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## MINIMAL MODELS OF HEYTING ARITHMETIC

IEKE MOERDIJK AND ERIK PALMGREN

**§1. Introduction.** In this paper, we give a constructive nonstandard model of intuitionistic arithmetic (Heyting arithmetic). We present two axiomatisations of the model: one finitary and one infinitary variant. Using the model these axiomatisations are proven to be conservative over ordinary intuitionistic arithmetic. The definition of the model along with the proofs of its properties may be carried out within a constructive and predicative metatheory (such as Martin-Löf's type theory). This paper gives an illustration of the use of sheaf semantics to obtain effective proof-theoretic results.

The axiomatisations of nonstandard intuitionistic arithmetic (to be called **HAI** and **HAI<sub>ω</sub>** respectively) as well as their model are based on the construction in [5] of a sheaf model for arithmetic using a site of filters. In this paper we present a "minimal" version of this model, built instead on a suitable site of provable filter bases. The construction of this site can be viewed as an extension of the well-known construction of the classifying topos for a geometric theory which uses "syntactic sites". (Such sites can in fact be used to prove semantical completeness of first order logic in a strictly constructive framework, see [6].)

We should mention that for classical nonstandard arithmetics there are several nonconstructive methods of proving conservativity over arithmetic, e.g. the compactness theorem, Mac Dowell–Specker's theorem [3]. A constructive conservation result was obtained by Dragalin [2] for the classical version of **HAI**. His argument used a boolean model construction. Alternatively, this can be proved by formalising definable ultrapowers in classical arithmetic. Coquand and Smith [1] proved conservativity of a weaker theory of nonstandard arithmetic (essentially our **HAI** without axioms (IV) and (VII)).

The paper is organised as follows. In Section 2 we present the nonstandard arithmetic and some of its extensions and properties. As a prelude to the central results we give in Section 3 a universal model of intuitionistic arithmetic using a full subcategory of the site of filter bases. The filter base model is defined and studied in Section 4.

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**§2. The nonstandard arithmetic.** The theory **HAI** is an extension of intuitionistic arithmetical **HA**. We expand the *arithmetical language* of **HA** with the unary predicate **St** and denote the resulting language  $L'$ . The intended meaning of  $\text{St}(x)$  is that  $x$  is a standard number. The seven axiom groups of **HAI** are:

- (I) The axioms of **HA** in the arithmetical language.
- (II)  $\text{St}(x) \wedge x = y \rightarrow \text{St}(y)$ .
- (III)  $\text{St}(x_1) \wedge \dots \wedge \text{St}(x_n) \rightarrow \text{St}(f(x_1, \dots, x_n))$ , for any function symbol  $f$  of **HA**.
- (IV)  $\neg\neg \text{St}(x) \rightarrow \text{St}(x)$ .
- (V)  $\exists x \neg \text{St}(x)$ .
- (VI) *The external induction schema.* For any formula  $A(\bar{x}, y)$  of the expanded language  $L'$  we assume

$$A(\bar{x}, 0) \wedge \forall^{\text{st}} y [A(\bar{x}, y) \rightarrow A(\bar{x}, s(y))] \rightarrow \forall^{\text{st}} y A(\bar{x}, y),$$

where  $\forall^{\text{st}} y \dots$  is shorthand for  $\forall x (\text{St}(x) \rightarrow \dots)$ .

- (VII) *The overspill principle.* For an arithmetical formula  $A(\bar{x}, y)$ :

$$\forall^{\text{st}} y A(\bar{x}, y) \rightarrow \exists y \neg \text{St}(y) \wedge (\forall u < y) A(\bar{x}, u).$$

From (I), (II) and (VI) it follows that the standard numbers constitute an initial segment:

$$(1) \quad \forall xy [\text{St}(y) \wedge x < y \rightarrow \text{St}(x)].$$

In a classical version the overspill principle would be redundant since it then follows from (1) and the induction schema of (I). The theory **HAI** is in fact consistent with a non-classical axiom

$$(NC) \quad \neg \forall x (\text{St}(x) \vee \neg \text{St}(x))$$

(see Theorem 4.10). The model to be constructed validates the following infinitary rule

$$(\omega\text{-rule}) \quad \frac{\Gamma(\bar{x}) \vdash A(\bar{x}, n) \quad \text{for all numerals } n}{\Gamma(\bar{x}) \vdash \forall^{\text{st}} y A(\bar{x}, y)}.$$

The infinitary calculus, **HAI** $_{\omega}$ , obtained by to **HAI** adding this rule will still be conservative over **HA** (Corollary 4.11).

One might also try to extend **HAI** with the following *transfer principle*

$$(TR) \quad \forall^{\text{st}} \bar{x} [A^{\text{st}}(\bar{x}) \longleftrightarrow A(\bar{x})]$$

for any arithmetical formula  $A(\bar{x})$ , where  $A^{\text{st}}$  denotes the formula resulting when restricting the quantifiers of  $A$  to **St**. However we have

**PROPOSITION 2.1.** **HAI** + (TR) *proves the principle of excluded middle for all arithmetical formulas.*

**PROOF.** We prove  $A \vee \neg A$  for arithmetical  $A$  by induction on the number of quantifiers. For quantifier free  $A$  this is immediate, since  $=$  is decidable. Suppose  $\forall x \forall \bar{y} (A(x, \bar{y}) \vee \neg A(x, \bar{y}))$ . By induction on  $z$  we obtain for all  $\bar{y}$

$$(2) \quad \forall z [(\exists x < z) A(x, \bar{y}) \vee (\forall x < z) \neg A(x, \bar{y})],$$

$$(3) \quad \forall z [(\exists x < z) \neg A(x, \bar{y}) \vee (\forall x < z) A(x, \bar{y})].$$

Let  $\bar{y}$  be standard numbers and let  $z$  be infinite. From (2) then follows  $\exists x A(x, \bar{y}) \vee \forall^{\text{st}} x \neg A(x, \bar{y})$ . By the transfer principle,  $\exists^{\text{st}} x A^{\text{st}}(x, \bar{y}) \vee \forall^{\text{st}} x \neg A^{\text{st}}(x, \bar{y})$ . Since  $\bar{y}$  were arbitrary, we have again by transfer and logic,

$$\forall \bar{y} [\exists x A(x, \bar{y}) \vee \neg \exists x A(x, \bar{y})].$$

Analogously, we obtain from (3)

$$\forall \bar{y} [\forall x A(x, \bar{y}) \vee \neg \forall x A(x, \bar{y})]. \quad \dashv$$

REMARKS 2.2. By inspecting the proof we see that assuming transfer for arithmetical  $\exists$ - $\wedge$ -formulas implies the principle of excluded middle for the very same formulas. However, the theory  $\mathbf{iHA}^\omega$  of [8] has such a restricted transfer principle, but induction only on standard natural numbers. This answers a question in [5, Remark 2.7].

A natural question is whether transfer holds in the model if we start out with Peano arithmetic ( $\mathbf{PA}$ ) instead of  $\mathbf{HA}$ . Remark 4.12 below answers the question in the negative.

**§3. The minimal model of arithmetic.** We construct a syntactic site from the formulas of arithmetic and then we build a universal (and minimal) model of  $\mathbf{HA}$  using sheaves over this site. For sequences of variables  $\bar{x} = x_1, \dots, x_n$  and  $\bar{y} = y_1, \dots, y_n$ , we let  $\bar{x} = \bar{y}$  abbreviate the formula  $x_1 = y_1 \wedge \dots \wedge x_n = y_n$ . As usual we write  $\varphi(\bar{x})$  when the free variables of  $\varphi$  are among  $\bar{x}$ .

DEFINITION 3.1. *The syntactic category S for HA.* The objects of the category are pairs  $(\varphi, \bar{x})$  where  $\varphi(\bar{x})$  is an arithmetical formula. The morphisms (or arrows) from  $(\varphi, \bar{x})$  to  $(\psi, \bar{y})$  are triples  $(\bar{u}; \theta; \bar{v})$  such that  $\theta(\bar{u}, \bar{v})$  is an arithmetical formula which is *provably functional in HA*:

- (a)  $\mathbf{HA} \vdash \forall \bar{s} [\varphi(\bar{s}) \rightarrow \exists \bar{w} \psi(\bar{w}) \wedge \theta(\bar{s}, \bar{w})]$ ,
- (b)  $\mathbf{HA} \vdash \forall \bar{s} \bar{w} \bar{z} [\varphi(\bar{s}) \wedge \theta(\bar{s}, \bar{w}) \wedge \theta(\bar{s}, \bar{z}) \rightarrow \bar{w} = \bar{z}]$ .

It is assumed that the substitutions are always done with fresh variables (computable from the arrows and objects). We use  $\varphi(\bar{x})$  as alternate notation for  $(\varphi, \bar{x})$ , and  $\theta(\bar{u}; \bar{v})$  for  $(\bar{u}; \theta; \bar{v})$ . Two such arrows  $\theta(\bar{u}; \bar{v})$  and  $\theta'(\bar{u}'; \bar{v}')$  are equal if

$$\mathbf{HA} \vdash \forall \bar{s} [\varphi(\bar{s}) \rightarrow \forall \bar{w} (\theta(\bar{s}, \bar{w}) \leftrightarrow \theta'(\bar{s}, \bar{w}))].$$

The identity arrow on  $(\varphi, \bar{x})$  is given by  $(\bar{x}; \bar{x} = \bar{y}; \bar{y})$ , where the sequences  $\bar{x}$  and  $\bar{y}$  have no common variable. The composition of arrows  $\theta(\bar{u}; \bar{v})$  and  $\rho(\bar{w}; \bar{s})$  is

$$(\bar{x}; \exists \bar{z} \theta(\bar{x}, \bar{z}) \wedge \rho(\bar{z}, \bar{y}); \bar{y}).$$

LEMMA 3.2. *The category S has pullbacks and terminal object, i.e., all finite limits.*

PROOF. This is essentially straightforward. A terminal object is, e.g.,  $(x = 0, x)$ . The construction of the pullback object of the arrows  $\theta(\bar{x}; \bar{y}) : \varphi(\bar{x}) \rightarrow \psi(\bar{y})$  and  $\rho(\bar{u}; \bar{y}) : \gamma(\bar{u}) \rightarrow \psi(\bar{y})$  is similar to the set-theoretic case:  $(\bar{s}, \bar{w}, \varphi(\bar{s}) \wedge \gamma(\bar{w}) \wedge \exists \bar{z} \theta(\bar{s}, \bar{z}) \wedge \rho(\bar{w}, \bar{z}))$ . \dashv

We define the site on S by giving its base of covering families [4, Definition III.2.2].

DEFINITION 3.3. The *syntactic site*  $S = (S, J)$  is given by the following base  $J$ . The family of morphisms  $\langle \alpha_i(\bar{x}_i, \bar{y}) : \varphi_i(\bar{x}_i) \rightarrow \psi(\bar{y}) \rangle_{i=1}^k$  is a *covering of  $\psi(\bar{y})$*  if

$$\mathbf{HA} \vdash \forall \bar{u} \left[ \psi(\bar{u}) \rightarrow \bigvee_{i=1}^k \exists \bar{x}_i \varphi_i(\bar{x}_i) \wedge \alpha_i(\bar{x}_i, \bar{u}) \right].$$

If an arrow by itself forms a covering family, it is called a *covering arrow*.

The following lemma provides plenty of sheaves over this site.

LEMMA 3.4. *The site  $(S, J)$  is subcanonical, i.e., every representable presheaf  $\text{Hom}_S(-, (\psi, \bar{y}))$  over  $S$  is a sheaf.*

PROOF. We omit the straightforward proof, which is analogous to that of Theorem 4.3 below. ⊥

We are now ready to define the *universal sheaf model* of  $\mathbf{HA}$  denoted  $\llbracket N \rrbracket_S$ .

- The type of natural numbers is interpreted as the representable sheaf

$$\llbracket N \rrbracket = \text{Hom}_S(-, (z = z, z)).$$

- The constant 0 is interpreted by the natural transformation  $1 \rightarrow \llbracket N \rrbracket$  given by  $\llbracket 0 \rrbracket_{(\varphi, \bar{x})}(\alpha) = (\bar{x}; z = 0; z)$ , where  $z \notin \bar{x}$ .

- A function symbol  $f$  of type  $N^n \rightarrow N$  is interpreted by a natural transformation  $\llbracket f \rrbracket : \llbracket N \rrbracket^n \rightarrow \llbracket N \rrbracket$ . It is defined by letting, for  $\alpha_1(\bar{x}_1; z_1), \dots, \alpha_n(\bar{x}_n; z_n) \in \llbracket N \rrbracket(\varphi, \bar{x})$ ,

$$\llbracket f \rrbracket_{(\varphi, \bar{x})}(\alpha_1(\bar{x}_1; z_1), \dots, \alpha_n(\bar{x}_n; z_n)) = \left( \bar{u}; \exists \bar{v} \bigwedge_{i=1}^n \alpha_i(\bar{u}, v_i) \wedge f(\bar{v}) = z; z \right)$$

where  $\bar{u}, z$  and  $\bar{v} = v_1, \dots, v_n$  are freshly chosen variables.

- For the equality relation define for  $\alpha(\bar{x}_1, z_1), \gamma(\bar{x}_2, z_2) \in \llbracket N \rrbracket(\varphi, \bar{x})$

$$\begin{aligned} \llbracket = \rrbracket_{(\varphi, \bar{x})}(\alpha(\bar{x}_1, z_1), \gamma(\bar{x}_2, z_2)) \\ \iff \mathbf{HA} \vdash \forall \bar{x} [\varphi(\bar{x}) \rightarrow \exists yz \alpha(\bar{x}, y) \wedge \gamma(\bar{x}, z) \wedge y = z]. \end{aligned}$$

This is indeed an  $(S, J)$ -relation, i.e.,  $\llbracket = \rrbracket$  is monotone and it has the *cover property*: if  $\langle \beta_i : \varphi_i(\bar{x}_i) \rightarrow \psi(\bar{y}) \rangle_{i=1}^k$  is a cover and  $\llbracket = \rrbracket_{(\varphi_i, \bar{x}_i)}(\alpha \circ \beta_i, \gamma \circ \beta_i)$  holds for all  $i = 1, \dots, k$ , then already  $\llbracket = \rrbracket_{(\varphi, \bar{y})}(\alpha, \gamma)$ .

It is straightforward to check that this constitutes a model. Let  $\Vdash$  be the forcing relation associated with the model. We will usually write  $\psi(\bar{x}) \Vdash A(\alpha)$  instead of the formally correct  $(\psi, \bar{x}) \Vdash A((\bar{x}; \alpha; y))$  since the relevant data can be read off from the shorter notation.

THEOREM 3.5. *Let  $A(\bar{y})$  be an arithmetical formula, where  $\bar{y} = y_1, \dots, y_n$ . Then for parameters  $\alpha_1(\bar{x}_1, z_1), \dots, \alpha_n(\bar{x}_n, z_n) \in \llbracket N \rrbracket(\psi, \bar{x})$  we have*

$$\psi(\bar{x}) \Vdash A(\alpha_1, \dots, \alpha_n) \iff \mathbf{HA} \vdash \forall \bar{u} \bar{y} \left[ \psi(\bar{u}) \wedge \bigwedge_{i=1}^n \alpha_i(\bar{u}, y_i) \rightarrow A(y_1, \dots, y_n) \right].$$

**PROOF.** By induction on the complexity of  $A$ . We do two illustrative cases.

Case  $A = \perp$ : By definition  $\psi(\bar{x}) \Vdash \perp(\alpha_1, \dots, \alpha_n)$  if, and only if,  $\psi(\bar{x})$  has an empty cover, i.e.,  $\mathbf{HA} \vdash \forall \bar{x} [\psi(\bar{x}) \rightarrow \perp]$ . Since each  $\alpha_i$  is functional on  $\psi(\bar{x})$  this is equivalent to

$$\mathbf{HA} \vdash \forall \bar{u}\bar{y} [\psi(\bar{u}) \wedge \bar{\alpha}(\bar{u}, \bar{y}) \rightarrow \perp],$$

where  $\bar{\alpha}(\bar{u}, \bar{y}) = \bigwedge_{i=1}^n \alpha_i(\bar{u}, y_i)$ .

Case  $A(y_1, \dots, y_n) = \exists z B(y_1, \dots, y_n, z)$ : We prove only the  $(\Leftarrow)$ -direction. Suppose

$$(4) \quad \mathbf{HA} \vdash \forall \bar{u}\bar{y} [\psi(\bar{u}) \wedge \bar{\alpha}(\bar{u}, \bar{y}) \rightarrow A(\bar{y})].$$

Let  $\psi'(\bar{u}, z) = \psi(\bar{u}) \wedge \exists \bar{y} \bar{\alpha}(\bar{u}, \bar{y}) \wedge B(\bar{y}, z)$ . By (4), the projection  $\pi_1 = (\bar{u}, z; \bar{u} = \bar{v}; \bar{v}) : \psi'(\bar{u}, z) \rightarrow \psi(\bar{x})$  is a covering arrow. Define a second projection  $\pi_2 : (\psi', \bar{u}, z) \rightarrow (w = w, w)$  by  $\eta = (\bar{u}, z; z = w; w)$ . It follows trivially that

$$\mathbf{HA} \vdash \forall \bar{u}z \bar{v}\bar{y} w [\psi'(\bar{u}, z) \wedge \bar{u} = \bar{v} \wedge \bar{\alpha}(\bar{v}, \bar{y}) \wedge z = w \rightarrow B(\bar{y}, w)].$$

So by the inductive hypothesis,

$$\psi'(\bar{u}, z) \Vdash B(\alpha_1 \circ \pi_1, \dots, \alpha_n \circ \pi_1, \pi_2).$$

Since  $\pi_1$  is a covering arrow we have by definition

$$\psi(\bar{u}) \Vdash A(\alpha_1, \dots, \alpha_n). \quad \dashv$$

**COROLLARY 3.6 (Completeness Theorem).** *A closed arithmetical formula is true in the model  $\llbracket N \rrbracket_S$  exactly when it is provable in  $\mathbf{HA}$ .*

**PROOF.** This follows from the previous theorem, by considering the equivalence at a terminal object in  $S$ .  $\dashv$

**REMARK 3.7.** The above completeness proof is entirely constructive and can be formalised in a predicative metatheory. It goes through for any first order theory which has at least one closed term. For more details see [6].

We recall the notion of elementary embedding between first order structures living in two different topoi, introduced in [5]. Let  $\mathcal{M}$  be an  $L$ -structure in  $\mathcal{E}$ , and let  $\mathcal{M}'$  be another one but in  $\mathcal{E}'$ . An elementary embedding  $(p, h) : (\mathcal{E}', \mathcal{M}') \rightarrow (\mathcal{E}, \mathcal{M})$  consists of a geometric morphism  $p : \mathcal{E}' \rightarrow \mathcal{E}$  and a homomorphism of  $L$ -structures  $h : p^*(\mathcal{M}) \rightarrow \mathcal{M}'$  such that for an  $L$ -formula  $A(x_1, \dots, x_n)$  and arbitrary arrows  $\alpha_1, \dots, \alpha_n : E \rightarrow M$  in  $\mathcal{E}$ ,

$$(5) \quad E \Vdash A(\alpha_1, \dots, \alpha_n) \iff p^*(E) \Vdash' A(h \circ p^*(\alpha_1), \dots, h \circ p^*(\alpha_n))$$

( $\Vdash$  and  $\Vdash'$  refer to the Beth–Kripke–Joyal semantics for  $\mathcal{E}$  and  $\mathcal{E}'$ , respectively.) Note that it is  $\mathcal{M}$  that is embedded in  $\mathcal{M}'$ . In case only the direction  $(\Rightarrow)$  of (5) holds, the pair  $(p, h)$  is called a weak elementary embedding. If  $\mathcal{E}$  is classical (boolean) and  $p$  is surjective, these notions of embedding are in fact the same.

**THEOREM 3.8 (Minimality).** *Let  $\mathcal{E}$  be a topos that contains a model  $\mathcal{N}$  of  $\mathbf{HA}$ . Then there exists a weak elementary embedding  $(\mathcal{E}, \mathcal{N}) \rightarrow (\mathbf{Sh}(S), \llbracket N \rrbracket_S)$ .*

PROOF (SKETCH). By a characterisation of geometric morphisms [4, Ch. VII] into a Grothendieck topos  $\text{Sh}(\mathcal{C})$  we know that every such is given by a left exact, continuous functor  $\mathcal{C} \rightarrow \mathcal{E}$ . Such a functor is required to preserve finite limits and to send covers to epimorphic families. Define  $q : \mathcal{S} \rightarrow \mathcal{E}$  on objects and morphisms as follows

$$\begin{aligned} q(\varphi(\bar{x})) &= \{\bar{x} \in \mathcal{N}^n : \varphi(\bar{x})\}^{\mathcal{E}} \\ q(\alpha(\bar{x}, \bar{y})) &= \{(\bar{x}, \bar{y}) \in \mathcal{N}^{n+m} : \alpha(\bar{x}, \bar{y})\}^{\mathcal{E}}, \end{aligned}$$

where  $\bar{x} = x_1, \dots, x_n$  and  $\bar{y} = y_1, \dots, y_m$ . The quantifiers in  $\varphi$  and  $\alpha$  range over  $\mathcal{N}$ . Since  $\mathcal{N}$  is a model of **HA**,  $q$  is easily seen to be a left exact, continuous functor. Let  $p = (p^*, p_*)$  be the geometric morphism associated with  $q$ . We then have a natural isomorphism  $p^* \mathbf{y} \xrightarrow{\sim} q$  where  $\mathbf{y}$  is the Yoneda functor ([4, Ch. VII]). In particular there is an isomorphism  $h : p^*(\llbracket N \rrbracket) \rightarrow \mathcal{N}$ . From the left exactness of  $p^*$  and the natural isomorphism it follows that  $(p, h)$  is a weak elementary embedding.  $\dashv$

REMARK 3.9. The above theorem states a *minimality* property of the universal model of **HA**. However, we note that it refers to the impredicative notion of topos. An appropriate reformulation in terms of sites would be necessary in a predicative framework.

§4. **A minimal model of nonstandard arithmetic.** The construction of the minimal model follows the same pattern as in Section 3. However we start out with a richer syntactic site consisting of filter bases.

A *provable filter base* is an arithmetical formula  $F(\bar{p}; \bar{x}) = F(p_1, \dots, p_m; x_1, \dots, x_n)$  such that

$$(6) \quad \forall \bar{p} \exists \bar{r} \quad \mathbf{HA} \vdash \forall \bar{x} [F(\bar{r}, \bar{x}) \rightarrow F(\bar{p}, \bar{x}) \wedge F(\bar{q}, \bar{x})]$$

where the *indices*  $\bar{p}$ ,  $\bar{q}$  and  $\bar{r}$  vary over sequences of numerals. Generally, when  $\bar{m}$ ,  $\bar{p}$ ,  $\bar{q}$  or  $\bar{r}$  occur under the provability sign, we adopt the convention that they always range over sequences of numerals. We often write  $\bar{x} \in F_{\bar{p}}$  for  $F(\bar{p}, \bar{x})$  and  $F_{\bar{p}} \subseteq_{\mathbf{HA}} F_{\bar{q}}$  for  $\mathbf{HA} \vdash \forall \bar{x} [\bar{x} \in F_{\bar{p}} \rightarrow \bar{x} \in F_{\bar{q}}]$ . Set-theoretic notation  $\cap, \cup, \dots$  with the obvious translation to formulas will also be used. A map between two filter bases  $F(\bar{p}; \bar{x})$  and  $G(\bar{q}; \bar{y})$  is a formula  $\alpha(\bar{x}, \bar{y})$  such that

$$\exists \bar{p} \quad \mathbf{HA} \vdash (\forall \bar{x} \in F_{\bar{p}}) \exists! \bar{y} \alpha(\bar{x}, \bar{y}).$$

The sequence of numerals  $\bar{p}$  is called a *functionality index* of  $\alpha$ . Two such maps  $\alpha(\bar{x}, \bar{y})$  and  $\beta(\bar{u}, \bar{v})$  are considered *equal* ( $\alpha \sim_{\mathbf{HA}} \beta$ ) if for some  $\bar{p}$  we have  $\mathbf{HA} \vdash (\forall \bar{x} \in F_{\bar{p}}) \forall \bar{y} [\alpha(\bar{x}, \bar{y}) \leftrightarrow \beta(\bar{x}, \bar{y})]$ . A map  $\alpha(\bar{x}, \bar{y})$  is *continuous* if

$$\forall \bar{q} \exists \bar{p} \quad \mathbf{HA} \vdash (\forall \bar{x} \in F_{\bar{p}}) (\exists \bar{y} \in G_{\bar{q}}) \alpha(\bar{x}, \bar{y}).$$

DEFINITION 4.1. *The category F of provable filter bases.* The objects are triples  $\mathcal{F} = (F, \bar{p}; \bar{x})$  such that  $F(\bar{p}; \bar{x})$  is a provable filter base. An arrow between  $\mathcal{F} = (F(\bar{p}; \bar{x}))$  and  $\mathcal{G} = (G(\bar{q}; \bar{y}))$  is a triple  $(\bar{x}; \alpha; \bar{y})$  such that  $\alpha(\bar{x}, \bar{y})$  is a continuous map  $F(\bar{p}; \bar{x}) \rightarrow G(\bar{q}; \bar{y})$ . Composition of arrows  $\alpha(\bar{x}, \bar{y}) : \mathcal{F} \rightarrow \mathcal{G}$  and  $\beta(\bar{u}, \bar{v}) : \mathcal{G} \rightarrow \mathcal{H}$  is given by  $(\beta \circ \alpha)(\bar{x}, \bar{v}) = \exists \bar{y} \alpha(\bar{x}, \bar{y}) \wedge \beta(\bar{y}, \bar{v})$ . The identity arrow on  $F(\bar{p}; \bar{x})$  is defined by  $\iota(\bar{x}, \bar{y}) \equiv (\bar{x} = \bar{y})$ .

Note that according to this definition, the formerly given category  $\mathcal{S}$  is in fact a full subcategory of  $\mathcal{F}$  consisting of the filter bases with no indices, i.e., trivial filter bases.

LEMMA 4.2. *The category  $\mathcal{F}$  has all finite limits.*

PROOF. The category axioms follows by intuitionistic logic, once we know that composition is well defined and respects  $\sim_{\mathbf{HA}}$ . This is left for the reader to check.

To construct the pullback of two arrows

$$\varphi(\bar{x}, \bar{z}) : F(\bar{p}; \bar{x}) \rightarrow H(\bar{r}; \bar{z}) \quad \psi(\bar{y}, \bar{z}) : G(\bar{q}; \bar{y}) \rightarrow H(\bar{r}, \bar{z})$$

define the filterbase

$$(F \times_H G)(\bar{p}, \bar{q}; \bar{x}, \bar{y}) \equiv F(\bar{p}, \bar{x}) \wedge G(\bar{q}, \bar{y}) \wedge \exists \bar{z} \varphi(\bar{x}, \bar{z}) \wedge \psi(\bar{y}, \bar{z}),$$

and two projections  $\pi_1 : \mathcal{F} \times_{\mathcal{F}} \mathcal{G} \rightarrow \mathcal{F}$  and  $\pi_2 : \mathcal{F} \times_{\mathcal{F}} \mathcal{G} \rightarrow \mathcal{G}$  by  $\pi_1(\bar{x}, \bar{y}, \bar{z}) \equiv (\bar{x} = \bar{z})$  and  $\pi_2(\bar{x}, \bar{y}, \bar{z}) \equiv (\bar{y} = \bar{z})$ .

A terminal object is, for instance,  $I(\bar{p}, y) \equiv (y = 0)$ . It is straightforward to check that these constructions satisfy the universal properties.  $\dashv$

We now define a site on  $\mathcal{F}$  by giving a base  $K$  of covering families. Let  $\mathcal{F} = F(\bar{q}; \bar{y})$  be a filter base. A finite family of morphisms  $\langle \alpha_i(\bar{x}_i, \bar{y}) : F^i(\bar{p}_i; \bar{x}_i) \rightarrow \mathcal{F} \rangle_{i=1}^n$  is a covering of  $\mathcal{F}$  if

$$\forall \bar{p}_1 \cdots \bar{p}_n \exists \bar{q} \quad \mathbf{HA} \vdash (\forall \bar{y} \in F_{\bar{q}}) \bigvee_{i=1}^n \exists \bar{x}_i \in F_{\bar{p}_i}^i \alpha_i(\bar{x}_i, \bar{y}).$$

THEOREM 4.3. *The site  $(\mathcal{F}, K)$  is subcanonical.*

PROOF. First we need to check that  $K$  indeed satisfies the axioms for a base of a site (cf. [4, p.111]). We check stability under pullbacks, and leave the other axioms to the reader. Suppose that  $\langle \alpha_i : \mathcal{F}^i \rightarrow \mathcal{F} \rangle_{i=1}^n$  covers  $\mathcal{F}$  and that  $\beta : \mathcal{G} \rightarrow \mathcal{F}$  is an arbitrary arrow. We have to show that  $\langle \pi_2^i : \mathcal{F}^i \times_{\mathcal{F}} \mathcal{G} \rightarrow \mathcal{G} \rangle_{i=1}^n$  is a covering. Let  $\mathcal{F}^i = F^i(\bar{p}_i; \bar{x}_i)$ ,  $\mathcal{F} = F(\bar{p}; \bar{y})$ ,  $\mathcal{G} = G(\bar{q}, \bar{v})$  and  $\alpha_i = \alpha_i(\bar{x}_i, \bar{y})$ . Let  $\bar{p}_i, \bar{q}_i$  be indices for  $\mathcal{F}^i \times_{\mathcal{F}} \mathcal{G}$  where  $i = 1, \dots, n$ . Since  $\langle \alpha_i \rangle$  is a cover, there is  $\bar{p}$  such that

$$F_{\bar{p}} \subseteq_{\mathbf{HA}} \alpha_1[F_{\bar{p}_1}^1] \cup \cdots \cup \alpha_n[F_{\bar{p}_n}^n].$$

By the continuity of  $\beta$  we find some  $\bar{q}$  such that  $\beta[G_{\bar{q}}] \subseteq_{\mathbf{HA}} F_{\bar{p}}$  and  $G_{\bar{q}} \subseteq_{\mathbf{HA}} G_{\bar{q}_1} \cap \cdots \cap G_{\bar{q}_n}$ . Thus

$$G_{\bar{q}} \subseteq_{\mathbf{HA}} \pi_2^1[(F^1 \times_F G)_{\bar{p}_1, \bar{q}_1}] \cup \cdots \cup \pi_2^n[(F^n \times_F G)_{\bar{p}_n, \bar{q}_n}].$$

To check subcanonicity write  $P = \text{Hom}_{\mathcal{F}}(-, \mathcal{G})$  and consider a covering  $\langle \alpha_k : \mathcal{F}^k \rightarrow \mathcal{F} \rangle_{k=1}^n$ . For each pair of maps in the covering family form the pullback square

$$\begin{array}{ccc} \mathcal{F}^k \times_{\mathcal{F}} \mathcal{F}^l & \xrightarrow{\pi_{kl}^2} & \mathcal{F}^l \\ \pi_{kl}^1 \downarrow & & \downarrow \alpha_l \\ \mathcal{F}^k & \xrightarrow{\alpha_k} & \mathcal{F} \end{array} \quad \alpha_k$$



and suppose that  $\beta_k \in P(\mathcal{F}^k)$  ( $k = 1, \dots, n$ ) satisfy the matching conditions

$$P(\pi_{k\ell}^1)(\beta_k) \sim_{\mathbf{HA}} P(\pi_{k\ell}^2)(\beta_\ell).$$

Thus we can choose  $\bar{p}_1, \dots, \bar{p}_n$  such that for any  $k, \ell$

$$(7) \quad \mathbf{HA} \vdash (\forall \bar{x} \in F_{\bar{p}_k}^k) (\forall \bar{y} \in F_{\bar{p}_\ell}^\ell) \forall \bar{z} [\alpha_k(\bar{x}, \bar{z}) \wedge \alpha_\ell(\bar{y}, \bar{z}) \rightarrow \forall \bar{v} (\beta_k(\bar{x}, \bar{v}) \leftrightarrow \beta_\ell(\bar{y}, \bar{v}))],$$

and such that  $\bar{p}_k$  is a functionality index for both  $\alpha_k$  and  $\beta_k$  ( $k = 1, \dots, n$ ). We define

$$\beta(\bar{z}, \bar{v}) \equiv \bigvee_{k=1}^n \exists \bar{x}_k \in F_{\bar{p}_k}^k \alpha_k(\bar{x}_k, \bar{z}) \wedge \beta_k(\bar{x}_k, \bar{v}).$$

Since  $\langle \alpha_k \rangle$  is a cover there is  $\bar{p}$  such that  $F_{\bar{p}} \subseteq_{\mathbf{HA}} \alpha_1[F_{\bar{p}_1}^1] \cup \dots \cup \alpha_n[F_{\bar{p}_n}^n]$ . By (7) it follows that  $\beta : \mathcal{F} \rightarrow \mathcal{G}$  is a map which is functional on  $F_{\bar{p}}$ . The continuity of  $\beta$  follows from the continuity of the  $\beta_k$ 's and that  $\langle \alpha_k \rangle$  is a cover. Again, by (7) it is clear that  $\beta \circ \alpha_k \sim_{\mathbf{HA}} \beta_k$ . It is straightforward to check that  $\beta$  indeed is the unique such arrow.  $\dashv$

We define an interpretation of the arithmetical language in  $\text{Sh}(\mathcal{F})$  and call the resulting model  $\llbracket N \rrbracket_{\mathcal{F}}$ .

- The type of natural numbers is interpreted by the representable sheaf

$$\llbracket N \rrbracket = \text{Hom}_{\mathcal{F}}(-, (x = x, p; x)).$$

- For every function symbol  $f : N^n \rightarrow N$  define a natural transformation  $\llbracket f \rrbracket : \llbracket N \rrbracket^n \rightarrow \llbracket N \rrbracket$  by

$$\llbracket f \rrbracket_{F(\bar{p}, \bar{x})}(\alpha_1, \dots, \alpha_n) = (\bar{x}; \exists s_1 \dots s_n \bigwedge_{i=1}^n \alpha_i(\bar{x}, s_i) \wedge f(s_1, \dots, s_n) = v; v).$$

- For the constant 0 define a natural transformation  $\llbracket 0 \rrbracket : 1 \rightarrow \llbracket N \rrbracket$  by

$$\llbracket 0 \rrbracket_{F(\bar{p}, \bar{x})}(\alpha) = (\bar{x}; 0 = v; v).$$

- For the equality = define an  $(F, K)$ -relation

$$\llbracket = \rrbracket_{F(\bar{p}, \bar{x})}(\alpha, \gamma) \iff \exists \bar{p} \quad \mathbf{HA} \vdash (\forall \bar{x} \in F_{\bar{p}}) \exists yz \alpha(\bar{x}, y) \wedge \gamma(\bar{x}, z) \wedge y = z.$$

Observe that  $\llbracket = \rrbracket_{\mathcal{F}}(\alpha, \gamma)$  iff  $\alpha \sim_{\mathbf{HA}} \gamma$ , by using the equality axioms. The interpretation is extended to terms in the usual way and the following lemma helps calculating it.

**LEMMA 4.4.** *Let  $t(\bar{x})$  be a term where  $\bar{x} = x_1, \dots, x_n$ . Then for any filter base  $\mathcal{F} = (F, \bar{p}; \bar{u})$  and any  $\alpha_1, \dots, \alpha_n \in \llbracket N \rrbracket(\mathcal{F})$ ,*

$$\llbracket t(\bar{x}) \rrbracket_{\mathcal{F}}(\alpha_1, \dots, \alpha_n) \sim_{\mathbf{HA}} \left( \bar{u}; \exists s_1 \dots s_n \bigwedge_{i=1}^n \alpha_i(\bar{u}, s_i) \wedge t(s_1, \dots, s_n) = v; v \right).$$

**PROOF.** By induction on  $t$ .  $\dashv$

Let  $\Vdash$  be the forcing relation associated with the interpretation given above.

**THEOREM 4.5.** *Let  $A(\bar{x})$  be an arithmetical formula where  $\bar{x} = x_1, \dots, x_n$ . Then for any filter base  $\mathcal{F} = F(\bar{p}, \bar{u})$  and any  $\alpha_1, \dots, \alpha_n \in \llbracket N \rrbracket(\mathcal{F})$*

$$\mathcal{F} \Vdash A(\alpha_1, \dots, \alpha_n)$$

$$\iff \exists \bar{p} \quad \mathbf{HA} \vdash (\forall \bar{u} \in F_{\bar{p}}) \exists y_1 \cdots y_n \bigwedge_{i=1}^n \alpha_i(\bar{u}, y_i) \wedge A(y_1, \dots, y_n).$$

**PROOF.** The proof proceeds by induction on  $A$  and it is analogous to that of Theorem 3.5 above and Lemma 2.2 in [5]. We do the case of universal quantification as an illustration. Let  $A(\bar{x}) = \forall y B(\bar{x}, y)$ . Suppose  $\mathcal{F} \Vdash A(\alpha_1, \dots, \alpha_n)$ . Let  $\mathcal{G}$  be the filter base  $G(\bar{p}; \bar{u}, v) \equiv F(\bar{p}; \bar{u})$ . There are obvious projections  $\pi_1 : \mathcal{G} \rightarrow \mathcal{F}$  and  $\pi_2 \in \llbracket N \rrbracket(\mathcal{G})$ . By the definition of forcing for the universal quantifier,  $\mathcal{G} \Vdash B(\alpha_1 \circ \pi_1, \dots, \alpha_n \circ \pi_1, \pi_2)$ . According to the inductive hypothesis this is equivalent to there existing some  $\bar{p}$  such that

$$(8) \quad \mathbf{HA} \vdash (\forall \bar{u} \in F_{\bar{p}}) \forall v \exists \bar{y} [\bar{\alpha}(\bar{u}, \bar{y}) \wedge B(\bar{y}, v)]$$

where  $\bar{y} = y_1, \dots, y_n$  and  $\bar{\alpha}(\bar{u}, \bar{y}) = \bigwedge_{i=1}^n \alpha_i(\bar{u}, y_i)$ . This yields the conclusion.

Conversely, suppose that (8) holds. Let  $\gamma : \mathcal{G} \rightarrow \mathcal{F}$  and  $\delta \in \llbracket N \rrbracket(\mathcal{G})$  be arbitrary, where  $\mathcal{G} = G(\bar{q}, \bar{v})$ . By the continuity of  $\gamma$ , there is some  $\bar{q}$  such that  $\gamma[G_{\bar{q}}] \subseteq_{\mathbf{HA}} F_{\bar{p}}$ . Thus

$$\mathbf{HA} \vdash (\forall \bar{v} \in G_{\bar{q}}) \exists \bar{y} \exists z [\gamma(\bar{v}, \bar{u}) \wedge \bar{\alpha}(\bar{u}, \bar{y}) \wedge \delta(\bar{v}, z) \wedge B(\bar{y}, z)].$$

By the inductive hypothesis,  $\mathcal{G} \Vdash B(\alpha_1 \circ \gamma, \dots, \alpha_n \circ \gamma, \delta)$ . ⊣

**COROLLARY 4.6.** *(Sh(F),  $\llbracket N \rrbracket_F$ ) is a universal model of  $\mathbf{HA}$ , i.e., a closed arithmetical formula holds in the model iff it is provable in  $\mathbf{HA}$ .* ⊣

We define an interpretation of the standard predicate. Let

$$\beta \in \text{St}^N(\mathcal{F}) \iff \exists m \exists \bar{p} \quad \mathbf{HA} \vdash (\forall \bar{u} \in F_{\bar{p}}) (\exists y \leq m) \beta(\bar{u}, y),$$

where  $\beta \in \llbracket N \rrbracket(\mathcal{F})$  and  $\mathcal{F} = F(\bar{p}; \bar{u})$ . It is easy to check that this indeed is an  $(F, K)$ -relation. In the model, universal quantification over standard numbers correspond in the following way to quantification over numerals.

**LEMMA 4.7.** *Let  $A(\bar{x}, y)$  be any  $L'$ -formula. Then  $\mathcal{F} \Vdash \forall^{\text{st}} y A(\bar{\alpha}, y)$  if, and only if, for all numerals  $n$  it is the case that  $\mathcal{F} \Vdash A(\bar{\alpha}, n)$ .*

**PROOF.** The direction  $(\Rightarrow)$  is clear.

$(\Leftarrow)$ : Let  $\beta : \mathcal{G} \rightarrow \mathcal{F}$  and let  $\delta \in \llbracket N \rrbracket(\mathcal{G})$  with  $\delta \in \text{St}^N(\mathcal{G})$ . Hence for some  $m$  and some  $\bar{q}$

$$(9) \quad \mathbf{HA} \vdash (\forall \bar{w} \in G_{\bar{q}}) (\exists y \leq m) \delta(\bar{w}, y).$$

Define filter bases  $\mathcal{E}^k$  for each numeral  $k = 0, \dots, m$ ,

$$\bar{w} \in G_{\bar{q}}^k \equiv \bar{w} \in G_{\bar{q}} \wedge \delta(\bar{w}, k).$$

We have inclusion morphisms  $\eta : \mathcal{E}^k \rightarrow \mathcal{G}$ . By (9) the family  $\langle \eta : \mathcal{E}^k \rightarrow \mathcal{G} \rangle_{k=0}^m$  is a covering. Thus it suffices to show

$$\mathcal{E}^k \Vdash A(\alpha_1 \circ \beta \circ \eta, \dots, \alpha_n \circ \beta \circ \eta, \delta \circ \eta).$$

This follows by monotonicity from the assumption, since  $\delta \circ \eta$  is constant  $k$  as a morphism in  $\llbracket N \rrbracket(\mathcal{F}^k)$ . ⊣

The following result states that formulas  $\forall^{\text{st}} y A$  are representable in case  $A$  is arithmetical.

**COROLLARY 4.8.** *For an element  $\alpha \in \llbracket N \rrbracket(\mathcal{F})$  and an arithmetical formula  $A(x, y)$ ,*

$$\mathcal{F} \Vdash \forall^{\text{st}} y A(\alpha, y) \text{ if, and only if, } \alpha : \mathcal{F} \rightarrow \mathcal{G} \text{ is an arrow,}$$

where  $\mathcal{G}$  is the filter base given by  $G(q, x) \equiv (\forall y \leq q) A(x, y)$ .

**PROOF.** By Lemma 4.7 and Theorem 4.5, we get that  $\mathcal{F} \Vdash \forall^{\text{st}} y A(\alpha, y)$  is equivalent to

$$\forall q \exists \bar{p} \quad \mathbf{HA} \vdash (\forall \bar{u} \in F_{\bar{p}}) \exists x \alpha(\bar{u}, x) \wedge (\forall y \leq q) A(x, y),$$

i.e.,  $\alpha : \mathcal{F} \rightarrow \mathcal{G}$  is an arrow. ⊣

The negation of the standard predicate is explained by

**LEMMA 4.9.** *For  $\mathcal{F} = F(\bar{p}; \bar{u})$  and  $\alpha \in \llbracket N \rrbracket(\mathcal{F})$ ,*

$$\mathcal{F} \Vdash \neg \text{St}(\alpha) \iff \forall n \exists \bar{p} \quad \mathbf{HA} \vdash (\forall \bar{u} \in F_{\bar{p}}) \exists y \alpha(\bar{u}, y) \wedge y \geq n.$$

**PROOF.** ( $\Leftarrow$ ): Suppose that  $\gamma : \mathcal{G} \rightarrow \mathcal{F}$  is an arrow with  $\mathcal{G} \Vdash \text{St}(\alpha \circ \gamma)$ , where  $\mathcal{G} = G(\bar{q}; \bar{w})$ . Hence for some  $n$  and some  $\bar{q}$ ,

$$\mathbf{HA} \vdash (\forall \bar{w} \in G_{\bar{q}}) \exists \bar{u} \exists y \gamma(\bar{w}, \bar{u}) \wedge \alpha(\bar{u}, y) \wedge y \leq n.$$

From the assumption we have  $\bar{p}$  such that

$$\mathbf{HA} \vdash (\forall \bar{u} \in F_{\bar{p}}) \exists y \alpha(\bar{u}, y) \wedge y \geq n + 1.$$

Assume  $\bar{p}$  is a functionality index of  $\alpha$ . By continuity of  $\gamma$ , there exists  $G_{\bar{r}} \subseteq_{\mathbf{HA}} G_{\bar{q}}$  such that

$$\mathbf{HA} \vdash (\forall \bar{w} \in G_{\bar{r}}) (\exists \bar{u} \in F_{\bar{p}}) \exists y \gamma(\bar{w}, \bar{u}) \wedge \alpha(\bar{u}, y) \wedge y \geq n + 1.$$

Thus  $\mathbf{HA} \vdash (\forall u \in G_{\bar{r}}) \perp$ , i.e.,  $\mathcal{G} \Vdash \perp$ .

( $\Rightarrow$ ): For every numeral  $n$  define a filter base  $\mathcal{F}^n$  by  $\bar{u} \in F_{\bar{p}}^n \equiv \bar{u} \in F_{\bar{p}} \wedge (\exists y < n) \alpha(\bar{u}, y)$ . The inclusion  $\eta : \mathcal{F}^n \rightarrow \mathcal{F}$  is an arrow. Clearly  $\mathcal{F}^n \Vdash \text{St}(\alpha \circ \eta)$ . Thus by assumption  $\mathcal{F}^n \Vdash \perp$ . Hence for some  $\bar{p}$ , we have  $\mathbf{HA} \vdash (\forall \bar{u} \in F_{\bar{p}}) [(\exists y < n) \alpha(\bar{u}, y) \rightarrow \perp]$ . By choosing  $\bar{p}$  to be a functionality index of  $\alpha$ , the desired conclusion follows. ⊣

The righthand side of the lemma can be read as:  $\alpha$  is an arrow from  $\mathcal{F}$  to the base of the Fréchet filter.

**THEOREM 4.10.** *The  $L'$ -structure  $(\text{Sh}(F), (\llbracket N \rrbracket_F, \text{St}))$  is a model of  $\mathbf{HAI}_\omega + (\text{NC})$ .*

**PROOF.** Axiom group (I) is valid by Theorem 4.5. Axioms (II) and (III) are straightforward to check.

To check the stability axiom (IV), suppose  $\mathcal{F} \Vdash \neg \neg \text{St}(\alpha)$ . Consider the filterbase  $\mathcal{G}$ :

$$\bar{u} \in G_{\bar{p}, n} \equiv \bar{u} \in F_{\bar{p}} \wedge (\exists y > n) \alpha(\bar{u}, y).$$

The natural projection  $\pi_1 : \mathcal{E} \rightarrow \mathcal{F}$  is an arrow. We have trivially

$$\forall m \exists \bar{p}, n \quad \mathbf{HA} \vdash (\forall \bar{u} \in F_{\bar{p}})[(\exists y > n) \alpha(\bar{u}, y) \rightarrow (\exists y \geq m) \alpha(\bar{u}, y)].$$

According to Lemma 4.9 this means  $\mathcal{E} \Vdash \neg \text{St}(\alpha \circ \pi_1)$ . By the assumption and monotonicity,  $\mathcal{E} \Vdash \perp$ , i.e.,  $\exists \bar{p} \exists n \quad \mathbf{HA} \vdash (\forall u \in F_{\bar{p}}) \exists y \alpha(\bar{u}, y) \wedge y \leq n$ , i.e.,  $\mathcal{F} \Vdash \text{St}(\alpha)$ .

To verify (V) consider the base for the Fréchet filter:  $\mathcal{E} \equiv C(p; x) \equiv x \geq p$ . Let  $\alpha \in \llbracket N \rrbracket(\mathcal{E})$  be given by  $\alpha(x, y) \equiv (x = y)$ . By Lemma 4.9, we have  $\mathcal{E} \Vdash \neg \text{St}(\alpha)$ . The constant map from  $\mathcal{E}$  to a terminal object  $\mathcal{F}$  is a covering, so  $\mathcal{F} \Vdash \exists x \neg \text{St}(x)$ .

The external induction schema (VI) follows readily from Lemma 4.7.

The most complicated schema to check is the overspill principle (VII). We check the equivalent formulation:  $\forall \bar{x} [\forall^{\text{st}} y A(\bar{x}, y) \rightarrow \exists y \neg \text{St}(y) \wedge A(\bar{x}, y)]$ . It is sufficient to prove that for all  $\bar{\alpha} = \alpha_1, \dots, \alpha_n \in \llbracket N \rrbracket(\mathcal{F})$  satisfying the assumption  $\mathcal{F} \Vdash \forall^{\text{st}} y A(\bar{\alpha}, y)$  we have  $\mathcal{F} \Vdash \exists y \neg \text{St}(y) \wedge A(\bar{\alpha}, y)$ . Let  $\bar{\alpha}(\bar{u}, \bar{x})$  be shorthand for  $\bigwedge_{i=1}^n \alpha_i(\bar{u}, x_i)$  and write  $\mathcal{F} = F(\bar{p}; \bar{u})$ . By the assumption and Lemma 4.7, it follows that  $\mathcal{F} \Vdash A(\bar{\alpha}, m)$  for any numeral  $m$ . Thus

$$(10) \quad \forall m \exists \bar{p} \quad \mathbf{HA} \vdash (\forall \bar{u} \in F_{\bar{p}}) \exists \bar{x} \bar{\alpha}(\bar{u}, \bar{x}) \wedge A(\bar{x}, m).$$

Define a filter base by

$$\bar{u}, v \in G_{\bar{p}, m} \equiv \bar{u} \in F_{\bar{p}} \wedge [m = 0 \vee \exists \bar{x} \bar{\alpha}(\bar{u}, \bar{x}) \wedge v \geq m - 1 \wedge A(\bar{x}, v)].$$

Then the natural projections  $\pi_1 : \mathcal{E} \rightarrow \mathcal{F}$  and  $\pi_2 : \mathcal{E} \rightarrow N$  are morphisms. We need to check that  $\pi_1$  is a covering map, i.e., that for all  $\bar{p}, m$  there exists  $\bar{q}$  with  $\mathbf{HA} \vdash (\forall \bar{u} \in F_{\bar{q}}) \exists v (\bar{u}, v \in G_{\bar{p}, m})$ . Suppose that  $m > 0$ . We have  $\mathbf{HA} \vdash (\forall \bar{u} \in F_{\bar{r}}) \exists \bar{x} \bar{\alpha}(\bar{u}, \bar{x}) \wedge A(\bar{x}, m)$ , for some  $\bar{r}$ . Now take  $\bar{q}$  such that  $F_{\bar{q}} \subseteq_{\mathbf{HA}} F_{\bar{r}} \cap F_{\bar{p}}$ . By Lemma 4.9,  $\mathcal{E} \Vdash \neg \text{St}(\pi_2)$ , so we need only show  $\mathcal{E} \Vdash A(\alpha_1 \circ \pi_1, \dots, \alpha_n \circ \pi_1, \pi_2)$ , i.e.,

$$\exists \bar{p} \exists m \quad \mathbf{HA} \vdash (\forall \bar{u}, v \in G_{\bar{p}, m}) \exists \bar{x} \bar{\alpha}(\bar{u}, \bar{x}) \wedge A(\bar{x}, v).$$

Taking  $m > 0$  and  $F_{\bar{p}}$  small this follows trivially.

The  $\omega$ -rule is evident by Lemma 4.7.

Finally (NC) is checked analogously to Proposition 2.5 of [5]. +

**COROLLARY 4.11.** *The theory  $\mathbf{HAI}_\omega + (\text{NC})$  is a conservative extension of  $\mathbf{HA}$ .* +

**REMARKS 4.12.** We note that all the constructions of this section work if we start out with  $\mathbf{PA}$  instead of  $\mathbf{HA}$  as the theory. The corresponding site of filter bases is denoted  $F_c$ . Let  $\mathbf{PAI}_\omega$  be as  $\mathbf{HAI}_\omega$  except that the principle of excluded middle is assumed for all *arithmetical* formulas. This theory extended with (NC) is modelled by  $\text{Sh}(F_c)$  and is hence a conservative extension of  $\mathbf{PA}$ . Note that this proves that (TR) does not hold in  $\text{Sh}(F_c)$ , since the infinitary rule would then yield the false conclusion that  $\mathbf{PA}$  is  $\omega$ -complete.

**THEOREM 4.13 (Minimality).** *Let  $\mathcal{E}$  be a topos with an  $L'$ -structure  $(\mathcal{N}, \text{St}')$  which is a model of  $\mathbf{HAI}$ . Suppose further that  $\text{St}' \subseteq \mathcal{N}$  is the natural numbers object in  $\mathcal{E}$ . Then there exists a weak elementary extension of  $L(\mathbf{HA})$ -structures*

$$(\mathcal{E}, \mathcal{N}) \longrightarrow (\text{Sh}(F), \llbracket N \rrbracket_F).$$

PROOF (SKETCH). As in Theorem 3.8 we find a left exact, continuous functor  $q : F \rightarrow \mathcal{E}$ . Define  $q$  by

$$\begin{aligned} q(F(\bar{p}; \bar{x})) &= \{\bar{u} \in \mathcal{N}^n : (\forall \bar{r} \in \text{St}') F(\bar{r}, \bar{u})\}^{\mathcal{E}} \\ q(\alpha(\bar{x}; \bar{y})) &= \{(\bar{u}, \bar{v}) \in \mathcal{N}^{n+m} : \alpha(\bar{u}, \bar{v})\}^{\mathcal{E}}. \end{aligned}$$

Note that since  $\text{St}'$  is the set of natural numbers in  $\mathcal{E}$ , it coincides with the interpretation of **HA**-numerals in  $\mathcal{N}$ . From this observation and the fact that  $(\mathcal{N}, \text{St}')$  is a model of **HAI** it follows easily that  $q$  is a functor which preserves finite limits. To prove that  $q$  sends covers to epimorphic families we use the overspill property of  $(\mathcal{N}, \text{St}')$ . Let  $\langle \alpha_i(\bar{x}_i, \bar{y}) : F^i(\bar{p}_i; \bar{x}_i) \rightarrow G(\bar{r}; \bar{y}) \rangle_{i=1}^n$  be a cover. Define new filter bases  $\mathcal{H}^i, i = 1, \dots, n$ , by

$$H^i(k, \bar{x}_i) \equiv (\forall \bar{p}_i \leq k) F^i(\bar{p}_i, \bar{x}_i).$$

It is easily seen that  $\langle \alpha_i : \mathcal{H}^i \rightarrow \mathcal{E} \rangle$  is also a cover. We use the convention that if  $\varphi(\bar{x})$  is an **HA**-formula, where  $\bar{x} = x_1, \dots, x_n$ , then  $\varphi^{\mathcal{E}}(\bar{x})$  denotes the  $m$ -ary relation in  $\mathcal{E}$  resulting from interpreting it in  $\mathcal{E}$ . By interpretation of the cover statement we have that

$$\begin{aligned} (\forall k_1, \dots, k_n \in \text{St}') (\exists \bar{s} \in \text{St}') (\forall v \in \mathcal{N}) \\ \left[ G^{\mathcal{E}}(\bar{s}, v) \Rightarrow \bigvee_{i=1}^n (\exists \bar{u}_i \in \mathcal{N}) (H^i)^{\mathcal{E}}(k_i, \bar{u}_i) \wedge \alpha_i^{\mathcal{E}}(\bar{u}_i, v) \right] \end{aligned}$$

holds in  $\mathcal{E}$ . Now reason in  $\mathcal{E}$ , and suppose that  $v \in q(G(\bar{r}; \bar{y}))$ . Hence for all  $k_1, \dots, k_n \in \text{St}'$ ,

$$(11) \quad \bigvee_{i=1}^n (\exists \bar{u}_i \in \mathcal{N}) (H^i)^{\mathcal{E}}(k_i, \bar{u}_i) \wedge \alpha_i^{\mathcal{E}}(\bar{u}_i, v).$$

It follows by the overspill principle that (11) holds for some infinite  $k_1, \dots, k_n$ . Hence we have some  $i$  and some  $\bar{u}_i \in \mathcal{N}$  such that  $(F^i)^{\mathcal{E}}(\bar{p}_i, \bar{u}_i)$  and  $\alpha_i^{\mathcal{E}}(\bar{u}_i, v)$  holds for any  $\bar{p}_i \in \text{St}'$ . Thus  $q(\alpha_i)(\bar{u}_i) = v$  and  $\bar{u}_i \in q(F^i(\bar{p}_i; \bar{x}_i))$ . This proves that  $\langle q(\alpha_i) : q(F^i) \rightarrow q(G) \rangle_{i=1}^n$  is an epimorphic family.  $\dashv$

We may combine Theorems 3.5 and 4.5, and the fact that **S** is a full subcategory of **F** to obtain an example of an (strong) elementary embedding. Note that with respect to this strong elementary embedding, the embedding constructed in Theorem 4.13 is a lifting of that of Theorem 3.8.

THEOREM 4.14. *There exists an elementary embedding*

$$(\text{Sh}(\mathbf{F}), \llbracket N \rrbracket_{\mathbf{F}}) \longrightarrow (\text{Sh}(\mathbf{S}), \llbracket N \rrbracket_{\mathbf{S}}). \quad \dashv$$

REFERENCES

[1] TH. COQUAND and J. M. SMITH, *An application of constructive completeness*, **Proceedings of the workshop TYPES '95, Torino, Italy, June 1995** (S. Berardi and M. Coppo, editors), Lecture Notes in Computer Science, vol. 1158, Springer-Verlag, Berlin, 1996.  
 [2] A. G. DRAGALIN, *An explicit Boolean-valued model for the non-standard arithmetic*, **Publications Mathematicae, Debrecen**, vol. 42 (1993), pp. 369–389.

- [3] R. KAYE, *Models of Peano arithmetic*, Clarendon Press, Oxford, 1991.  
[4] S. MAC LANE and I. MOERDIJK, *Sheaves in geometry and logic*, Springer-Verlag, 1992.  
[5] I. MOERDIJK, *A model for intuitionistic non-standard arithmetic*, *Annals of Pure and Applied Logic*, vol. 73 (1995), pp. 37–51.  
[6] E. PALMGREN, *Constructive sheaf semantics*, *Mathematical Logic Quarterly*, to appear.  
[7] ———, *A note on 'Mathematics of infinity'*, this JOURNAL, vol. 58 (1993), pp. 1195–1200.  
[8] ———, *A constructive approach to nonstandard analysis*, *Annals of Pure and Applied Logic*, vol. 73 (1995), pp. 297–325.

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