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A completeness theorem for open maps

A. Joyal\(^a\), I. Moerdijk\(^b,\,*\)

\(^a\) Université du Québec à Montréal, Montréal, Que., Canada H3C 3P8
\(^b\) Mathematisch Instituut, Universiteit Utrecht, Postbus 80.010, 3508 TA Utrecht, Netherlands

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Abstract

This paper provides a partial solution to the completeness problem for Joyal’s axiomatization of open and etale maps, under the additional assumption that a collection axiom (related to the set-theoretical axiom with the same name) holds.

In many categories of geometric objects, there are natural classes of open maps, of proper maps, and of etale maps. Some of the formal properties of these classes in the category of schemes were isolated by Grothendieck [S]. In the late 70’s, an attempt was made by the first author to give a full axiomatization of classes of open (and of etale) maps in topoi: along with a list of axioms, he suggested some concrete candidates for “universal” classes of open (etale) maps in path-topoi such as \( \mathcal{F}^2 \) and \( \mathcal{F}^1 \) which could be used to test the completeness of the axioms.

In 1988 E. Dubuc proved a completeness theorem for these axioms for etale maps relative to the Sierpinski topos \( \mathcal{F}^2 \), under some special assumptions on the site; see [2].

In this paper we prove a completeness theorem for any class \( R \) of open maps (in any topos \( \mathcal{E} \)) which satisfies an additional axiom called the Collection Axiom. More precisely, for any topos \( \mathcal{F} \), the class of morphisms in \( \mathcal{F}^2 \) which are quasi-pullback squares (in \( \mathcal{F} \)) satisfies the axioms for open maps as well as this Collection Axiom. Given \( \mathcal{E} \) and \( R \), we construct a suitable topos \( \mathcal{F} \) and a geometric morphism \( \phi: \mathcal{F}^2 \to \mathcal{E} \), with the property that an arrow \( f \) in \( \mathcal{E} \) belongs to the class \( R \) iff its inverse image \( \phi^*(f) \) is a quasi-pullback. We also prove a completeness theorem of the following form: for a topos \( \mathcal{E} \) and a class \( R \) of open maps satisfying the Collection Axiom, there exists a suitable geometric realization functor from \( \mathcal{E} \) into a category of locales, with the property that a map \( f \) in \( \mathcal{E} \) belongs to \( R \) iff its geometric realization is an open (in the usual sense) map of locales. Analogous results hold for pretopoi and exact categories. Furthermore, we show that these completeness theorems yield similar theorems for classes of etale maps, thereby improving Dubuc’s result.

* Corresponding author.
It should be noted that these results are only partial solutions to the original problem, since the Collection Axiom does not hold in all examples (see Section 6) below.

Our work also provides a new ("synthetic") approach to modal logic, since open maps in a topos relate to interior operators, and hence to modal operators for the internal logic of the topos.

The axioms for open maps presented in this paper are closely related to axioms for "small" maps and the categorical foundations of the cumulative hierarchy and ordinal numbers (see [10]). In particular, our Collection Axiom is related to the axiom of set theory bearing the same name.

The results of this paper were first announced in [9]. We would like to thank Eduardo Dubuc and Peter Johnstone for their comments on an earlier version of this paper.

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1. Axioms for open maps and for etale maps

A class $R$ of arrows in a topos $\mathcal{E}$ is said to be a class of open maps if it satisfies the following axioms A1–A6:

(A1) Any isomorphism belongs to $R$, and $R$ is closed under composition.

(A2) ("Stability axiom") In any pullback square

$$
\begin{array}{ccc}
Y' & \xrightarrow{f'} & Y \\
\downarrow & & \downarrow \phi \\
X' & \xrightarrow{\pi} & X,
\end{array}
$$

if $f$ belongs to $R$ then so does $f'$.

(A3) ("Descent axiom") In any pullback square (1), if $f'$ belongs to $R$ and $p$ is epi then $f$ belongs to $R$.

(A4) For any set $I$, the unique map $\sum_{i \in I} 1 \rightarrow 1$ belongs to $R$.

(A5) For any family of arrows $Y_i \rightarrow X_i (i \in I)$ in the class $R$, their sum $\sum Y_i \rightarrow \sum X_i$ belongs to $R$.

(A6) ("Quotient axiom") In any commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow \phi & & \downarrow f \\
B & \xrightarrow{\pi} & X,
\end{array}
$$

if $p$ is epi and $g$ belongs to $R$ then so does $f$.

A class $R$ of arrows in a topos $\mathcal{E}$ is said to be a class of etale maps if it satisfies the axioms A1–A5 above, and the following two axioms A7, A8:
(A7) For any arrow \( f: Y \to X \) in \( R \), its diagonal \( \Delta_f: Y \to Y \times_X Y \) also belongs to \( R \).

(A8) In any commutative diagram of the form (2) above, if \( p \) is epi while both \( g \) and \( p \) belong to \( R \), then \( f \) belongs to \( R \).

With obvious modifications, these axioms apply to categories other than topoi. For example, for a pretopos \( \mathcal{E} \) one defines the notion of a class of open (or etale) maps by the same axioms, except that the index-set \( I \) occurring in the axioms (A4) and (A5) is required to be finite.

In this paper, we shall be interested in an additional "collection axiom" (which is related to the axiom of collection in Zermelo–Fraenkel set theory). Before we state it we recall that, in any category with pullbacks, a commutative square of the form (1) above is said to be a quasi-pullback square if the evident arrow \( Y' \to X' \times_X Y \) is an epimorphism. Note that if we combine axioms A3 and A6, we see that for any quasi-pullback square of the form (1), if \( f' \) belongs to \( R \) and \( p \) is epi then \( f \) belongs to \( R \).

(A9) ("Collection axiom") For any two arrows \( p: Y \to X \) and \( f: X \to A \) where \( p \) is epi and \( f \) belongs to \( R \), there exists a quasi-pullback square of the form

\[
\begin{array}{ccc}
Z & \rightarrow & Y \\
\downarrow^{a} & \nearrow^{f} & \downarrow^{p} \\
B & \rightarrow & X \\
\end{array}
\]

where \( h \) is epi and \( g \) belongs to \( R \).

It is sometimes useful to say that the collection axiom holds for \( f \in R \) if for every epi \( p: Y \to X \) there is a quasi-pullback square as above. It then follows from the axioms for a class of open maps that the collection axiom holds for a given \( f: X \to A \) if and only if it holds for every pullback \( X \times_A C_i \to C_i \) of \( f \) along the members of a covering \( (C_i \to A)_{i \in I} \) of the object \( A \).

We now consider some elementary examples:

**Example 1.1.** Let \( C \) be a small category, and consider the associated topos \( \hat{C} \) of contravariant Set-valued functors on \( C \). One obtains classes of open maps and of etale maps in \( \hat{C} \) by defining a map \( f: G \to F \) to be open (respectively etale) if for any arrow \( \alpha: D \to C \) in \( C \), the square

\[
\begin{array}{ccc}
G(C) & \rightarrow & G(D) \\
\downarrow^{G(\alpha)} & \nearrow^{f_{\alpha}} \downarrow_{G(\alpha)} \\
F(C) & \rightarrow & F(D) \\
\end{array}
\]

is a quasi-pullback (respectively a pullback) of sets.

**Example 1.2.** Let \( \mathbf{2} \) be the category with two distinct objects 0 and 1, and exactly one non-identity arrow \( 0 \to 1 \). Then as a particular case of the previous example, there are
A class of open maps and a class of etale maps in the presheaf category \( \mathbb{2} = \text{Sets}^{\mathbb{2}^0} \). More generally, if \( \mathcal{E} \) is any topos (or pretopos), there are canonical classes of open maps and of etale maps in \( \mathcal{E}^{\mathbb{2}^0} \), where a map \( Y \to X \) in \( \mathcal{E}^{\mathbb{2}^0} \) is defined to be open (respectively etale) if the corresponding square

\[
\begin{array}{ccc}
Y_1 & \to & X_1 \\
\downarrow & & \downarrow \\
Y_0 & \to & X_0
\end{array}
\]

is a quasi-pullback (respectively a pullback) in \( \mathcal{E} \). Our completeness theorem (Theorem 2.1 below) states that this example is in some sense “universal”.

**Example 1.3.** Given a collection of classes of open (or etale) maps in a (pre)topos, their intersection is evidently again a class of open (etale) maps. In particular, there is always a smallest class of open (etale) maps containing some given arrows. In a pretopos \( \mathcal{E} \), the smallest class of etale maps is the class of finite covering maps, i.e. maps \( f: F \to E \) for which there exists an epimorphism \( \sum_{i=1}^m U_i \to E \) and for each \( k \) a pullback of the form

\[
\begin{array}{ccc}
\sum_{i=1} U_k & \to & F \\
\downarrow & & \downarrow \\
U_k & \to & E.
\end{array}
\]

For a presheaf topos \( \mathcal{C} \), the class of open maps described in Example 1.1 is the smallest class of open maps in \( \mathcal{C} \). Indeed, if \( f: G \to F \) is a member of the class described in Example 1.1, there \( f \) fits into a quasi-pullback square

\[
\begin{array}{ccc}
\sum_{C, y \in G(C)} C(-, C) & \to & G \\
\downarrow & & \downarrow \\
\sum_{C, x \in F(C)} C(-, C) & \to & F
\end{array}
\]

where the summations are over all objects \( C \in \mathcal{C} \) and all elements \( y \in G(C) \) (resp. \( x \in F(C) \)). The left-hand vertical arrow in this square sends the summand indexed by \( C, y \) via the identity to the summand indexed by \( C, f_C(y) \). It readily follows that this left-hand arrow, and hence also \( f \), must belong to the smallest class of open maps.

**Example 1.4.** Familiar notions of open (or flat) maps and of etale maps from topology and algebraic geometry yield classes of open maps and of etale maps in the corresponding gros topos, as will be apparent from the discussion of sites in Section 5.

**Remark.** In all of the above examples, the Collection Axiom (A9) is satisfied. In the case of \( \mathcal{E}^{\mathbb{2}^0} \) (Example 1.2), for a given epi \( p: Y \to X \) and an open map \( f: X \to A \) as in the statement of (A9) this follows readily by taking the cover \( h: B \to A \) in (A9) to be
$B_1 + B_2 \to A$, where for $A = (A_0 \to A_1)$ we let $B_1 = (\phi \to A_1)$ and $B_2 = (A_0 \to A_0)$.

For the presheaf category $\mathcal{C}$ (Example 1.1), observe that for an open map $f: G \to F$ and a surjection $p: H \to G$, there is a quasi-pullback diagram

$$
\begin{array}{ccc}
C(-, C) \to H & \to & G \\
\downarrow \pi & & \downarrow f \\
\sum_{C, z \in F(C)} C(-, C) & \to & F,
\end{array}
$$

where the map $\pi$ sends the $C, z$-summand to the $C, f(z)$-summand via the identity. As in Example 1.3, the map $\pi$ must be open. We leave the verification of the collection axiom for the finite covering maps in a pretopos (Example 1.3) to the reader, and postpone the discussion of Example 1.4 to Section 5.

**Example 1.5.** Let $\mathcal{O}$ be a class of monomorphisms in a Grothendieck topos, satisfying the following three axioms:

1. **(M0)** Every isomorphism belongs to $\mathcal{O}$, and $\mathcal{O}$ is closed under composition;
2. **(M1)** If $U_i \to X$ belongs to $\mathcal{O}$ for every index $i \in I$, then $\bigvee U_i \to X$;
3. **(M2)** If $U \to X$ belongs to $\mathcal{O}$, then so does its pullback $U \times_X Y \to Y$ along any morphism $Y \to X$.

Now define $R$ to be the class of morphisms $f: Y \to X$ with the property that for any map $X' \to X$ and any $U \to X \times_X Y$ in $\mathcal{O}$, the image $\pi_3(U) \to X'$ along the projection $X' \times_X Y \to X'$ is again in $\mathcal{O}$. Then $R$ is a class of open maps. In general, classes of open maps obtained in this way need not satisfy the collection axiom. In particular, not every class of open maps is obtained from a class $\mathcal{O}$ of monomorphisms in this way: the etale topos provides a typical counterexample.

Any class of open maps gives rise to a class of etale maps:

**Proposition 1.6.** For any class of open maps, the class of open maps $Y \to X$ with open diagonal $Y \to Y \times_X Y$ is a class of etale maps.

**Proof.** The proof is elementary. By way of illustration, we verify Axioms A1 (composition) and A8. For A1, suppose $g: Z \to Y$ and $f: Y \to X$ are open maps with open diagonal. Then $f \circ g$ is open since open maps are closed under composition. Moreover, the diagonal of $f \circ g$ is the composition of the open diagonal $Z \to Z \times_Y Z$ of $g$ and the map $Z \times_Y Z \to Z \times_X Z$; the latter map is the pullback of the diagonal of $f$, as in

$$
\begin{array}{ccc}
Z \times_Y Z & \to & Y \\
\downarrow \text{p.b.} & & \downarrow \\
Z \times_X Z & \to & Y \times_X Y,
\end{array}
$$
hence is open. Thus the diagonal \( Z \to Z \times_X Z \) of \( f \circ g \) is the composition of open maps, hence is open. For Axiom A8, consider a diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow g & & \downarrow f \\
B & \xleftarrow{p} & \\
\end{array}
\]
where \( p \) and \( g \) are etale maps and \( p \) is epi. Then \( f \) is open by Axiom A6 for open maps. Furthermore, since \( g \) is open so is the map \( Y \times_B Y \to X \times_B X \) (this is a composition of pullbacks of \( g \)). But then by Axiom A6 the diagram
\[
\begin{array}{ccc}
Y & \to & X \\
\downarrow \text{open} & & \downarrow \\
Y \times_B Y & \xrightarrow{\text{open}} & X \times_B X \\
\end{array}
\]
shows the diagonal of \( f \) is open. Thus \( f \) is etale. \( \square \)

Conversely, we have:

**Proposition 1.7.** (a) For any given class of etale maps satisfying the collection axiom, the class of those maps \( f: Y \to X \) which fit into a quasi-pullback square
\[
\begin{array}{ccc}
V & \to & Y \\
\downarrow g & & \downarrow f \\
U & \to & X \\
\end{array}
\]
with \( p \) epi and \( g \) etale is a class of open maps, satisfying the collection axiom.

(b) Moreover, the given class of etale maps then consists exactly of those open maps whose diagonal is also open (as in Proposition 1.6).

**Proof.** (a) This is again elementary. Collection is used to show that the open maps are closed under composition; by way of illustration, we give the details for this. Suppose \( h: Z \to Y \) and \( f: Y \to X \) are open, so that there are a quasi-pullback square (3) for \( f \), and a similar one (4) for \( h \):
\[
\begin{array}{ccc}
B & \to & Z \\
\downarrow k & & \downarrow h \\
A & \xrightarrow{q} & Y \\
\end{array}
\]
with \( q \) epi and \( k \) etale. Form the pullback \( V \times_Y A \to V \); this map is epi since \( A \to Y \) is. By collection, there exists a quasi-pullback square of the form
\[
\begin{array}{ccc}
W & \to & V \times_Y A \to V \\
\downarrow \iota & & \downarrow \\
U' & \longrightarrow & U \\
\end{array}
\]
where $\ell$ is etale. Then the composite etale map $W \times_A B \to W \to U'$ fits into a quasi-pullback

\[
\begin{array}{ccc}
W \times_A B & \to & Z \\
\downarrow & & \downarrow \ell \\
U' & \to & X,
\end{array}
\]

showing that $\ell \circ f$ is again open.

(b) We postpone the proof of part (b) and list some elementary properties of etale maps first. \(\square\)

**Proposition 1.8.** In a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & Y \\
g \downarrow & & f \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

if $g$ is etale and $f$ is mono or etale then $h$ is etale.

**Proof.** If $f$ is mono, the pullback square

\[
\begin{array}{ccc}
Z & \to & Z \\
\downarrow & & \downarrow g \\
Y & \xrightarrow{f} & X
\end{array}
\]

shows that $h$ is etale whenever $g$ is. If $f$ is etale, then so is its diagonal $Y \to Y \times_X Y$ (Axiom (A7)), and thus the pullback square

\[
\begin{array}{ccc}
Z & \to & Y \\
(1, h) \downarrow & & \downarrow \\
Z \times_X Y & \to & Y \times_X Y
\end{array}
\]

shows that $(1, h)$ is etale. Furthermore, the projection $\pi_2 : Z \times_X Y \to Y$ is the pullback of the etale map $g$, hence is itself etale. Thus $f = \pi_2 \circ (1, h)$ is the composition of etale maps, hence etale. \(\square\)

**Proposition 1.9.** In a quasi-pullback diagram

\[
\begin{array}{ccc}
V & \to & Y \\
g \downarrow & & f \downarrow \\
U & \xrightarrow{p} & X
\end{array}
\]

where $p$ is epi and $f$ is mono, $f$ is etale whenever $g$ is.
Proof. Proposition 1.8 applied to the diagram

\[
\begin{array}{ccc}
V & \rightarrow & U \\ \\
\downarrow & & \downarrow \\ \\
\rightarrow & & \\ \\
U & \rightarrow & U \\
\end{array}
\]

yields that \( V \rightarrow U \times_Y V \) is etale. By Axiom A8, it follows that \( U \times_Y V \rightarrow U \) is etale. Thus by descent (A3), \( f: Y \rightarrow X \) must be etale. \( \Box \)

**Proposition 1.10.** In a quasi-pullback square of the form (5), if \( p \) is epi while both \( g \) and \( \Delta_f: Y \times_Y V \) are etale, then \( f \) is etale.

Proof. First assume that \( p: V \rightarrow X \) is the identity, so that we have a diagram

\[
\begin{array}{ccc}
V & \rightarrow & Y \\ \\
\downarrow & & \downarrow \\ \\
\rightarrow & & \\ \\
X & \rightarrow & Y \\
\end{array}
\]

where \( q \) is epi while \( g \) and \( \Delta_f \) are etale. Then the pullback

\[
\begin{array}{ccc}
V \times_Y V & \rightarrow & Y \\ \\
\downarrow & & \downarrow \\ \\
\rightarrow & & \\ \\
V \times_Y V & \rightarrow & Y \times_Y V \\
\end{array}
\]

shows that \( V \times_Y V \rightarrow V \times_Y V \) is etale. Moreover, since \( q \) is etale so is the projection \( V \times_Y V \rightarrow V \). Thus the projection \( V \times_Y V \rightarrow V \) is a composition of etale maps, hence etale. Since \( V \rightarrow Y \) is epi, it follows by descent that \( V \rightarrow Y \) must be etale. Axiom (A8) now yields that \( f \) is etale. This proves the special case where \( p \) is the identity. For an arbitrary epi \( p: V \rightarrow X \), the special case applied to the diagram

\[
\begin{array}{ccc}
V & \rightarrow & U \times_Y V \\ \\
\downarrow & & \downarrow \\ \\
\rightarrow & & \\ \\
U & \rightarrow & U \\
\end{array}
\]

yields that \( \pi_1: U \times_Y V \rightarrow U \) is etale (since its diagonal is the pullback of the diagonal of \( f \) along \( U \rightarrow X \), hence is etale). By descent, it follows that \( f \) is etale. \( \Box \)

**Proof of Proposition 1.7(b).** Suppose both \( f: Y \rightarrow X \) and \( \Delta_f: Y \rightarrow Y \times_Y Y \) are open. Since \( \Delta_f \) is mono, the definition of open maps from etale maps and Proposition 1.9 yield that \( \Delta_f \) is etale. Then Proposition 1.10 applied to a quasi-pullback square which witnesses that \( f \) is open yields that \( f \) is etale. \( \Box \)

**Remark 1.11.** The classes of open maps and of etale maps in Examples 1.1 and 1.2 above are obtained from each other as in Propositions 1.6 and 1.7. As we shall see later, this also holds for most of the gros topoi mentioned in Example 1.4.
2. The completeness theorem

In this section we state several versions of our completeness theorem, the proof of which will be given in a later section.

Observe first that for a morphism of topoi $\phi : \mathcal{F} \rightarrow \mathcal{E}$, any class of open (or etale) maps in $\mathcal{F}$ gives rise to an induced such class in $\mathcal{E}$, consisting of exactly those maps $f : Y \rightarrow X$ in $\mathcal{E}$ for which the inverse image $\phi^*(f) : \phi^*Y \rightarrow \phi^*X$ is open (or etale) in $\mathcal{F}$. Indeed, the fact that the inverse image functor $\phi^*$ preserves colimits and finite limits immediately implies that if the given class in $\mathcal{F}$ satisfies the axioms, then so does this induced class in $\mathcal{E}$.

In Example 1.2 of the previous section, we have seen that for any topos $\mathcal{F}$, the topos $\mathcal{F}^{20}$ of arrows in $\mathcal{F}$ supports a canonical class of open maps, consisting of those maps in $\mathcal{F}^{20}$ which form quasi-pullback squares in $\mathcal{F}$. This class is universal, in the following sense.

**Theorem 2.1** (Completeness Theorem). For any class $R$ of open maps satisfying the collection axiom in a topos $\mathcal{E}$, there exists a topos $\mathcal{F}$ and a geometric morphism $\phi : \mathcal{F}^{20} \rightarrow \mathcal{E}$ such that $R$ is induced by the canonical class of open maps in $\mathcal{F}^{20}$.

In other words, the morphism $\phi$ in the statement of the theorem has the property that a map $Y \rightarrow X$ is open in $\mathcal{E}$ iff $\phi^*Y \rightarrow \phi^*X$ is a quasi-pullback square in $\mathcal{F}$.

As a consequence, one obtains a similar completeness theorem for etale maps, in terms of the canonical class of etale maps in $\mathcal{F}^{20}$ given by the pullback squares:

**Corollary 2.2.** Any class of etale maps satisfying the collection axiom in a topos $\mathcal{E}$ is induced along a geometric morphism $\phi : \mathcal{F}^{20} \rightarrow \mathcal{E}$ by the canonical class in $\mathcal{F}^{20}$.

**Proof.** Write $R$ for the given class of etale maps in $\mathcal{E}$, and let $R'$ be the associated class of open maps, as in Proposition 1.7. By the completeness theorem, there exists a geometric morphism $\phi : \mathcal{F}^{20} \rightarrow \mathcal{E}$ such that $R'$ is induced from the canonical class of open maps in $\mathcal{F}^{20}$. But by Proposition 1.7(b) the given class $R$ of etale maps consists precisely of the open maps with open diagonal, while by the remark at the end of the previous section the canonical class of etale maps in $\mathcal{F}^{20}$ similarly consists of open maps in $\mathcal{F}^{20}$ with open diagonal. It follows that $R$ is induced from this canonical class of etale maps in $\mathcal{F}^{20}$. \[\Box\]

For any object $B$ in a topos $\mathcal{E}$ equipped with a class of etale maps, we write $\mathbf{Et}/B$ for the full subcategory of $\mathcal{E}/B$ consisting of etale maps. The following Corollary of the Completeness Theorem was also discussed in [2], with essentially the same proof (cf. [2, 12.5 and 8.3]).

**Corollary 2.3.** For any class of etale maps satisfying collection in a topos $\mathcal{E}$, and for any object $B$ of $\mathcal{E}$, the category $\mathbf{Et}/B$ is a topos and the inclusion $\mathbf{Et}/B \hookrightarrow \mathcal{E}/B$ is the inverse image of a geometric morphism $\mathcal{E}/B \rightarrow \mathbf{Et}/B$. 

Proof. Observe first that the given class of etale maps in $\mathcal{E}$ defines an evident class of etale maps in $\mathcal{E}/B$ which again satisfies collection. It thus suffices to consider the case where $B$ is the terminal object 1. Let $\varphi : \mathcal{F}_2^\varphi \to \mathcal{E}$ be the geometric morphism given by the Corollary 2.2 above, and let $\delta_1 : \mathcal{F}_2^\varphi \to \mathcal{F}$ be the canonical geometric morphism given by

$$(\delta_1)_* (Z \to Y) = Z, \quad \delta_1^* (I) = (I \to 1).$$

Now construct the pushout square of Grothendieck topoi

$$\mathcal{F}_2^\varphi \xrightarrow{\varphi} \mathcal{E}$$
$$\delta_1 \downarrow \quad \downarrow \mu$$
$$\mathcal{F} \xrightarrow{\gamma} \mathcal{P}.$$

By [12], this pushout topos $\mathcal{P}$ indeed exists, and is constructed as the (pseudo-) pullback of categories and inverse image functors. In this particular case, this means that $\mathcal{P}$ is the category of triples $(T, E, 0)$, where $T$ is an object of $\mathcal{F}$ and $E$ is an object of $\mathcal{E}$ while $0 : \delta_1^* T \rightrightarrows \varphi^* E$. Furthermore, the inverse image functors $v^*$ and $\mu^*$ are the evident projections. We claim that this category $\mathcal{P}$ is equivalent to $\text{Et}/1$, by an equivalence of categories which identifies $\mu^*$ with the inclusion $\text{Et}/1 \subseteq \mathcal{E}$. Indeed, by Corollary 2.2, for any object $E$ of $\mathcal{E}$ the unique map $E \to 1$ is etale in $\mathcal{E}$ iff $\varphi^* E \to \varphi^* 1$ is a pullback in $\mathcal{F}$, i.e. iff the object $\varphi^* E$ of $\mathcal{F}_2^\varphi$ is an isomorphism when viewed as an arrow in $\mathcal{F}$. One now easily obtains the desired equivalence $\mathcal{P} \cong \text{Et}/1$. $\square$

Next we will give an equivalent formulation of the Completeness Theorem in terms of extension of models. Recall that if $\mathcal{E}$ is the classifying topos of a geometric theory $T$, then a model of $T$ is the same thing as a morphism of topoi $M : \text{Sets} \to \mathcal{E}$; that is, a point of $\mathcal{E}$. An extension of such a model $M$ is a morphism of models $h : M \to N$, or in other words, a natural transformation $h : M^* \to N^*$ of inverse image functors $M^*$, $N^* : \mathcal{E} \to \text{Sets}$. For a class of open maps $R$ in $\mathcal{E}$ such an extension is said to be anodyne (with respect to $R$) if for each open map $Y \to X$ in $\mathcal{E}$, the square

$$\begin{array}{ccc}
M^*(Y) & \xrightarrow{h^*} & N^*(Y) \\
\downarrow & & \downarrow \\
M^*(X) & \xrightarrow{\pi^*} & N^*(X)
\end{array}$$

is a quasi-pullback of sets. Similarly, if $R'$ is a class of etale maps in $\mathcal{E}$, the extension $h : M \to N$ is said to be anodyne with respect to $R'$ if, for each etale map $Y \to X$ in $\mathcal{E}$, the square (6) is a pullback of sets.

Observe that if the open class $R$ is obtained from the etale class $R'$ as in Proposition 1.7, then an extension of models is anodyne with respect to $R$ iff it is so with respect to $R'$.

More generally, for an arbitrary topos $\mathcal{F}$ the geometric morphisms $M : \mathcal{F} \to \mathcal{E}$ are models of $\mathcal{E}$ in $\mathcal{F}$, and one can define anodyne extensions of such models $M \to N$ in precisely the same way, using (quasi-) pullback squares in $\mathcal{F}$.
We will see later that for suitable classes of open (or etale) maps in various classifying topoi, the concept of anodyne extension reduces to familiar notions such as that of a local extension of local rings, of an algebraically closed extension of algebras, of an existentially closed extension in model theory, etc.

The completeness theorem can now be stated in a different but equivalent form:

**Corollary 2.4.** Let R be a class of open maps, satisfying the collection axiom, in a topos $\mathcal{E}$. Then a map $Y \to X$ in $\mathcal{E}$ is open whenever for any anodyne extension $h : M \to N$ of models of $\mathcal{E}$, the corresponding square

$$
\begin{array}{ccc}
M^*Y & \to & N^*Y \\
\downarrow & & \downarrow \\
M^*X & \to & N^*X
\end{array}
$$

is a quasi-pullback.

In the statement of this corollary, the models of $\mathcal{E}$ are models in an arbitrary topos $\mathcal{F}$. However, in many of our examples, it suffices to consider models in $\text{Sets}$.

**Proof** (of Corollary 2.4 from Theorem 2.1). Let $\varphi : \mathcal{T}^g \to \mathcal{E}$ be a geometric morphism such that $R$ is induced from the canonical class in $\mathcal{T}^g$, as in Theorem 2.1. This morphism $\varphi$ can be viewed as a pair of morphisms $\varphi_0, \varphi_1 : \mathcal{T} \to \mathcal{E}$ together with a natural transformation $h : \varphi_1^* \to \varphi_0^*$. The completeness theorem then asserts that a map $Y \to X$ in $\mathcal{E}$ is open iff the square

$$
\begin{array}{ccc}
\varphi_1^*Y & \to & \varphi_0^*Y \\
\downarrow & & \downarrow \\
\varphi_1^*X & \to & \varphi_0^*X
\end{array}
$$

is a quasi-pullback. Thus, $h$ is an anodyne extension of models in $\mathcal{T}$, and Corollary 2.4 follows by considering just this anodyne extension. $\square$

From Corollary 2.4 one obtains a similar result for classes of etale maps, by exactly the same argument as for Corollary 2.2 above.

**Corollary 2.5.** Let $R'$ be a class of etale maps satisfying the collection axiom in a topos $\mathcal{E}$. Then a map $Y \to X$ in $\mathcal{E}$ is etale whenever for any anodyne extension (w.r.t. $R'$) of models of $\mathcal{E}$ the corresponding square (7) is a pullback.

3. Proof of the completeness theorem

In this section, we will present a proof of Theorem 2.1. Throughout, $\mathcal{E}$ will be a Grothendieck topos equipped with a class $R$ of open maps. The proof will proceed
by giving an explicit site description for the classifying topos for anodyne extensions (with respect to $R$) of models of $\mathcal{E}$. By definition, such a classifying topos $\mathcal{A}$ is a topos for which there exists a natural equivalence between geometric morphisms $\mathcal{F} \to \mathcal{A}$ and anodyne extensions $h : M \to N$ of models of $\mathcal{E}$ in $\mathcal{F}$, for any topos $\mathcal{F}$. To construct such a topos $\mathcal{A}$, write $S$ for the Sierpinski space, and $\mathcal{S}(S)$ for the associated topos of sheaves. Thus for any topos $\mathcal{F}$, the arrow category $\mathcal{F}^\circ$ is the product of topoi

$$\mathcal{F}^\circ = \mathcal{S}(S) \times \mathcal{F}.$$ 

This product topos $\mathcal{S}(S) \times \mathcal{F}$ has the universal property that a geometric morphism $f : \mathcal{S}(S) \times \mathcal{F} \to \mathcal{G}$ into another topos $\mathcal{G}$ corresponds to a pair of geometric morphisms $f_0, f_1 : \mathcal{F} \to \mathcal{G}$ together with a natural transformation $\theta : f_1^* \to f_0^*$. Now, for the given topos $\mathcal{E}$, let $\mathcal{E}\mathcal{S}(S)$ be the exponential topos. (This topos exists for general reasons (see [7]), but we shall give an explicit construction of it shortly.) Furthermore, let $e : \mathcal{S}(S) \times \mathcal{E}\mathcal{S}(S) \to \mathcal{E}$ be the evaluation morphism, which corresponds to two geometric morphisms $e_0, e_1$ and a transformation $\theta$:

$$e_0, e_1 : \mathcal{E}\mathcal{S}(S) \to \mathcal{E}, \quad \theta : e_1^* \to e_0^*.$$

Let $j$ be the Grothendieck topology on $\mathcal{E}\mathcal{S}(S)$ which for each open map $Y \to X$ in $\mathcal{E}$ forces the corresponding square

$$\begin{array}{ccc}
e_1^* Y & \rightarrow & e_0^* Y \\
\downarrow & & \downarrow \\
e_1^* X & \rightarrow & e_0^* X
\end{array}$$

to become a quasi-pullback. In other words, $j$ is the smallest Grothendieck topology on $\mathcal{E}\mathcal{S}(S)$ for which the associated sheaf functor $\mathcal{E}\mathcal{S}(S) \to (\mathcal{E}\mathcal{S}(S))_j$ turns the map $e_1^*(Y) \to e_0^*(Y) \times_{e_0^*(X)} e_0^*(X)$ into an epi in $(\mathcal{E}\mathcal{S}(S))_j$ for each open map $Y \to X$ in $\mathcal{E}$. Write $i : (\mathcal{E}\mathcal{S}(S))_j \to \mathcal{E}\mathcal{S}(S)$ for the inclusion. Then by the very definition of the topology $j$, the map $i^*(\theta) : (e_1 i)^* \to (e_0 i)^*$ is an anodyne extension between the two models $e_1 i$ and $e_0 i$ of $\mathcal{E}$ in $(\mathcal{E}\mathcal{S}(S))_j$. Furthermore, the definition of $j$ and the universal property of the exponential topos $\mathcal{E}\mathcal{S}(S)$ readily show that this anodyne extension is universal. Thus $A := (\mathcal{E}\mathcal{S}(S))_j$ classifies anodyne extensions of $\mathcal{E}$-models.

Towards our more explicit description of this classifying topos, we shall first consider sites for exponential topos $\mathcal{E}\mathcal{S}(S)$. Let $C$ be any site with finite limits for $\mathcal{E}$, and let $C^2$ be the associated arrow category. There are two natural Grothendieck topologies $J_c$ and $J_d$ on $C^2$ which one obtains from the given topology on $C$. In the topology $J_c$ (the subscript “c” is for “codomain”) the basic covers of an object $(Y \to X)$ of $C^2$ are of the form

$$J_c : \left( \begin{array}{ccc} X \times_X Y & \rightarrow & Y \\
\downarrow \phi & & \downarrow \phi \\
X & \rightarrow & X \\
(\alpha \in A), \quad \phi \in (\alpha \check{\alpha})_c \end{array} \right)$$
where \( \{ h_x : X_x \to X \}_{x \in A} \) is any cover of \( X \) in the site \( C \). In \( J_d \), the basic covers of such an object are families of the form

\[
J_d : \left( \begin{array}{c} Y_eta \\ \downarrow \psi \\ X \end{array} \right) \xrightarrow{(\psi, 1)} \left( \begin{array}{c} Y \\ \downarrow \\ X \end{array} \right), \quad \beta \in B,
\]

where \( \{ g_\beta : Y_\beta \to Y \}_{\beta \in B} \) is any cover of \( Y \) in the site \( C \). Observe that these basic covers for \( J_c \) and \( J_d \) are both stable under composition and under pullback. Let \( J_c \cup J_d \) be the union of these two Grothendieck topologies: it is the smallest topology for which the families in \( J_c \) and those in \( J_d \) both cover; it consists of precisely those covers which can be obtained from \( J_c \)-covers and \( J_d \)-covers by composition of covers.

**Lemma 3.1.** For any site \( C \) for \( \mathcal{E} \), closed under finite limits, the category \( C^2 \) equipped with the union topology \( J_c \cup J_d \) is a site for the exponential topos \( \mathcal{E}^{\text{Sh}(S)} \).

**Proof.** For any topos \( \mathcal{F} \), we need to establish a natural equivalence between morphisms \( \mathcal{F} \to \text{Sh}(C^2, J_c \cup J_d) \) and morphisms \( \mathcal{F} \to \mathcal{E}^{\text{Sh}(S)} \). The latter correspond to morphisms \( \text{Sh}(S) \times \mathcal{F} \to \mathcal{E} \), or equivalently, to triples \( (f_0, f_1, \theta) \) consisting of two morphisms \( f_0, f_1 : \mathcal{F} \to \mathcal{E} \) and a natural transformation \( \theta : f_1^* \to f_0^* \). By Diaconescu's theorem, we thus need to establish a natural equivalence between left exact functors \( F : C^2 \to \mathcal{F} \) which are continuous with respect to \( J_c \cup J_d \), and triples \( (F_0, F_1, \theta) \) consisting of two left exact continuous functors \( F_0 \) and \( F_1 : C \to \mathcal{F} \) and a natural transformation \( \theta : F_1 \to F_0 \).

Now for any two categories \( C \) and \( \mathcal{F} \) with finite limits there is a natural equivalence between left-exact functors

\[
F : C^2 \to \mathcal{F}
\]

and triples

\[
\begin{array}{c}
F_0 \\
C
\end{array} \xrightarrow{\theta} \begin{array}{c}
F_1 \\
\mathcal{F}
\end{array}
\]

with \( F_0 \) and \( F_1 \) left exact: Given \( F \), one simply defines

\[
F_1(X) = F(id : X \to X), \quad F_0(X) = F(X \to 1),
\]

while \( \theta_X : F_1(X) \to F_0(X) \) is the image under \( F \) of the evident map \( (X = X) \to (X \to 1) \) in \( C^2 \). Conversely, given \( F_0, F_1 \) and \( \theta \), for any object \( (f : Y \to X) \) of \( C^2 \) one defines \( F(Y \to X) \) to be the pullback,

\[
\begin{array}{c}
F(Y \to X) \xrightarrow{\theta_Y} F_1(X) \\
\downarrow \quad \downarrow \theta_x \\
F_0(Y) \xrightarrow{F_0(f)} F_0(X)
\end{array}
\]

while \( \theta_X : F_1(X) \to F_0(X) \) is the image under \( F \) of the evident map \( (X = X) \to (X \to 1) \) in \( C^2 \). Conversely, given \( F_0, F_1 \) and \( \theta \), for any object \( (f : Y \to X) \) of \( C^2 \) one defines \( F(Y \to X) \) to be the pullback,
It thus suffices to show that $F$ is continuous w.r.t. the topology $J_c \lor J_d$ on $C^2$ iff $F_0$ and $F_1$ are both continuous w.r.t. the given topology on $C$. Now given $F$, the functor $F_0$ defined in (8) will be continuous precisely when for every cover $\{h_x : X_x \to X\}_{x \in A}$, the functor $F$ sends the family
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
X_x
\end{array}
\end{array}
\downarrow (h_x, h_x)
\begin{array}{c}
\begin{array}{c}
X
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
X
\end{array}
\end{array}
\end{array}, \quad x \in A
\]
(10)
to a cover in the topos $\mathcal{F}$. And similarly, the functor $F_1$ defined in (9) will be continuous precisely when for any cover $\{g_{\beta} : Y_{\beta} \to Y\}_{\beta \in B}$ in $C$, the functor $F$ sends the family
\[
\begin{array}{c}
\begin{array}{c}
Y_{\beta}
\end{array}
\downarrow
\begin{array}{c}
\begin{array}{c}
Y
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}, \quad \beta \in B
\]
(11)
to a cover in $\mathcal{F}$. But the families obtained by pullback along $(Y \to X) \to (X = X)$ from families of the form (10) are exactly the covers in $J_c$, while the covers in $J_d$ are similarly obtained by pullback from families of the form (11). Thus $F_0$ and $F_1$ are continuous precisely when $F$ send covers of both $J_c$ and $J_d$ to covers in $\mathcal{F}$, as required.

We shall now work with the (large) site for $\mathcal{E}$ given by $\mathcal{E}$ itself, and the associated site
\[
(\mathcal{E}^2, J_c \lor J_d)
\]
(12)
for $\mathcal{E}^{Sh(S)}$. Consider on $\mathcal{E}^2$ a third Grothendieck topology $J_\alpha$ ("o" for open), the covers of which are singleton families of the form
\[
J_\alpha(Y, g, Y') \quad (1, p) \quad \downarrow \quad \downarrow f
\]
\[
\begin{array}{c}
\begin{array}{c}
y
\end{array}
\end{array} \quad \downarrow \quad \begin{array}{c}
\begin{array}{c}
y
\end{array}
\end{array} \quad \downarrow \quad \begin{array}{c}
\begin{array}{c}
Z
\end{array}
\end{array}
\]
(13)
where $p \circ g = f$ and $p$ is an open map in $\mathcal{E}$. Notice that, as for $J_c$ and $J_d$, the covers in $J_\alpha$ are stable under composition and pullback, since composition and pullback of open maps in $\mathcal{E}$ yield open maps (Axioms A1, A2). We shall be interested in the topology $J_c \lor J_d \lor J_\alpha$ on $\mathcal{E}^2$ generated by all the covers of $J_c$, $J_d$, and $J_\alpha$ together.

**Proposition 3.2.** The topos $Sh(\mathcal{E}^2, J_c \lor J_d \lor J_\alpha)$ classifies anodyne extensions of $\mathcal{E}$-models.

**Proof.** The pair of morphisms $e_0, e_1 : \mathcal{E}^{Sh(S)} \to \mathcal{E}$ and the natural transformation $\theta : e_0^* \to e_1^*$ which correspond to the evaluation morphism $e : Sh(S) \times \mathcal{E}^{Sh(S)} \to \mathcal{E}$ (as above) are given by the left exact continuous functors of large sites:
\[
E_0, E_1 : \mathcal{E} \to \mathcal{E}^2
\]
defined by
\[
E_1(X) = (id : X \cong X), \quad E_0(X) = (X \to 1).
\]
while $\theta$ is the evident transformation. Thus for a map $Z \to X$ in $\mathcal{E}$, the square

$$
\begin{array}{ccc}
E^*_\alpha(Z) & \to & E^*_\alpha(X) \\
\downarrow & & \downarrow \\
E^*_\beta(Z) & \to & E^*_\beta(X)
\end{array}
$$

is the image under the Yoneda embedding $\mathcal{E}^2 \to \text{Sh}(\mathcal{E}^2, J_c \vee J_d)$ of the square

$$
\begin{array}{ccc}
E_\alpha(Z) & \to & E_\alpha(X) \\
\downarrow & & \downarrow \\
E_\alpha(Z) \times_{E_\alpha(X)} E_\alpha(X) & \to & E_\alpha(X).
\end{array}
$$

Consequently, the first square is a quasi-pullback if and only if the map $E_\alpha(Z) \to E_\alpha(Z) \times_{E_\alpha(X)} E_\alpha(X)$ obtained from the second square is a cover in the site $\mathcal{E}^2$. But this map in $\mathcal{E}^2$, when written as a square in $\mathcal{E}$, is precisely

$$
\begin{array}{ccc}
Z & \to & Z \\
\parallel & & \downarrow \\
Z & \to & X.
\end{array}
$$

(14)

So the topology $j$ on $\mathcal{E}^{\text{Sh}(S)}$ considered above, which forces open maps in $\mathcal{E}$ to go to quasi-pullbacks, is precisely the topology generated by the covers of the form (14). But any map of the form (13) is the pullback of the map (14) along $(Y \to X) \to (Z \to X)$. Thus the topology $j$ is given by adding the covers of $J_\alpha$ to the covers of $J_c \vee J_d$ which define the exponential topos. In other words, the topos $\text{Sh}(\mathcal{E}^2, J_c \vee J_d \vee J_\alpha)$ is precisely the classifying topos $\mathcal{E}^{\text{Sh}(S)}$ for anodyne extensions. $\square$

The covers in $J_c \vee J_d \vee J_\alpha$ are obtained by taking compositions of covers in $J_c$, in $J_d$ and in $J_\alpha$. Call a cover of an object $(Y \to X)$ of $\mathcal{E}^2$ in the topology $J_c \vee J_d \vee J_\alpha$ of “standard form” if it is a composition of covers constructed as: a cover of $(Y \to X)$ in $J_c$, followed by a cover of this cover in $J_\alpha$, followed by a cover of this last cover in $J_\alpha$, all as in the following diagram:

$$
\begin{array}{ccc}
Z^\alpha_\beta & \to & X_a \times_X Y \to Y \\
\downarrow & & \downarrow \\
U^\alpha_\beta_{\text{open}}(X_a) & \to & X_a \times_{\text{cover}} X.
\end{array}
$$

(15)

Here we have started with a cover $\{X_a \to X\}_{x \in A}$ of $X$, then for each $x$ a cover $\{Z^\alpha_{\beta_1} \to X_a \times_X Y\}_{\beta \in B_1}$ of $X_a \times_X Y$, and finally for each $x$ and each $\beta$ a cover of $(Z^\alpha_{\beta_2} \to X_a)$ of type (13), given by the open map $U^\alpha_\beta \to X_a$. The family (15) is thus a family of composed arrows in $\mathcal{E}^2$, indexed by all pairs $(x, \beta)$ with $x \in A$ and $\beta \in B_2$.

Our aim is to apply the collection axiom and show that every cover in $J_c \vee J_d \vee J_\alpha$ can be refined by a cover of this “standard form”. For this, it will be convenient to introduce some notation: we will use an asterisk $\ast$ for composition of covers, so that the diagram (15) represents a cover of type $J_\alpha \ast J_d \ast J_c$. Furthermore, we write e.g.
Lemma 3.3. If the class $R$ of open maps in $\mathscr{E}$ satisfies the collection axiom, then the Grothendieck topologies $J_\circ$, $J_d$ and $J_0$ on the arrow-category $\mathscr{E}^2$ have the following commutation properties:

(i) $J_\circ \cdot J_d \supseteq J_d \cdot J_\circ$;
(ii) $J_\circ \cdot J_d \supseteq J_\circ \cdot J_d \cdot J_\circ$;
(iii) $J_\circ \cdot J_d \supseteq J_\circ \cdot J_d \cdot J_\circ$.

Proof. (i) A cover of $(Y \to X)$ of type $J_d \cdot J_\circ$ is given by a family of diagrams of the form

\[
\begin{array}{ccc}
Y_a & \to & Y \\
\downarrow & & \downarrow \\
U & \xrightarrow{open} & X
\end{array}
\]

where $U \to X$ is open and $\{Y_a \to Y\}$ is a cover of $Y$ in $\mathscr{E}$. As a family of composite arrows $(Y_a \to U) \to (Y \to X)$ in $\mathscr{E}^2$, this is identical to the family given by the diagrams

\[
\begin{array}{ccc}
Y_a & \to & Y \\
\downarrow & & \downarrow \\
U & \xrightarrow{open} & X \\
\end{array}
\]

This family is a cover of $(Y \to X)$ of type $J_\circ \cdot J_d$, so (i) holds.

(ii) A cover of $(Y \to X)$ of type $J_\circ \cdot J_d$ is given by a cover $\{Y_a \to Y\}_{a \in A}$ of $Y$, and then for each $a$ a cover $\{X_{a\beta} \to X\}_{\beta \in B_a}$ of $X$, to give a family of composites in $\mathscr{E}^2$ of the form

\[
\begin{array}{ccc}
X_{a\beta} & \times_X Y_a & \to & Y \\
\downarrow & & \downarrow & & \downarrow \\
X_a & \to & X \\
\end{array}
\]

Now apply the collection axiom to the epimorphism $\sum_a \sum_{\beta} X_{a\beta} \to \sum_a X$ and the open map $\sum_a X \to X$ (given by the identity on each summand). This gives a quasi-pullback square, together with a factorization indicated by the dotted arrow,
Using this factorization, we can write \( W = \sum_{\alpha} \sum_{\beta} W_{\beta}^{\alpha} \), where each summand \( W_{\beta}^{\alpha} \) is mapped by the dotted arrow into the summand \( X_{\beta}^{\alpha} \). Since the square in (17) is a quasi-pullback, we have for each index \( \alpha \) a cover \( \{ W_{\beta}^{\alpha} \to X' \}_\beta \). Pulling this cover back along the projection \( X' \times_X Y_a \to Y_a \) thus yields for each \( \alpha \) a cover \( \{ W_{\beta}^{\alpha} \times_X Y_a \to X' \times_X Y_a \}_\beta \). Furthermore, since \( \{ Y_a \to Y \}_a \) is a cover, so is \( \{ X' \times_X Y_a \to X' \times_X Y \}_a \). By composition of covers, the family of composite arrows

\[
W_{\beta}^{\alpha} \times_X Y_a \to X' \times_X Y_a \to X' \times_X Y,
\]

for all \( \alpha \in A \) and \( \beta \in B_a \), thus covers \( X' \times_X Y \). Now consider for each \( \alpha \) and \( \beta \) the diagram

\[
\begin{array}{ccc}
W_{\beta}^{\alpha} \times_X Y_a & \xrightarrow{\text{cover}} & X' \times_X Y_a \\
\downarrow & & \downarrow \\
W_{\beta}^{\alpha} & \xrightarrow{\text{open}} & X'
\end{array}
\]

This diagram represents an arrow \( (W_{\beta}^{\alpha} \times_X Y_a \to W_{\beta}^{\alpha}) \to (Y \to X) \) in \( \mathfrak{S}^2 \), and the family of all these arrows (for all \( \alpha \in A \) and \( \beta \in B_a \)) is evidently a cover of \( (Y \to X) \) of type \( J_a \ast J_a \ast J_c \). Furthermore, this cover refines the given cover (16), as shown for each \( \alpha \) and \( \beta \) by the arrow in \( \mathfrak{S}^2 \) represented by the commutative square

\[
\begin{array}{ccc}
W_{\beta}^{\alpha} \times_X Y_a & \to & X_{\beta}^{\alpha} \times_X Y_a \\
\downarrow & & \downarrow \\
W_{\beta}^{\alpha} & \to & X_{\beta}^{\alpha}
\end{array}
\]

This proves part (ii).

(iii) A cover of \( (Y \to X) \) of type \( J_a \ast J_a \ast J_c \) is given by an open map \( U \to X \) and a cover \( \{ U_a \to U \}_a \), and can be represented by the family of diagrams

\[
\begin{array}{ccc}
U_a \times_U Y & \to & Y \\
\downarrow & & \downarrow \\
U_a & \xrightarrow{\text{cover}} & U \xrightarrow{\text{open}} X,
\end{array}
\]

In order to refine this cover by a cover of the standard form, first apply collection, to obtain a quasi-pullback and a dotted factorization, as in

\[
\begin{array}{ccc}
\sum U_a & \to & U \\
\downarrow \text{open} & \quad \downarrow \text{q.p.b.} & \downarrow \text{open} \\
X' & \to & X.
\end{array}
\]

By this factorization, we can write \( V \) as a sum \( V = \sum V_a \), where each summand \( V_a \) is mapped into \( U_a \). Since the square is a quasi-pullback, the map \( V \to X' \times_X U \) is epi, so the family \( \{ V_a \to X' \times_X U \}_a \) is a cover in \( \mathfrak{S} \). By pullback along \( X' \times_X Y \to X' \times_X U \) we
thus obtain a cover \( \{ V_x \times_U Y \rightarrow X' \times_U Y \} \). Now consider for each \( \alpha \) the diagram
\[
\begin{array}{c}
V_x \\
\downarrow
\end{array}
\begin{array}{c}
\rightarrow \hspace{1cm}
\rightarrow
\end{array}
\begin{array}{c}
V' \\
\downarrow
\end{array}
\begin{array}{c}
X' \\
\downarrow
\end{array}
\begin{array}{c}
\rightarrow \hspace{1cm}
\rightarrow
\end{array}
\begin{array}{c}
X \\
\downarrow
\end{array}
\begin{array}{c}
\text{cover}
\end{array}
\]
\( \alpha \in A. \)

The family of all the arrows in \( S^2 \) represented by these diagrams is evidently a cover of \( (Y \rightarrow X) \) of type \( J_o \ast J_d \ast J_c \). Furthermore, the square
\[
\begin{array}{c}
V_x \\
\downarrow
\end{array}
\begin{array}{c}
\rightarrow \hspace{1cm}
\rightarrow
\end{array}
\begin{array}{c}
V' \\
\downarrow
\end{array}
\begin{array}{c}
U_x \\
\downarrow
\end{array}
\begin{array}{c}
U' \\
\downarrow
\end{array}
\begin{array}{c}
U \\
\downarrow
\end{array}
\begin{array}{c}
\text{cover}
\end{array}
\]
will show that this cover (19) refines the given cover (18). This proves part (iii), and thus completes the proof of the lemma. 

The following lemma is an immediate consequence.

**Lemma 3.4.** If the class \( R \) satisfies collection, then every cover in the Grothendieck topology \( J_c \vee J_d \vee J_o \) on the arrow category \( S^2 \) can be refined by a cover of the standard form \( J_o \ast J_d \ast J_c \).

**Proof.** Since the covers in \( J_o, J_d \) and \( J_c \) are each stable under pullback, it suffices to show that the composition of two standard covers can be refined by a standard cover. Such a composition is of the type \( J_o \ast J_d \ast J_c \ast J_o \ast J_d \ast J_c \), and by Lemma 3.3 we have
\[
J_o \ast J_d \ast J_c \ast J_o \ast J_d \ast J_c \geq J_o \ast J_d \ast J_o \ast J_d \ast J_c \ast J_d \ast J_c \quad \text{(by part (iii))}
\]
\[
\geq J_o \ast J_d \ast J_o \ast J_d \ast J_o \ast J_d \ast J_c \ast J_c \quad \text{(by (ii))}
\]
\[
\geq J_o \ast J_o \ast J_d \ast J_d \ast J_d \ast J_c \ast J_c \ast J_c,
\]
the latter by several applications of part (i). But the covers of each of the types \( J_o, J_d \) and \( J_c \) are closed under composition, so that \( J_o \ast J_o \geq J_o \), and similarly for \( J_c \) and \( J_d \). Thus we can continue the above sequence of inequalities by
\[
J_o \ast J_o \ast J_o \ast J_d \ast J_d \ast J_d \ast J_c \ast J_c \geq J_o \ast J_d \ast J_c.
\]
This shows that any composite of standard covers, of type \( (J_o \ast J_d \ast J_c) \ast (J_o \ast J_d \ast J_c) \), can be refined by a standard cover, of type \( J_o \ast J_d \ast J_c \), as required. 

**Proof of Theorem 2.1.** Consider the universal anodyne extension given by
\[
e_0, e_1: (\mathcal{S}^{Sh(S)})_j \rightarrow \mathcal{S}
\]
and the natural transformation \( \theta: e_1^* \rightarrow e_0^* \), all as in the beginning of this section. If we take the large site \( (\mathcal{S}^2, J_c \vee J_d \vee J_o) \) for the topos \( (\mathcal{S}^{Sh(S)})_j \), then the inverse image
functors $e_0^*$ and $e_1^*$ are the composites

$$
\mathcal{E} \xrightarrow{E_0} \mathcal{E}^2 \xrightarrow{\text{yon}} \text{Sh}(\mathcal{E}^2, J_c \vee J_d \vee J_o),
$$

where \text{yon} is the Yoneda embedding followed by sheafification. Now consider an arrow $f: Z \to X$ in $\mathcal{E}$ for which the square

$$
\begin{array}{ccc}
E_0^*Z & \to & E_1^*X \\
\theta_z & \downarrow & \theta_x \\
E_0^*Z & \to & E_1^*X
\end{array}
$$

is a quasi-pullback in $\mathcal{E}^{\text{Sh}((\mathcal{S}))} = \text{Sh}(\mathcal{E}^2, J_c \vee J_d \vee J_o)$. As in the proof of Proposition 3.2 (see (14) above), this means that the map

$$
\begin{pmatrix}
Z \\
\downarrow \\
Z
\end{pmatrix}
\xrightarrow{(1,f)}
\begin{pmatrix}
Z \\
\downarrow \\
X
\end{pmatrix}
$$

(20)

is a cover in $\mathcal{E}^2$ for the topology $J_c \vee J_d \vee J_o$. By Lemma 3.4, there must then be a standard cover of $(Z \to X)$ which refines this covering map (20). This standard cover is of the form (15), repeated here (with $Z$ in place of $Y$) as

$$
\begin{array}{ccc}
Z^*_\beta \xrightarrow{\text{cover}} X_x \times_X Z & \to & Z \\
\downarrow & \downarrow & \downarrow \\
U^*_\beta \xrightarrow{\text{open}} X_x & \xrightarrow{\text{cover}} & X.
\end{array}
$$

(21)

For each pair of indices $\alpha$ and $\beta$, the composite rectangle, consisting of the two right-hand squares, is a quasi-pullback. Therefore so is their sum

$$
Z' = \sum_\beta Z^*_\beta \to Z
$$

$$
X' = \sum_\alpha X_\alpha \to X.
$$

Thus if we write $U = \sum U^*_\beta$, we obtain a commutative diagram of the form

\[\begin{array}{ccc}
Z' & \to & Z \\
\downarrow & & \downarrow \\
U & \xrightarrow{\text{open}} & X \\
\downarrow & & \downarrow \\
\text{Z} & \to & \text{X}
\end{array}\]
where the $Z'$, $X'$, $Z$, $X$-square is a quasi-pullback. This last diagram can be rewritten in the form

$$
\begin{array}{ccc}
Z' & \rightarrow & U \\
\downarrow & \text{open} & \downarrow \\
X' & \rightarrow & X
\end{array}
$$

in which the outer rectangle is quasi-pullback. It follows that the inner right-hand “square” is also a quasi-pullback, so that $U \rightarrow X' \times_X Z$ is epi. But $U \rightarrow X$ is open, so by the quotient-axiom (A6) the projection $X' \times_X Z \rightarrow X'$ must be open as well. Since $X' \rightarrow X$ is epi, it follows by the descent axiom (A3) that the map $f: Z \rightarrow X$ that we started out with must also be open. This completes the proof of the completeness theorem.

In a completely analogous way (but with finite covers and sums instead of arbitrary ones) one obtains a completeness theorem for classes of open maps in a pretopos, as defined in section 1:

**Theorem 3.5.** Let $R$ be a class of open maps satisfying collection in a pretopos $\mathcal{E}$. Then there exists a pretopos $\mathcal{T}$ and a morphism of pretopoi

$$
\Phi: \mathcal{E} \rightarrow \mathcal{T}^2
$$

such that a map $Y \rightarrow X$ in $\mathcal{E}$ belongs to $R$ iff $\Phi Y \rightarrow \Phi X$ is a quasi-pullback square in $\mathcal{T}$.

We note that by (the categorical version of) Gödel's completeness theorem, it follows that $\mathcal{T}$ can be taken to be a product $\prod_{i \in I} \text{Sets}$ of copies of the pretopos of sets.

**Proof.** Analogous to preceding argument, we consider $\mathcal{E}$ as a coherent site, and construct a new Grothendieck topology $J_c \vee J_d \vee J_o$ on the arrow category $\mathcal{E}_2^2$. This Grothendieck topology consists of finite covering families, since the topology on the pretopos $\mathcal{E}$ is coherent. There are maps of coherent sites $E_0, E_1: \mathcal{E} \rightarrow \mathcal{E}_2^2$ and a transformation $\theta: E_1 \rightarrow E_0$ defined exactly as before, and the preceding proof shows that a map $Y \rightarrow X$ in $\mathcal{E}$ is open iff the square

$$
\begin{array}{ccc}
E_1 Y & \rightarrow & E_1 X \\
\downarrow & & \downarrow \\
E_0 Y & \rightarrow & E_0 X
\end{array}
$$

is a “quasi-pullback” in the coherent site $(\mathcal{E}_2^2, J_c \vee J_d \vee J_o)$, in the sense that $E_1 Y \rightarrow E_0 Y \times_{E_0 X} E_1 X$ is a cover. Let $\mathcal{T}$ be the pretopos generated by this last coherent site. Then $E_0, E_1, \theta$ together with the Yoneda-embedding followed by sheafification $y: \mathcal{E}_2^2 \rightarrow \mathcal{T}$ define a map of pretopoi

$$
\Phi: \mathcal{E} \rightarrow \mathcal{T}^2, \quad \Phi(X) = y(\theta_\varepsilon: E_1 X \rightarrow E_0 X),
$$

for which the conclusion of Theorem 3.5 holds. □
**Remark.** One can define a class of open maps in an exact category in a similar way, but deleting any reference to sums and replacing "epi" by "regular epi" in the formulation of the axioms. There is an analogous completeness theorem for such classes of open maps in exact categories. We leave the formulation of similar results for weaker contexts, such as regular categories, to the reader.

### 4. Geometric realization

Let $\mathcal{E}$ be a topos, and let $B$ be a fixed base locale. A geometric realization for $\mathcal{E}$ (over $B$) is a functor

$$[-]: \mathcal{E} \to (\text{locales}/B), \quad X \mapsto |X|,$$

which preserves finite limits and arbitrary colimits. Given such a realization functor, one can consider the class of those maps in $\mathcal{E}$ the realization of which is an open map of locales. We will show in this section that this class of open maps in $\mathcal{E}$, and that any given class $R$ of open maps in a topos $\mathcal{E}$ which satisfies the collection axiom is induced in this way, for a suitable realization functor.

**Proposition 4.1.** For any geometric realization functor (22), the class of maps $f: Y \to X$ for which $|f|: |Y| \to |X|$ is an open map of locales is a class of open maps in $\mathcal{E}$.

**Proof.** Apart from the Descent Axiom (A3), all the axioms (A1–6) for open maps in a topos follow immediately from standard properties of open maps of locales (cf. [11]), together with the fact that $|\cdot|$ preserves finite limits and arbitrary colimits. For descent, consider a pullback square

$$\begin{array}{ccc}
Z \times_X Y & \xrightarrow{q} & Z \\
g \downarrow & & \downarrow f \\
Y & \xrightarrow{p} & X
\end{array}$$

in $\mathcal{E}$, where $p$ is epi and $|g|$ is an open map of locales. Then $q$ is also epi in $\mathcal{E}$. Since epimorphisms in a topos are coequalizers of their kernel pairs, the preservation properties of $|\cdot|$ imply that in the diagram of locales

$$\begin{array}{ccc}
|Y\times_X Y\times_X Z| & \xrightarrow{\pi_1}\ & |Y\times_X Z| \\
\downarrow & & \downarrow g \\
|Y\times_X Y| & \xrightarrow{\pi_2}\ & |Y| \\
& & \downarrow p \\
& & |X|
\end{array}$$

all three squares are pullbacks and both rows are coequalizer-kernel pair diagrams. (For ease of notation, in (23) and below we delete the realization symbol $|\cdot|$ for maps.)

To see that $f: |Z| \to |X|$ is open, consider an open $U \subseteq |Z|$. Since $g: |Y\times_X Z| \to |Y|$ is open by assumption, the image $gq^{-1}(U)$ is open in $|Y|$. Furthermore, since both squares on the left are pullbacks, the map $\pi_1, \gamma: |Y\times_X Y\times_X Z| \to |Y\times_X Y|$ is open and
the Beck–Chevalley condition holds for both squares. Thus
\[
\pi_1^{-1}q^{-1}(U) = \pi_1\pi_3^{-1}q^{-1}(U) \quad \text{(Beck–Chevalley)}
\]
\[
= \pi_2\pi_3^{-1}q^{-1}(U) \quad \text{(since } q\pi_1 = q\pi_3)\]
\[
= \pi_2^{-1}g^{-1}(U) \quad \text{(Beck–Chevalley)}.
\]

Since the lower row in (23) is a coequalizer of locales, it follows that there is a unique open \(V \subseteq |X|\) with \(p^{-1}(V) = gq^{-1}(U)\). We claim that this \(U\) is the image \(f(V)\) of \(V\).

First, note that by the adjunction \(g \dashv g^{-1}\) and the definition of \(V\),
\[
q^{-1}U \leq g^{-1}gq^{-1}(U) = g^{-1}p^{-1}(V) = q^{-1}f^{-1}(V).
\]
Since \(q\) is surjective, it follows that \(U \leq f^{-1}(V)\), or \(f(U) \subseteq V\). Thus \(U \subseteq |Z| \rightarrow |X|\) factors through \(V \subseteq |X|\). By the commutativity of the diagram
\[
\begin{array}{ccc}
qu^{-1}U & \rightarrow & U \\
\downarrow & & \downarrow f \\
gq^{-1}U & \rightarrow & V \\
\end{array}
\]
this factorization \(U \rightarrow V\) must be surjective, so that \(f(U) = V\). This shows that \(f:|Z| \rightarrow |X|\) is open, and completes the proof of the proposition. \(\square\)

For a geometric realization functor \([-\]: \mathcal{E} \rightarrow (\text{locales}/B)\), we will call the class of open maps in \(\mathcal{E}\) as defined in Proposition 1 the class induced by this realization functor. From our Completeness Theorem 2.1, one now obtains the following result.

**Theorem 4.2.** Let \(\mathcal{E}\) be a topos, and let \(R\) be a class of open maps in \(\mathcal{E}\) satisfying the collection axiom. Then there exists a locale \(B\) and a geometric realization functor \([-\]: \mathcal{E} \rightarrow (\text{locales}/B)\) which induces the class \(R\).

For the proof of this theorem, we recall first the construction of the “Alexandrov locale” \(A(P)\) associated to any poset \(P\). By definition, the opens of \(A(P)\) are the downwards closed subsets of \(P\). Classically \(A(P)\) is the space of points of \(P\) equipped with the Alexandrov topology. For any locale \(L\), there is a bijective correspondence between locale maps \(A(P) \rightarrow L\) and maps of posets \(P \rightarrow \text{Points}(L)\). Thus the functor \(A\) is a left adjoint, and hence preserves all colimits. It also preserves finite products (see [11]), but it does not preserve all equalizers. The construction is also stable under base-change, in the sense that for any morphism of toposes \(f: \mathcal{F} \rightarrow \mathcal{E}\) and any preorder \(P\) in \(\mathcal{E}\), the canonical map
\[
A(f^*P) \rightarrow f^#A(P)
\]
is an isomorphism of locales in \(\mathcal{F}\). For a map of posets \(\varphi: P \rightarrow Q\), the induced map \(A(\varphi): A(P) \rightarrow A(Q)\) is explicitly described as follows: for an open \(V \subseteq A(Q)\), i.e.
a downsegment $V$ in $Q$,

$$A(\varphi)^{-1}(V) = \varphi^{-1}(V).$$

The right adjoint $A(\varphi)_*$ is described on downsegments $U$ in $P$ by

$$A(\varphi)_*(U) = \{q \in Q \mid \forall p \in P (\varphi p \leq q \Rightarrow p \in U)\}.$$ 

The map $A(\varphi)^{-1}$ also has a left adjoint $A(\varphi)!$, described by

$$A(\varphi)! (U) = \{q \in Q \mid \exists p(q \leq \varphi p \text{ and } p \in U)\}. \quad (25)$$

Using these explicit formulas, one readily checks that $A(\varphi)$ is an inclusion (respectively a surjection) of locales whenever $\varphi$ is 1–1 (respectively surjective). Say $\varphi$ has path-lifting if for any $p \in P, q \in Q$,

$$q \leq \varphi(p) \Rightarrow \exists r \in P (\varphi(r) = q \text{ and } r \leq p). \quad (26)$$

Lemma 4.3. A map $\varphi : P \to Q$ of posets has path-lifting iff $A(\varphi) : A(P) \to A(Q)$ is an open map of locales.

Proof. ($\Leftarrow$) To check (26), suppose $q \leq \varphi(p)$ and consider the open sets $\downarrow (p) \subseteq A(P)$ and $\downarrow (q) \subseteq A(Q)$. Since $A(f)$ is open, we have

$$A(\varphi)!((\downarrow p) \cap A(\varphi)^{-1}(\downarrow q)) = A(\varphi)!((\downarrow p) \cap (\downarrow q)). \quad (27)$$

But $A(\varphi)!((\downarrow p) \cap (\downarrow q)) = ((\downarrow \varphi(p)) \cap (\downarrow q)) = (\downarrow q)q$. Hence by (27) also $q \in A(\varphi)!((\downarrow p) \cap A(\varphi)^{-1}(\downarrow q))$. Thus there exists an $r \in (\downarrow p) \cap A(\varphi)^{-1}(q)$ with $q \leq \varphi(r)$. Or in other words, $r \leq p$ and $\varphi(r) \leq q \leq \varphi(r)$, exactly as required in (26).

($\Rightarrow$) Since $A(\varphi)!^{-1}$ always has a left adjoint $A(\varphi)!$, it suffices to show for open $U \subseteq A(P)$ and $V \subseteq A(Q)$ that the “Frobenius identity”

$$A(\varphi)! (U \cap A(\varphi)^{-1}(V)) = A(\varphi)! (U) \cap V \quad (28)$$

holds. The inclusion $\subseteq$ in (28) holds by adjointness. For the reverse inclusion, suppose $q \in A(\varphi)! (U \cap A(\varphi)^{-1}(V))$. Then $q \in V$ and $q \leq \varphi(p)$ for some $p \in U$. By path-lifting, $q = \varphi(r)$ for some $r \leq p$. Then $r \in U \cap A(\varphi)^{-1}(V)$, so $q \in A(\varphi)! (U \cap A(\varphi)^{-1}(V))$. This proves the lemma. \(\square\)

Of course, the locale $A(P)$ is (isomorphic to) the localic reflection of the presheaf topos $\text{Sets}^{\text{op}}$, and for a map $\varphi : P \to Q$ of posets, $A(\varphi)$ is the localic reflection of the geometric morphism $\text{Sets}^{\text{op}} \to \text{Sets}^{\text{op}}$ induced by $\varphi$. Thus the preceding lemma is really a special case of Proposition 2.6 in [6].

Next we will show that the Alexandrov functor $A$ preserves certain special equalizers. To this end, recall that for any locale $L$ there is bijective correspondence between locale maps $\varphi : L \to A(P)$ and flat functors $\Phi : P \to \mathcal{C}(L)$, given by
\( \Phi(p) = \varphi^{-1}(\downarrow p) \) and \( \varphi^{-1}(U) = \bigvee_{p \in U} \Phi(p) \). Flatness of \( \Phi \) means that the two conditions

\[
L = \bigvee_{p \in P} \Phi(p),
\]

\[
\Phi(p) \land \Phi(p') = \bigvee \{ \Phi(r) | r \leq p \text{ and } r \leq p' \}
\]

hold. (Here we have identified \( L \) with the top-element of the lattice \( \mathcal{C}(L) \).)

Now consider for a map of sets \( \alpha: X_1 \to X_0 \) the graph \( G(\alpha) \), which is the poset \( X_0 + X_1 \) with order defined by

\[ x \leq y \text{ iff } x = y, \text{ or } x \in X_1, y \in X_0 \text{ and } \alpha(x) = y. \]

The functor \( G: \text{Sets}^2 \to (\text{posets}) \) is readily seen to preserve pullbacks and colimits (\( G \) is a leaf adjoint). Define the mapping cylinder of \( \alpha \) to be the locale

\[ C_\alpha = AG(\alpha). \]

\( C_\alpha \) is like the ordinary mapping cylinder, except that the interval \([0, 1]\) has been replaced by the Sierpinski-space \( S = C_1 \) (where \( 1 \) denotes the terminal object in \( \text{Sets}^2 \)).

**Lemma 4.4.** The mapping cylinder functor \( \text{Sets}^2 \to (\text{locales}) \) preserves equalizers.

**Proof.** Consider for a parallel pair \( f, g: (Y_1 \xrightarrow{\beta} Y_0) \to (X_1 \xrightarrow{\delta} X_0) \) of maps in \( \text{Sets}^2 \) their equalizer

\[ (E, \gamma) \xleftarrow{\iota} (Y, \beta) \xrightarrow{f} (X, \alpha), \]

where \( E_j = \{ y \in Y_j | f(y) = g(y) \} \) for \( j = 0, 1 \) and \( i \) is the inclusion map. The functor \( G \) preserves monos and the functor \( A \) sends monos to inclusions of locales, so in the diagram of mapping cylinders

\[ C_\gamma \xleftarrow{i} C_\beta \xrightarrow{f} C_{\alpha} \]

the map (again denoted by) \( i \) is an inclusion of locales. Now suppose \( \varphi: L \to C_\beta \) is a map of locales such that \( f \varphi = g \varphi \). We will show that \( \varphi \) factors through \( C_\gamma \). This factorization is necessarily unique since \( i \) is an inclusion. The map \( \varphi \) corresponds to a flat map of posets

\[ \Phi: G(\beta) \to L, \quad \Phi(y) = \varphi^{-1}(\downarrow y). \]

We show first that for any fixed \( y \in Y = Y_0 + Y_1 \),

\[ \Phi(y) = \bigvee \{ \Phi(y) | f(y) = g(y) \} \]

(31)
(the \(y\) on the r.h.s. of (31) is the same fixed \(y\); so this is the supremum over a subset of the singleton \(\{\Phi(y)\}\)). To prove (31), write
\[
\Phi(y) = \varphi^{-1}(\downarrow y) \leq \varphi^{-1}f^{-1}(\downarrow fy)
\]
\[
= \varphi^{-1}g^{-1}(\downarrow fy)
\]
\[
= \varphi^{-1}\{z \in Y \mid g(z) \leq f(y)\}
\]
\[
= \sqrt{\{\Phi(z) \mid g(z) \leq f(y)\}}.
\]

Therefore
\[
\Phi(y) = \Phi(y) \wedge \sqrt{\{\Phi(z) \mid g(z) \leq f(y)\}}
\]
\[
= \sqrt{\{\Phi(y) \wedge \Phi(z) \mid g(z) \leq f(y)\}}
\]
\[
= \sqrt{\{\Phi(w) \mid w \leq y \text{ and } g(w) \leq f(y)\}},
\]

the latter since \(\Phi\) is flat; cf. (30)). Thus to prove (31) it suffices to show that
\[
\sqrt{\{\Phi(w) \mid w \leq y \text{ and } g(w) \leq f(y)\}} = \sqrt{\{\Phi(y) \mid f(y) = g(y)\}}. \tag{32}
\]

The inequality \(\geq\) clearly holds. For the converse, consider \(w \leq y\) with \(g(w) \leq f(y)\). In case \(y \in Y_1\) we have \(w = y\) and hence \(g(y) = g(w) = f(y)\). And in case \(y \in Y_0\), we have either \(w = y\) whence again \(g(y) = f(y)\), or \(\beta(w) = y\) in which case \(g(y) = g\beta(w) = \alpha g(w) = f(y)\) (the latter since \(g(w) \leq f(y)\)). Thus \(\Phi(w) \leq \Phi(y) \leq \sqrt{\{\Phi(y) \mid f(y) = g(y)\}}\). This proves (32).

Now let \(\Psi : G(y) \to \mathcal{O}(L)\) be the restriction of \(\Phi\). We claim that \(\Psi\) is flat. Indeed, by definition of \(\Psi\) and of \(E\) as an equalizer,
\[
\sqrt{\{\Phi(y) \mid y \in Y \text{ and } f(y) = g(y)\}} = \sqrt{\{\Phi(y) \mid f(y) = g(y)\}}
\]
\[
= \sqrt{\{\Phi(y) \mid y \in Y\}} \quad \text{(by (31))}
\]
\[
= L \quad \text{(since } \Phi\text{ is flat).}
\]

This shows that \(\Psi\) satisfies the first condition (29) for flatness. For the other condition (30) for flatness, pick \(y\) and \(z\) in \(E\). Then
\[
\Phi(y) \wedge \Phi(z) = \sqrt{\{\Phi(w) \mid w \in Y \text{ and } w \leq y, z\}} \quad \text{(} \Phi\text{ is flat)}
\]
\[
= \sqrt{\{\Phi(w) \mid w \in Y \text{ and } w \leq y, z, \text{ and } f(w) = g(w)\}} \quad \text{(by (31))}
\]
\[
= \sqrt{\{\Phi(w) \mid w \leq Y, z \text{ and } w \in E\}},
\]

which shows that the restriction \(\Psi\) of \(\Phi\) to \(E\) satisfies (30). This proves that \(\Psi : G(y) \to \mathcal{O}(L)\) is flat. It follows that \(\Psi\) defines a map of locales \(\phi : L \to C_y\), with \(\varphi^{-1}(\downarrow e) = \Psi(e)\) for any \(e \in E\). It follows again readily from (31) that \(i \circ \psi = \phi\), as required. This proves the lemma. \(\square\)
Remark 4.5. For a map of sets \( \alpha : X_1 \to X_0 \) as above, the locale \( AG(\alpha) \) can also be constructed as the localic reflection of the topos, obtained by Artin Glueing \([14]\) along the direct functor of the induced geometric morphism \( \alpha : \text{Sets}/X_1 \to \text{Sets}/X_0 \). In other words, the topos \( \text{Sh}(AG(\alpha)) \) is equivalent to the comma category \( (\text{Sets}/X_0)/\alpha^{*} \).

Peter Johnstone pointed out to us that from this point of view, Lemma 4.4 is a special case of the more general result that Artin Glueing preserves equalizers (and pullbacks). This more general result can easily be proved, once it has been observed that geometric morphisms from a topos \( \mathcal{X} \) into a glued topos \( \mathcal{F}/L \) obtained by glueing along a left exact functor \( L : \mathcal{X} \to \mathcal{F} \), correspond to decompositions of \( \mathcal{X} \) into an open subtopos \( i : X_0 \to \mathcal{X} \) and its complementary closed subtopos \( j : X_c \to \mathcal{X} \), together with two maps \( X_0 \to \mathcal{X} \) and \( X_c \to \mathcal{X} \) which are compatible with the fringe functors \( j^{*} = i^{*} \) and \( L \).

Proof of Theorem 4.2. Consider the given class \( R \) of open maps in the topos \( \mathcal{E} \). By assumption, \( R \) satisfies the collection axiom, so by our Completeness Theorem there is topos \( \mathcal{F} \) and a morphism of topoi

\[
\varphi : \mathcal{F}^{\gamma} \to \mathcal{E}
\]

such that a map \( f : Y \to X \) in \( \mathcal{E} \) belongs to \( R \) iff \( \varphi^{*}(f) \) is a quasi-pullback in \( \mathcal{F} \). Let \( p : \text{Sh}(L) \to \mathcal{F} \) be a cover of the topos \( \mathcal{F} \) by a locale \( L ([11], [111], \text{etc.}) \), and let \( B = L \times S \) where \( S \) is the Sierpinski space. Define a realization functor

\[
| - | : \mathcal{E} \to (\text{locales}/B) \tag{33}
\]

as follows: for an object \( X \in \mathcal{E} \), \( p^{*}\varphi^{*}(X) \) is a map \( p^{*}\varphi^{*}_{\gamma}(X) \to p^{*}\varphi^{*}_{\delta}(X) \) in \( \text{Sh}(L) \), and its mapping cylinder – constructed inside \( \text{Sh}(L) \) – is a locale \( C(X) = AG(p_{\gamma}^{*}X \to p^{*}\varphi^{*}_{\delta}(X)) \) over the Sierpinski locale \( S = C(1) \). By the correspondence between internal locales in \( \text{Sh}(L) \) and (external) locales over \( L \), this mapping cylinder corresponds to a locale over \( L \times S \), which we define to be \( |X| \). This defines the realization functor \((33)\). This functor indeed preserves finite limits and arbitrary colimits, since both inverse image functors \( p^{*} \) and \( \varphi^{*} \) do, as does the mapping cylinder functor \( C \) (when defined as taking values in locales over \( S \), by the preservation properties of the functors \( A \) and \( G \) and the preceding lemma). Furthermore, for a map \( f : Y \to X \) in \( \mathcal{E} \), the realization \( |Y| \to |X| \) thus defined is open iff \( C(p^{*}\varphi^{*}Y) \to C(p^{*}\varphi^{*}X) \) is an open map of locales in \( \text{Sh}(L) \). By Lemma 4.3, this is the case exactly when the map of posets \( G(p^{*}\varphi^{*}Y) \to G(p^{*}\varphi^{*}X) \) has path-lifting. This means precisely that the square

\[
\begin{array}{ccc}
p^{*}\varphi^{*}_{\gamma}Y & \to & p^{*}\varphi^{*}_{\delta}X \\
\downarrow & & \downarrow \\
p^{*}\varphi^{*}_{\delta}Y & \to & p^{*}\varphi^{*}_{\delta}X
\end{array}
\]

is a quasi-pullback in \( \text{Sh}(L) \). Since the functor \( p^{*} \) is faithful (by construction), the similar square without the \( p^{*} \)’s is also a quasi-pullback. By choice of \( \varphi \), this means that
Y → X belongs to R. This shows that the class induced by this realization functor (33) coincides with the given class R, and thus proves the theorem.

Exactly as in the case of Corollary 2.2, the analogue of Theorem 4.2 for etale maps is a consequence:

**Corollary 4.6.** Let R be a class of etale maps in a topos $\mathcal{E}$, satisfying the collection axiom. Then there are a locale B and a geometric realization functor $|-|: \mathcal{E} \to (\text{locales}/B)$ such that a map $Y \to X$ in $\mathcal{E}$ belongs to the class R iff its realization $|Y| \to |X|$ is a local homeomorphism of locales.

5. Sites with open or etale maps

In this section we will discuss how classes of open or etale maps in a site give rise to similar classes in the associated topos of sheaves. For such classes in the topos, the collection axiom will be seen to hold, so that our Completeness Theorem applies.

Consider a small category C with finite limits, and with finite sums which are disjoint and universal. We shall assume that C is equipped with a subcanonical Grothendieck pretopology all whose covers are finite, and which respects the finite sums, in the sense that for any family $\{A_i \to B \mid i = 1, \ldots, n\}$,

$$\{A_i \to B\}_{i=1}^n$$ is a covering family ⇔ $\sum A_i \to B$ is a cover.

Here and below, “cover” or “covering family” only refers to the Grothendieck petropology. We will call a map $B \to A$ a surjection if there exists a cover $C \to A$ factoring through it,

$$\begin{array}{c}
B \\
\downarrow \\
C \longrightarrow A.
\end{array}$$

We remark that our finiteness hypotheses on the site C are by no means essential; the same arguments will apply to any site when closed under sums of the same cardinality as the covering families.

A class of open maps in C is a class of arrows R satisfying the following axioms:

1. (O1) R is closed under composition, and contain all isomorphisms.
2. (O2) R is stable under pullback.
3. (O3) For any family of arrows $\{A_i \to B \mid i = 1, \ldots, n\}$, for any $n \geq 0$, the sum $\sum A_i \to B$ belongs to R iff each $A_i \to B$ does.
4. (O4) If a composite $f \circ p: C \to B \to A$ belongs to R and p is surjection then f belongs to R.
(O5) Every cover can be refined by an R-cover; i.e., for any cover $C \to B$ there is an arrow $D \to C$ such that the composite $D \to B$ is a cover which also belongs to $R$.

The axiom (O5) together with (O3) imply that $C$ is equivalent to a site in which all covering families consist of R-maps. (Dubuc refers to this axiom (O5) as "ETC" in [2].)

**Remark.** In practice, axiom (O4) is difficult to check because it refers to all surjections, rather than just the covers in the site. However, if $R_\alpha$ is a class of maps in $C$ which satisfies axioms (O1–3) and (O5), then one obtains a class $R$ of open maps in $C$ by defining: an arrow $f: B \to A$ belongs to $R$ iff $f \circ p = g$ for some $g: C \to A$ in $R_\alpha$ and some surjection $p$.

A class $R$ of maps in a site $C$ is said to be a class of *etale maps* if it satisfies the axioms (O1)–(O3), (O5), and the following two axioms:

- **(O6)** If $f: B \to A$ belongs to $R$ then so does its diagonal $B \to B \times_A B$.
- **(O7)** In a commutative diagram

\[
\begin{array}{ccc}
C & \to & B \\
\downarrow & & \downarrow \\
A & \to & A
\end{array}
\]

where $C \to B$ is a cover which belongs to $R$ and $C \to A$ belongs to $R$, the map $B \to A$ also belongs to $R$.

**Proposition 5.1.** Let $S$ be a class of etale maps in a site $C$. Define the class $R$ of maps in $C$ to consist of those $B \to A$ for which there exists a quasi-pullback square

\[
\begin{array}{ccc}
B' & \to & B \\
\downarrow & & \downarrow \\
A' & \to & A
\end{array}
\]

where $A' \to A$ is a cover and $B' \to A'$ belongs to $S$. Then $R$ is a class of open maps in $C$. Furthermore, a map $B \to A$ belongs to the given class $S$ iff $B \to A$ and $B \to B \times_A B$ both belong to $R$.

Here and below, we call a commutative square (1) in a site $C$ a quasi-pullback if the map $B' \to A' \times_A B$ is a surjection. The proof of Proposition 5.1 is completely analogous to that of Proposition 1.7.

**Remark.** As an analogue of Proposition 1.6, for a given class $R$ of open maps in a site $C$ the class $S$ of maps $B \to A$ in $R$ for which $B \to B \times_A B$ also belongs to $R$ satisfies the axioms for etale maps, except possibly the axiom (O5).
For a class of open maps $R$ in a site $\mathbf{C}$, one can define a similar class $\tilde{R}$ of maps of sheaves: say $Y \to X$ belongs to $\tilde{R}$ iff for each $C \in \mathbf{C}$ and each arrow $C \to X$ there is a family $\{D_i \to C \mid i \in I\}$ of maps in $R$ which fit into a quasi-pullback square

$$
\begin{array}{c}
\sum D_i \to Y \\
\downarrow \quad \downarrow \\
C \to X
\end{array}
$$

in the topos $\text{Sh}(\mathbf{C})$. (Here and below we identify an object $C \in \mathbf{C}$ with its image under the Yoneda embedding $C \to \text{Sh}(\mathbf{C})$.)

**Theorem 5.2.** For any class of open maps $R$ in a site $\mathbf{C}$, the induced class $\tilde{R}$ is a class of open maps satisfying the collection axiom in the topos $\text{Sh}(\mathbf{C})$. It is the smallest such class for which the Yoneda embedding preserves open maps. Moreover, a map $D \to C$ in $\mathbf{C}$ belongs to $\tilde{R}$ whenever its image under the Yoneda embedding belongs to $R$.

**Proof.** The proof that $\tilde{R}$ satisfies the axioms for open maps in a topos, including collection, is elementary. By way of example, we verify the descent axiom (A3) and the collection axiom (A9).

For (A3), consider a pullback square

$$
\begin{array}{ccc}
Y' & \xrightarrow{a} & Y \\
\downarrow f' & \downarrow \downarrow f & \downarrow \downarrow f \\
x' & \xrightarrow{p} & X
\end{array}
$$

in the topos, where $p$ is epi and $f'$ belongs to $\tilde{R}$. Consider any map $x: C \to X$ from an object $C \in \mathbf{C}$. Since $p$ is epi, there is a cover $\beta: D \to C$ such that $p \circ \beta$ factors as $p \circ \gamma$ for some arrow $\gamma: D \to X'$. Moreover, by choosing a refinement of this cover $\beta$ if necessary, axiom (O5) allows us to assume that $\beta$ also belongs to $R$. Since $f'$ belongs to $\tilde{R}$, there is a family of $R$-maps $\delta_i: E_i \to D$ such that the left-hand square below is a quasi-pullback.

$$
\begin{array}{ccc}
\sum E_i & \to & Y' \to Y \\
\downarrow \quad \downarrow \\
D & \xrightarrow{\beta} & X' \xrightarrow{p} X
\end{array}
$$

Since $\beta$ is a cover, it follows that the square on the far right is also a quasi-pullback. Moreover, each composite $\beta \circ \delta_i$ is open. This shows that $f$ belongs to $\tilde{R}$.

For collection (A9), consider a map $f: Y \to X$ in $\tilde{R}$ and an epi $p: Z \to Y$. Since $f$ belongs to $\tilde{R}$, there is for each arrow $z: C \to X$ a quasi-pullback of the form (35). Now...
for each index i (as in (35)) form the pullback $D_i \times_Y Z \rightarrow D_i$. This is an epi of sheaves, so there is a cover $E_i \rightarrow D_i$ (which by (05) we can assume to belong to $R$) which factors through $D_i \times_Y Z$. Taking the sum over all arrows $\alpha: C \rightarrow X$ from representable sheaves into $X$ yields a diagram

\[
\begin{array}{ccc}
\sum E_i & \rightarrow & Z \\
\downarrow & & \downarrow \\
\sum D_i & \rightarrow & Y \\
\downarrow & \text{q.p.b.} & \downarrow \\
\sum C & \rightarrow & X \\
\end{array}
\]

where each composite $E_i \rightarrow D_i \rightarrow C$ belongs to $R$. As we will show in a moment, these composites then also belong to $\bar{R}$. By axioms (A4) and (A5) for $\bar{R}$ (the verification of which we omitted) the left hand vertical composite in the diagram above also belongs to $\bar{R}$. Since the ‘square’ with vertices $\sum E_i$, $Z$, $X$ and $\sum C$ is also a quasi-pullback, this shows that collection holds.

Next, if $D \rightarrow C$ belongs to $R$, then by (02) so does its pullback along any map $C' \rightarrow C$ in $C$. Thus when viewed as a map of representable sheaves, $D \rightarrow C$ belongs to $\bar{R}$. This shows that the Yoneda embedding $y: C \rightarrow Sh(C)$ preserves open maps, a fact which we already used in the proof of the collection axiom for $R$. It is also clear from the definition of $\bar{R}$ that $\bar{R}$ is the smallest class of open maps in $Sh(C)$ for which $y: C \rightarrow Sh(C)$ preserves open maps.

Finally, suppose $D \rightarrow C$ is a map in $C$ which, as a map of sheaves, belongs to $\bar{R}$. Then for the identity map $C \rightarrow C$ there is a family of $R$-maps $\{D_i \rightarrow C | i \in I\}$ for which

\[
\begin{array}{ccc}
\sum D_i & \rightarrow & D \\
\downarrow & & \downarrow \\
C & = & C \\
\end{array}
\]

is a quasi-pullback of sheaves. Thus the family $\{D_i \rightarrow D | i \in I\}$ is surjective, while by axiom (O3) the map $\sum D_i \rightarrow C$ belongs to $R$, so axiom (O4) applied to the diagram

\[
\begin{array}{ccc}
\sum D_i & \rightarrow & D \\
\downarrow & \rightarrow & \\
& C & \\
\end{array}
\]

shows that $D \rightarrow C$ also belongs to $R$. This completes the proof of the theorem. \qed

**Remark.** Suppose we are given a class of maps $R_0$ in the site $C$ which satisfy axioms (O1–3,5), as in the Remark preceding Proposition 5.1. Then one obtains a class $\bar{R}$ of
open maps in the topos of sheaves, consisting of those maps $Y \to X$ which fit into a quasi-pullback (2) with each $D_i \to C$ in $R_0$. The restriction $\tilde{R} \cap C$ of this class to the representable sheaves is precisely the class $R$ of open maps in $C$ obtained from $R_0$, as in the Remark preceding Proposition 5.1.

Now let $S$ be a class of etale maps in a site $C$, and let $R$ be the associated class of open maps in $C$ as in Proposition 5.1. Let $\tilde{R}$ be the associated class of open maps between sheaves. Thus $\tilde{R}$ can be described in terms of $S$, as the class of those maps $Y \to X$ which fit into a quasi-pullback square of the form (35) with each $D_i \to C$ in $S$. Define the class $\tilde{S}$ of maps in $Sh(C)$ to consists of those maps $Y \to X$ in $\tilde{R}$ for which the diagonal $Y \to Y \times_X Y$ also belongs to $\tilde{R}$.

**Corollary 5.3.** For a class $S$ of etale maps in $C$, the class $\tilde{S}$ is a class of etale maps in the topos $Sh(C)$, satisfying the collection axiom. It is the smallest such class for which the Yoneda embedding $C \to Sh(C)$ preserves etale maps. Moreover, a map $D \to C$ in $C$ belongs to $S$ whenever its image under the Yoneda embedding belongs to $\tilde{S}$.

**Proof.** Since $\tilde{R}$ is a class of open maps by Theorem 5.2, it follows by Proposition 1.7 that $\tilde{S}$ is a class of etale maps in $Sh(C)$. Moreover, the proof of the collection axiom (A9) for $\tilde{R}$ above shows that $\tilde{S}$ also satisfies collection. Suppose now $T$ is any other class of etale maps satisfying collection in $Sh(C)$, and containing the image of $S$ under the Yoneda embedding. Let $T'$ be the associated class of open maps, as in Proposition 1.7(a). Then $T'$ contains the image of $R$, so by Theorem 5.2 above $\tilde{R} \subseteq T'$. Hence by Proposition 1.7(b) and the definition of $\tilde{S}$ from $\tilde{R}$, it follows that $\tilde{S} \subseteq T$. This shows that $\tilde{S}$ is the smallest class, as claimed in the Corollary. Finally, if $D \to C$ in $C$ is a map such that its image under the Yoneda embedding belongs to $\tilde{S}$, then this image as well as its diagonal belong to $\tilde{R}$. Hence by Theorem 5.2 $D \to C$ belongs to $S$. This proves the Corollary 5.3. \(\square\)

Let $S$ be a class of etale maps in a site $C$, and let $\tilde{S}$ be the associated class of etale maps between sheaves, all as before. Write $Et(C)$ for the full subcategory of $Sh(C)$ consisting of etale sheaves, i.e. sheaves $Y$ for which the unique map $Y \to 1$ belongs to $\tilde{S}$. By Corollary 2.3, $Et(C)$ is a Grothendieck topos, and there is a connected geometric morphism

$$p : Sh(C) \to Et(C),$$

whose inverse image $p^\ast$ is the inclusion functor. Let $et_C$ be the full subcategory of $C$ consisting of objects $C \in C$ for which $C \to 1$ belongs to $S$. Then (via the Yoneda embedding) $et_C$ is a full subcategory of $Et(C)$. Furthermore, if $Y$ is an object of $Et(C)$ then by definition of the class $\tilde{S}$ there exists a cover $\sum D_i \to Y$ for which each $D_i \to 1$ belongs to $S$ (take diagram (35) with $X = C = 1$). In other words, every object in $Et(C)$ is covered by objects in the etale site $et_C$. It follows by Giraud's theorem that $et_C$ is
a site for the topos $Et(C)$. The inclusion functor

$$i : et_C \to C$$

is a morphism of sites, and induces a geometric morphism

$$\varphi : Sh(C) \to Sh(et_C).$$

Under the equivalence $Et(C) \cong Sh(et_C)$, this geometric morphism $\varphi$ corresponds to $p$.

(One may also reverse this argument, and first show directly that $\varphi$ is a connected geometric morphism with the property that the counit $\varphi^* p_! X \to X$ is an isomorphism iff $X$ is an etale object in $Sh(C)$. It thus follows (without invoking Corollary 2.3) that $Et(C)$ is a Grothendieck topos.)

The following result is due to Dubuc [2]; he also shows that it implies a version of the completeness theorem.

**Proposition** (Dubuc). For a class $S$ of etale maps in a site $C$ the inclusion of $Et(C)$ into $Sh(C)$ induces a local geometric morphism

$$p : Sh(C) \to Et(C).$$

Recall here [8] that a geometric morphism $p$ is local if it is connected (i.e., $p^*$ is full and faithful) and $p_*$ has a right adjoint. Any such $p$ induces isomorphisms in cohomology. For applications of this result the theory of spectra, see Dubuc [2].

**Proof.** By the identification of $Et(C)$ and $Sh(et_C)$, it suffices to observe that the morphism of sites $i : et_C \to C$ satisfies the conditions from Theorem 1.8 of [8], viz. that this morphism is left exact, full and faithful, and continuous, and has the covering lifting property (as defined there). \qed

### 6. Path-lifting and open maps

Let $I = [0, 1]$ be the unit interval. A map $f : Y \to X$ between sheaves on $I$ is said to have *path-lifting* (towards the future) if for any $0 \leq t_0 \leq t_1 \leq 1$, any section $x : [t_0, t_1] \to X$ and any $y_0 \in f^{-1}(x(t_0))$, there exists a lifting $\tilde{x} : [t_0, t_1] \to Y$ with $\tilde{x}(t_0) = y_0$ and $f \circ \tilde{x} = x$. Notice that since $X$ and $Y$ are sheaves, it is enough to consider rational points $t_0$ and $t_1$. There is an evident symmetric notion of path-lifting into the past: If $\tau : I \to I$ is the map $\tau(x) = 1 - x$, then $f$ has path-lifting into the past iff $\tau^*(f)$ has path-lifting into the future. We shall see shortly that the class of maps which have path-lifting is a class of open maps in the topos $Sh(I)$.

There is an associated class of maps $Y \to X$ in $Sh(I)$, with the property that both $Y \to X$ and its diagonal $Y \to Y \times_Y Y$ have path-lifting towards the future; these are the maps with *unique* path-lifting. A sheaf $X$ on $I$ is said to have (unique) path-lifting if the map $X \to 1$ does. Let $T$ be the interval $[0, 1]$, topologized by taking as open sets the
intervals \([t, 1]\), and let \(p: I \to T\) be the evident surjection. Then a sheaf \(X\) on \(I\) has unique path-lifting towards the future iff the count \(p*p_X X \to X\) is an isomorphism. In particular, the sheaves on \(I\) with unique path-lifting form a topos, equivalent to \(Sh(T)\).

A sheaf \(X\) on \(I\) gives rise to a category \(X_{\text{rat}}\) of rational points of \(X\). Regarding \(X\) as a local homeomorphism \(p: X \to I\), this category has as objects the points \(x \in X\) with \(p(x)\) rational, and as arrows \(x \to x'\) all sections on the interval \([p(x), p(x')]\), provided \(p(x) \leq p(x')\). A map of sheaves \(Y \to X\) defines an evident functor \(Y_{\text{rat}} \to X_{\text{rat}}\) of small categories. In particular, the unique map \(X \to 1\) gives a functor \(X_{\text{rat}} \to I_{\text{rat}}\), where \(I_{\text{rat}}\) is the ordered set of rationals between 0 and 1, viewed as a category. Thus one obtains a left-exact functor

\[
(-)_{\text{rat}}: Sh(I) \to \text{Cat/}I_{\text{rat}}.
\]

Say a functor between small categories \(q: D \to C\) has arrow-lifting if for any object \(D \in D\) and for any arrow \(c: q(D) \to C\) in \(C\) there exists a lifting \(\beta: D \to D'\) in \(D\) with \(q(\beta) = c\). The following property is immediate from the definitions:

**Proposition 6.1.** A map \(Y \to X\) of sheaves on \(I\) has path-lifting towards the future iff the functor \(Y_{\text{rat}} \to X_{\text{rat}}\) has arrow-lifting, and \(Y \to X\) has unique path-lifting iff \(Y_{\text{rat}} \to X_{\text{rat}}\) is a discrete opfibration.

All this relativizes to an arbitrary topos \(\mathcal{E}\): there is a class of arrows in the topos \(\mathcal{F} \times Sh(I)\) of \(\mathcal{F}\)-internal sheaves on \(I\) which have (unique) path-lifting towards the future. The objects of \(\mathcal{F} \times Sh(I)\) with unique path-lifting form a topos \(\mathcal{F} \times Sh(T)\). And a map \(Y \to X\) in \(\mathcal{F} \times Sh(I)\) has path-lifting iff the induced functor \(Y_{\text{rat}} \to X_{\text{rat}}\) between internal categories in \(\mathcal{F}\) has arrow-lifting.

More generally, if \(\mathcal{E}\) is a topos, each family of paths in \(\mathcal{E}\), i.e. each morphism of topoi

\[
\mathcal{F} \times Sh(I) \to \mathcal{E},
\]

gives a class \(R\) of maps in \(\mathcal{E}\), consisting of exactly those maps \(Y \to X\) with the property that \(\varphi^* Y \to \varphi^* X\) has path-lifting along the paths in \(\mathcal{F}\), or, as we will say, with respect to \(\mathcal{F}\) (or \(\varphi\)).

**Theorem 6.2.** For any family of paths \(\mathcal{F} \times Sh(I) \to \mathcal{E}\) in a topos \(\mathcal{E}\), the class of maps in \(\mathcal{E}\) which have path-lifting with respect to this family is a class of open maps.

**Proof.** It suffices to show that the class \(R\) of maps in \(\mathcal{F} \times Sh(I)\) which have path-lifting (towards the future) is a class of open maps, since the class in \(\mathcal{E}\) is induced by \(R\) along the given geometric morphism \(\mathcal{F} \times Sh(I) \to \mathcal{E}\). To this end, we apply a relative version of Proposition 6.1, which states that a map \(Y \to X\) in \(\mathcal{F} \times Sh(I)\), i.e. a map \(Y \to X\) of \(\mathcal{F}\)-internal sheaves on \(I\), belongs to \(R\) iff the functor \(Y_{\text{rat}} \to X_{\text{rat}}\) between categories in \(\mathcal{F}\) has arrow lifting. The axioms for open maps can then easily be verified, using elementary properties of functors with arrow-lifting. For the descent axiom (A3), one uses that if \(Y \to X\) is epi, then the functor \(Y_{\text{rat}} \to X_{\text{rat}}\) is "surjective" in the following
sense: (1) it is surjective on objects; (2) it is locally surjective on arrows in the sense that each arrow $x \to x'$ in the codomain of the functor has a factorization $x = x_0 \to x_1 \to \cdots \to x_n = x'$ into a composite of $n$ arrows each of which is in the image of the functor. Further details are elementary, and left to the reader. □

Since a map $Y \to X$ has unique path-lifting iff both this map and its diagonal $Y \to Y \times_X Y$ have path-lifting, the preceding theorem together with Proposition 1.6 yields:

**Corollary 6.3.** For any family $\mathcal{F} \times \text{Sh}(I) \to \mathcal{E}$ of paths in a topos $\mathcal{E}$, the family of maps in $\mathcal{E}$ which have unique path-lifting with respect to this family is a class of etale maps.

Analogous to Corollary 2.3 we have

**Theorem 6.4.** For any family of paths $\varphi : \mathcal{F} \times \text{Sh}(I) \to \mathcal{E}$ in a topos $\mathcal{E}$, the objects of $\mathcal{E}$ which have unique path-lifting form a topos.

**Proof.** Let $T$ be the unit interval with the course topology given by open sets $(t, 1]$, as before. The quotient map $p : I \to T$ induces a geometric morphism $p : \mathcal{F} \times \text{Sh}(I) \to \mathcal{F} \times \text{Sh}(T)$ and, as observed before, an object $F$ of $\mathcal{F} \times \text{Sh}(I)$ has unique path-lifting iff the counit $p^*p_* F \to F$ is an isomorphism; or equivalently, iff $F$ is isomorphic to an object in the image of $p^*$. It follows by the explicit description of pushout topos of $\mathcal{M}$, as in the proof of Corollary 2.3, that the subcategory $\text{Et}_\mathcal{E}$ of $\mathcal{E}$ consisting of objects with unique path-lifting is equivalent to the pushout topos of the diagram $\mathcal{F} \times \text{Sh}(T) \leftarrow \mathcal{F} \times \text{Sh}(I) \to \mathcal{E}$. □

**Remark 6.5.** If $p : \mathcal{G} \to \mathcal{F}$ is a surjective geometric morphism, then a functor $D \to C$ between small categories in $\mathcal{F}$ has arrow-lifting iff its inverse image $p^*D \to p^*C$ has arrow-lifting in $\mathcal{G}$. Furthermore, for an object $X \in \mathcal{F} \times \text{Sh}(I)$, i.e. an $\mathcal{F}$-internal sheaf $X$ on $I$, the construction of the category $X_{\text{rat}}$ of rational points is natural in $\mathcal{F}$, in the sense that

$p^*(X_{\text{rat}}) \cong p^*(X)_{\text{rat}}$.

It follows by (the relative form of) Proposition 6.1 that if $\varphi : \mathcal{F} \times \text{Sh}(I) \to \mathcal{E}$ is a family of paths in $\mathcal{E}$, the class of open maps in $\mathcal{E}$ defined by $\varphi$ coincides with that defined by $\varphi \circ p$. In particular, if the class $R$ in $\mathcal{E}$ is induced by a family $\varphi : \mathcal{F} \times \text{Sh}(I) \to \mathcal{E}$, then the same class $R$ is also induced by a subtopos $\mathcal{P}$ of the path-topos $\mathcal{E}^{\text{Sh}(I)}$. Indeed, for $\mathcal{P}$ one simply takes the image $\mathcal{P}$ of the transposed map $\hat{\phi} : \mathcal{F} \to \mathcal{E}^{\text{Sh}(I)}$, with its associated family of paths $\mathcal{P} \times \text{Sh}(I) \to \mathcal{E}$, which is the transposed of the inclusion $\mathcal{P} \subset \mathcal{E}^{\text{Sh}(I)}$.

Every class of open maps in a topos which satisfies the collection axiom is induced by a class of paths as in Theorem 6.2:
Theorem 6.6. For any class $R$ of maps satisfying the collection axiom in a topos $\mathcal{E}$, there exists a family of paths $\mathcal{F} \times \text{Sh}(I) \rightarrow \mathcal{E}$ such that $R$ consists precisely of the maps which have path-lifting w.r.t. $\mathcal{F}$.

Proof. This follows immediately from the Completeness Theorem. Indeed, let $\pi : [0, 1] \rightarrow S$ be the surjection onto the Sierpinski space defined by $\pi(t) = 0$ if $t < 1$ and $\pi(1) = 1$. This map $\pi$ induces a surjection $\pi : \mathcal{F} \times \text{Sh}(I) \rightarrow \mathcal{F} \times \text{Sh}(S) = \mathcal{F}^2$, with the property that a morphism $Y \rightarrow X$ in $\mathcal{F}^{2}$ belongs to the canonical class of open maps in $\mathcal{F}^2$ iff $\pi^* Y \rightarrow \pi^* X$ has path-lifting towards the future. Thus for $\varphi : \mathcal{F}^2 \rightarrow \mathcal{E}$ as in the Completeness Theorem, the given class $R$ is induced by the family of paths $\varphi \circ \pi : \mathcal{F} \times \text{Sh}(I) \rightarrow \mathcal{E}$. □

Remark 6.7. The class of open maps obtained from a family of paths, as in Theorem 6.2, in general doesn’t satisfy the collection axiom. For example, in the topos $\text{Sh}(I)$, the class of maps with path-lifting towards the future cannot satisfy collection; this follows from our Completeness Theorem since every morphism of topoi $\mathcal{F} \times \text{Sh}(S) \rightarrow \mathcal{F}^2 \times \text{Sh}(I)$ factors through the projection $\mathcal{F} \times \text{Sh}(S) \rightarrow \mathcal{F}$ (because $I$ is Hausdorff).

Open maps in a topos are related to interior operators, and thus provide modal operators for the internal logic of the topos; cf. [3, 4, 13]. For example, one may think of a particular topos $\mathcal{E}$ as a topos of events, and a class of paths $\mathcal{P} \subseteq \mathcal{E}^{\text{Sh}(I)}$ (cf. Remark 6.5 above) as the class of segments of world-lines. Then it is natural to assume that $\mathcal{P}$ is closed under restriction and composition. From $\mathcal{P}$ one can define a modal operator “it will always be the case that ...” (necessity towards the future $\square_f$), and, in case the end-point map $\partial_0 : \mathcal{P} \subseteq \mathcal{E}^{\text{Sh}(I)} \rightarrow \mathcal{E}$ is open, also an operator “it may have been the case that ...” (possibility in the past $\Diamond_p$), which is left-adjoint to necessity in the future. All this is most easily visualized in the case of topological spaces: Let $B$ be a topological space (of events), and let $P \subseteq B^I$ be a subspace closed under restriction and composition of paths. Then for a sheaf $X$ on $B$ and a subsheaf $U \subseteq X$ (a “predicate” on $X$), we obtain another subsheaf

$$\star \Diamond_f U = \text{Int}\{x \in U \mid \text{for all paths } \alpha : [0, 1] \rightarrow B \text{ in } P \text{ and all liftings}\}$$

$$\beta : [0, 1] \rightarrow X \text{ of } \alpha \text{ with } \beta(0) = x \text{ it holds that } \beta(1) \in U\}.$$ 

Here we view $X$ and $U$ as etale spaces over $B$, and “Int” is the ordinary topological interior operator. If the map $\partial_0 : P \rightarrow B$, which sends a path $\alpha$ to its starting point $\alpha(0)$, is an open map, then for $U \subseteq X$ as before the possibility-operator is described as

$$\Diamond_p U = \{x \in X \mid \exists \text{ path } \alpha : [0, 1] \rightarrow B \text{ in } P \text{ and a lifting } \beta : [0, 1] \rightarrow X$$

$$\text{with } \beta(0) = x \text{ and } \beta(1) \in U \}.$$ 

This is indeed an open subset of $X$, since $\partial_0 : P \rightarrow B$ is assumed to be open.
In this way, every class of paths in a topos $\mathcal{E}$ provides $\mathcal{E}$ with tense-logical modal operators.

References