# Representing topoi by topological groupoids

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**Abstract**

It is shown that every topos with enough points is equivalent to the classifying topos of a topological groupoid.

## 1 Definitions and statement of the result

We recall some standard definitions ([1, 5, 9]). A topos is a category $\mathcal{E}$ which is equivalent to the category of sheaves of sets on a (small) site. Equivalently, $\mathcal{E}$ is a topos iff it satisfies the Giraud axioms ([1], p. 303). The category of sets $\mathcal{S}$ is a topos, and plays a role analogous to that of the one–point space in topology. In particular, a *point* of a topos $\mathcal{E}$ is a topos morphism $p: \mathcal{S} \to \mathcal{E}$. It is given by a functor $p^*: \mathcal{E} \to \mathcal{S}$ which commutes with colimits and finite limits. For an object (sheaf) $E$ of $\mathcal{E}$, the set $p^*(E)$ is also denoted $E_p$, and called the *stalk* of $E$ at $p$. The topos $\mathcal{E}$ is said to have *enough points* if these functors $p^*$, for all points $p$, are jointly conservative (see [9], p. 521, [1]). Almost all topoi arising in practice have enough points. This applies in particular to the presheaf topos $\hat{\mathbb{C}}$ on an arbitrary small category $\mathbb{C}$, and to the étale topos associated to a scheme. In fact, any "coherent" topos has enough points (see Deligne, Appendix to Exposé VI in [1]).

We describe a particular kind of topos with enough points. Recall that a *groupoid* is a category in which each arrow is an isomorphism. Such a groupoid is thus given by a set $X$ of objects, and a set $G$ of arrows, together with structure maps

$$\begin{tikzcd} G \times_X G \arrow[r, "m"] & G \arrow[loop below, "i"] \arrow[r, shift left=3, "s"] \arrow[r, shift right=3, "t"'] & X. \arrow[l, "u"'] \end{tikzcd} \tag{1}$$

Here $s$ and $t$ denote the source and the target, $u(x) \in G$ is the identity at $x \in X$, $i(g) = g^{-1}$ is the inverse, and $m(g,h) = g \circ h$ is the composition. A *topological groupoid* is such a groupoid in which $X$ and $G$ are each equipped with a topology, for which all the structure maps in (1) are continuous.

Given such a topological groupoid, a *$G$–sheaf* is a sheaf on $X$ equipped with a continuous $G$–action. Thus a $G$–sheaf consists of a local homeomorphism $p: E \to X$ together with a continuous action map $E \times_X G \to E$, defined for all $e \in E_x$ and $g: y \to x$ in $G$, and denoted $e, g \mapsto e \cdot g$; this map should satisfy the usual identities for an action.

The category $\mathrm{Sh}_G(X)$ of all such $G$–sheaves, and action preserving maps between them, is a topos. It is called the *classifying topos* of the groupoid $G \rightrightarrows X$. Such a classifying topos always has enough points. In fact, any (ordinary) point $x \in X$ defines a point $\bar{x}: \mathcal{S} \to \mathrm{Sh}_G(X)$, by

$$\bar{x}^*(E) = E_x = p^{-1}(x).$$

The collection of all these points $\bar{x}$ is jointly conservative.

Our main aim is to prove that every topos with enough points is, up to equivalence, the classifying topos of some topological groupoid:

**Theorem 1.1** *Let $\mathcal{E}$ be any topos with enough points. There exists a topological groupoid $G \rightrightarrows X$ for which there is an equivalence of topoi*

$$\mathcal{E} \cong \mathrm{Sh}_G(X).$$

We end this introductory section with some comments on related work. Representations of categories of sheaves by groupoids go back to Grothendieck's Galois theory ([4]). In [8], a general theorem was proved, which is similar to our result, and which states that for every topos $\mathcal{E}$ (not necessarily with enough points) there is a groupoid $G \rightrightarrows X$ in the category of locales ("pointless spaces") for which there is an equivalence $\mathcal{E} \cong \mathrm{Sh}_G(X)$. This theorem was sharpened, again in the context of locales, in [7]. The basic idea for our construction comes from the latter paper.

We wish to point out, however, that our result for topoi with enough points is not a formal consequence of any of these theorems. Moreover, our proof is different. The proofs in [8] and [7] depend essentially on change–of–base techniques, the internal logic of a topos, and the behaviour of locales in this context. These techniques cannot be applied to the present situation. In fact, we believe that the proof of our theorem is much more accessible and direct.

# 2 Description of the groupoid

Let $\mathcal{E}$ be a topos with enough points. We recall the definition of the space $X = X_{\mathcal{E}}$ from [2], §2, and show that it is part of a groupoid $G \rightrightarrows X$. First, although the collection of all points of $\mathcal{E}$ is in general a proper class, there will always be a *set* of points $p$ for which the functors $p^*$ are already jointly conservative [5], Corollary 7.17. Fix such a set, and call its members *small* points of $\mathcal{E}$. Next, let $S$ be an object of $\mathcal{E}$ with the property that the subobjects of powers of $S$, i.e., all sheaves $B \subset S^n$ for $n \geq 0$, together generate $\mathcal{E}$. For example, $S$ can be the disjoint sum of all the objects in some small site for $\mathcal{E}$. Let $I$ be an infinite set, with cardinality so large that

$$\mathrm{card}(S_p) \leq \mathrm{card}(I)$$

for all small points $p$ of $\mathcal{E}$.

In general, if $A$ is any set with $\mathrm{card}(A) \leq \mathrm{card}(I)$, we call an *enumeration* of $A$ a function $\alpha: D = \mathrm{dom}(\alpha) \to A$, where $D \subset I$ and $\alpha^{-1}(a)$ is infinite for each $a \in A$. These enumerations carry a natural topology, whose basic open sets are the sets

$$V_u = \{\alpha \mid u \subset \alpha\}; \tag{2}$$

here $u$ is any function $\{i_1, \ldots, i_n\} \to A$ defined on a finite subset of $I$, and $u \subset \alpha$ means that $i_k \in \mathrm{dom}(\alpha)$ and $\alpha(i_k) = u(i_k)$, for $k = 1, \ldots, n$. Leaving the index set $I$ implicit, we denote this topological space by

$$\mathrm{En}(A),$$

and call it the enumeration space of $A$.

The space $X$, involved in the groupoid, is defined by gluing several of these enumeration spaces together. A point of $X$ is an equivalence class of pairs $(p, \alpha)$, where $p$ is a small point of $\mathcal{E}$ and $\alpha \in \mathrm{En}(S_p)$ is an enumeration of the stalk $S_p$. Two such pairs $(p, \alpha)$ and $(p', \alpha')$ are equivalent, i.e., define the same point of $X$, if there exists a natural isomorphism $\tau: p^* \to p'^*$ for which $\alpha' = \tau_S \circ \alpha$. (Note that for such a $\tau$, its component $\tau_S$ is uniquely determined by $\alpha$ and $\alpha'$, because $\alpha$ is surjective.) In what follows, we will generally simply denote a point of $X$ by $(p, \alpha)$, and we will not distinguish such pairs from their equivalence classes whenever we can do so without causing possible confusion. The topology on the space $X$ is given by the basic open sets $U_{i_1,\ldots,i_n,B}$, defined for any $i_1, \ldots, i_n \in I$ and any $B \subset S^n$, as

$$U_{i_1,\ldots,i_n,B} = \{(p, \alpha) \mid (\alpha(i_1), \ldots, \alpha(i_n)) \in B_p\}. \tag{3}$$

Observe that this is well–defined on equivalence classes; i.e., if $(p, \alpha) \sim (p', \alpha')$ by an isomorphism $\tau$ as above, then $\alpha(i) \in B_p$ iff $\alpha'(i) \in B_{p'}$, where we write $\alpha(i) = (\alpha(i_1), \ldots, \alpha(i_n))$ and similarly for $\alpha'$.

Next, we define the space $G$ of arrows. The points of $G$ are equivalence classes of quintuples

$$(p, \alpha) \xrightarrow{\theta} (q, \beta),$$

where $(p, \alpha)$ and $(q, \beta)$ are points of $X$ as above, and $\theta: p^* \to q^*$ is a natural isomorphism. (We do *not* require that $\beta = \theta_S \circ \alpha$.) Two such $(p, \alpha) \xrightarrow{\theta} (q, \beta)$ and $(p', \alpha') \xrightarrow{\theta'} (q', \beta')$ represent the same point of $G$ whenever there are isomorphisms $\tau: p^* \to p'^*$ and $\sigma: q^* \to q'^*$ such that $\alpha' = \tau_S \circ \alpha$ and $\beta' = \sigma_S \circ \beta$, while in addition $\sigma\theta = \theta'\tau$. The topology on $G$ is given by the basic open sets $V_{i_1,\ldots,i_n,B,j_1,\ldots,j_n,C} = V_{i,B,j,C}$ defined by

$$V_{i,B,j,C} = \{(p, \alpha) \xrightarrow{\theta} (q, \beta) \mid \alpha(i) \in B_p, \beta(j) \in C_q, \text{ and } \theta(\alpha(i)) = \beta(j)\}.$$

Here we have again used the shorter notation $\alpha(i)$ for $(\alpha(i_1), \ldots, \alpha(i_n))$, etc. Note, as above, that these basic open sets are well–defined on equivalence classes. It remains to define the structure maps (§1(1)) of the groupoid. For an arrow $g = [(p, \alpha) \xrightarrow{\theta} (q, \beta)]$, its source and target are defined by

$$s(g) = (p, \alpha) \qquad \text{and} \qquad t(g) = (q, \beta).$$

The maps $s$ and $t$ are well–defined on equivalence classes, and are easily seen to be continuous for the topologies on $X$ and $G$ as just defined. For two arrows $g = [(p, \alpha) \xrightarrow{\theta} (q, \beta)]$ and $h = [(q', \beta') \xrightarrow{\rho} (r, \gamma)]$ for which $[q, \beta] = [q', \beta']$ as points of $X$, the composition $h \circ g$ in $G$ is defined as follows: since $(q, \beta) \sim (q', \beta')$, there is an isomorphism $\tau: q^* \to q'^*$ so that $\beta' = \tau_S \circ \beta$. Define $h \circ g$ to be the equivalence class of

$$(p, \alpha) \xrightarrow{\rho \circ \tau_S \circ \theta} (r, \gamma).$$

It is easy to check that his definition does not depend on the choice of $\tau$, is again well–defined on equivalence classes, and is continuous for the given topology on $G$ and the fibred product topology on $G \times_X G$. Finally, the identity $u: X \to G$ and the inverse $i: G \to G$ are the obvious maps $u(p, \alpha) = [(p, \alpha) \xrightarrow{\mathrm{id}} (p, \alpha)]$ and $i[(p, \alpha) \xrightarrow{\theta} (q, \beta)] = [(q, \beta) \xrightarrow{\theta^{-1}} (p, \alpha)]$.

This completes the definition of the topological groupoid $G \rightrightarrows X$.

# 3 Review of locally connected maps

Before we turn to some basic properties of the groupoid $G \rightrightarrows X$, we need to recall some elementary properties of locally connected (or "locally 0-acyclic" [10]) maps between topological spaces. These properties are all analogous to well-known properties of locally connected maps of topoi. For spaces, however, the definitions and proofs are much simpler, and it seems worthwhile to give an independent presentation.

A continuous map $f: Z \to Y$ of topological spaces is called *locally connected* (l.c.) if $f$ is an open map, and $Z$ has a basis of open sets $\mathcal{B}$ with the property that for any $y \in Y$, and any basic open set $B \in \mathcal{B}$, the fibre $B_y = f^{-1}(y) \cap B$ is connected or empty.

**Lemma 3.1** *(i) The composition of two locally connected maps is l.c.*

*(ii) In a pullback square*

$$\begin{array}{ccc} Z' & \longrightarrow & Z \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y, \end{array}$$

*if $f$ is l.c. then so is $f'$.*

*(iii) Any local homeomorphism (sheaf projection) is l.c.*

*(iv) If the composition $f = p \circ g: Z \to E \to Y$ is l.c. and $p: E \to Y$ is a local homeomorphism then $g$ is l.c.*

*Proof.* These are all elementary. We just remark that for (i), one first proves that for a l.c. $f: Z \to Y$ with basis $\mathcal{B}$ as above, $f(S) \cap B$ is connected for every open $B \in \mathcal{B}$ and any connected subset $S \subset f(B)$. Then, if $g: Y \to W$ is another l.c. map with basis $\mathcal{A}$ for $Y$, the sets $B \cap f^{-1}(A)$, for $B \in \mathcal{B}$ and $A \in \mathcal{A}$ with $A \subset f(B)$, form a basis for $Z$ witnessing that $g \circ f$ is l.c. $\square$

**Proposition 3.2** *For any l.c. map $f: Z \to Y$ there exists a unique (up to homeomorphism) factorization*

$$Z \overset{c}{\twoheadrightarrow} \pi_0(f) \xrightarrow{p} Y, \qquad f = p \circ c,$$

*where $p$ is a local homeomorphism and $c$ is a l.c. map with connected fibres.*

*Proof.* We define the space $\pi_0(f)$: the points are pairs $(y, C)$ where $y \in Y$ and $C$ is a connected component of $f^{-1}(y)$. To define the topology on $\pi_0(f)$, let $\mathcal{B}$ be the collection of all those open sets $B \subset Z$ for which $B_y$ is empty or connected $(\forall y \in Y)$. Then $\mathcal{B}$ is a basis for $Z$. The basic open sets of $\pi_0(f)$ are now defined to be the sets

$$B^* = \{(y, [B_y]) \mid y \in f(B)\},$$

where $[B_y]$ is the connected component of $f^{-1}(y)$ which contains $B_y = f^{-1}(y) \cap B$, and $B$ ranges over all elements of $\mathcal{B}$.

To see that this is a basis, suppose $(y, C) \in B^* \cap A^*$. Thus, $A, B \in \mathcal{B}$ and $C \supset A_y \cap B_y$. Since $C$ is connected, there is a chain of basic open sets

$$A = B_0, B_1, \ldots, B_n = B$$

in $Z$ with the property that $B_i \cap B_{i+1} \cap C \neq \emptyset$ $(i = 0, \ldots, n-1)$. Now let

$$D = (B_0 \cup \cdots \cup B_n) \cap f^{-1}\Big(\bigcap_{i=0}^{n-1} f(B_i \cap B_{i+1})\Big).$$

Then $D \in \mathcal{B}$, and $(y, C) \in D^* \subset B^* \cap A^*$.

Now $f: Z \to Y$ factors into a map $c: Z \to \pi_0(f)$, $c(z) = (f(z), [z])$ where $[z]$ is the component of $f^{-1}f(z)$ containing $z$, and a map $p: \pi_0(f) \to Y$, $p(y, C) = y$. This map $p$ restricts to a homeomorphism $B^* \to f(B)$ for each $B \in \mathcal{B}$, hence is a local homeomorphism. Thus $c$ is a locally connected map by 3.1(iv) and the fibers of $c$ are evidently connected, since $c^{-1}(y, C) = C \subset f^{-1}(y)$.

The uniqueness of this factorization is easy, and we omit the proof. $\square$

**Corollary 3.3** *(i) Let $f: Z \to Y$ be a l.c. map. Then the pullback functor of sheaves*

$$f^*: \mathrm{Sh}(Y) \to \mathrm{Sh}(Z)$$

*has a left adjoint $f_!: \mathrm{Sh}(Z) \to \mathrm{Sh}(Y)$.*

*(ii) For any pullback square of topological spaces*

$$\begin{tikzcd} Z' \arrow[r, "b"] \arrow[d, "f'"'] & Z \arrow[d, "f"] \\ Y' \arrow[r, "a"'] & Y \end{tikzcd}$$

*and $f$ l.c. the projection formula $a^* f'_! = f_! b^*$ holds.*

One also says, that the "Beck–Chevalley condition" holds for this square.

*Proof.* In the proof, we identify the category of sheaves $E$ over $Z$ with that of local homeomorphisms $p: E \to Z$.

Ad (i). For a local homeomorphism $p: E \to Z$, the composite $f \circ p$ is l.c. by Lemma 3.1(i) and (iii), so factors uniquely as $f \circ p = (E \twoheadrightarrow \pi_0(f \circ p) \to Y)$ as in Proposition 3.2. Define $f_!(E)$ to be the sheaf $\pi_0(f \circ p) \to Y$. Thus, by construction, the stalk of $f_!(E)$ at $y$ is the set of connected components of $(f \circ p)^{-1}(y)$,

$$f_!(E)_y = \pi_0((fp)^{-1}(y)). \tag{4}$$

For adjointness, let $q: F \to Y$ be a sheaf on $Y$, and let $\alpha: E \to f^*(F) = F \times_Y Z$ be a map. Then $\bar{\alpha} = \pi_1 \circ \alpha: E \to F$ is a map over $Y$, i.e., $q \circ \bar{\alpha} = f \circ p$. Since the fibres of $q$ are discrete, $\bar{\alpha}$ is constant on the connected components of each fibre $(fp)^{-1}(y)$, hence factors uniquely as a map $f_!(E) \to F$.

Ad (ii). For a sheaf $E$ on $Z$, the adjointness of part (i) provides a canonical map $f_!b^*(E) \to a^*f'_!(E)$. It follows from (4) that this map is an isomorphism on each stalk, hence an isomorphism of sheaves. □

# 4 Properties of the groupoid

In this section, we present some properties of the groupoid $G \rightrightarrows X$, defined in §2, which will enter into the proof of the theorem, to be given in the next section.

Before going into this, we recall from [2] that the enumeration spaces $\mathrm{En}(A)$ described in §2 are all connected and locally connected; in fact, each basic open set of the form $V_u$ is connected. For a (small) point $p: \mathcal{S} \to \mathcal{E}$, the enumeration space $\mathrm{En}(S_p)$ is contained in our space $X$, via the obvious map

$$i_p: \mathrm{En}(S_p) \to X, \qquad i_p(\alpha) = (p, \alpha).$$

This map is a continuous injection (but not necessarily an embedding).

**Proposition 4.1** *The fibres of the source and target maps $s, t: G \to X$ are enumeration spaces: for a point $(p, \alpha) \in X$, there are homeomorphisms $s^{-1}(p, \alpha) \cong \mathrm{En}(S_p) \cong t^{-1}(p, \alpha)$.*

*Proof.* It suffices to prove this for the source map. Fix $(p, \alpha) \in X$. Note first that any point $(p, \alpha) \xrightarrow{\theta} (q, \beta)$ in $G$ is equivalent to the point $(p, \alpha) \xrightarrow{\mathrm{id}} (p, \theta_S^{-1} \circ \beta)$; in other words, each equivalence class has a representation of the form

$$(p, \alpha) \xrightarrow{\mathrm{id}} (p, \beta). \tag{5}$$

Thus, the evident map $j_{(p,\alpha)}: \mathrm{En}(S_p) \to G$ defined by $\beta \mapsto [(p, \alpha) \xrightarrow{\mathrm{id}} (p, \beta)]$ is a bijection into $s^{-1}(p, \alpha)$. Now consider a basic open set $V$ in $G$ as described in §2, of the form

$$V = \{(p, \alpha) \xrightarrow{\theta} (q, \beta) \mid \alpha(i) \in B_p, \beta(j) \in C_q, \theta(\alpha(i)) = \beta(j)\}.$$

Representing equivalence classes in the form (5), we can also write

$$V = \{(p,\alpha) \xrightarrow{\mathrm{id}} (p,\beta) \mid \alpha(i) \in B_p, \beta(j) \in C_q, \alpha(i) = \beta(j)\}.$$

Thus, for $(p,\alpha)$ fixed, and for $s_1 = \alpha(i_1), \ldots, s_n = \alpha(i_n)$, we see that

$$j^{-1}_{(p,\alpha)}(V) = \{\beta \in \mathrm{En}(S_p) \mid s_1 = \beta(i_1), \ldots, s_n = \beta(i_n)\}, \tag{6}$$

if $(s_1, \ldots, s_n) \in B_p \cap C_p$, and empty otherwise. But the right hand side of (6) exactly describes a standard basic open set (§2(2)) in the enumeration space $\mathrm{En}(S_p)$. □

**Proposition 4.2** *The source and target maps are locally connected.*

*Proof.* Again, it suffices to do one of the two, say the source map. Consider a basic open set $V \subset G$ as in the previous proof,

$$V = \{(p,\alpha) \xrightarrow{\mathrm{id}} (p,\beta) \mid \alpha(i) \in B_p, \beta(j) \in C_q, \alpha(i) = \beta(j)\}.$$

Then $s(V) = \{(p,\alpha) \mid \alpha(i) \in B_p$ and $\exists \beta \in \mathrm{En}(S_p)(\alpha(i) = \beta(j) \in C_p)\} = \{(p,\alpha) \mid \alpha(i) \in B_p \cap C_p\}$, and this is the union of basic open sets $V_u$ in $X$, where $u$ ranges over all partial functions $u: \{i_1, \ldots, i_n\} \to S_p$ with $u(i) = (u(i_1), \ldots, u(i_n)) \in B_p \cap C_p$. Thus $s$ is an open map. Furthermore, the proof of Proposition 4.1 shows that under the identification $\mathrm{En}(S_p) \cong s^{-1}(p,\alpha)$, the set $V \cap s^{-1}(p,\alpha)$ corresponds to a standard basic open set in $\mathrm{En}(S_p)$. In particular, $V \cap s^{-1}(p,\alpha)$ is connected. This shows that $s$ is a locally connected map. □

Next, we recall from [2] that there is a topos morphism

$$\varphi: \mathrm{Sh}(X) \to \mathcal{E},$$

described at the level of the stalks by

$$\varphi^*(E)_{(p,\alpha)} = E_p,$$

for each point $(p,\alpha) \in X$. The following lemma was proved in [2]:

**Lemma 4.3** *The functor $\varphi^*: \mathcal{E} \to \mathrm{Sh}(X)$ has a left adjoint $\varphi_!$. Furthermore, for each (small) point $p: \mathcal{S} \to \mathcal{E}$, there is a commutative (up to isomorphism) square*

$$\begin{tikzcd} \mathrm{Sh}(\mathrm{En}(S_p)) \arrow[r, "i_p"] \arrow[d, "\pi"'] & \mathrm{Sh}(X) \arrow[d, "\varphi"] \\ \mathcal{S} \arrow[r, "p"] & \mathcal{E} \end{tikzcd}$$

*for which the projection formula*

$$\pi_! i_p^* = p^* \varphi_!$$

*holds.*

*(Here $\mathcal{S} = \mathbf{Sets} = \mathrm{Sh}(\mathrm{pt})$, while $\pi$ and $i_p$ are induced by the continuous maps of spaces* $\mathrm{pt} \xleftarrow{\pi} \mathrm{En}(S_p) \xrightarrow{i_p} X$.*)*

**Proposition 4.4** *The source and target maps $s, t: G \rightrightarrows X$ fit into a square of topos morphisms*

$$\begin{tikzcd} \mathrm{Sh}(G) \arrow[r, "t"] \arrow[d, "s"'] & \mathrm{Sh}(X) \arrow[d, "\varphi"] \\ \mathrm{Sh}(X) \arrow[r, "\varphi"] & \mathcal{E} \end{tikzcd}$$

*which commutes up to a canonical isomorphism $\sigma: s^*\varphi^* \cong t^*\varphi^*$. Moreover, the projection formula holds for this square, i.e., the induced natural transformation*

$$\beta: s_! t^* \to \varphi^* \varphi_!$$

*is an isomorphism.*

*Proof.* For a point $g = [(p, \alpha) \xrightarrow{\theta} (q, \beta)]$ of $G$, and for any object $E$ of $\mathcal{E}$, we have

$$\begin{aligned} s^*\varphi^*(E)_g &= E_p, \\ t^*\varphi^*(E)_g &= E_q, \end{aligned}$$

and the stalk of the isomorphism $\sigma_E: s^*\varphi^*(E) \to t^*\varphi^*(E)$ at the point $g$ is defined to be the isomorphism $\theta_E: E_p \to E_q$. (It is easy to check that $\sigma_E$ is continuous, using the explicit description of the topology on $\varphi^*(E)$ given in [2].)

Next, we prove for each sheaf $F$ on $X$ that $s_! t^*(F) = \varphi^*\varphi_!(F)$ (or more precisely, that the canonical map $s_! t^*(F) \to \varphi^*\varphi_!(F)$ is an isomorphism.) It suffices to check that $s_! t^*(F)_x = \varphi^*\varphi_!(F)_x$ for the stalks at an arbitrary point $x = (p, \alpha)$ in $X$. Consider for this the diagram

$$\begin{tikzcd} \mathrm{Sh}(\mathrm{En}(S_p)) \arrow[r, "j_{(p,\alpha)}"] \arrow[d, "\pi"'] & \mathrm{Sh}(G) \arrow[r, "t"] \arrow[d, "s"] & \mathrm{Sh}(X) \arrow[d, "\varphi"] \\ \mathcal{S} \arrow[r, "x"] & \mathrm{Sh}(X) \arrow[r, "\varphi"] & \mathcal{E} \end{tikzcd} \tag{7}$$

Here the left-hand square comes from a pullback of topological spaces (Proposition 4.1), and $j_{(p,\alpha)}$ is as defined in the proof of 4.1. Since $s$ is locally connected by Proposition 4.2, Corollary 3.3(ii) gives the projection formula

$$x^* s_! = \pi_! j^*_{(p,\alpha)} \tag{8}$$

for the left hand square in (7). Moreover, since $t \circ j_{(p,\alpha)} = i_p$ and $\varphi \circ x = p$, Lemma 4.3 gives that

$$(\varphi \circ x)^* \varphi_! = \pi_! (t \circ j_{(p,\alpha)})^* \tag{9}$$

for the composed rectangle. Thus

$$\begin{aligned} s_! t^*(F)_x &= x^* s_! t^*(F) \\ &= \pi_! j^*_{(p,\alpha)} t^*(F) \quad \text{(by (8))} \\ &= x^* \varphi^* \varphi_!(F) \quad \text{(by (9))} \\ &= \varphi^* \varphi_!(F)_x, \end{aligned}$$

□

# 5 Proof of the theorem

We will now prove the theorem stated in §1, and repeated here in the following form.

**Theorem 5.1** *The functor* $\varphi^*: \mathcal{E} \to \mathrm{Sh}(X)$ *induces an equivalence of categories* $\mathcal{E} \cong \mathrm{Sh}_G(X)$.

*Proof.* By definition of the set of small points $p$ of $\mathcal{E}$, the functor $\varphi^*$ is faithful. It follows that $\varphi^*$ induces an equivalence between $\mathcal{E}$ and the category of coalgebras for the comonad $\varphi^*\varphi_*$ on $\mathrm{Sh}(X)$ (see e.g. [9]). By standard category theory ([3, 9]) the latter category is in turn equivalent to that of algebras for the monad $\varphi^*\varphi_!$ on $\mathrm{Sh}(X)$. Thus, to prove the theorem, it suffices to show that for any sheaf $F$ on $X$, algebra structures

$$\tau: \varphi^*\varphi_!(F) \to F \tag{10}$$

are in bijective correspondence to groupoid actions

$$\mu: F \times_X G \to F. \tag{11}$$

By the projection formula $\varphi^*\varphi_!(F) = s_!t^*(F)$, maps $\tau$ as in (10) correspond to maps $s_!t^*(F) \to F$ over $X$, and hence to maps $\tilde{\tau}: t^*(F) \to s^*(F)$ over $G$, since $s_!$ is left adjoint to $s^*$. By composing with the projection $s^*(F) = G \times_X F \to F$, these maps $\tilde{\tau}$ correspond to maps

$$\tau^*: F \times_X G = s^*(F) \to F.$$

For an arrow $g$ in $G$ and a point $\xi \in F_{s(g)}$, we write

$$\xi \cdot g =_{\mathrm{def}} \tau^*(\xi, g). \tag{12}$$

To prove of the theorem, it now suffices to verify that $\tau$ in (10) satisfies the unit and associativity axioms for an algebra structure if and only if the corresponding multiplication (12) satisfies the unit and associativity laws for an action.

To this end we first make the correspondence between (10) and (12) more explicit: For a point $(p, \alpha) \in X$, we have by 4.3 and 4.4,

$$\begin{aligned} \varphi^*\varphi_!(F)_{(p,\alpha)} &= \pi_! i_p^*(F) \\ &= \text{the set of connected components of } i_p^*(F). \end{aligned}$$

So for a point $(p, \beta) \in X$, any $\xi \in F_{(p,\beta)}$ defines a connected component $[\xi] \in i_p^*(F)$, and $\tau_{(p,\alpha)}([\xi])$ then defines a point in $F_{(p,\alpha)}$. Now let $g$ be any arrow in $G$. Since $(p, \alpha) \xrightarrow{\theta} (q, \beta)$ is equivalent to (i.e., defines the same point of $G$ as) $(p, \alpha) \xrightarrow{\mathrm{id}} (p, \theta_S \circ \beta)$ we may represent $g$ in the form

$$g = [(p, \alpha) \xrightarrow{\mathrm{id}} (p, \beta)].$$

Then, for $\xi \in F_{(p,\beta)}$, the action (12) is defined from $\tau$ by

$$\xi \cdot g = \tau_{(p,\alpha)}([\xi]). \tag{13}$$

For $\tau$, the laws for an algebra structure assert that for any $\zeta \in F_{(p,\gamma)}$,

$$\tau_{(p,\gamma)}([\zeta]) = \zeta \tag{14}$$

$$\tau_{(p,\alpha)}([\tau_{(p,\beta)}([\zeta])]) = \tau_{(p,\alpha)}([\zeta]). \tag{15}$$

But clearly, (14) states that $\zeta \cdot 1 = \zeta$, where 1 is the identity arrow $[(p,\gamma) \xrightarrow{\text{id}} (p,\gamma)]$ in $G$, while (15) states that $(\zeta \cdot h) \cdot g = \zeta \cdot (h \circ g)$, where $g$ and $h$ are the arrows in $G$ represented by

$$[(p,\alpha) \xrightarrow{\text{id}} (p,\beta)] \qquad \text{and} \qquad [(p,\beta) \xrightarrow{\text{id}} (p,\gamma)],$$

respectively. Since any composable pair of arrows in $G$ can be represented in this form, (15) is equivalent to the associativity condition for the action by $G$ on $F$, and the theorem is proved. $\square$

## 6 The action by the group $\mathrm{Aut}(I)$

We conclude this paper with some remarks on the action by the group $H = \mathrm{Aut}(I)$ of bijections $\pi\colon I \to I$, from the set $I$ to itself. There is a natural continuous action of $H$ on the space $X$, defined explicitly by

$$(p,\alpha) \cdot \pi = (p, \alpha \circ \pi),$$

and which is well–defined on equivalence classes.

Let $\mathrm{Sh}_H(X)$ denote the topos of $H$–equivariant sheaves on $X$. We observe first that each sheaf $\varphi^*(E)$ on $X$ carries a natural action by $H$, so that from $\varphi$ one obtains a topos morphism $\psi$:

$$\psi\colon \mathrm{Sh}_H(X) \to \mathcal{E}.$$

Explicitly, $\psi^*(E)$ is the same sheaf on $X$ as $\varphi^*(E)$,

$$\psi^*(E) = \{(p,\alpha,e) \mid (p,\alpha) \in X,\ e \in E_p\}$$

and $H$ acts on $\psi^*(E)$ by acting trivially in the $e$–coordinate.

We will prove the following proposition:

**Proposition 6.1** *The morphism $\psi\colon \mathrm{Sh}_H(X) \to \mathcal{E}$ has the following properties:*

*(i) $\psi^*$ is full and faithful.*

*(ii) $\psi^*$ commutes with exponentials, i.e., the canonical map*

$$\psi^*(F^E) \to \psi^*(F)^{\psi^*(E)}$$

*is an isomorphism, for any two sheaves $E$ and $F$ in $\mathcal{E}$.*

*(iii) $\psi^*$ is bijective on subsheaves, i.e., for any sheaf $E$ in $\mathcal{E}$, $\psi^*$ induces an isomorphism*

$$\mathrm{Sub}_{\mathcal{E}}(E) \to \mathrm{Sub}_H(\psi^*(E)).$$

Here $\mathrm{Sub}_{\mathcal{E}}(E)$ is the set of subsheaves of $E$ in $\mathcal{E}$, while $\mathrm{Sub}_H(\psi^*(E))$ is the set of $H$–invariant subsheaves of $\psi^*(E)$.

Property (i) is actually a consequence of property (iii). Using standard terminology ([6]), (i) expresses that $\psi$ is connected, (iii) that it is hyperconnected. Since any hyperconnected (in fact, any open) morphism with property (ii) is locally connected, we can rephrase Proposition 6.1 as

**Corollary 6.2** *The morphism $\psi\colon \mathrm{Sh}_H(X) \to \mathcal{E}$ is locally connected and hyperconnected.*

Note that $\mathrm{Sh}_H(X)$ is an étendue ([1]). Thus any topos with enough points admits a locally connected hyperconnected cover from an étendue. This is related to a result of [12], stating that any topos (not necessarily with enough points) admits a hyperconnected morphism from a ("localic") étendue.

*Proof of Proposition 6.1.* As said, (i) is a consequence of (iii). For (ii), note first that $\varphi$ naturally factors as

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[r, "\rho"] \arrow[dr, "\varphi"'] & \mathrm{Sh}_H(X) \arrow[d, "\psi"] \\ & \mathcal{E}. \end{tikzcd}$$

The inverse image $\rho^*$ simply 'forgets' the $H$–action. In particular, $\rho^*$ is conservative (i.e., reflects isomorphisms). Moreover, since $H$ is a discrete group, $\rho^*$ preserves exponentials. (In fact, $\rho$ is an 'atomic' morphism.) To show that $\psi^*$ preserves exponentials, it therefore suffices to show that $\varphi^*$ does. We prove this in a separate lemma.

**Lemma 6.3** *The functor $\varphi^*\colon \mathcal{E} \to \mathrm{Sh}(X)$ preserves exponentials.*

*Proof.* This follows from Theorem 5.1 and [11], Theorem 3.6(b), because $s, t\colon G \rightrightarrows X$ are locally connected. Alternatively, from the explicit description of $\varphi_!$ in [2] together with Lemma 4.3, one easily checks that the canonical map $\varphi_!(S \times \varphi^* E) \to \varphi_!(S) \times E$ is an isomorphism for each sheaf $S$ on $X$ and each $E \in \mathcal{E}$. It then follows in the standard way by adjointness that $\varphi^*$ preserves exponentials. $\square$

We return to Proposition 6.1, and prove part (iii). Let $E \in \mathcal{E}$, and let $S \subset \varphi^*(E)$ be an arbitrary subsheaf. We have to show that if $S$ is $H$–invariant, then $S = \varphi^*(U)$ for a (necessarily unique) subsheaf $U \subset E$. By Theorem 5.1, it suffices to prove that if $S$ is $H$–invariant then it is also $G$–invariant. To this end, consider any arrow $g$ in $G$. As in the the proof of 5.1, it can be represented in the form $g = [(p, \alpha) \xrightarrow{\mathrm{id}} (p, \beta)]$.

Let $e \in E_p$ and assume that $s = (p, \beta, e) \in S_{(p,\beta)} \subset E_{(p,\beta)} \simeq E_p$. We have to show that $s \cdot g = (p, \alpha, e) \in S_{(p,\alpha)}$. Choose a section $\sigma: U_{i,C} \to S$ through $s$, defined on a basic open neighbourhood $U_{i,C}$ of $(p, \beta)$. By the description of the topology on $\varphi^*(E)$ in [2], we may assume that $\sigma$ is of the form $\sigma(q, \gamma) = f_q(\gamma(i))$ where $f: C \to E$ in $\mathcal{E}$. In particular, $s = \sigma(p, \beta) = f_p(\beta(i))$.

Now choose $\pi \in H$ so that $\alpha \circ \pi(i) = \beta(i)$. Then $(p, \alpha \circ \pi(i)) \in U_{i,C}$, so $(p, \alpha \circ \pi, e) = \sigma(p, \alpha \circ \pi) \in S_{(p,\alpha\circ\pi)}$. By invariance of $S$ under the action by $\pi$, also $(p, \alpha, e) \in S_{(p,\alpha)}$, as was to be shown. $\square$

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