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A model for intuitionistic non-standard arithmetic

Ieke Moerdijk

Mathematisch Instituut, Universiteit Utrecht, P.O. Box 80.010, 3508 TA Utrecht, Netherlands

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Voor Dirk van Dalen ter gelegenheid van zijn zestigste verjaardag

Abstract

This paper provides an explicit description of a model for intuitionistic non-standard arithmetic, which can be formalized in a constructive metatheory without the axiom of choice.

0. Introduction

It is with great pleasure that I am responding to the invitation of the editors to write something on the occasion of Van Dalen's birthday. Unlike most other contributors to this volume, I am not a former Ph.D.-student of Van Dalen (but of the other half of the Dutch Constructive Conscience). Nonetheless Van Dalen has had a considerable influence, both directly and indirectly, on my mathematical development. I remember that back in 1980, as an undergraduate, I was disappointed in logic, and was thinking of shifting to topology. Then Van Dalen came along and gave a course at the University of Amsterdam on sheaves and their relation to logic (the first such course in Holland), and subsequently organised a stimulating seminar on the subject. A course of lectures on Kripke–Joyal semantics by Michael Fourman formed part of this seminar. I was immediately fascinated by the subject, and still am.

The case is typical. As another example (before my time) I might mention Van Dalen's arranging of a course of lectures by Smorynski on non-standard models of arithmetic at the University of Utrecht in 1978 (cf. [12]), which had a considerable impact on the Dutch logic community.

Last but not least, Van Dalen's invitation to Reyes to spend his sabbatical year at the University of Utrecht started a long-term collaboration resulting in [9].

This paper relates to Van Dalen's own work on forcing models for intuitionistic analysis [13], and at the same time builds on the three topics – sheaves and logic, non-standard arithmetic, and my work with Reyes, mentioned before. I will describe a sheaf model for intuitionistic non-standard arithmetic and analysis. The model is

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somewhat similar to the Basel topos discussed in [9], except that it concentrates on non-standard integers rather than non-standard (infinitesimal) reals. As a result, properties of ideals of smooth functions do not play a role here, and the model to be described is much simpler than the ones discussed in [9].

In this model, there is an extension N of the "standard" natural numbers $\mathbb{N}$, containing infinitely large integers. In an appropriate sense (see Proposition 2.2, Corollary 2.3 and Remark 2.7 below) these non-standard numbers form an elementary extension of the standard ones. On the other hand, the validity of the "overspill principle" (Proposition 2.6) provides an ample supply of infinitely large numbers.

The description of the model is purely constructive, as are the proofs of its basic properties (unless explicitly stated otherwise). In particular, ultrafilters do not play a role here.

For first-order arithmetic, the model can in principle be described concretely by an explicit forcing relation. Nonetheless I have chosen to present the model within the common framework of sheaves and Kripke–Joyal semantics. Thus it is possible to use general methods and results of the theory of sheaves and topoi (exposed, e.g. in [5, 6]). In particular, I might mention that the non-standard extension of arithmetic discussed here automatically extends to higher types, so as to give a conservative extension of the theory HHA (higher-order Heyting arithmetic). The fact that the construction of the model and the proofs of its properties are purely constructive implies that they can be done in HHA itself. In particular, the construction of the model can be performed inside (or, relative to) any other topos (or model of HHA). Using this method of relativization (which is reminiscent of iterated forcing in set theory) one automatically obtains variations of the non-standard model, in which e.g. Church's thesis for functions $N \to N$ or the continuity principle for functions $N^N \to N$ are valid.

This paper is a slightly revised version of the paper [8] which was submitted in October 1992. Since then, my attention has been drawn to other approaches to constructive non-standard arithmetic and its possible relevance for the theory of computing; see e.g. [7, 10, 11]. Dragalin [2] also considered the problem of constructively modelling non-standard arithmetic. A. Blass pointed out to me that the category of filters $\mathbb{F}$ on which my model is based occurs already in [4], and is discussed in detail in [1]. Blass also kindly drew my attention to an error in the earlier version [8]. In addition, I would like to thank J. van Oosten and E. Palmgren for helpful discussions.

1. A category of filters

We define a category $\mathbb{F}$ of filters on subsets of $\mathbb{N}^k$ ($k \geqslant 0$), where $\mathbb{N}$ denotes the set of natural numbers. An object of the category $\mathbb{F}$ is a pair $\underline{A} = (A, \mathscr{F})$, where $A \subseteq \mathbb{N}^k$ is any subset and $\mathscr{F}$ is a filter of subsets of A. For two such objects $\underline{A} = (A, \mathscr{F})$ and $\underline{B} = (B, \mathscr{G})$, a partial function $\underline{A} \to \underline{B}$ is a function $\alpha : F \to B$ defined on a member F of $\mathscr{F}$. Such a function α is said to be continuous if $\alpha^{-1}(G) \in \mathscr{F}$ for every $G \in \mathscr{G}$. Two continuous partial functions $\alpha : F \to B$ and $\alpha' : F' \to B$ are *equivalent* if α and α' coincide on some member $E \subseteq F \cap F'$ of $\mathscr{F}$. An arrow $\underline{A} \to \underline{B}$ in the category $\mathbb{F}$ is an equivalence class $[\alpha]$ of continuous partial functions. For two such arrows $[\alpha]: \underline{A} \to \underline{B}$ and $[\beta]: \underline{B} \to \underline{C} = (C, \mathscr{H})$, represented by continuous partial functions $\alpha: F \to B$ and $\beta: G \to C$, there is a well-defined composition $[\beta] \circ [\alpha]: \underline{A} \to \underline{C}$, represented by the partial function $\beta \circ \alpha: F \cap \alpha^{-1}(G) \to C$. This defines the category $\mathbb{F}$. In the sequel, we will usually denote the arrow $[\alpha]$, represented by a continuous partial function α as above, simply by α again.

For an object $\underline{A} = (A, \mathscr{F})$ in $\mathbb{F}$, any subset $U \subseteq A$ gives a new object $\underline{U} = (U, \mathscr{F} | U)$, where $\mathscr{F} | U$ is the filter $\{F \cap U \mid F \in \mathscr{F}\}$. There is an evident arrow $\underline{U} \to \underline{A}$, represented by the inclusion $U \subsetneq A$. Arrows of this form will be called *embeddings*. More generally, if $i: U \to A$ is any one-to-one function, and $i^{-1}(\mathscr{F})$ denotes the filter $\{i^{-1}(F) \mid F \in \mathscr{F}\}$, then $i: (U, i^{-1}(\mathscr{F})) \to (A, \mathscr{F})$ is also called an embedding. Note that this arrow i is an isomorphism in the category $\mathbb{F}$ whenever $i(U) \in \mathscr{F}$.

It is perhaps helpful to think of the category $\mathbb{F}$ (and similar categories to be discussed below) in a "geometric" way, as a category of map-germs. This can be made more precise by means of a functor c from $\mathbb{F}$ into the category *Germs* of pointed topological spaces and germs of continuous mappings. More explicitly, the objects of this category are pairs (X, x_0) where X is a topological space and $x_0 \in X$ is a basepoint; the arrows $(X, x_0) \to (Y, y_0)$ are equivalence classes of basepoint preserving continuous functions $f: U \to Y$, defined on some open neighbourhood U of x_0 in X; two such f and f' are equivalent if they agree on a sufficiently small neighbourhood of x_0. The equivalence class $[f]$ is called the germ of f at x_0. The functor

$$c: \mathbb{F} \to \mathit{Germs}$$

takes an object $(A, \mathscr{F})$ of $\mathbb{F}$ to the space $c(A, \mathscr{F}) = A \cup \{\infty\}$, where A is given the discrete topology while a fundamental system of neighbourhoods of ∞ in the space $c(A, \mathscr{F})$ consists of the sets $F \cup \{\infty\}$ where $F \in \mathscr{F}$. This functor c is full and faithful, and preserves much of the structure of the category $\mathbb{F}$ to be discussed.

Any subset $S \subseteq \mathbb{N}^k$ defines an object of $\mathbb{F}$, given by the set S and the trivial filter $\{S\}$; we will usually just write S for this object $(S, \{S\})$. Objects of this form will be called *simple objects* of $\mathbb{F}$.

We will now review some very elementary properties of the category $\mathbb{F}$. (For a more extensive discussion, see [1].) First, $\mathbb{F}$ has finite products: the terminal object is the simple object $1 = \{0\}$. For two objects $\underline{A} = (A, \mathscr{F})$ and $\underline{B} = (B, \mathscr{G})$, their product $\underline{A} \times \underline{B}$ is constructed as

$$\underline{A} \times \underline{B} = (A \times B, \mathscr{F} \times \mathscr{G}),$$

where $\mathscr{F} \times \mathscr{G}$ is the filter generated by the product sets $F \times G$, for all $F \in \mathscr{F}$ and all $G \in \mathscr{G}$. The pullback of two arrows $\alpha: \underline{A} \to \underline{B}$ and $\beta: \underline{C} \to \underline{B}$ is constructed similarly: if $\underline{C} = (C, \mathscr{H})$ as before, while $\alpha: F_0 \to B$ and $\beta: H_0 \to B$ are representing functions, then

$$\underline{A} \times_{\underline{B}} \underline{C} = (F_0 \times_B H_0, \mathscr{F} \times_B \mathscr{H}),$$

where $\mathscr{F} \times_B \mathscr{H}$ is the filter generated by all fibered products $F \times_B H$, for all $F \in \mathscr{F}$ with $F \subseteq F_0$, and all $H \in \mathscr{H}$ with $H \subseteq H_0$. From products and pullbacks one can construct equalizers. Explicitly, the equalizer of two arrows $\alpha, \beta: \underline{A} \to \underline{B}$, represented by $\alpha: F \to B$ and $\beta: F' \to B$, is the embedding $\underline{U} \subsetneq \underline{A}$ given by the set $U = \{x \in A \mid x \in F \cap F'$ and $\alpha x = \beta x\}$. This describes all finite limits in $\mathbb{F}$.

The category $\mathbb{F}$ also has all finite sums. The initial object 0 in $\mathbb{F}$ is the simple object given by the empty set. The sum of two objects $\underline{A} = (A, \mathscr{F})$ and $\underline{B} = (B, \mathscr{G})$ in $\mathbb{F}$ is constructed as

$$\underline{A} + \underline{B} = (A + B, \mathscr{F} + \mathscr{G}),$$

where $A + B$ is the disjoint sum and $\mathscr{F} + \mathscr{G} = \{F + G \mid F \in \mathscr{F}, G \in \mathscr{G}\}$.

An arrow $\alpha: (A, \mathscr{F}) \to (B, \mathscr{G})$ in the category $\mathbb{F}$ is said to be *covering* if $\alpha(F) \in \mathscr{G}$ for any $F \in \mathscr{F}$. Combining this with the continuity of α, we see that the map α is covering iff for all $G \subseteq B$:

$$G \in \mathscr{G} \Leftrightarrow \alpha^{-1}(G) \in \mathscr{F}.$$

Note also that the map $\alpha: (A, \mathscr{F}) \to (B, \mathscr{G})$ is covering in this sense iff it induces an open map germ $c(A, \mathscr{F}) \to c(B, \mathscr{G})$ between pointed spaces.

We now wish to introduce a Grothendieck topology on $\mathbb{F}$, for which the covering families of any given object $\underline{B}$ are all the finite families $\{\underline{A}_i \to \underline{B}\}_{i=1}^n$ with the property that the induced arrow $\underline{A}_1 + \cdots + \underline{A}_n \to \underline{B}$ is covering. The following lemma shows that these coverings satisfy the conditions for a Grothendieck topology.

Lemma 1.1. (i) *If $\underline{A} \to \underline{B}$ and $\underline{B} \to \underline{C}$ are coverings in $\mathbb{F}$ then so is their composition $\underline{A} \to \underline{C}$.*

(ii) *If $\underline{A} \to \underline{B}$ is a covering arrow in $\mathbb{F}$, then for any other arrow $\underline{C} \to \underline{B}$ the pullback $\underline{C} \times_{\underline{B}} \underline{A} \to \underline{C}$ is again covering.*

(iii) *For any family of arrows $\{\underline{A}_i \to \underline{B}\}_{i=1}^n$ and any arrow $\underline{C} \to \underline{B}$ in $\mathbb{F}$, the canonical arrow $(\underline{C} \times_{\underline{B}} \underline{A}_1) + \cdots + (\underline{C} \times_{\underline{B}} \underline{A}_n) \to \underline{C} \times_{\underline{B}} (\underline{A}_1 + \cdots + \underline{A}_n)$ is an isomorphism.*

Proof. Elementary verification, using the explicit description of pullbacks and sums given above. □

This lemma shows that the category of filters $\mathbb{F}$ is indeed equipped with a well-defined Grothendieck topology. The associated topos of sheaves will be denoted Sh($\mathbb{F}$). (It is a coherent topos. So by Deligne's theorem it has many ("enough") points; can these points be described explicitly?)

Lemma 1.2. *For any covering arrow $\alpha: \underline{A} \to \underline{B}$ in $\mathbb{F}$, the diagram*

$$\underline{A} \times_{\underline{B}} \underline{A} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \underline{A} \xrightarrow{\alpha} \underline{B}$$

is a coequalizer in $\mathbb{F}$.

(Note: arbitrary coequalizers cannot be constructed in $\mathbb{F}$; see [1].)

Proof. Write $\underline{A} = (A, \mathscr{F})$ and $\underline{B} = (B, \mathscr{G})$ as before, and suppose α is represented by a partial function $\alpha: F_0 \to B$ where $F_0 \in \mathscr{F}$. Then, as described above, the pullback is constructed as

$$\underline{A} \times_{\underline{B}} \underline{A} = (F_0 \times_B F_0, \mathscr{F} \times_B \mathscr{F}).$$

To see that the diagram in the statement of the lemma is a coequalizer, let $\gamma: \underline{A} \to \underline{C} = (C, \mathscr{H})$ be any arrow in $\mathbb{F}$ so that $\gamma\pi_1 = \gamma\pi_2$. Then we may assume that γ is represented by a function $F_1 \to C$ defined on a member $F_1 \subseteq F_0$ of $\mathscr{F}$, where F_1 is chosen so small that $\gamma\pi_1 = \gamma\pi_2$ on $F_1 \times_B F_1$. Write $G_1 = \alpha(F_1) \subseteq B$. Then

$$F_1 \times_B F_1 \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} F_1 \overset{\alpha}{\to} G_1$$

is a coequalizer in the category of sets. Therefore, there is a unique well-defined function $\delta: G_1 \to C$ so that $\delta \circ \alpha = \gamma: F_1 \to C$. Now observe that δ represents an arrow $\delta: \underline{A} \to \underline{C}$ in $\mathbb{F}$. Indeed, the domain $G_1 = \alpha(F_1)$ belongs to the filter $\mathscr{G}$ since $F_1 \in \mathscr{F}$ and α is covering; furthermore, δ is continuous, since for any $H \in \mathscr{H}$ we have $\alpha^{-1}\delta^{-1}(H) = \gamma^{-1}(H) \cap F_1 \in \mathscr{F}$, whence $\delta^{-1}(H) \in \mathscr{G}$ since α is covering.

To see that δ is unique with this property, suppose $\varepsilon: \underline{B} \to \underline{C}$ is any other arrow in $\mathbb{F}$ so that $\varepsilon \circ \alpha = \gamma$. Then there is an $F_2 \in \mathscr{F}$, chosen so small that (i) ε is represented by a function $F_2 \to C$, (ii) $\varepsilon(\alpha(x)) = \gamma(x)$ for all $x \in F_2$, and (iii) $F_2 \subseteq F_1$. But then $\varepsilon\alpha(x) = \delta\alpha(x)$ for all $x \in F_2$, or $\varepsilon(y) = \delta(y)$ for all $y \in \alpha(F_2)$. Since α is covering, $\alpha(F_2)$ belongs to $\mathscr{G}$, so $\varepsilon = \delta$ as arrows $\underline{B} \to \underline{C}$ in $\mathbb{F}$.

This proves the lemma. □

It follows from Lemma 1.2 that the Grothendieck topology on $\mathbb{F}$ is sub-canonical. This means that for any object $\underline{A}$ in $\mathbb{F}$, the representable functor

$$\mathbb{F}(-, \underline{A}): \mathbb{F}^{\mathrm{op}} \to \mathrm{Sets}$$

is a sheaf. In particular, if $S \subseteq \mathbb{N}^k$ is any subset, the corresponding simple object $S \in \mathbb{F}$ defines a "simple" sheaf. An important special case is the one where S is the set $\mathbb{N}$ of natural numbers. The corresponding simple sheaf is then denoted N. Thus, for any object $(A, \mathscr{F})$ of $\mathbb{F}$, the sheaf N is defined by

$$N(A, \mathscr{F}) = \mathbb{F}((A, \mathscr{F}), \mathbb{N}) = \varinjlim_{F \in \mathscr{F}} \mathrm{Hom}(F, \mathbb{N}), \tag{1}$$

where $\mathrm{Hom}(F, \mathbb{N})$ denotes the set of all functions from F into $\mathbb{N}$. It is this sheaf N that will occupy most of our attention in the next section.

The global sections functor

$$\Gamma: \mathrm{Sh}(\mathbb{F}) \to \mathrm{Sets}$$

is defined for each sheaf $\mathscr{E}$ on $\mathbb{F}$ by

$$\Gamma(\mathscr{E}) = \mathscr{E}(1),$$

where 1 is the terminal object of $\mathbb{F}$. As for any Grothendieck topos, this functor Γ has a left adjoint

$$\Delta : \mathrm{Sets} \to \mathrm{Sh}(\mathbb{F}).$$

To describe Δ explicitly, notice first that any set S gives rise to a "simple" sheaf $\hat{S}$, defined as

$$\hat{S}(A, \mathscr{F}) = \varinjlim_{F \in \mathscr{F}} \mathrm{Hom}(F, S)$$

(just as for the sheaf N above). Call a function $\varphi : F \to S$ *bounded* if $\varphi(F)$ is contained in a finite subset of S. Define $\Delta(S) \subseteq \hat{S}$ to be the subsheaf of $\hat{S}$ given by equivalence classes of bounded functions:

$$\Delta(S)(A, \mathscr{F}) = \varinjlim_{F \in \mathscr{F}} \mathrm{Hom}_{\mathrm{bounded}}(F, S).$$

This construction is clearly functorial in S. As a justification for the notation $\Delta(S)$, we state that the functor Δ thus defined is indeed left adjoint to Γ.

Lemma 1.3. *For any sheaf $\mathscr{E}$ and any set S there is a natural bijective correspondence*

$$\mathrm{Hom}(\Delta S), \mathscr{E}) \cong \mathrm{Hom}(S, \Gamma(\mathscr{E})).$$

The verification of this lemma is left to the reader.

In particular, as for any Grothendieck topos, the natural numbers object of $\mathrm{Sh}(\mathbb{F})$ is the sheaf $\Delta(\mathbb{N})$. It is a proper subsheaf of its "non-standard" extension N defined in (1) above.

2. Arithmetic in the topos of sheaves on $\mathbb{F}$

We will describe some elementary properties of the logic of the topos $\mathrm{Sh}(\mathbb{F})$ of sheaves on the filter-category $\mathbb{F}$. Any topos models a standard type theory with a type S for any object in the topos, and standard constructions of product types $S \times T$, function types T^S and power types $\mathscr{P}(S)$; the type of truth-values is then denoted Ω and the type of natural numbers by $\mathbb{N}$. For such a type theory associated with an arbitrary topos, the rules of higher-order intuitionistic arithmetic (sometimes referred to as HHA) are valid; this is discussed in detail in [5]. Truth in a particular Grothendieck topos such as $\mathrm{Sh}(\mathbb{F})$ can be calculated using sheaf semantics, defined in terms of a "Kripke–Joyal" forcing relation

$$\underline{A} \Vdash \varphi(\alpha_1, \ldots, \alpha_n).$$

Here $\underline{A}$ is an object of the site $\mathbb{F}$ of the topos, while $\varphi(x_1, \ldots, x_n)$ is a formula of the type theory associated to this topos $\mathrm{Sh}(\mathbb{F})$, with free variables x_i of sort S_i, and

$\alpha_i \in S_i(\underline{A})$ (or equivalently, $\alpha_i : \underline{A} \to S_i$ in Sh($\mathbb{F}$)). The rules defining this forcing relation are given in [6, Section VI.7].

For the topos Sh($\mathbb{F}$) there is a particular type N, given by the simple sheaf N corresponding to the set of natural numbers:

$$N(\underline{A}) = \mathrm{Hom}_{\mathbb{F}}(\underline{A}, \mathbb{N}).$$

There are evident arrows $0: 1 \to N$ and $+, \cdot : N \times N \to N$, as well as a successor $s: N \to N$, and an order relation $<$ on $N \times N$ (given by the evident simple subsheaf of $N \times N$). Thus N models the language of first-order Peano arithmetic. For a formula $\varphi(x_1, \dots, x_n)$ of arithmetic, and arrows $\alpha_1, \dots, \alpha_n : \underline{A} \to N$ in $\mathbb{F}$, define a new forcing relation $\Vdash_0$ by

$$\underline{A} \Vdash_0 \varphi(\alpha_1, \dots, \alpha_n) \text{ iff } \exists F \in \mathscr{F} \ \forall x \in F \varphi(\alpha_1(x), \dots, \alpha_n(x)). \tag{2}$$

Here on the right we suppose F to be chosen so small that the arrows $\alpha_1, \dots, \alpha_n$ are represented by continuous partial functions defined on F. Then it makes sense to require that $\varphi(\alpha_1(x), \dots, \alpha_n(x))$ is true for all $x \in F$, as expressed on the right of (2). Furthermore, this does not depend on the functions $\alpha_1, \dots, \alpha_n$ chosen to represent the given arrows $\alpha_1, \dots, \alpha_n$.

Lemma 2.1. *For any formula $\varphi(x_1, \dots, x_n)$ of first-order arithmetic, for any object $\underline{A}$ in $\mathbb{F}$ and any $\alpha_1, \dots, \alpha_n : \underline{A} \to N$ as above,*

$$\underline{A} \Vdash \varphi(\alpha_1, \dots, \alpha_n) \text{ iff } \underline{A} \Vdash_0 \varphi(\alpha_1, \dots, \alpha_n).$$

Proof. Induction on φ. For illustration, we treat the cases of implication and existential quantification. Also we take $n = 1$ for convenience.

Case of implication. Suppose the lemma holds for $\varphi(x)$ and $\psi(x)$, and choose any object $\underline{A} = (A, \mathscr{F})$ and any arrow $\alpha: \underline{A} \to N$ represented by a function $\alpha: F_0 \to \mathbb{N}$ where $F_0 \in \mathscr{F}$.

($\Rightarrow$) Assume $\underline{A} \Vdash \varphi(\alpha) \to \psi(\alpha)$. Let $F_1 = \{x \in A \mid x \in F_0$, and if $\varphi(\alpha(x))$ then also $\psi(\alpha(x))\}$. We claim that $F_1 \in \mathscr{F}$. To this and, let $G = \{x \in A \mid x \in F_0$ and $\varphi(\alpha(x))\}$, and let $\mathscr{F}_\varphi = \{F \cap G \mid F \in \mathscr{F}\}$. Then there is an evident arrow

$$i: (A, \mathscr{F}_\varphi) \to (A, \mathscr{F})$$

in $\mathbb{F}$, represented by the identity function on A. By induction hypothesis, the lemma holds for φ, so $(A, \mathscr{F}_\varphi) \Vdash \varphi(\alpha \circ i)$. Hence, since $\underline{A} \Vdash \varphi(\alpha) \to \psi(\alpha)$, also $(A, \mathscr{F}_\varphi) \Vdash \psi(\alpha \circ i)$. Again by induction hypothesis, this means that there exists an $H \in \mathscr{F}$ so that $\forall x \in H \cap G$: $\psi(\alpha(x))$; or by definition of G,

$$\forall x \in H \cap F_0 \text{: if } \varphi(\alpha(x)) \text{ then } \psi(\alpha(x)).$$

But then $H \cap F_0 \subseteq F_1$, so $F_1 \in \mathscr{F}$ as required.

($\Leftarrow$) Assume now $\underline{A} \Vdash_0 \varphi(\alpha) \to \psi(\alpha)$, so that F_1 (as defined above) belongs to $\mathscr{F}$. We must show that $\underline{A} \Vdash \varphi(\alpha) \to \psi(\alpha)$. To this end, let $\beta: \underline{B} \to \underline{A}$ be any arrow in $\mathbb{F}$ such that

$\underline{B} \Vdash \varphi(\alpha\beta)$. Write $\underline{B} = (B, \mathscr{G})$ and represent β by a function $\beta: G_1 \to A$ where $G_1 \in \mathscr{G}$. Using the induction hypothesis that the lemma holds for φ, we may choose G_1 so small that $\forall y \in G_1$: $\varphi(\alpha\beta(y))$. Furthermore, using continuity of β we may choose G_1 so small that $G_1 \subseteq \beta^{-1}(F_1)$; i.e.,

$$\forall y \in G_1: \text{ if } \varphi(\alpha\beta(y)) \text{ then } \psi(\alpha\beta(y)).$$

But then also $\forall y \in G_1$: $\psi(\alpha\beta(y))$. So, using the induction hypothesis for ψ, we conclude that $\underline{B} \Vdash \psi(\alpha\beta)$. This shows that $\underline{A} \Vdash \varphi(\alpha) \to \psi(\alpha)$, as desired.

Case of existential quantification. Suppose the lemma holds for $\varphi(x_1, x_2)$, and choose $\underline{A} = (A, \mathscr{F})$ and $\alpha: \underline{A} \to N$ represented by $\alpha: F_0 \to \mathbb{N}$, as before.

($\Rightarrow$) Suppose $\underline{A} \vdash \exists x_2 \varphi(\alpha, x_2)$. Then there exists a covering family $\{\beta_i: \underline{B} \to \underline{A}\}_{i=1}^n$ and arrows $\delta_i: \underline{B}_i \to \underline{A}$ such that $\underline{B}_i \Vdash \varphi(\alpha\beta_i, \delta_i)$, for each $i = 1, \ldots, n$. By induction hypothesis, this means that for each $\underline{B}_i = (B_i, \mathscr{G}_i)$ there is some $G_i \in \mathscr{G}_i$ so that

$$\forall y \in G_i: \ \varphi(\alpha\beta_i(y), \delta_i(y)).$$

(We assume that G_i is chosen so small that β_i and δ_i are represented by functions defined on G_i.) Since the $\beta_i: \underline{B}_i \to \underline{A}$ induce a covering $\underline{B}_1 + \cdots + \underline{B}_n \to \underline{A}$, we have

$$\beta_1(G_1) \cup \cdots \cup \beta_n(G_n) \in \mathscr{F}.$$

Thus also $K := F_0 \cap (\beta_1(G_1) \cup \cdots \cup \beta_n(G_n)) \in \mathscr{F}$. But if $x \in K$ then $x = \beta_i(y)$ for some i and some $y \in G_i$, whence $\varphi(\alpha(x), \delta_i(y))$. Thus $\forall x \in K \exists m \varphi(\alpha(x), m)$. This shows $\underline{A} \Vdash_0 \exists x_2 \varphi(\alpha, x_2)$.

($\Leftarrow$) For the converse, suppose now $\underline{A} \Vdash_0 \exists x_2 \varphi(\alpha, x_2)$. Then there is some $F_1 \subseteq F_0$ in $\mathscr{F}$ so that $\forall x \in F_1 \exists m \varphi(\alpha, m)$. Let $B = A \times \mathbb{N}$, and define a filter $\mathscr{G}$ on B generated by all sets of the form

$$\tilde{F} = \{(x, m) \mid x \in F \text{ and } \varphi(\alpha(x), m)\},$$

where F ranges over all sets $F \in \mathscr{F}$ for which $F \subseteq F_0$ (so that $\alpha(x)$ is defined). This defines an object $\underline{B} = (B, \mathscr{G})$ of $\mathbb{F}$, and the two projections from $A \times \mathbb{N}$ to A and $\mathbb{N}$ define arrows

$$\underline{A} \overset{\pi_1}{\leftarrow} \underline{B} \overset{\pi_2}{\rightarrow} N.$$

Clearly, by definition of the filter $\mathscr{G}$,

$$\underline{B} \Vdash_0 \varphi(\alpha \circ \pi_1, \pi_2).$$

Hence, since by induction hypothesis the lemma holds for φ, also

$$\underline{B} \Vdash \varphi(\alpha \circ \pi_1, \pi_2).$$

Furthermore, we claim that $\pi_1: \underline{B} \to \underline{A}$ is covering. Indeed, if $F \subseteq A$ is such that $\pi_1^{-1}(F) \in \mathscr{G}$, then there exists an $F' \in \mathscr{F}$ so that $F' \subseteq F_0$ and

$$\pi_1^{-1}(F) \supseteq \{(x, m) \mid x \in F' \text{ and } \varphi(\alpha(x), m)\}.$$

But $F_1 \in \mathscr{F}$ is such that $\forall x \in F_1 \exists m \varphi(\alpha(x), m)$, so $F' \cap F_1 \subseteq F$. This shows $F \in \mathscr{F}$. Hence $\pi_1 : \underline{B} \to \underline{A}$ is indeed covering, as claimed. It thus follows from $\underline{B} \Vdash \varphi(\alpha\pi_1, \pi_2)$ that $\underline{A} \Vdash \exists x_2 \varphi(\alpha, x_2)$.

This completes the case of the existential quantifier. □

From this lemma, one immediately obtains the following proposition.

Proposition 2.2. *For any formula* $\varphi(x_1, \dots, x_n)$ *of first-order arithmetic, the formula* $\forall x_1 \dots x_n \in N \varphi(x_1, \dots, x_n)$ *is valid in* Sh($\mathbb{F}$) *iff* $\varphi(m_1, \dots, m_n)$ *is true for all n-tuples of natural numbers* $m_1, \dots, m_n$.

In other words, the object N of Sh($\mathbb{F}$) models true arithmetic; in particular, N satisfies the Peano axioms. I wish to emphasize that the proof of Lemma 2.1 and (hence) that of Proposition 2.2 are completely constructive. In particular, the proof can be formalized within higher-order Heyting arithmetic (HHA).

In a suitable sense, Proposition 2.2 expresses that N is an elementary extension of the standard model $\mathbb{N}$ in Sets. To be more precise, we need a general definition. Consider an arbitrary first-order language L (in the present case L is the language of arithmetic).

Definition. For an L-structure M in a topos $\mathscr{E}$, and another one M' in a topos $\mathscr{E}'$, a *morphism* $(\mathscr{E}', M') \to (\mathscr{E}, M)$ is a pair consisting of a geometric morphism $p : \mathscr{E}' \to \mathscr{E}$ and a homomorphism of L-structures $h : p^*(M) \to M'$. Such a morphism (p, h) is an *elementary extension* if for any object E in $\mathscr{E}$, any L-formula $\varphi(x_1, \dots, x_n)$ with free variables as indicated, and any arrows $\alpha_1, \dots, \alpha_n : E \to M$ in $\mathscr{E}$,

$$E \Vdash \varphi(\alpha_1, \dots, \alpha_n) \text{ iff } p^*(E) \Vdash \varphi(h \circ p^*(\alpha_1), \dots, h \circ p^*(\alpha_n)).$$

(On the left, $\Vdash$ refers to the Kripke–Joyal semantics for $\mathscr{E}$, and on the right to that for $\mathscr{E}'$; cf. [6, Section VI.6].)

Notice that in case $\mathscr{E}$ = Sets and p is the identity, this reduces to the usual notion of elementary extension in model theory.

Returning to our example, the canonical morphism Sh($\mathbb{F}$) → Sets together with the inclusion $\Delta(\mathbb{N}) \subseteq N$, described at the end of the previous section, give a morphism (Sh($\mathbb{F}$), N) → (Sets, $\mathbb{N}$) into the "standard model" (Sets, $\mathbb{N}$). The following corollary is then essentially a reformulation of Proposition 2.2.

Corollary 2.3. *The canonical morphism* (Sh($\mathbb{F}$), N) → (*Sets*, $\mathbb{N}$) *is an elementary extension.*

Proposition 2.4. *In* Sh($\mathbb{F}$), *the following form of the axiom of choice is valid, for any formula* φ *of arithmetic*:

$$\forall z \in N [\forall x \in N \; \exists y \in N \varphi(x, y, z) \to \exists f \in N^N \; \forall x \in N \varphi(x, fx, z)].$$

Proof. Fix $\underline{A} = (A, \mathscr{F})$ and a parameter $\alpha: \underline{A} \to N$. Assume $\underline{A} \Vdash \forall x \in N \exists y \in N \varphi(x, y, z)$. Then $\underline{A} \times N \Vdash \exists y \in N \varphi(\pi_2, y, \alpha\pi_1)$. By Lemma 2.1, this means that there is an $F_0 \in \mathscr{F}$ such that for all $v \in F_0$ and for all $w \in \mathbb{N}$ it holds that $\exists y \in \mathbb{N} \varphi(w, y, \alpha(v))$. Pick any function $\beta: F_0 \times \mathbb{N} \to \mathbb{N}$ so that $\varphi(w, \beta(v, w), \alpha(v))$ holds for all $(v, w) \in F_0 \times \mathbb{N}$. Then β represents an arrow $\beta: \underline{A} \times N \to N$, or $\hat{\beta}: \underline{A} \to N^N$, and by Lemma 2.1 again, $\underline{A} \times N \vdash \varphi(\pi_2, \beta, \alpha\pi_1)$; or equivalently, $\underline{A} \times N \Vdash \varphi(\pi_2, \hat{\beta}(\pi_2), \alpha\pi_2)$. Since π_2 is "generic" [6, pp. 305, 317] this gives $\underline{A} \Vdash \forall x \in N \varphi(x, \hat{\beta}(x), \alpha)$, and the proposition is proved. □

Note that unlike the proof of Proposition 2.2, that of 2.4 is not entirely constructive: we assume that "externally" (in the metatheory) the axiom of choice holds for the formula $\forall x \exists y \varphi(x, y, \alpha(z))$, where φ is arithmetic but α is a possibly non-arithmetic (partial) function defined on a possibly non-arithmetic set. (However, see Section 3.)

We now compare these non-standard numbers N to the natural numbers object of the topos $\mathrm{Sh}(\mathbb{F})$. As is well known, in any Grothendieck topos the natural numbers object is given by the constant sheaf $\Delta\mathbb{N}$ corresponding to the set of natural numbers. Recall from Section 1 that for our topos $\mathrm{Sh}(\mathbb{F})$, the sheaf $\Delta\mathbb{N}$ has as value at an object $\underline{A} = (A, \mathscr{F})$ the set of equivalence classes of bounded functions $F \to \mathbb{N}$, where $F \in \mathscr{F}$. This sheaf $\Delta\mathbb{N}$ is a subsheaf of the representable sheaf N. Thus one can introduce a predicate "$\mathrm{St}(x)$" (for: x is a standard number) with a free variable x of type N, and define its interpretation in $\mathrm{Sh}(\mathbb{F})$ to be the subsheaf $\Delta\mathbb{N} \subseteq N$. Thus for $\alpha: \underline{A} \to N$,

$$\underline{A} \Vdash \mathrm{St}(\alpha) \quad \text{iff} \quad \exists F \in \mathscr{F}: \alpha(F) \text{ is bounded subset of } \mathbb{N}. \tag{3}$$

There is also a subsheaf $I \subseteq N$ of infinitely large natural numbers, defined as the representable sheaf given by the Fréchet filter $\mathscr{I}$, which has as a basis all the tails $[n, \infty) \subseteq \mathbb{N}$ for all $n \in \mathbb{N}$:

$$I = \mathbb{F}(-, (\mathbb{N}, \mathscr{I})).$$

So for $\alpha: \underline{A} \to N$ as before,

$$\underline{A} \Vdash \alpha \in I \quad \text{iff} \quad \forall n \in \mathbb{N} \exists F \in \mathscr{I} \forall x \in F: \alpha(x) \geq n. \tag{4}$$

Proposition 2.5. *The following formulas are valid in the topos* $\mathrm{Sh}(\mathbb{F})$:

(i) $\forall x \in N(x \in I \Leftrightarrow \neg \mathrm{St}(x)$ *and* $\forall x \in N(\mathrm{St}(x) \Leftrightarrow \neg(x \in I))$.

So I and St are each others pseudo-complement; in particular, they are both $\neg\neg$*-stable.*

(ii) $\neg \forall x \in N(x \in I \vee \mathrm{St}(x))$.

(iii) $\exists x \in N(x \in I)$.

(iv) $\forall x, y \in N(x \in I \wedge x < y \to y \in I)$.

(v) $\mathrm{St}(0) \wedge \forall x \in N(\mathrm{St}(x) \to \mathrm{St}(x + 1))$.

(vi) $\forall x, y \in \mathbb{N}(\mathrm{St}(x) \wedge y < x \to \mathrm{St}(y))$.

(vii) $\forall x, y(x \in I \wedge \mathrm{St}(y) \to y < x)$.

Proof. (i) Consider any object $\underline{A} = (A, \mathscr{F})$ of $\mathbb{F}$, and any arrow $\alpha: \underline{A} \to N$, represented by a function α which we may assume to be defined on all of A. Clearly we cannot have both $\underset{\sim}{A} \Vdash \alpha \in I$ and $\underset{\sim}{A} \Vdash \mathrm{St}(\alpha)$, unless $\emptyset \in \mathscr{F}$. Thus $\underset{\sim}{A} \Vdash \alpha \in I \rightarrow \neg\mathrm{St}(\alpha)$ and $\underset{\sim}{A} \Vdash \mathrm{St}(\alpha) \rightarrow \neg(\alpha \in I)$.

For the reverse implications, suppose first $\underset{\sim}{A} \Vdash \neg(\alpha \in I)$. Let $\bar{\mathscr{F}} \supseteq \mathscr{F}$ be the filter generated by all sets of the form

$$\mathscr{F}[n] = \{x \in F \mid \alpha(x) \geqslant n\},$$

where $F \in \mathscr{F}$ and $n \in \mathbb{N}$. This defines an object $(A, \bar{\mathscr{F}})$ of $\mathbb{F}$, together with an evident arrow $i:(A, \bar{\mathscr{F}}) \rightarrow (A, \mathscr{F})$, represented by the identity function on A. Now $(A, \bar{\mathscr{F}}) \Vdash \alpha \circ i \in I$, while by assumption $\underset{\sim}{A} \Vdash \neg(\alpha \in I)$, so $\emptyset \in \bar{\mathscr{F}}$. Thus there exists an $n_0 \in \mathbb{N}$ and an $F_0 \in \mathscr{F}$ so that $\{x \in F_0 \mid \alpha(x) \geqslant n_0\} = \emptyset$. Then $\alpha(F_0)$ is a bounded subset of $\mathbb{N}$, so $\underset{\sim}{A} \Vdash \mathrm{St}(\alpha)$.

Finally, suppose $\underset{\sim}{A} \Vdash \neg\mathrm{St}(\alpha)$. We must show $A \Vdash \alpha \in I$. To this end, pick $n \in \mathbb{N}$. Let $\mathscr{F}_n \supseteq \mathscr{F}$ be the filter generated by all sets of the form $\{x \in F \mid \alpha(x) < n\}$ where $F \in \mathscr{F}$. Let $j:(A, \mathscr{F}_n) \rightarrow (A, \mathscr{F})$ be the evident map. Then $(A, \mathscr{F}_n) \Vdash \mathrm{St}(\alpha \circ j)$. Since $\underset{\sim}{A} \Vdash \neg\mathrm{St}(\alpha)$, also $(A, \mathscr{F}_n) \Vdash \neg\mathrm{St}(\alpha \circ j)$. Therefore $\emptyset \in \mathscr{F}_n$. So there exists an $F \in \mathscr{F}$ such that $\{x \in F \mid \alpha(x) < n\} = \emptyset$. In other words, $\forall x \in F\ \alpha(x) \geqslant n$. Since $n \in \mathbb{N}$ was arbitrary, this shows $\underset{\sim}{A} \Vdash \alpha \in I$.

(ii) Suppose that $\underset{\sim}{A} = (A, \mathscr{F})$ is any object so that $\underset{\sim}{A} \Vdash \forall x \in N(x \in I \vee \mathrm{St}(x))$. We must show $\emptyset \in \mathscr{F}$. Consider the product $\underset{\sim}{A} \times N$, given by the set $A \times \mathbb{N}$ and the filter $\mathscr{F} \times \mathbb{N} = \{F \times \mathbb{N} \mid F \in \mathscr{F}\}$. Since $\underset{\sim}{A} \Vdash \forall x \in N(x \in I \vee \mathrm{St}(x))$ by assumption, we must have $\underset{\sim}{A} \times N \Vdash \pi_2 \in I \vee \mathrm{St}(\pi_2)$. It follows that there is a covering family

$$\{\beta_i : \underset{\sim}{B}_i \rightarrow \underset{\sim}{A} \times N\}_{i=1}^{n} \tag{5}$$

in $\mathbb{F}$ so that for each index i,

$$\underset{\sim}{B}_i \Vdash \pi_2\beta_i \in I \quad \text{or} \quad \underset{\sim}{B}_i \Vdash \mathrm{St}(\pi_2\beta_i). \tag{6}$$

Write $\underset{\sim}{B}_i = (B_i, \mathscr{G}_i)$, and assume β_i is represented by a function $\beta_i : B_i \rightarrow A \times \mathbb{N}$. The fact that (5) is a covering family then means that for any sets $G_i \in \mathscr{G}_i$ $(i = 1, \ldots, n)$:

$$\exists F \in \mathscr{F} \quad [F \times \mathbb{N} \subseteq \beta_1(G_1) \cup \cdots \cup \beta_n(G_n)]. \tag{7}$$

I will now give two arguments arriving at the desired conclusion. The first one is easy, but uses classical logic. The second argument is purely constructive, but somewhat involved.

For the classical argument, split the covering family (5) into two groups, say (after reindexing) that $\underset{\sim}{B}_i \Vdash \pi_2\beta_i \in I$ for $i = 1, \ldots, k_0$ and $\underset{\sim}{B}_i \Vdash \mathrm{St}(\pi_2\beta_i)$ for each $i = k_0 + 1, \ldots, n$. Then for $i > k_0$ there is a $G_i \in \mathscr{G}$ and a $b_i \in \mathbb{N}$ so that $\pi_2\beta_i(G_i) \subseteq [0, b_i]$. Let $b = \max\{b_i \mid i = k_0 + 1, \ldots, n\}$. Since $\underset{\sim}{B}_i \Vdash \pi_2\beta_i \in I$ for $i \leqslant k_0$ there are $G_i \in \mathscr{G}_i$ for $i = 1, \ldots, k_0$ so that $\pi_2\beta_i(G_i) \subseteq [b + 2, \infty)$. For this choice of $G_1, \ldots, G_n$ one thus has

$$\beta_1(G_1) \cup \cdots \cup \beta_n(G_n) \subseteq A \times (\mathbb{N} - \{b_{n+1}\}).$$

So if $F \in \mathscr{F}$ is such that $F \times \mathbb{N} \subseteq \beta_1(G_1) \cup \cdots \cup \beta_n(G_n)$, as in (7), then $F \times \{b_{n+1}\} = \emptyset$, hence $F = \emptyset$. Thus $\emptyset \in \mathscr{F}$ as desired.

This argument is non-constructive, since intuitionistically one cannot split the family (5) into two such groups – or, what comes down to the same problem – one cannot take the maximum $b = \max b_i$ when i ranges over a subfinite set, in case $\{i \mid \underline{B}_i \Vdash \mathrm{St}(\pi_2\beta_i)\}$.

A constructive proof goes as follows. Using (5) and (6), we show by induction on $j = 1, \ldots, n+1$ that

$$\exists k_j \forall G_j \in \mathscr{G}_j \ldots \forall G_n \in \mathscr{G}_n \forall l \exists F \in \mathscr{F}: \quad F \times [k_j, l] \subseteq \beta_j(G_j) \times \cdots \times \beta_n(G_n). \qquad (8_j)$$

For $j = 1$, we can take $k_j = 0$ and (8_j) is an evident consequence of (7). For the induction step suppose that (8_j) holds for some number k_j, and consider the following two cases (a) and (b):

(a) $B_j \Vdash \mathrm{St}(\pi_2\beta_j)$. Then for some $\bar{G}_j \in \mathscr{G}_j$ and some $b \in \mathbb{N}$ we have $\pi_2\beta_j(\bar{G}_j) \subseteq [0, b]$. Let $k_{j+1} = \max(b, k_j + 1)$. Then for any sets $G_{j+1} \in \mathscr{G}_{j+1}, \ldots, G_n \in \mathscr{G}_n$, the induction hypothesis (8_j) gives

$$\forall l \, \exists F \in \mathscr{F}: \quad F \times [k_{j+1}, l] \subseteq \beta_{j+1}(G_{j+1}) \cup \cdots \cup \beta_n(G_n).$$

(b) $\underline{B}_j \Vdash \pi_2\beta_j \in I$. In this case the same k_j that witnesses (8_j) also works for (8_{j+1}). Indeed, pick $G_{j+1} \in \mathscr{G}_{j+1}, \ldots, G_n \in \mathscr{G}_n$, and $l \in \mathbb{N}$. Since $\underline{B}_j \Vdash \pi_2\beta_j \in I$ we can furthermore find a $\bar{G}_j \in \mathscr{G}_j$ so that $\beta_j(\bar{G}_j) \subseteq [l+1, \infty)$. Now by induction hypothesis (8_j),

$$\exists F \in \mathscr{F}: \quad F \times [k_j, l] \subseteq \beta_j(\bar{G}_j) \cup \beta_{j+1}(G_{j+1}) \cup \cdots \cup \beta_n(G_n).$$

Since $\beta_j(\bar{G}_j) \cap F \times [k_j, l] = \emptyset$, then also

$$\exists F \in \mathscr{F}: \quad F \times [k_j, l] \subseteq \beta_{j+1}(G_{j+1}) \cup \cdots \cup \beta_n(G_n).$$

This completes the induction. At the final stage $j = n+1$ at which we arrive, (8_j) states

$$\exists k \forall l \; \exists F \in \mathscr{F} \quad F \times [k, l] = \emptyset.$$

For $l = k$ this gives the desired conclusion $\emptyset \in \mathscr{F}$. This completes the (second, constructive) proof of part (ii).

(iii) It will be enough to show that $1 \Vdash \exists x \in N(x \in I)$. Consider the object $(\mathbb{N}, \mathscr{T})$ of $\mathbb{F}$ representing the sheaf I. There is an evident map $i: (\mathbb{N}, \mathscr{T}) \to (\mathbb{N}, \{\mathbb{N}\}) = N$ represented by the identity function on $\mathbb{N}$. Clearly

$$(\mathbb{N}, \mathscr{T}) \Vdash i \in I. \qquad (9)$$

Furthermore, the unique arrow $(\mathbb{N}, \mathscr{T}) \to 1$ forms a singleton covering family. Hence from (9) we conclude $1 \Vdash \exists x \in N(x \in I)$.

I will omit the easy proofs of parts (iv)–(vii) of the Proposition. □

Note that it follows from Proposition 2.5 that the new predicate St is not arithmetically definable (for then by (v) and induction it would follow $\forall x \in N \mathrm{St}(x)$ is valid, contradicting (iii)).

Proposition 2.6. *The following "overspill" principle is valid in the topos* Sh($\mathbb{F}$), *for any arithmetical formula* φ:

$$\forall x \in N[\forall y \in N(\mathrm{St}(y) \to \varphi(x, y)) \to \exists z \in I \forall y \in N(y < z \to \varphi(x, y))].$$

Proof. Suppose $\alpha: \underline{A} \to N$ is such that $\underline{A} \Vdash \forall y \in N(\mathrm{St}(y)) \to \varphi(\alpha, y))$. Write $\varphi'(x, y)$ for $\forall w < y \varphi(x, w)$. By Proposition 2.5 (vi), also $\underline{A} \Vdash \forall y \in N(\mathrm{St}(y) \to \varphi'(\alpha, y))$. In particular $\underline{A} \Vdash \varphi'(\alpha, \bar{n})$ for each $n \in \mathbb{N}$, and the corresponding constant function $\bar{n}: \underline{A} \to N$. Therefore by Proposition 2.2

$$\forall n \in \mathbb{N} \exists F \in \mathscr{F} \forall x \in F \varphi'(\alpha(x), n). \tag{$*$}$$

For each $n \geqslant 0$ write

$$G_n = \{(x, i) \in A \times \mathbb{N} \mid i \geqslant n \text{ and } \varphi'(\alpha(x), i)\}.$$

Consider the filter $\mathscr{G}$ on $A \times \mathbb{N}$ generated by all $F \times \mathbb{N}$ with $F \in \mathscr{F}$, and by all $G_n (n \geqslant 0)$. Write

$$\underline{B} = (B, \mathscr{G}) = (A \times \mathbb{N}, \mathscr{G}).$$

Then there are maps

$$N \overset{\pi_2}{\leftarrow} \underline{B} \overset{\pi_1}{\rightarrow} \underline{A}$$

in $\mathbb{F}$, and one readily verifies (using ($*$)) that π_1 is a covering map. Furthermore, $\underline{B} \Vdash \pi_2 \in I$ and $\underline{B} \Vdash \varphi'(\alpha\pi_1, \pi_2)$. Hence $\underline{A} \Vdash \exists z \in I \varphi'(\alpha, z)$, as required. □

Remark 2.7. So far each arithmetical formula $\varphi(x_1, \ldots, x_n)$ has been interpreted as a formula of the language of the topos Sh($\mathbb{F}$), by interpreting all variables and quantifiers as ranging over the representable sheaf N. Of course φ can also be interpreted as a formula for the standard natural numbers object $\Delta\mathbb{N}$, or equivalently, one may consider the restricted formula

$$\varphi^{St}(x_1, \ldots, x_n)$$

obtained from φ by replacing all quantifiers $\forall x \in N$ and $\exists x \in N$ over the representable sheaf N by their restrictions $\forall x \in N(\mathrm{St}(x) \to \ldots)$ and $\exists x \in N(\mathrm{St}(x) \wedge \ldots)$ to standard numbers. In the topos Sh($\mathbb{F}$), the following formula is classically valid, for any formula $\varphi(x_1, \ldots, x_n)$ of first-order arithmetic:

$$\forall x_1 \ldots x_n \in N(\mathrm{St}(x_1) \wedge \cdots \wedge \mathrm{St}(x_n) \to (\varphi^{St}(x_1, \ldots, x_n) \Leftrightarrow \varphi(x_1, \ldots, x_n))) \tag{10}$$

This expresses that in the topos Sh($\mathbb{F}$), the standard numbers $\Delta\mathbb{N}$ form an "elementary submodel" of the non-standard ones N. However, unlike Corollary 2.3, the proof that (10) holds for all arithmetic φ uses classical logic. It would be of interest to describe explicitly the (or a) class of arithmetical formulas φ for which (10) holds constructively, and compare this to results of Palmgren [11].

3. Variations: Church's thesis and continuity

The topos described in Section 1 is the simplest example of an entire class of similar topoi, with arithmetical properties of the kind described in Section 2. In this section we will briefly outline the construction of some of these topoi.

There are two evident types of variation on the construction of the topos Sh($\mathbb{F}$). Firstly, one may study sites analogous to $\mathbb{F}$, and their sheaves. Secondly, one may construct sheaves on the site $\mathbb{F}$ relative to some base topos other than the topos of sets, such as Hyland's effective topos [3] or the free topos [5]. For this second type of variation, it is important that the treatment of Sh($\mathbb{F}$) given so far is constructive. A combination of these two types of variation is also worth investigating.

An evident variation on the site $\mathbb{F}$ is to consider a minimal site for which the properties presented in Section 2 still hold. For this, let $\mathbb{F}_{ar} \subseteq \mathbb{F}$ be the subsite given by arithmetically definable objects and functions: An object $\underline{A} = (A, \mathscr{F})$ belongs by definition to the smaller site $\mathbb{F}_{ar}$ if A is an arithmetically definable subset of $\mathbb{N}^k$ ($k \geqslant 0$) and $\mathscr{F}$ has a basis $\{F_n\}$ which is arithmetically definable. An arrow $\underline{A} \to \underline{B} = (B, \mathscr{G})$ between two such objects of $\mathbb{F}_{ar}$ is an equivalence class of arithmetically definable partial functions $F \to B$ defined on some $F \in \mathscr{F}$, and continuous as in Section 1. The topology on $\mathbb{F}_{ar}$ is similar to the one on $\mathbb{F}$, and is again subcanonical.

The properties stated for Sh($\mathbb{F}$) in Lemma 2.1, Proposition 2.2, Corollary 2.3, Propositions 2.4–2.6 are all valid in Sh($\mathbb{F}_{ar}$) also. Furthermore, the proof of Proposition 2.4 for Sh($\mathbb{F}_{ar}$) only requires the axiom AC-NN of choice for arithmetical formulas.

One can construct this site $\mathbb{F}_{ar}$ inside the effective topos Eff, to obtain a new topos Eff[$\mathbb{F}_{ar}$] of internal sheaves on $\mathbb{F}_{ar}$. This gives a new model $N \in$ Eff[$\mathbb{F}_{ar}$] of arithmetic. Validity in this model can be described as an iteration of Kleene realizability and Kripke–Joyal forcing over the site $\mathbb{F}_{ar}$. Since the proofs of properties 2.1–2.6 only use principles (such as AC-NN) which are all valid in Eff, one obtains the following result.

Proposition 3.1. *For the topos* Eff($\mathbb{F}_{ar}$) *the properties, stated for* Sh($\mathbb{F}$) *in Propositions* 2.2, 2.4–2.6, *are all valid, provided that in Proposition* 2.2 *one reads "Kleene-realizable" for "true"; in addition, Church's thesis*

$$\forall f \in N^N \exists x \in N \forall y \in N \quad f(y) = \{x\}(y)$$

is valid in Eff($\mathbb{F}_{ar}$).

As another variation one can construct a site $\mathbb{C}$ analogous to that of $\mathbb{F}$, by replacing the natural numbers by the Baire space $\mathbb{B} = \mathbb{N}^{\mathbb{N}}$. An object of the site $\mathbb{C}$ is then a pair $\underline{A} = (A, \mathscr{F})$ where A is a closed subset of some product $\mathbb{B}^k$, and $\mathscr{F}$ is a filter of closed subsets of A. Arrows between two such objects $(A, \mathscr{F}) \to (B, \mathscr{G})$ are equivalence classes of continuous mappings $f: F \to B$ defined or some $F \in \mathscr{F}$, and with the property that $f^{-1}(G) \in \mathscr{F}$ for any $G \in \mathscr{G}$. Call such a map $f: (A, \mathscr{F}) \to (B, \mathscr{G})$ a covering map if $f(F) \in \mathscr{G}$, if $f: F \twoheadrightarrow f(F)$ is an open surjection, and if moreover for any $G \subseteq B$ it holds that $G \in \mathscr{G}$ whenever $f^{-1}(G) \in \mathscr{F}$. Define a Grothendieck topology on $\mathbb{C}$ by taking as covering families those finite families $(B_i, \mathscr{G}_i) \to (A, \mathscr{F})$ for which $\sum (B_i, \mathscr{G}_i) \to (A, \mathscr{F})$ is a covering map. This topology is subcanonical. The set of natural numbers may be considered as an object $(\mathbb{N}, \{\mathbb{N}\})$ of $\mathbb{C}$, and gives rise to a representable sheaf $N \in \mathrm{Sh}(\mathbb{C})$.

Proposition 3.2. *For the object N of* $\mathrm{Sh}(\mathbb{C})$, *the properties* 2.1–2.6 *are all valid. Moreover, the statement expressing that (internally) all functions $N^N \to N^N$ are continuous, is also valid in* $\mathrm{Sh}(\mathbb{C})$.

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