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CLASSIFYING SPACES FOR TOPOSES WITH ENOUGH POINTS

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ABSTRACT. This paper provides an introduction to and a survey of recent work with C. Butz. The central construction is that of a “classifying space” for any Grothendieck topos with enough points. It is proved that this space has the same cohomology and homotopy groups as the topos. The construction also has applications in mathematical logic, where it yields new topological completeness theorems.

This paper is an extended version of a lecture I gave at the “Seminario Matematico e Fisico” in Milan. I would first of all like to thank the organisers of the Seminario for inviting me to present the lecture as well as this extended written version.

Like the lecture, this paper mainly describes some recent joint work with C. Butz [BM1,2] on the relation between Grothendieck toposes and topological spaces. Our construction applies to toposes satisfying a certain technical condition, viz. that of “having enough points”. This condition is quite innocent, however, since it is satisfied by most toposes arising in geometric practice [D].

For any topos $\mathcal{E}$ satisfying this condition, we construct a topological space $X_{\mathcal{E}}$ having the same cohomology as $\mathcal{E}$ (Theorem 1 below). We also show how $\mathcal{E}$ can be completely described in terms of equivariant sheaves on this space (Theorems 2 and 3). The description given here of the space $X_{\mathcal{E}}$ can be thought of as a very general classifying space construction. Indeed, in some concrete cases, it is related to more standard classifying spaces. For example, if $\mathcal{E}$ is the topos of presheaves on

a small category $\mathbb{C}$, the space $X_{\mathcal{E}}$ is (weakly) homotopy equivalent to the usual classifying space $B\mathbb{C}$. I would like to point out that, although the construction of the space $X_{\mathcal{E}}$ applies to a much more general context, the properties of this space are in general less convenient than those of more standard classifying spaces like $B\mathbb{C}$. I hope that further work will show that for certain classes of topoi, the space $X_{\mathcal{E}}$ can be replaced by a naturally defined cell complex.

The properties of the construction of $X_{\mathcal{E}}$ from $\mathcal{E}$ also have applications to topological models in logic. Some recent work in this direction [AB, BM3] is described at the end of this paper.

1. Preliminary definitions

Recall [SGA4, MM, ...] that a *topos* is (a category equivalent to) the category of sheaves on a small site. More explicitly, a *site* is a pair $(\mathbb{C}, J)$, where $\mathbb{C}$ is a small category with pullbacks, while J assigns to each object C in $\mathbb{C}$ a collection $J(\mathbb{C})$ of “covering families” satisfying certain axioms:

- (i) The singleton family $\{C \xrightarrow{id} C\}$ belongs to $J(C)$.
- (ii) If $\{C_i \rightarrow C\}$ belongs to $J(C)$ then for any arrow $D \rightarrow C$ the pullback family $\{C_i \times_C D \rightarrow D\}_i$ belongs to $J(D)$.
- (iii) If $\{C_i \rightarrow C\}_i$ belongs to $J(C)$ and, for each i , the family $\{D_{ij} \rightarrow C_i\}_j$ belongs to $J(C_i)$, then the family $\{D_{ij} \rightarrow C_i \rightarrow C\}_{i,j}$ of all compositions belongs to $J(C)$.

(There is also a version of these axioms which applies to the cases where $\mathbb{C}$ does not necessarily have pullbacks.)

A *sheaf* S on such a site $(\mathbb{C}, J)$ is a functor $S : \mathbb{C}^{op} \rightarrow \mathbf{Sets}$, such that for any covering family $\{C_i \rightarrow C\}$ in $J(C)$ the sequence

$$S(C) \rightarrow \prod_i S(C_i) \rightrightarrows \prod_{i,j} S(C_i \times_C C_j) \quad (1)$$

is an equalizer. The category (topos) of all these sheaves, and all natural transformations between them, is denoted $Sh(\mathbb{C}, J)$.

The leading example is of course that of sheaves on a topological space (for a space X , the topos of sheaves is denoted $Sh(X)$). Other important sites and their sheaves arise in algebraic geometry, e.g. the étale site on a scheme (see e.g. [Mi]).

It should be pointed out immediately that many categories of sheaves which are in fact toposes are most naturally described in terms other than these sheaves on sites. This applies in particular to the following example.

Recall that a *groupoid* $\mathcal{G}$ is a (small) category all of whose arrows are invertible. Thus, $\mathcal{G}$ consists of a set G_0 of objects, a set G_1 of arrows, and structure maps

$$G_1 \times_{G_0} G_1 \xrightarrow{\circ} G_1 \xrightleftharpoons[d_0]{d_1} G_0 \xrightarrow{u} G_1 \xrightarrow{i} G_1.$$

(d_0, d_1 for domain and codomain, u for identities or units, i for inverse, and $\circ$ for composition). A *topological groupoid* is a small groupoid for which G_0 and G_1 are each equipped with a topology, making all structure maps continuous. For example, if a topological group G acts continuously on a space X , one can form a groupoid $X \times G$ with X as space of objects, and $X \times G$ as space of arrows: each pair (x, g) defines an arrow $x \rightarrow g \cdot x$. For many examples and uses of topological groupoids, see [B,C,M,W] and references cited there.

For a general topological groupoid $\mathcal{G}$, a $\mathcal{G}$ -*sheaf* is a sheaf on G_0 equipped with a (right) action by $\mathcal{G}$. If we write the sheaf as an étale space $p : E \rightarrow G_0$, the action is a continuous map $E \times_{G_0} G_1 \rightarrow E$, assigning to each $e \in E_x$ and $g : y \rightarrow x$ an element $e \cdot g \in E_y$, and satisfying the usual identities for an action. The category of all such $\mathcal{G}$ -sheaves and action preserving maps between them is a topos, denoted $Sh(\mathcal{G})$. In the special case where $\mathcal{G} = X \times G$ as above, $Sh(\mathcal{G})$ is the category of all G -equivariant sheaves on X , and is also denoted $Sh_G(X)$.

Going back to general toposes, we recall that a *morphism* $f : \mathcal{F} \rightarrow \mathcal{E}$ between toposes $\mathcal{E}$ and $\mathcal{F}$ is a functor $f^* : \mathcal{E} \rightarrow \mathcal{F}$ (in the opposite direction!) which commutes with finite limits and arbitrary colimits. For example, for a groupoid $X \times G$, the functor “forget the action”

$Sh_G(X) \rightarrow Sh(X)$ defines a morphism

$$q : Sh(X) \rightarrow Sh_G(X). \quad (2)$$

An similarly, forgetting the action defines for any topological groupoid $\mathcal{G}$ a morphism

$$q : Sh(G_0) \rightarrow Sh(\mathcal{G}). \quad (3)$$

These morphisms in fact exhibit $Sh(\mathcal{G})$ and $Sh_G(X)$ as “generalized” (i.e., topos theoretic) orbit spaces of G_0 and X .

Among toposes, the role of the *point* is played by the topos of sets (this is the category of sheaves on the one-point space). Thus, it is natural to define a *point* of a topos $\mathcal{E}$ as a morphism

$$p : \mathbf{Sets} \rightarrow \mathcal{E}.$$

One says that the topos $\mathcal{E}$ has “*enough points*” if the collection of functors $p^* : \mathcal{E} \rightarrow \mathbf{Sets}$, for all points p , is jointly faithful. This means that a map $E \rightarrow E'$ between objects of $\mathcal{E}$ (i.e., between sheaves) is an isomorphism iff, for each point p , the map of sets $p^*(E')$ is an isomorphism.

Obviously, the topos $Sh(X)$ of sheaves on a topological space X has enough points. Since the functors q^* for the “quotient” morphisms q in (2) and (3) are faithful, $Sh_G(X)$ and $Sh(\mathcal{G})$ have enough points as well.

A topos $\mathcal{E}$ is said to be *coherent* if it can be described as the category of sheaves on a site $(\mathbf{C}, J)$ for which $\mathbf{C}$ has (pullbacks and) a terminal object and all covering families are finite (i.e., *members* of $J(C)$ are finite for each object C , not $J(C)$ itself). Many toposes arising naturally in algebraic geometry are coherent. A classical result due to Deligne [D] states that any coherent topos has enough points.

Coherent toposes also arise in logic, as classifying toposes of finitary geometric theories ([MR,TT,MM]). In fact, any coherent topos is the classifying topos for such a theory. A similar relation between arbitrary toposes and infinitary theories will be described and used in the next section.

2. Approximation of toposes by topological spaces

In this section we will present several results which express in different ways how a given topos $\mathcal{E}$ can be approximated by a topos of sheaves on a topological space X – this can be the topos of all sheaves, or a more restricted topos of equivariant sheaves.

To describe the topological spaces involved, it will be convenient to use the theory of classifying toposes. We briefly recall the main aspects of this theory, and refer the reader to the books cited at the end of the previous section for a detailed exposition. For any topos $\mathcal{E}$ one can find a “theory” (in the sense of first order logic) $\mathbb{T}_{\mathcal{E}}$ such that $\mathcal{E}$ is a “classifying topos” for $\mathbb{T}_{\mathcal{E}}$; this means that the points p of $\mathcal{E}$ correspond exactly to models M_p of this theory $\mathbb{T}_{\mathcal{E}}$. (For example, the Zariski topos over a ground field k is a classifying topos for the theory of local k -algebras, while the topos of simplicial sets is a classifying topos for linear orders.) More generally, for any other topos $\mathcal{F}$, morphisms $\mathcal{F} \rightarrow \mathcal{E}$ correspond to models of $\mathbb{T}_{\mathcal{E}}$ in $\mathcal{F}$. This applies in particular to the topos $Sh(Y)$ of sheaves on a space Y , so as to give an equivalence of categories, between morphisms $f : Sh(Y) \rightarrow \mathcal{E}$ on the one hand and sheaves of $\mathbb{T}_{\mathcal{E}}$ -models on Y on the other:

$$Sh(Y) \rightarrow \mathcal{E} \Leftrightarrow \text{sheaf of } \mathbb{T}_{\mathcal{E}}\text{-models on } Y. \quad (4)$$

The axioms of the theory $\mathbb{T}_{\mathcal{E}}$ can be expressed in a special, so-called geometric, form, and may contain infinite disjunctions. The size of the formal language needed to formulate the theory $\mathbb{T}_{\mathcal{E}}$ depends directly on the size of a given site for $\mathcal{E}$. The theory $\mathbb{T}_{\mathcal{E}}$ satisfies a version of the “downward” Löwenheim-Skolem theorem which implies that, in order to check derivability in the theory, it suffices to consider models of a bounded size, say of cardinality at most κ . This property is related to the fact that, if a topos $\mathcal{E}$ has enough points, there is always a small set of points $p : \mathbf{Sets} \rightarrow \mathcal{E}$ which is already jointly faithful. (Note that both the collections of all objects of $\mathcal{E}$ and that of all points of $\mathcal{E}$ are in general proper classes.)

So, for a given topos $\mathcal{E}$ with enough points, let us fix such an infinite cardinal κ so that the set of those points $p : \mathbf{Sets} \rightarrow \mathcal{E}$ for which the

corresponding model M_p of $\mathbb{T}_{\mathcal{E}}$ has cardinality $\leq \kappa$ is jointly faithful. We will think of κ as a set, viz. of all ordinals strictly less than κ , as usual.

Next, define an *enumerated model* to be a pair (M, α) where M is a model of $\mathbb{T}_{\mathcal{E}}$ and $\alpha : D \rightarrow M$ is a map defined on a subset $D \subseteq \kappa$, with the property that $\alpha^{-1}(m)$ is infinite for every $m \in M$ (we will call such an α an “infinite-to-one” map). Two enumerated models (M, α) and (N, β) are *equivalent* if there exists an isomorphism of $\mathbb{T}_{\mathcal{E}}$ -models $\theta : M \rightarrow N$ with $\beta\theta = \alpha$ (in particular, α and β must have the same domain D). Below, we will often simply write (M, α) when it is clear that the equivalence class of (M, α) is meant.

Let $X_{\mathcal{E}}$ be the set of all equivalence classes of enumerated models. This set $X_{\mathcal{E}}$ carries a natural topology: Each formula $\varphi(x_1, \dots, x_n)$ which is a conjunction of atomic formulas, and each sequence $\xi = (\xi_1, \dots, \xi_n)$ of ordinals $\xi_i < \kappa$, together define a basic open set

$$V_{\varphi, \xi} = \{(M, \alpha) \mid \xi_1, \dots, \xi_n \in \text{domain}(\alpha) \\ \text{and } \varphi(\alpha(\xi_1), \dots, \alpha(\xi_n)) \text{ holds in } M\}.$$

This completes the definition of the topological space $X_{\mathcal{E}}$ about which we already spoke in the introduction.

The enumerated models form a natural sheaf $\mathcal{M}$ on this space $X_{\mathcal{E}}$, with as stalk at the point $x = (M, \alpha)$ the underlying set of the model M . In other words, $\mathcal{M}$ is the space of all equivalence classes of triples (M, α, m) with $m \in M$ and $(M, \alpha) \in X_{\mathcal{E}}$. Here two triples (M, α, m) and (N, β, n) are equivalent if (M, α) is equivalent to (N, β) via an isomorphism $\theta : M \rightarrow N$ as above, such that $\theta(m) = n$. The topology on $\mathcal{M}$ is defined by the basic open sets

$$W_{\varphi, \xi, \eta} = \{(M, \alpha, m) \mid (M, \alpha) \in V_{\varphi, \xi}, m = \alpha(\eta)\},$$

for φ, ξ as above and any $\eta < \kappa$. For this topology, the natural projection $\pi : \mathcal{M} \rightarrow X_{\mathcal{E}}$ is a local homeomorphism, so that $\mathcal{M}$ is indeed a sheaf on X . The fiber of $\pi : \mathcal{M} \rightarrow X_{\mathcal{E}}$ over a point (M, α) is the $\mathbb{T}_{\mathcal{E}}$ -model M , and $\mathcal{M}$ is in fact a sheaf of $\mathbb{T}_{\mathcal{E}}$ -models. By the correspondence (4) above, $\mathcal{M}$ thus defines a morphism of toposes

$$f : Sh(X_{\mathcal{E}}) \rightarrow \mathcal{E}. \quad (5)$$

As an example, consider the topos $(G\text{-sets})$ of sets with an action by a fixed *finite* group G . This topos classifies G -torsors; in particular, for any topological space Y the correspondence (4) specializes to an equivalence between morphisms $Sh(Y) \rightarrow (G\text{-sets})$ and G -torsors over Y (i.e, covering projections with group G). Thus X consists of pairs (M, α) where M is a set with a free and transitive G -action and $\alpha : D \rightarrow M$ is an infinite-to-one map from a subset $D \subseteq \mathbb{N}$ of the natural numbers. Up to isomorphism, M is G itself, acting on itself by (left) multiplication. Thus, the space $X_{(G\text{-sets})}$ can be inscribed as the orbit space of the space $\tilde{X}_{(G\text{-sets})}$ of all infinite-to-one maps $\alpha : D \rightarrow G$ (where $D \subseteq \mathbb{N}$), equipped with the action by G defined by pointwise right multiplication: $(\alpha \circ g)(d) = \alpha(d) \cdot g$. The topology on $\tilde{X}_{(G\text{-sets})}$ has as subbasic open sets all sets of the form

$$W_{n,g} = \{(\alpha : D \rightarrow G) \mid n \in D, \alpha(n) = g\},$$

and $X_{(G\text{-sets})}$ has the quotient topology. It can be shown that $\tilde{X}_{(G\text{-sets})}$ is a contractible space with a free and proper G -action (so $\tilde{X} \rightarrow X$ is a covering projection). Thus $X_{(G\text{-sets})}$ is a model of the classifying space BG and $\tilde{X}_{(G\text{-sets})}$ of the universal bundle EG .

Much more generally, for any topos $\mathcal{E}$ with enough points the space $X_{\mathcal{E}}$ always approximates $\mathcal{E}$ from the cohomological point of view:

THEOREM 1 ([BM1]) – *For any topos $\mathcal{E}$ with enough points, the morphism $f : Sh(X_{\mathcal{E}}) \rightarrow \mathcal{E}$ induces an isomorphism in cohomology*

$$H^*(\mathcal{E}, A) \xrightarrow{\sim} H^*(X_{\mathcal{E}}, f^*A)$$

for any abelian group A in $\mathcal{E}$.

Another way to express essentially the same result is that $f^* : \mathcal{E} \rightarrow Sh(X_{\mathcal{E}})$ induces a full and faithful functor $D^+(\mathcal{E}) \rightarrow D^+(X_{\mathcal{E}})$ at the level of derived categories.

We now consider how f^* maps $\mathcal{E}$ itself, rather than the derived category, to the sheaves on $X_{\mathcal{E}}$.

PROPOSITION – *The functor $f^* : \mathcal{E} \rightarrow Sh(X_{\mathcal{E}})$ is full and faithful, and preserves the operations of first order logic (i.e. f is open), as well as exponentials (internal homs).*

One can obtain stronger results concerning the presentation of the topos structure of $\mathcal{E}$ if one allows the action of a group or groupoid on the space $X_{\mathcal{E}}$.

First, notice that the group $\pi = \text{Aut}(\kappa)$ of permutations of κ acts continuously on $X_{\mathcal{E}}$, in the obvious way:

$$a \cdot (M, \alpha) = (M, \alpha \circ a^{-1}), \quad \text{for any } a \in \pi.$$

Let $Sh_{\pi}(X_{\mathcal{E}})$ be the corresponding topos of equivariant sheaves. It is not difficult to see that the morphism $f : Sh(X_{\mathcal{E}}) \rightarrow \mathcal{E}$ can be factored through the quotient morphism $q : Sh(X_{\mathcal{E}}) \rightarrow Sh_{\pi}(X_{\mathcal{E}})$ (cf. (2)), to give a map

$$g : Sh_{\pi}(X_{\mathcal{E}}) \rightarrow \mathcal{E}.$$

Indeed, here the theory of classifying topoi is useful again. The required factorization can be thought of as being given by a natural isomorphism

$$\rho_a : f^* \rightarrow \tilde{a}^* \circ f^*$$

for any automorphism $a \in \text{Aut}(\kappa)$ and the induced operation $\tilde{a} : X_{\mathcal{E}} \rightarrow X_{\mathcal{E}}$ (satisfying a multiplicativity property relating ρ_{ab} to ρ_a and ρ_b). By the equivalence (4), such an isomorphism ρ_a corresponds to an isomorphism of models $\rho_a : M \rightarrow \tilde{a}^*(M)$ on $X_{\mathcal{E}}$. For a point $x = (M, \alpha)$ of $X_{\mathcal{E}}$, the stalks of the two sheaves involved are the same: $M_x = M$ and $\tilde{a}^*(M)_x = M_{\tilde{a}(x)} = M$, and the stalk of the required automorphism $(\rho_a)_x : M \rightarrow M$ is defined to be the identity. (Thus, this action witnesses in some way that the sheaf M is constant on each subspace of all points (M, α) where M is kept fixed but α varies.)

THEOREM 2 — *The morphism $g : Sh_{\pi}(X_{\mathcal{E}}) \rightarrow \mathcal{E}$ has the property that g^* is a full and faithful embedding which preserves all the “elementary” topos structure of $\mathcal{E}$ (limits, colimits, exponentials, sub-object classifier).*

It is even possible to obtain an equivalence of categories, at the cost of replacing the group π by a groupoid $\mathcal{G}$ with $X_{\mathcal{E}}$ as space G_0 of objects. In this groupoid, an arrow from a point (M, α) to another

point (M, β) is an isomorphism $h : M \rightarrow N$ of $\mathbf{T}_{\mathcal{E}}$ -models. (We do *not* require that $\beta h = \alpha$.) More precisely, since points of $X_{\mathcal{E}}$ are really equivalence classes of such pairs (M, α) , the arrows are also defined as suitable equivalence classes. We stipulate that if (M, α) is equivalent to (M', α') via $\theta : M \rightarrow M'$ (an isomorphism such that $\alpha\theta = \alpha'$) and (N, β) is similarly equivalent to (N', β') via τ , then $h : (M, \alpha) \rightarrow (N, \beta)$ is equivalent to $\tau h \theta^{-1} : (M', \alpha') \rightarrow (N', \beta')$.

There is a natural topology on the space G_1 of all these arrows, for which the basic open sets are of the form

$$V_{\varphi, \xi, \psi, \varsigma} = \{(M, \alpha \xrightarrow{h} (N, \rho) \mid (M, \alpha) \in V_{\varphi, \xi}, (N, \beta) \in V_{\psi, \varsigma}, \text{ and} \\ h(\alpha(\xi_i)) = \beta(\varsigma_i), \quad i = 1, \dots, n\}.$$

Here, as before, $\varphi(x_1, \dots, x_n)$ and $\psi(y_1, \dots, y_n)$ are conjunctions of atomic formulas from the language defining $\mathbf{T}_{\mathcal{E}}$, while $\xi = \xi_1, \dots, \xi_n$ and $\varsigma = \varsigma_1, \dots, \varsigma_n$ are two lists of ordinals $< \kappa$.

For this topology on the space G_1 , the domain and codomain maps $d_0, d_1 : G_1 \rightrightarrows G_0 = X_{\mathcal{E}}$ are continuous, as are all the other structure maps for the groupoid (unit, inverse, composition).

THEOREM 3 ([BM2]) – *The morphism $f : Sh(X_{\mathcal{E}}) \rightarrow \mathcal{E}$ induces an equivalence between $\mathcal{E}$ and the topos $Sh(\mathcal{G})$ of equivariant sheaves for the topological groupoid $\mathcal{G}$.*

This result is proved in detail in [BM2]. Here, we just note that $f^* : \mathcal{E} \rightarrow Sh(X_{\mathcal{E}})$ indeed maps into the category of G -equivariant sheaves. As for Theorem 2, this can be seen by observing that the two sheaves of models $d_0^*(\mathcal{M})$ and $d_1^*(\mathcal{M})$ are isomorphic, by an isomorphism defined explicitly in terms of the arrows of the groupoid. Indeed, at a point $g = [g : (M, \alpha) \rightarrow (N, \beta)]$ in G_1 , the stalk of $d_0^*(\mathcal{M})$ is $\mathcal{M}$ and that of $d_1^*(\mathcal{M})$ is N , and g itself defines the required isomorphism $d_0^*(\mathcal{M})_g \rightarrow d_1^*(\mathcal{M})_g$.

3. Topological models of classical logic

In this section two applications to logic are described.

Let $\mathbf{T}$ be a theory in ordinary, classical predicate logic, formulated in a first order language L . For any topological space Y , one can define in a standard way [MM] the notion of a topological (sheaf) model of $\mathbf{T}$ over the space Y . The universe of such a model is a sheaf $\mathcal{S}$ on Y ; each n -ary relation symbol is interpreted as a subsheaf of $\mathcal{S} \times \dots \times \mathcal{S} = \mathcal{S}^n$, and each n -ary function symbol as a morphism $\mathcal{S}^n \rightarrow \mathcal{S}$. One can then inductively define a subsheaf $[[\varphi(x_1, \dots, x_n)]] \subseteq \mathcal{S}^n$ for each formula $\varphi(x_1, \dots, x_n)$ with free variables among $x_1, \dots, x_n$, using the operations $\wedge, \vee, \Rightarrow, \neg$ on subsheaves. To be a model of $\mathbf{T}$, one furthermore requires that $\mathbf{T}$ -provable formulas are true, in the sense that $[[\varphi(x_1, \dots, x_n)]] = \mathcal{S}^n$ whenever $\mathbf{T} \vdash \forall x_1 \dots \forall x_n \varphi(x_1, \dots, x_n)$. Note that, in particular, it follows that each such subsheaf $[[\varphi(x_1, \dots, x_n)]]$ is *complemented*.

This notion of model can be extended to Henkin type theories [H], where one has not just one universe but several “types” $S, T, \dots$, and for each two types S and T a product type $S \times T$ and a function type T^S . Moreover there is a basic type B of Boolean truth values. There are axioms relating these types, such as a function abstraction axiom relating functional relations $\varphi(x, y)$ on $S \times T$ to elements of type T^S , etc. A (*standard*) *topological model* $\mathcal{M}$ of such a type theory consists of a sheaf $\mathcal{S} = \mathcal{M}(S)$ which interprets the type S , for each type S ; this interpretation has to commute with the product and exponential of sheaves: $\mathcal{M}(S \times T) \simeq \mathcal{M}(S) \times \mathcal{M}(T)$ and $\mathcal{M}(T^S) \simeq \mathcal{M}(T)^{\mathcal{M}(S)}$. Moreover, B is interpreted as the constant sheaf $2 = 1 + 1$. Relations, functions symbols and more general formulas are then interpreted just as for first order theories. Furthermore, all $\mathbf{T}$ -provable formulas should be true, as before.

COMPLETENESS THEOREM ([AB]) – *For any type theory $\mathbf{T}$ there exists a standard topological model $\mathcal{M}$ over a space Y such that*

- (i) (*completeness*) *A sentence φ is provable in $\mathbf{T}$ iff $[[\varphi]] = Y$;*
- (ii) (*functional completeness*) *Any morphism $\mathcal{M}(S) \rightarrow \mathcal{M}(T)$ between interpretations of types is definable by a provably functional relation.*

It is possible to sketch the idea of the proof on the basis of the results of the previous section. Let $\mathbb{T}$ be the given type theory. We can view $\mathbb{T}$ (somewhat artificially) as a first order many sorted (geometric) theory, so that it has a classifying topos $\mathcal{E}$. Consider the space $X_{\mathcal{E}}$ and the morphism $f : Sh(X_{\mathcal{E}}) \rightarrow \mathcal{E}$ of the Proposition in Section 2. The universal model $\mathcal{U}$ of $\mathbb{T}$ in $\mathcal{E}$ pulls back to a model $\mathcal{M} = f^*(\mathcal{U})$ of $\mathbb{T}$ in $X_{\mathcal{E}}$. Since the universal model is known to have the properties of the theorem, and f^* preserves these properties by the Proposition, the model $\mathcal{M}$ still has the properties (i), (ii). As observed by Awodey and Butz, the points of the space Y can be viewed as classical Henkin models.

The definability of functions as stated in part (ii) can be extended to arbitrary subsheaves if one takes a group action into account.

DEFINABILITY THEOREM ([BM3]) – *For $\mathbb{T}, Y$ and $\mathcal{M}$ as in the completeness theorem, there exists a group π , acting on the space Y as well as on the model $\mathcal{M}$, such that, for any type S and any π -invariant complemented subsheaf $A \subseteq \mathcal{M}(S)$, there exists a formula $\varphi(x)$ with free variable of type S such that $A = [[\varphi(x)]]$.*

This theorem is proved in [BM3] for first order theories, but the proof also applies to type theories. As for the completeness theorem, it is again based on the construction of the classifying topos $\mathcal{E}$ for $\mathbb{T}$, and the morphism $f : Sh(X_{\mathcal{E}}) \rightarrow \mathcal{E}$, and can be outlined as follows: for the group π as in Theorem 2, the morphism $g : Sh_{\pi}(X_{\mathcal{E}}) \rightarrow \mathcal{E}$ preserves all topos structure. It follows that the fully faithful functor g^* is also full on subobjects. Thus, the given complemented π -invariant subsheaf $A \subseteq \mathcal{U}(S)$ must be of the form $g^*(R)$ for a complemented subsheaf $R \subset \mathcal{U}(S)$ in the universal model. Since $\mathcal{U}(S)$ is coherent and R is complemented, R is again coherent, hence definable.

For more details, the interested reader is referred to the references [AB] and [BM3].

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