category of \( r \)-invariant sheaves form an étendue topos. In this paper, we will consider the notion of a local equivalence relation \( r \) on a locale \( M \). We will show that if \( r \) is locally simply connected (in an appropriate sense), then the category of \( r \)-invariant sheaves on \( M \) is a topos, and in fact an étendue. (We will also explain, in Example 2.3, how this result relates to a similar statement for local equivalence relations on topological spaces in [16].)

Our main result is that every étendue can be obtained this way. Indeed, in Theorem 7.1 we will show that for any étendue \( \mathcal{T} \), there exists a local equivalence relation...
relation $r$ on some locale $M$ for which there is an equivalence of topoi

\[ \text{Sh}(M, r) = \text{Sh}. \]

Moreover, this local equivalence relation is locally simply connected in the sense referred to above. The construction of $M$ and $r$ is based on the observation that étendue topoi in some sense ‘classify’ local equivalence relations: every locally connected geometric morphism from a topos of sheaves on a locale $M$ into an étendue $\mathcal{T}$ gives rise to a canonical local equivalence relation on $M$. Furthermore, essential use is made of the construction from [6] of a localic cover with contractible fibres of any given topos.

For an étendue $\mathcal{T}$ with enough points, there exists a topological space with a local equivalence relation $r$ for which there is an equivalence of form (*), but this space has to be obtained by a completely different construction, cf. [10, 11]; here the reader will also find a discussion of the relation between étendues, foliations, holonomy and monodromy.

Our main result is a presentation theorem for étendues; we wish to point out that this result bears no relation to the type of presentation considered in [9].

1. Equivalence relations on locales

Let $X$ be a locale. By an equivalence relation on $X$ we shall always mean a sublocale $R \subseteq X \times X$ satisfying the usual conditions of reflectivity, symmetry, and transitivity, and in addition having the property that the two projection maps

\[ d_0, d_1 : R \rightarrow X \]

are open maps. This implies that the coequalizer $\pi_R : X \rightarrow X/R$ of $d_0$ and $d_1$ is also an open map [14]. Note that, unlike the case of topological spaces, $R$ need not coincide with the kernel pair of $\pi_R$, cf. [8]. Since $R$ is reflexive and transitive, there is a (truncated) simplicial complex of locales

\[ R \times_X R \cong R \Rightarrow X. \]

By applying the functor $\text{sh}(-)$, we obtain a similar diagram of topoi and geometric morphisms. We write $\text{sh}(X; R)$ for the associated descent topos. So the objects of $\text{sh}(X; R)$ are sheaves $E$ on $X$ equipped with ‘descent data’ $\theta_E : d_0^* E \rightarrow d_1^* E$ satisfying a unit and cocycle condition. (This construction will be discussed in greater generality in Section 3.) Equivalently, $\theta$ can be given in the form of an action by $R$ on $E$, or of a transport on $E$ along $R$, i.e. a map $R \times_X E \rightarrow E$ satisfying usual associativity and unit laws. The localic reflection of this topos $\text{sh}(X; R)$ is (the topos of sheaves on) the quotient locale $X/R$.

In the context of topological spaces, it is well known that $\text{sh}(X; R)$ coincides with $\text{sh}(X/R)$ in case the map $d_0 : R \rightarrow X$ has ‘enough local sections’ (cf. [1, p.
In fact, it is enough to require $d_0$ to be an open map; the argument also works for locales:

**Proposition 1.1.** For any equivalence relation $R$ on a locale $X$ (with $d_0$ and $d_1$ open), the map $\text{sh}(X; R) \to \text{sh}(X/R)$ is an equivalence of topoi.

**Proof.** It is enough to show that the topos $\text{sh}(X; R)$ is generated by subobjects of the terminal object $1$. Such subobjects are $R$-saturated open sublocales of $X$, i.e. sublocales of the form $d_1d_0^{-1}(U) \subseteq X$, where $U \subseteq X$ is any open sublocale. Such $R$-saturated sublocales carry a unique action by $R$, hence are objects of $\text{sh}(X; R)$. Now consider an arbitrary object $E$ of $\text{sh}(X; R)$, given as an étale map $p : E \to X$ with an action $\theta : R \times_X E \to E$. Let $s : W \to E$ be any section of $E \to X$ over an open $W \subseteq X$. We wish to show that $W$ is covered by opens $W_i \subseteq W$ with the property that $s|_{W_i} : W_i \to E$ can be extended to an $R$-equivariant section

$$\tilde{s}_i : d_1d_0^{-1}(W_i) \to E.$$ (1.3)

Thus each such section $\tilde{s}_i$ is a map in the topos $\text{sh}(X; R)$ from a subobject of $1$ into $E$. All these sections $\tilde{s}_i$, for all possible sections $s : W \to E$, cover $E$ since $p : E \to X$ is a local homeomorphism. So this indeed shows that $\text{sh}(X; R)$ is generated by subobjects of $1$.

To construct these local extensions $\tilde{s}_i$ from the given section $s : W \to E$, consider first the pullback $d_1^*(E|_W) = E|_W \times_W R$ of $E|_W \to W$ along $d_1 : R|_W = R \cap (W \times W) \to W$, as in

$$
\begin{array}{ccc}
d_1^*(E|_W) & \longrightarrow & E|_W \\
p' \downarrow & & \downarrow p \\
R|_W = R \cap (W \times W) & \longrightarrow & W
\end{array}
$$ (1.4)

The map $p'$ in this diagram has two sections induced by $s : W \to E$, namely

$$s_1 = \theta(\text{id}, sd_0) : R|_W \to R \times_W E \to E, \quad s_2 = sd_1.$$ (In point-set notion, $d_1^*(E|_W) = \{(x, y, e) \mid (x, y) \in R, e \in p^{-1}(y)\}$, and $s_1(x, y) = \theta((x, y), s(x)), s_2(x, y) = s(y)$.) These two sections agree on the diagonal $\Delta : W \to R|_W$. Since $p'$ is étale, it follows that they must agree on a neighbourhood $N$ of the diagonal. We may assume that this neighbourhood is of the form

$$N = \bigcup_i R|_{W_i} = \bigcup_i R \cap (W_i \times W_i),$$

for some open cover $W = \bigcup_i W_i$. By definition of $s_1$ and $s_2$, this means that each restriction $s|_{W_i} : W_i \to E$ is $(R|_{W_i})$-equivariant. It follows that $s|_{W_i}$ can be extended to the $R$-saturation $d_1d_0^{-1}(W_i)$ of $W_i$. Indeed, let $P$ be the kernel pair of
\[ d_1 : d_0^{-1}(W_i) \rightarrow d_1 d_0^{-1}(W_i), \] as in the diagram

\[
\begin{array}{c}
\pi_0 \quad d_0^{-1}(W_i) \\
\pi_1 \quad d_1 d_0^{-1}(W_i)
\end{array}
\]

\[ (1.5) \]

Since \( d_1 : R \rightarrow X \) is an open surjection by assumption, this diagram is a coequalizer, cf. [7]; furthermore, since \( s|_{W_i} \) is \((R|_{W_i})\)-equivariant, the cocycle condition for the action \( \theta \) by \( R \) on \( E \) implies that the map \( s_i : d_0^{-1}(W_i) \rightarrow E \), given in point-set terms by \( s_i(x, y) = \theta((x, y), s(x)) \), satisfies the identity \( s_i \pi_1 = s_i \pi_2 \). Thus \( s_i \) factors through the coequalizer \((1.5)\), to give the desired section \( \tilde{s}_i : d_1 d_0^{-1}(W_i) \rightarrow E \). \( \square \)

An equivalence relation \( R \) on a locale \( X \), as above, is said to be connected (respectively locally connected) if \( d_0 \) and \( d_1 \) are connected (respectively locally connected) maps of locales, i.e. if the corresponding geometric morphisms \( d_0, d_1 : \text{sh}(R) \rightarrow \text{sh}(X) \) are connected, respectively locally connected.

**Proposition 1.2.** If \( d_0, d_1 \) are connected (respectively locally connected) maps, the quotient map \( X \rightarrow X/R \) is a connected (respectively locally connected) map.

**Proof.** For the locally connected case, if \( d_0, d_1 : R \Rightarrow X \) are locally connected, then by [15] so is the geometric morphism \( \text{sh}(X) \rightarrow \text{sh}(X; R) \), and hence by Proposition 1.1, \( X \rightarrow X/R \) is a locally connected map. For the connected case, assume that \( d_0, d_1 : R \Rightarrow X \) are connected. Again by Proposition 1.1, it suffices to see that \( \text{sh}(X) \rightarrow \text{sh}(X; R) \) is a connected geometric morphism. Consider two \( R \)-equivariant sheaves \((E, \theta)\) and \((F, \mu)\), and a map \( \phi : E \rightarrow F \) in \( \text{sh}(X) \). We must prove that \( \phi \) is \( R \)-equivariant, i.e. a map in \( \text{sh}(X; R) \). Consider the two maps \( \alpha, \beta : d_0^* E \rightarrow d_0^* F \) described, in point-set notation, for \((x, y) \in R \) and \( e \in E_x\), by

\[
\alpha((x, y), e) = ((x, y), \mu((y, x), \phi((x, y), e)))) ,
\]

\[
\beta((x, y), e) = ((x, y), \phi(e)) .
\]

Thus \( \beta = d_0^*(\phi) \), and since \( d_0^* \) is full and faithful, \( \alpha = d_0^*(\alpha') \) for a unique map \( \alpha' : E \rightarrow F \) in \( \text{sh}(X) \). By the unit-condition for the actions \( \theta \) and \( \mu \), we have \( \Delta^*(\alpha) = \Delta^*(\beta) \), where \( \Delta : X \rightarrow R \) is the diagonal. Hence \( \alpha' = \wedge^* d_0^* \alpha' = \Delta^* \alpha = \Delta^* \beta = \phi \), and thus, applying \( d_0^* \), \( \alpha = \beta \). This identity expresses that \( \phi \) is an \( R \)-equivariant map. This proves the proposition. \( \square \)

As a consequence, we obtain the following:

**Proposition 1.3.** Let \( R \) be a connected equivalence relation on a locale \( X \). Then for any sheaf \( E \) on \( X \), there is at most one action by \( R \) on \( E \).

(If there is such an action, we call \( E \) an \( R \)-invariant sheaf.)
Proof. Let $\theta$ and $\theta'$ be two $R$-actions on $E$. Since the forgetful functor $\text{sh}(X; R) \to \text{sh}(X)$ is full and faithful, the identity map on $E$ in $\text{sh}(X)$ must also be an $R$-equivariant map $(E, \theta) \to (E, \theta')$, thus $\theta = \theta'$. □

2. Local equivalence relations and sheaves

For a locale $M$, consider for each open $U \subseteq M$ the set $E_M(U)$ of equivalence relations $R$ on $U$, as defined in Section 1. For open sublocales $V \subseteq U \subseteq M$ there is an evident restriction map $E_M(U) \to E_M(V)$, making $E_M$ into a presheaf on $M$. By definition [1, p. 485], a local equivalence relation on $M$ is a global section of the associated sheaf $\tilde{E}_M$. An equivalence relation $R$ on any locale $U$ gives rise to a local equivalence relation $L(R)$ on $U$. Let $r$ be a local equivalence relation on $M$. An equivalence relation $R$ on an open $U \subseteq M$ will be called a chart for $r$ if $L(R)$ agrees with the restriction of $r$ to $U$; if $V \subseteq U$, then $(V, R|_V)$ is also a chart for $r$; we call it a subchart of $(U, R)$. An atlas for $r$ is a family $\{(U_i, R_i)\}$ of charts for $r$ such that the $U_i$'s cover $M$. A family $\{(U_i, R_i)\}$ will be an atlas for some local equivalence relation iff for any two indices $i$ and $j$, $U_i \cap U_j$ is covered by open $W$ such that $R_i|_W = R_j|_W$. An atlas is a refinement of another if each chart of the former is a subchart of some chart of the latter.

By our conventions in Section 1, it follows that any local equivalence relation $r$ has an atlas consisting of charts $(U, R)$ for which $R = Z_U$ are open maps. Furthermore, $r$ is said to be locally connected if any atlas for $r$ can be refined by an atlas consisting of connected and locally connected charts, i.e. charts $(U, R)$ for which $R = Z_U$ are connected and locally connected maps. Such an atlas will be called a connected atlas for $r$.

Following [1], we now define, for a local equivalence relation $r$ on a locale $M$ and a sheaf $F$ on $M$, the notion of an $r$-transport on $F$. Consider for an open $U \subseteq M$ the set $T_F(U)$ of pairs $(R, \theta)$, where $R$ is an equivalence relation on $U$ and $\theta : R \times_U (F|_U) \to (F|_U)$ is an action by $R$ on $F|_U$ (as in Section 1). With the obvious restrictions maps $T_F(U) \to T_F(V)$ for opens $V \subseteq U \subseteq M$, this gives a presheaf $T_F$ on $M$, with a projection map $\pi : T_F \to E_M$. Passing to the associated sheaves, we obtain a map $\tilde{\pi} : \tilde{T}_F \to \tilde{E}_M$. An $r$-transport on the sheaf $F$ is by definition a global section $t$ of $\tilde{T}_F$ such that $\tilde{\pi}(t) = r$. A sheaf equipped with an $r$-transport is called an $r$-invariant sheaf, or an $r$-sheaf. Such an $r$-transport is thus given by an open cover $\bigcup U_i = M$, equivalence relations $R_i$ on $U_i$, and actions $\theta_i$ of $R_i$ on $F|_{U_i}$ all locally compatible on intersections $U_i \cap U_j$. As before, we call $\mathcal{U} = \{(U_i, R_i, \theta_i)\}$ an atlas for $t$, and each of its members a chart for $t$.

Any atlas or chart for $t$ has an evident underlying atlas or chart for $r$. We note that if $\mathcal{U}$ is an atlas for $r$ with underlying atlas $\mathcal{U}$ for $r$, and $\mathcal{V}$ is another atlas for $r$ which refines $\mathcal{U}$, then $\mathcal{V}$ can be refined by an atlas $\mathcal{V}'$ for $t$ which has the given $\mathcal{V}$ as underlying atlas for $r$. It follows that for two sheaves with $r$-transport $(F, t)$ and $(F', t')$, there exists atlases for $t$ and $t'$ with identical underlying atlas for $r$. It
also follows that if \( r \) is locally connected, any atlas for \( t \) can be refined by an atlas whose underlying atlas for \( r \) is connected.

In this paper, we shall only consider local equivalence relations \( r \) which are locally connected. For such an \( r \), it follows readily from Proposition 1.2 that for a given sheaf \( F \) on \( M \), there is at most one \( r \)-transport \( t \) on \( F \). Thus (the existence of) an \( r \)-transport on \( F \) is a property, rather than an additional structure. For a locally connected \( r \), we therefore define the category \( \text{sh}(M, r) \) to be the full subcategory of \( \text{sh}(M) \) consisting of sheaves on \( M \) which admit an \( r \)-transport (necessarily unique).

**Remark 2.1.** The property of being an \( r \)-sheaf on \( X \) is a local property. More explicitly, if \( q : Y \rightarrow X \) is an étale map (a local homeomorphism), then any local equivalence relation \( r \) on \( X \) induces, in an evident way, a local equivalence relation on \( Y \), which we denote \( q^* r \); if \( r \) is locally connected, then so is \( q^* r \); and conversely, provided \( q \) is surjective. In this case, it is clear that if \( E \in \text{sh}(X) \), then \( E \) is an \( r \)-sheaf iff \( q^* E \) is an \( q^* r \)-sheaf.

**Remark 2.2.** More generally, for an arbitrary local equivalence relation \( r \) on a locale \( M \) and two sheaves with \( r \)-transport \((F, t)\) and \((F', t')\), there is a straightforward definition of transport-preserving map \( F \rightarrow F' \), so that one obtains a category \( \text{sh}(M, r) \). Using the remarks in Section 1, one can easily show that in case \( r \) is locally connected, any sheaf map \( F \rightarrow F' \) is transport-preserving, so that for such \( r \), the fact that the forgetful functor \( \text{sh}(M, r) \rightarrow \text{sh}(M) \) is full and faithful is a result, rather than a definition.

**Example 2.3.** For any locale \( M \), there is a ‘maximal’ local equivalence relation \( r_{\text{max}} \) on \( M \), given by the single chart \((U, R)\), where \( U = M \) and \( R = M \times M \). If \( M \) is a locally connected locale, then \( r_{\text{max}} \) is also locally connected. The category \( \text{sh}(M, r_{\text{max}}) \) is exactly the category of locally constant sheaves on \( M \). This category is not in general a Grothendieck topos. For example [3, p. 314] when \( M \) is the Hawaiian earring, \( \text{sh}(M, r_{\text{max}}) \) is not closed under infinite sums; on the other hand, if \( \text{sh}(M, r_{\text{max}}) \) is a Grothendieck topos, it must have infinite sums, and these sums must be preserved by the forgetful functor, cf. loc. cit., Theorem 6; cf. also [10]. The fact that \( \text{sh}(M, r_{\text{max}}) \) is not a Grothendieck topos disproves Theorem 4.14 in [16].

**Example 2.4.** Let \( r \) be a (locally connected) local equivalence relation on a locale \( M \). For any locale \( T \), there is a sheaf \( T^{(r)} \) on \( M \) of germs of \( r \)-invariant maps \( M \rightarrow T \). A typical section of \( T^{(r)} \) over an open \( U \subseteq M \) is a map \( s : U \rightarrow T \) which has the property that \( U \) is covered by \( r \)-charts \((U_i, R_i)\) such that each restriction \( s|_{U_i} : U_i \rightarrow T \) factors through the quotient map \( U_i \rightarrow R_i/R_i \). This sheaf \( T^{(r)} \) has \( r \)-transport, hence is an object of \( \text{sh}(M, r) \). When \( T \) is the Sierpinski space, \( T^{(r)} \) is a subobject classifier for \( \text{sh}(M, r) \), and \( \text{sh}(M, r) \) is an elementary topos. This is discussed more fully in [10].
3. Simplicial topoi and descent

Recall that a simplicial topos is a simplicial object \( \mathcal{E} \) in the category of (Grothendieck) topoi, except that the simplicial identities are required to hold only up to coherent isomorphisms. Thus a simplicial topos consists of a sequence of topoi \( \mathcal{E}_n \) (\( n \geq 0 \)), and for each nondecreasing function \( \alpha : [n] \to [m] \) (where \( [n] = \{0, 1, \ldots, n\} \)) a geometric morphism

\[
\mathcal{E}(\alpha) : \mathcal{E}_m \to \mathcal{E}_n ;
\]

furthermore, for each such \( \alpha : [n] \to [m] \) and \( \beta : [m] \to [k] \), there is given an isomorphism \( \theta_{\alpha, \beta} : \mathcal{E}(\alpha) \circ \mathcal{E}(\beta) \to \mathcal{E}(\beta \alpha) \), and these \( \theta \)'s are required to satisfy suitable coherence conditions. (Thus, a simplicial topos is a homomorphism of bicategories from the category \( \Delta^{op} \) into the bicategory of Grothendieck topoi.)

We adopt the standard notation from simplicial sets; for example, we write \( d_j : \mathcal{E}_n \to \mathcal{E}_{n-1} \) for \( \mathcal{E}(\partial_j) \), where \( \partial_j : [n-1] \to [n] \) is the strictly increasing function which omits \( j \) (for \( 0 \leq j \leq n \)).

For each simplicial topos \( \mathcal{E} \), one can construct a universal augmentation \( D(\mathcal{E}) \), as in

\[
\cdots \Rightarrow \mathcal{E}_2 \Rightarrow \mathcal{E}_1 \Rightarrow \mathcal{E}_0 \to D(\mathcal{E}) .
\] (3.1)

The category \( D(\mathcal{E}) \) can be explicitly described in various equivalent ways; e.g. as the category of descent objects: thus an object of \( D(\mathcal{E}) \) is a pair \((\mathcal{E}, \mu)\) where \( \mathcal{E} \) is an object of \( \mathcal{E}_0 \) and \( \mu : d^n_0 \mathcal{E} \to d^n_1 \mathcal{E} \) is an isomorphism satisfying the appropriate unit and cocycle conditions (cf. [15, section 3]); the arrows in \( D(\mathcal{E}) \) between two such objects \((\mathcal{E}, \mu)\) and \((\mathcal{E}', \mu')\) are arrows \( \mathcal{E} \to \mathcal{E}' \) in \( \mathcal{E}_0 \) which are compatible with the ‘descent data’ \( \mu \) and \( \mu' \). It follows from the general existence theorem for colimits of Grothendieck topoi ([15, Section 2] and [12]) that \( D(\mathcal{E}) \) is a Grothendieck topos, and is the colimit of the diagram \( \mathcal{E} \). The augmentation geometric morphism \( a : \mathcal{E}_0 \to D(\mathcal{E}) \) has as its inverse image the forgetful functor \( a^* : D(\mathcal{E}) \to \mathcal{E}_0 \), so that \( a^*(\mathcal{E}, \mu) = \mathcal{E} \).

The following is part of [15, Theorem 3.6]:

**Lemma 3.1.** For a simplicial topos \( \mathcal{E} \), if all the face maps \( d_j : \mathcal{E}_n \to \mathcal{E}_{n-1} \) are open (respectively locally connected, or atomic), then so is the augmentation \( a : \mathcal{E}_0 \to D(\mathcal{E}) \). \( \square \)

In particular, if \( X \) is a simplicial locale, we obtain a simplicial topos \( \text{sh}(X) \) by constructing the topos of sheaves \( \text{sh}(X_n) \) on each locale \( X_n \), and hence a descent topos \( D(\text{sh}(X)) \), and Lemma 3.1 gives the following:

**Lemma 3.2.** For a simplicial locale \( X \), in which all the face maps \( d_j : X_n \to X_{n-1} \)
are étale, the augmentation \( \text{sh}(X_i) \rightarrow \mathcal{D}(\text{sh}(X_i)) \) is an atomic geometric morphism, and \( \mathcal{D}(\text{sh}(X_i)) \) is an étendue.

**Proof.** Since the \( d_j \) are étale, the induced geometric morphisms \( d_j : \text{sh}(X_n) \rightarrow \text{sh}(X_{n-1}) \) are atomic. By Lemma 3.1, the augmentation \( \text{sh}(X_0) \rightarrow \mathcal{D}(\text{sh}(X_0)) \), which is evidently surjective, must also be atomic. Since this augmentation is also clearly a localic geometric morphism, it must be a slice, and thus \( \mathcal{D}(\text{sh}(X_0)) \) is an étendue. □

A map of simplicial topoi \( f : \mathcal{F} \rightarrow \mathcal{E} \) is given by geometric morphisms \( f_n : \mathcal{F}_n \rightarrow \mathcal{E}_n \) for each \( n \geq 0 \), together with, for each \( \alpha : [n] \rightarrow [m] \), an isomorphism

\[
f_\alpha : f_n \circ \mathcal{F}(\alpha) \rightarrow \mathcal{E}(\alpha) \circ f_m,
\]

and these isomorphisms are required to be compatible with the isomorphisms \( \theta_{\alpha, \beta} \) for \( \mathcal{E} \) and \( \mathcal{F} \). Such a map \( f : \mathcal{F} \rightarrow \mathcal{E} \) induces a geometric morphism \( \mathcal{D}(f) : \mathcal{D}(\mathcal{F}) \rightarrow \mathcal{D}(\mathcal{E}) \) between descent topoi, which is compatible with the augmentations in the sense that the square

\[
\begin{array}{c}
\mathcal{F}_0 \\
\downarrow a_f
\end{array}
\begin{array}{c}
\mathcal{E}_0 \\
\downarrow a_e
\end{array}
\begin{array}{c}
\mathcal{D}(\mathcal{F}) \\
\downarrow \mathcal{D}(f)
\end{array}
\begin{array}{c}
\mathcal{D}(\mathcal{E})
\end{array}
\]

(3.2)

commutes up to canonical isomorphism. Later we will use the following lemma concerning connected geometric morphisms (these are morphisms whose inverse image functor is full and faithful).

**Lemma 3.3.** Let \( f : \mathcal{F} \rightarrow \mathcal{E} \) be a map of simplicial topoi. If \( f_0 \) is connected and \( f_1 \) is surjective, then the induced geometric morphism \( \mathcal{D}(f) : \mathcal{D}(\mathcal{F}) \rightarrow \mathcal{D}(\mathcal{E}) \) is again connected.

**Proof.** Consider two objects \( (E, \mu) \) and \( (E', \mu') \) in \( \mathcal{D}(\mathcal{E}) \). We wish to show that arrows \( (E, \mu) \rightarrow (E', \mu') \) in \( \mathcal{D}(\mathcal{E}) \) correspond bijectively to arrows \( \mathcal{D}(f)^*(E, \mu) \rightarrow \mathcal{D}(f)^*(E, \mu') \) in \( \mathcal{D}(\mathcal{F}) \). Since \( f_0^* \) is full and faithful by assumption, it evidently suffices to show that for an arrow \( \alpha : E \rightarrow E' \) in \( \mathcal{E}_0 \), \( \alpha \) is compatible with descent data \( \mu : d_0^* \rightarrow d_0^* \) and \( \mu' : d_0^* \rightarrow d_0^* \) (in \( \mathcal{E}_1 \)) iff \( f_0^*(\alpha) \) is compatible with the induced descent data (in \( \mathcal{F}_1 \))

\[
f_1^*(\mu) : d_0^* f_0^*(E) \cong f_1^* d_0^*(E) \rightarrow f_1^* d_0^*(E) \cong d_1^* f_0^*(E)
\]

on \( f_0^*(E) \) and (similarly) \( f_1^*(\mu') \) on \( f_0^*(E') \). But this readily follows by the assumption that \( f_1^* : \mathcal{E}_1 \rightarrow \mathcal{F}_1 \) is a faithful functor. □
Recall that by the existence theorem for colimits [12, 15], already used in the construction of descent topoi $\mathcal{D}(\mathcal{C})$, the pushout topos $\mathcal{B} \cup_{\mathcal{A}} \mathcal{C}$ of any two geometric morphisms $f: \mathcal{A} \to \mathcal{B}$ and $g: \mathcal{A} \to \mathcal{C}$ between Grothendieck topoi exists,

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{g} & \mathcal{C} \\
\downarrow{f} & & \downarrow{v} \\
\mathcal{B} & \xrightarrow{u} & \mathcal{B} \cup_{\mathcal{A}} \mathcal{C}
\end{array}
$$

and can be constructed simply as follows: the objects of $\mathcal{B} \cup_{\mathcal{A}} \mathcal{C}$ are triples $(B, C, u)$, where $B$ is an object of the topos $\mathcal{B}$ and $C$ one of $\mathcal{C}$, while $v: f^*(B) \to g^*(C)$ is an isomorphism in the topos $\mathcal{A}$. An arrow $(B, C, v) \to (B', C', v')$ in the pushout topos $\mathcal{B} \cup_{\mathcal{A}} \mathcal{C}$ is given by a pair of arrows $\beta: B \to B'$ in $\mathcal{B}$ and $\gamma: C \to C'$ in $\mathcal{C}$ such that $v' \circ f^*(\beta) = g^*(\gamma) \circ v$ in $\mathcal{A}$. In the square (3.3), the inverse images $u^*$ and $v^*$ of the indicated geometric morphisms are the evident forgetful functors.

One can easily verify that for a pushout square, $u^*$ is full and faithful whenever $g^*$ is; in other words, we have the following:

**Lemma 3.4.** The pushout of a connected geometric morphism along any other geometric morphism is again connected (‘connectedness is preserved under co-base-change’).

Slightly more involved is the following lemma:

**Lemma 3.5.** Let $f: \mathcal{F}_1 \to \mathcal{E}_1$ be a map of simplicial topoi, with induced geometric morphism $\mathcal{D}(f): \mathcal{D}(\mathcal{F}_1) \to \mathcal{D}(\mathcal{E}_1)$. If $f_1: \mathcal{F}_1 \to \mathcal{E}_1$ is connected and $f_2: \mathcal{F}_2 \to \mathcal{E}_2$ is surjective, then the square (3.2) is a pushout.

**Proof.** Let us write $\mathcal{P}$ for the pushout topos. Then the objects of $\mathcal{P}$ are of the form

$$(F, \mu, E, v),$$

where $F$ is an object of $\mathcal{F}_0$ with descent data $\mu: d^*_0 F \to d^*_1 F$, while $E$ is an object of $\mathcal{E}_0$ and $v: F \to f_0^* E$ is an isomorphism. This gives an arrow $f_1^* d^*_0 E \to f_1^* d^*_1 E$ in $\mathcal{F}_1$: the broken arrow in the following diagram

$$
\begin{array}{ccc}
f_1^* d^*_0 E & \cong & d^*_0 f_0^* E \\
\downarrow{\cong} & & \downarrow{\mu} \\
f_1^* d^*_1 E & \cong & d^*_1 f_0^* E
\end{array}
$$
Since \( f_*^* \) is full and faithful by assumption, this arrow comes from a unique arrow \( \sigma : d_0^* E \to d_T^* E \). This arrow \( \sigma \) satisfies the cocycle condition in \( \mathbb{E}_2 \) because in \( \mathbb{F}_2 \), the map \( \mu \), and hence also \( f_1^*(\sigma) \), does, while \( f_2^* : \mathbb{E}_2 \to \mathbb{F}_2 \) is faithful by assumption. The arrow \( \sigma \) also satisfies the unit condition in \( \mathbb{E}_0 \) for a similar reason, since \( f_0^* : \mathbb{E}_0 \to \mathbb{F}_0 \) is again faithful (in fact \( f_0^* : \mathbb{E}_0 \to \mathbb{F}_0 \) is a retract of \( f_i : \mathbb{E}_i \to \mathbb{F}_i \), so \( f_0 \) is connected since \( f_i \) is). This shows that from an object \((3.4)\) in the pushout \( \mathcal{P} \), one can construct an object in \( \mathcal{D}(\mathbb{E}_i) \).

Conversely, any object \((E, \sigma)\) in \( \mathcal{D}(\mathbb{E}_i) \) gives an object \((F, \mu, E, \nu)\) in the pushout, where \( F = f_0^* E \) and \( \mu \) is defined as

\[
\mu : d_0^* F = d_0^* f_0^* E \cong f_1^* d_0^* E \xrightarrow{f_1^* \mu} f_1^* d_T^* E \cong d_T^* f_0^* E \cong d_T^* F,
\]

and \( \nu : F \to f_0^* E \) is defined to be the identity.

These constructions establish a suitable equivalence of categories \( \mathcal{D}(\mathbb{E}_i) \cong \mathcal{P} \), proving the lemma. \( \square \)

### 4. The topos defined by an atlas

This section is of auxiliary character. It defines a topos \( \text{sh}(M, \mathcal{U}) \) out of an atlas \( \mathcal{U} \) for a local equivalence relation \( r \) on the locale \( M \), and \( \text{sh}(M, \mathcal{U}) \) in general will depend on the choice of \( \mathcal{U} \) (and even, in the most general case, on some further choice of a ‘hypercovering’).

For any atlas \( \mathcal{U} = \{(U_i, R_i)\}_{i \in I} \) for a local equivalence relation, we construct a simplicial locale \( U \). (a hypercovering of \( M \), in fact): the locale \( U_0 \) of vertices is the disjoint sum

\[
U_0 = \coprod_{j \in I} U_j ,
\]

while the space \( U_1 \) of 1-simplices is defined as

\[
U_1 = \coprod_{i,j \in I} \coprod_{k \in K_{i,j}} U_{ijk} ,
\]

where \( K_{i,j} \) is an index set for some open covering \( U_{ijk} \) of \( U_i \cap U_j \) by sublocales on which \( R_i \) and \( R_j \) agree. The simplicial operators

\[
U_1 \Rightarrow U_0
\]

are defined in the obvious way (if we assume, as we may, that \( K_{i,i} = \{\ast\} \), a one-point set, and that \( U_{j,j,\ast} = U_j \)). We now define \( U \) as the coskeleton of the truncated simplicial locale (4.3).

\[
U_\ast = \text{Csk}(U_1 \Rightarrow U_0) .
\]
Thus, \( U_2 \) is a coproduct with an index set whose typical element is given by data \(((i_0, i_1, i_2), (k_0, k_1, k_2))\) with the \( i \)'s in \( I \), and \( k_2 \in K_{i_0 i_1} \), etc., and the summand corresponding to this index is

\[
U_{i_0 i_1 i_2} \cap U_{i_1 i_2 i_0} \cap U_{i_2 i_0 i_1}.
\]

The simplicial locale \( U_\ast \) has an evident augmentation \( a \) to \( M \) given by the inclusions \( U_i \rightarrow M \) \((i \in I)\). All maps in the diagram

\[
\cdots U_3 \rightarrow U_2 \rightarrow U_1 \rightarrow U_0 \rightarrow M
\]  

are étale, so \( U_\ast \) is a simplicial sheaf on \( M \).

**Lemma 4.1.** The descent topos \( \mathcal{D}(\text{sh}(U_\ast)) \) is equivalent to the topos \( \text{sh}(M) \) of sheaves on \( M \), by an equivalence compatible with the augmentations (3.1) and (4.5).

**Proof.** We view \( U_\ast \) as a simplicial sheaf on \( M \). Since \( U_1 \rightarrow U_0 \times U_0 \) is surjective and \( U_\ast \) is defined as a coskeleton, \( U_\ast \) is clearly a hypercover of \( M \) (i.e. an internal contractible simplicial set inside \( \text{sh}(M) \)). By standard theory of simplicial covering spaces [5, Appendix] applied in \( \text{sh}(M) \), an object of \( \mathcal{D}(\text{sh}(U_\ast)) \) can be identified with a covering projection into \( U_\ast \). But by contractibility of \( U_\ast \), each such is a trivial covering projection, i.e. it corresponds to a sheaf of \( M \). This proves the lemma. \( \square \)

The sum of the equivalence relations \( R_i \) defines an equivalence relation \( R_0 \) on the sum \( U_0 \) (cf. (4.1)); similarly, the sum of the equivalence relations \( R_{i_1} \) on the sum \( U_1 \) (cf. (4.2)), and on \( U_2 \), etc. By the evident compatibilities, we get a morphism of simplicial locales

\[
q_n : U_n \rightarrow U_n/R_n \quad (n = 0, 1, 2, \ldots)
\]

and hence a morphism of their respective descent topos; we denote the descent topos for the simplicial topos \( \left(\text{sh}(U_n/R_n)\right)_n \) by \( \text{sh}(M, U_\ast) \). All this is depicted in the following diagram (utilizing Lemma 4.1 for the descent of the left-hand column):

\[
\begin{array}{cccc}
\text{sh}(U_\ast) & \rightarrow & \text{sh}(U_\ast/R_\ast) & \\
\downarrow & & \downarrow & \\
\text{sh}(U_0) & \rightarrow & \text{sh}(U_0/R_0) & \\
\downarrow & & \downarrow & \\
\text{sh}(M) & \rightarrow & \text{sh}(M, U_\ast) & \\
\end{array}
\]  

(4.6)
Lemma 4.2. For any open atlas $\mathcal{U}$ and choice of hypercovering $U$, the topos $\text{sh}(M, U)$ is an étendue.

Proof. If $(W, R)$ is an open chart, and $V \subseteq W$ is an open sublocale, then one obtains an inclusion of an open sublocale $V/(R|_V) \to W/R$. In the right-hand column of (4.6) each map $U_n/R_n \to U_{n-1}/R_{n-1}$ is a sum of such inclusions, hence is étale. By Lemma 3.2, the descent topos $\text{sh}(M, U)$ is an étendue. □

The situation simplifies considerably for the case where $r$ is an (open and) locally connected local equivalence relation. If $\mathcal{U}$ is a connected atlas for $r$, we may choose the $U_{ij,k}$ so small that the charts $(U_{ij,k}, R_{ij,k})$ are all connected (and open, locally connected, of course). If this is the case, we say that the hypercovering $U$ is connected; then the geometric morphisms

$$\text{sh}(U_n) \to \text{sh}(U_n/R_n) \quad (n = 0, 1)$$

are connected geometric morphisms. Consequently, we have by Lemmas 3.5 and 3.4, the following lemma:

Lemma 4.3. For a connected atlas $\mathcal{U}$ and any connected hypercovering $U$, associated to it,

$$\text{sh}(U_n) \to \text{sh}(U_n/R_n) \quad (4.7)$$

is a push-out, and the geometric morphism $\pi : \text{sh}(M) \to \text{sh}(M, U)$ is again connected. □

(The geometric morphism $\pi$ is also locally connected.)

Thus, $\text{sh}(M, U)$ may be identified, via $\pi^*$, with a full subcategory of $\text{sh}(M)$, and since the remaining parts of the diagram (4.7) do not depend on the choice of $U$, it follows that $\text{sh}(M, U)$ only depends on the atlas $\mathcal{U}$ itself, not on the choice of hypercovering $U$, as long as $U$ is taken to be connected. Therefore, we may write $\text{sh}(M, \mathcal{U})$ for $\text{sh}(M, U)$. It is an étendue, by Lemma 4.2. The objects of $\text{sh}(M, \mathcal{U})$ we call $\mathcal{U}$-sheaves.

We already observed that for $r$ locally connected, $\text{sh}(M, r)$ is a full subcategory of $\text{sh}(M)$, so we may compare it with the $\text{sh}(M, \mathcal{U})$'s. It is clear from Proposition 1.1 that if the structure of $r$-sheaf on a sheaf $E$ is given by an atlas $\mathcal{U}$, then $E \in \text{sh}(M, \mathcal{U})$; and conversely, every $\mathcal{U}$-sheaf is an $r$-sheaf, so that $\text{sh}(M, r)$ is the union of all the subcategories $\text{sh}(M, \mathcal{U})$ as $\mathcal{U}$ ranges over the connected atlases for $r$. This union is actually a filtered one; for, any two connected atlases for $r$
have a common refinement, and it is easy to see that if \( \mathcal{U}' \) refines \( \mathcal{U} \), then \( \text{sh}(M, \mathcal{U}) \subseteq \text{sh}(M, \mathcal{U}') \).

5. Simply connected maps and étendues

Let \( f : Y \rightarrow X \) be a map of locales. We shall call \( f \) simply connected if

(i) \( f \) is connected (i.e. \( f^* : \text{sh}(X) \rightarrow \text{sh}(Y) \) is full and faithful),

(ii) for every sheaf \( E \) on \( Y \), if there exists an open cover \( \bigcup U_i = Y \) of \( Y \) and sheaves \( D_i \) on \( X \) such that \( E|_{U_i} \cong f^*(D_i)|_{U_i} \), then there exists a sheaf \( D \) on \( X \) such that \( E \cong f^*(D) \).

Condition (ii) expresses that if a sheaf \( E \) on \( Y \) is locally in the image of \( f^* \), then it is in the image of \( f^* \) (up to isomorphism). (The conditions together express the intuitive idea that \( f \) is a map with simply connected fibers, in a very weak way, but sufficient for our purposes in this paper. Surely for a general theory of simply connected maps, one should use a stronger notion, which is stable under pullback.)

Examples. (a) The unique map \( Y \rightarrow 1 \) is simply connected if every locally constant sheaf on the locale \( Y \) is constant. In particular, if a path-connected topological space \( T \) is simply connected in the usual sense (defined in terms of paths), then the unique map \( T \rightarrow 1 \) is simply connected.

(b) If \( T \rightarrow B \) is a locally connected map of topological spaces with connected and simply connected fibers (in the usual topological sense), then as a map of locales, \( f \) is simply connected in the sense just defined. (This is not trivial; a detailed proof is given in [lo, Lemma 3.21 and [ll].)

(c) The standard argument that a locally constant sheaf on the (localic) unit interval \( I \) is constant will (when applied internally in \( \text{sh}(X) \)) show that the projection \( X \times I \rightarrow X \) is simply connected, for every locale \( X \).

(d) Let \( Y \) be a connected and locally connected locale, and suppose that the restriction map \( Y^\triangle \rightarrow Y^{a\triangle} \) is a stable surjection (here \( \triangle \) is the standard 2-simplex, and \( a\triangle \) is its boundary). Then example (c) and [6, Lemma 3.41] show that the map \( Y \rightarrow 1 \) is simply connected.

(e) The previous example can be relativized: A connected and locally connected map of locales \( f : Y \rightarrow X \) is simply connected, in the sense defined above, whenever \( (Y^\triangle)_X \rightarrow (Y^{a\triangle})_X \) is a stable surjection. (Here, for any locale \( A \), \( (Y^A)_X \) denotes the relative exponential 'of maps \( A \rightarrow Y \) which become constant when composed with \( f : Y \rightarrow X^* \), i.e. the locale defined by the pull-back diagram

\[
\begin{array}{ccc}
(Y^A)_X & \longrightarrow & Y^A \\
\downarrow & & \downarrow f^A \\
X & \longrightarrow & X^A
\end{array}
\]
where the map $X \to X^d$ is the exponential adjoint of the projection map $X \times A \to A$.

The notion of simply connected map given here is related to local equivalence relations in the following way. For any open map $f : Y \to X$, its kernel pair $Ker(f) = Y \times_X Y$ defines an equivalence relation on $Y$. The induced local equivalence relation on $Y$, given by the atlas consisting of the single chart $(Y, Ker(f))$, is called the local kernel pair of $f$, and denoted $Lker(f)$. If the map $f$ is locally connected, then so is this local equivalence relation on $Y$, and we have a category $sh(Y; Lker(f))$, together with an evident factorization of $f^* : sh(X) \to sh(Y)$ through the forgetful functor $sh(Y; Lker(f)) \to sh(Y)$.

The following is now obvious from the definition, and from Proposition 1.1.

**Lemma 5.1.** A locally connected map $f : Y \to X$ is simply connected iff $f^*$ induces an equivalence of categories $sh(X) \cong sh(Y; Lker(f))$. □

An equivalence relation $R$ on a locale $X$ is said to be simply connected if the quotient map $X \to X/R$ is a simply connected map. (If $d_0, d_1 : R \to X$ are locally connected, it can be shown that $X \to X/R$ is simply connected whenever $d_0, d_1$ are; but we will neither use nor prove this here.) Moreover, an atlas for a local equivalence relation is called simply connected if all its charts are; and a local equivalence relation $r$ is called locally simply connected if every atlas for $r$ can be refined by a simply connected atlas (this implies that $r$ is locally connected).

**Lemma 5.2.** Let $r$ be a locally connected local equivalence relation on the locale $M$, and let $\mathcal{U}$ be a simply connected atlas for $r$. Then the inclusion functor $sh(M, \mathcal{U}) \to sh(M, r)$ is an equivalence of categories.

**Proof.** The inclusion functor is a functor between full subcategories of $sh(M)$, hence is full and faithful. To see that it is essentially surjective, consider a sheaf $E$ on $M$ with $r$-transport. We have to show that there exists an atlas for this $r$-transport with underlying $r$-atlas the given atlas $\mathcal{U}$. By the uniqueness of transport, this means that we have to show that for any chart $(U, R)$ of $\mathcal{U}$, the restricted sheaf $E|_U$ is isomorphic to $\pi^*(D)$ for some sheaf $D$ on $U/R$ (where $\pi$ is the quotient map $U \to U/R$). Since $E$ has $r$-transport, there exists an atlas $V$ for $r$, whose charts $(V_i, K_i)$ act on $E|_{V_i}$, so for the given $U$, there exists a covering $\bigcup V_i$ of $U$ such that for each index $i$ there exist a sheaf $D_i$ on $V_i/R_i$ with $E|_{V_i} \cong \pi_i^*(D_i)$, where $\pi_i : V_i \to V_i/R_i$ is the quotient map. Let $\mu_i : V_i \to U$ and $\nu_i : V_i/R_i \to U/R$ be the inclusions, so that $\nu_i \pi_i = \pi \mu_i$. Then $D_i \cong \nu_i^* \nu_i^*(D_i)$, so $E|_{V_i} \cong \pi_i^* \nu_i^*(\nu_i^*(D_i)) \cong \mu_i^* \pi^*(\nu_i^*D_i) \equiv \pi^*(\nu_i^*D_i)|_{V_i}$. Thus $E|_U$ is in the image of $\pi^*$, up to isomorphism. Since by assumption the quotient map $\pi : U \to U/R$ is simply connected, it follows that $E|_U$ is isomorphic to $\pi^*(D)$ for some sheaf $D$ on $U/R$, as required. □

This lemma, together with Lemma 4.2, yields the following theorem:
Theorem 5.3. Let \( r \) be a local equivalence relation on a locale \( M \). If \( r \) is locally simply connected, then \( \text{sh}(M, r) \) is an étendue topos. \( \square \)

6. Maps from locales into étendues

Let \( \mathcal{T} \) be a fixed étendue topos. In this section, we will show how for any locale \( M \), a locally connected geometric morphism \( a : \text{sh}(M) \to \mathcal{T} \) gives rise to a local equivalence relation on \( M \). Recall that for locally connected \( a \), the inverse image functor \( a^* \) has a left adjoint \( a_! : \text{sh}(M) \to \mathcal{T} \).

Lemma 6.1. The locale \( M \) has a basis of open sublocales \( U \subseteq M \) with the property that \( \mathcal{T}/a_*U \) is a localic topos.

Proof. Let \( G \) be an object of \( \mathcal{T} \) for which \( \mathcal{T}/G \) is a localic topos, and construct the locale \( B \) by the pull-back

\[
\begin{array}{ccc}
\text{sh}(B) & \longrightarrow & \mathcal{T}/G \\
q \downarrow & & \downarrow p \\
\text{sh}(M) & \longrightarrow & \mathcal{T}
\end{array}
\]

Then, by construction of \( B \), the topos \( \text{sh}(B) \) is equivalent over \( \text{sh}(M) \) to \( \text{sh}(M)/a^*(G) \). And \( q \) is induced by an étale map (a local homeomorphism) of locales, also denoted \( q : B \to M \). The required basis for \( M \) consists of those open \( U \subseteq M \) over which \( q \) has a section. Indeed, let \( s : U \to B \) be a section of \( q \). This section can be viewed as a map \( s : U \to a^*(G) \) in \( \text{sh}(M) \), and hence corresponds by adjunction to a map \( \hat{s} : a_!(U) \to G \) in \( \mathcal{T} \). But then the topos \( \mathcal{T}/a_!(U) = (\mathcal{T}/G)/\hat{s} \) is localic since \( \mathcal{T}/G \) is. \( \square \)

By the lemma, any open \( U \subseteq M \) in this basis for \( M \) gives rise to a locale \( a_!(U) \) and a map \( \varepsilon_U : U \to a_!(U) \), for which there is an equivalence of topoi under \( \text{sh}(U) \), as in

\[
\begin{array}{ccc}
\text{sh}(U) & \xrightarrow{\varepsilon_U} & \text{sh}(a_!(U)) \\
\downarrow & & \downarrow \\
\text{sh}(M) & \longrightarrow & \mathcal{T}
\end{array}
\]

Notice that by construction, \( \varepsilon_U \) is a connected and locally connected map of locales. Thus, since connected locally connected maps are stable under pullback,

\[
R_U := \text{Ker}(\varepsilon_U) \subseteq U \times U
\]
is a connected and locally connected equivalence relation on $U$. We shall prove the following:

**Lemma 6.2.** The charts $(U, R_U)$, for all open $U \subseteq M$ for which $\mathcal{T}/a, (U)$ is localic, form an atlas for a local equivalence relation on $M$.

We will call this local equivalence relation the *local kernel of $a$*, and denote it by $\text{Lker}(a)$. (This is compatible with the similar notation used in Section 5.) Clearly $\text{Lker}(a)$ is locally connected.

**Proof.** For two such open $V \subseteq U \subseteq M$, it is enough to show that $\text{Lker}(e_U)V = \text{Lker}(e_V)$. Consider the diagram

$$
\begin{array}{ccc}
V & \xymatrix{ & U \ar[dl]_{e_V} \ar[dr]^{e_U} } \\
\end{array}
\xymatrix{ a_*V & a_*U \ar[l]_{a_*(i)} }
$$

obtained from the inclusion $i : V \subseteq U$. Since $\mathcal{T}/a,V \to \mathcal{T}/a,U$ is a map of slice topoi over $\mathcal{T}$, the corresponding map of locales $a_*V \to a_*U$ is étale. Thus

$$
\text{Lker}(e_V)V = \text{Lker}(e_U)i \\
= \text{Lker}(a_*(i)e_V) \\
= \text{Lker}(e_V),
$$

where the latter equality holds by the following lemma.

**Lemma 6.3.** Let $f : Y \to X$ and $e : X \to B$ be maps of locales, where $e$ is étale. Then $\text{Lker}(f) = \text{Lker}(ef)$.

**Proof.** Consider an open $U \subseteq X$ such that $e|_U$ is a homeomorphism $U \cong e(U)$. Then

$$
\text{Ker}(f)|_{f^{-1}U} = \text{Ker}(f|_{f^{-1}U}) \\
= \text{Ker}(ef|_{f^{-1}U}) \\
= \text{Ker}(ef)|_{f^{-1}U}.
$$

Since this holds for all such $U$, $\text{Ker}(f)$ and $\text{Ker}(ef)$ agree on an open cover of $Y$. Hence $\text{Lker}(f) = \text{Lker}(ef)$. □

This construction of the local equivalence relation $\text{Lker}(a)$ on $M$ from the
locally connected geometric morphism $a : \text{sh}(M) \to \mathcal{T}$ enjoys various naturality properties. We single out the following. Recall the #-construction of Section 2 for lifting a local equivalence back along an étale map. Then the following holds:

**Lemma 6.4.** For a pull-back square

\[
\begin{array}{ccc}
\text{sh}(N) & \xrightarrow{b} & \mathcal{R} \\
\downarrow{q} & & \downarrow{p} \\
\text{sh}(M) & \xrightarrow{a} & \mathcal{T}
\end{array}
\]

where $a$ (and hence $b$) are locally connected geometric morphisms and $p$ is an étale map between étendues, we have $q^* (\text{Lker}(a)) = \text{Lker}(b)$.

**Proof.** Let $V \subseteq N$ be an open sublocale of $N$, so small that $q|_V$ is a homeomorphism $V \cong q(V)$, and moreover so small that both $\mathcal{R}/b|_V$ and $\mathcal{T}/a|_V (q(V))$ are localic topoi. (Note that this property is inherited by smaller open sublocales.) Then there is an induced map $b_* (V) \to a_* (q(V))$ which is étale (in fact a homeomorphism) since $p$ is. The result follows from Lemma 6.3. \(\square\)

**Remark 6.5** (which we shall not use). If $r$ is a local equivalence relation on a locale $M$, and $\mathcal{U}$ is a connected atlas for $r$, then there is an induced geometric morphism $a : \text{sh}(M) \to \text{sh}(M, \mathcal{U})$, as in Section 4. The local equivalence relation $\text{Lker}(a)$ is in general larger than $r$. It coincides with $r$ if $r$ has an atlas consisting of charts $(U, R)$ with the property that $R$ is the kernel pair of $U \to U/R$.

### 7. The main theorem

We now prove the result announced in the title of the paper:

**Theorem 7.1.** For every étendue $\mathcal{T}$, there exists a locale $M$ and a locally simply connected local equivalence relation $r$ on $M$ for which there exists an equivalence of topoi $\text{sh}(M, r) \simeq \mathcal{T}$.

In the proof, we shall use the following construction from [6]: for any topos $\mathcal{E}$, there exists a locale $X = X_{\mathcal{E}}$ in $\mathcal{E}$ such that $X$ is (internally) contractible and locally contractible, and moreover such that the topos $\mathcal{E}[X]$ of $\mathcal{E}$-internal sheaves on $X$ is (externally) localic. In particular, this locale $X$ has (internally in $\mathcal{E}$) a basis, containing $X$ itself, and consisting of open $U \subseteq X$ which are connected and locally connected, and 'simply connected' in the sense that $U^\triangleright \to U^*\triangleright$ is a stable surjection of locales in $\mathcal{E}$ (cf. Example (d) in Section 5 for notation). Moreover, these properties of the internal locale $X$ are stable under pull-back along an
arbitrary geometric morphism $f : \mathcal{T} \to \mathcal{C}$. In the special case where $f : \mathcal{T} \to \mathcal{C}$ is such that both $\mathcal{T}$ and $\mathcal{T}[f^*(X)]$ are localic, then $\mathcal{T}[f^*(X)] \to \mathcal{T}$ corresponds to an (external) map of locales $b : B \to A$, and the stable internal properties of $X$ just listed can be rephrased as follows: $b$ is an open surjection, and $B$ has a basis, containing $B$ itself, which consists of open sublocales $U \subseteq B$ with the property that each restriction $b|_U : U \to b(U)$ satisfies the conditions for the map $f : T \to X$ in Example (e) in Section 5; in particular, each such restriction is a simply connected map.

For the proof of the theorem, consider for the given étendue $\mathcal{T}$ such a locale $X = X_\mathcal{T}$ in $\mathcal{T}$. Since $\mathcal{T}[X_\mathcal{T}]$ is localic, there is a locale $M$ and a geometric morphism $a : \text{sh}(M) \to \mathcal{T}$ for which $\text{sh}(M) \simeq \mathcal{T}[X_\mathcal{T}]$, over $\mathcal{T}$. Moreover, $a$ is a connected and locally connected geometric morphism, since $X_\mathcal{T}$ is a connected and locally connected locale in $\mathcal{T}$.

Now let $G$ be an object with full support in $\mathcal{T}$ for which $\mathcal{T}/G$ is a localic topos (such a $G$ exists since $\mathcal{T}$ is an étendue). Thus there is a locale $A$ and an étale surjection $p : \text{sh}(A) \to \mathcal{T}$ such that $\mathcal{T}/G \simeq \text{sh}(A)$, over $\mathcal{T}$. It follows that the pull-back of $p$ along $a : \text{sh}(M) \to \mathcal{T}$ is again étale. Hence this pull-back is a localic topos, say $\text{sh}(B)$ as in

$$
\begin{array}{ccc}
\text{sh}(B) & \xrightarrow{b} & \text{sh}(A) \\
\downarrow^{q} & & \downarrow^{p} \\
\text{sh}(M) & \xrightarrow{a} & \mathcal{T}
\end{array}
$$

The local equivalence relation $r$ on $M$ in the statement of the theorem will be $\text{Lker}(a)$, as constructed in Section 6. A comparison of the pushout squares in the following two lemmas will now prove the equivalence of topoi $\text{sh}(M, r) \simeq \mathcal{T}$ asserted in the theorem.

**Lemma 7.2.** The pull-back square (7.1) of topoi is also a pushout square.

**Proof.** The maps $p$ and $q$ are étale surjections, hence descent maps [7, 13]. In other words, $\text{sh}(M)$ is obtained by descent from the simplicial topos of sheaves on the simplicial locale

$$B_. = (\cdots B \times_M B \times_M B \rightrightarrows B \times_M B \rightrightarrows B),$$

and $\mathcal{T}$ is similarly obtained from the simplicial topos

$$\text{sh}(A_.) = (\cdots \text{sh}(A) \times_{\mathcal{T}} \text{sh}(A) \rightrightarrows \text{sh}(A)).$$

Moreover, all the components of the simplicial map $\text{sh}(B_.) \to \text{sh}(A_.)$ are pull-
backs of the map \( a : \text{sh}(M) \to \mathcal{T} \), hence are connected and locally connected. The lemma thus follows from Lemma 3.5. \( \square \)

**Lemma 7.3.** With \( A, B, \) etc. as above, there is a pushout square of topoi

\[
\begin{array}{ccc}
\text{sh}(B) & \xrightarrow{b} & \text{sh}(A) \\
\downarrow{q} & & \downarrow \\
\text{sh}(M) & \longrightarrow & \text{sh}(M, \text{Lker}(a))
\end{array}
\]

**Proof.** Note first that by the properties of the \([6]\)-construction listed above, the map \( b : B \to A \) is connected, locally connected, and simply connected. Hence by Lemma 5.1, the map \( b : B \to A \) induces an equivalence of categories \( \text{sh}(A) \cong \text{sh}(B, \text{Lker}(b)) \). In particular, the latter is a topos. Write \( \mathcal{P} \) for the pushout of \( \text{sh}(B) \to \text{sh}(B, \text{Lker}(b)) \) and \( q : \text{sh}(B) \to \text{sh}(M) \). By the explicit description of pushouts of topoi given in Section 3, \( \mathcal{P} \) is the category of triples \((F, F', \sigma)\), where \( F' \) is a \( \text{Lker}(b) \)-sheaf on \( B \), \( F \) is a sheaf on \( M \), and \( \sigma \) is an isomorphism \( q^*(F) \cong F' \) of sheaves on \( B \). In other words, \( \mathcal{P} \) is (equivalent to) the category of sheaves \( F \) on \( M \) such that \( q^*(F) \) is an \( \text{Lker}(b) \)-sheaf. But, by Lemma 6.4, \( \text{Lker}(b) = q^* \text{Lker}(a) \), so by Remark 2.1, we conclude that the pushout \( \mathcal{P} \) is equivalent to the category \( \text{sh}(M, \text{Lker}(a)) \). This proves the lemma. \( \square \)

Note that we have proved that \( \text{sh}(M, \text{Lker}(a)) \) is a topos without invoking Theorem 5.3 and the (as yet unproved) fact that \( \text{Lker}(a) \) is locally simply connected, as we asserted in the theorem. To prove this fact, it suffices to show that \( \text{Lker}(b) \) is locally simply connected, since \( q : B \to M \) is \'etale and \( \text{Lker}(b) = q^* \text{Lker}(a) \). But as explained above, the properties of the \([6]\)-construction imply that \( B \) has a basis of open sublocales \( U \subset B \) for which \( U \to b(U) \) is a connected, locally connected, and simply connected map, so that \( \text{Ker}(b|_U) \) is a simply connected (and locally connected) equivalence relation on \( U \). Since the collection of these \( U \subset B \) form a basis for \( B \), the local equivalence relation \( \text{Lker}(b) \) must be locally simply connected.

This completes the proof of the theorem. \( \square \)

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