Wellfounded trees in categories

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1 Introduction

This is the first of a series of papers in which we study type-theoretic constructs in categories. Our general purpose is to exhibit various analogies between categorical logic and type theory. More specifically, one of our goals is to show how methods of topos theory (such as sheaf and realizability of interpretations of higher order logic) apply to Martin-Löf type theories of the kind presented in [15] and yield similar inner models of such systems of weak proof-theoretic strength. Earlier work in this direction has been done, e.g., by Grayson [6] and lately by Coquand and the second author [2]. Another, related, goal is to describe the constructions of such predicative type theories in categorical terms, so as to arrive at a notion of "predicative topos" which bears the same relation to such type theories as elementary toposes do to extensions of (impredicative) full higher order intuitionistic arithmetic.

We will take as our starting point Seely's correspondence [18] between locally cartesian closed categories and a rudimentary version of Martin-Löf type theory with dependent sums and products. (See also [10].) This correspondence is not very precise, in fact there are coherence problems related to the interpretation of substitution, but there are various ways to avoid these problems, e.g., by explicitly interpreting substitution operations (Curien [3]) or modifying the locally cartesian closed category to obtain a split fibration (Bénabou [1], Hofmann [7]). Adding first order logic, binary sums and quotients of equivalence relations, one obtains a type theory which corresponds to the notion of a locally cartesian closed pretopos, or pretopos with dependent products. Such pretoposes will be our basic structures, in the context of which we will discuss additional type theoretic constructions from [15]. In particular, in [16] we will discuss Martin-Löf's theory of universes and categorical models of Aczel's constructive set theory CZF, while in the present paper we will concentrate on so-called W-types. In [17], we will discuss how these two constructions behave under (a categorical version of) the passage from intensional to extensional type theories.

We recall that in type theory, the W-type construction defines the type of wellfounded trees with a given branching type. In this paper, we give an abstract categorical characterization of W-types. We calculate these W-types explicitly in some categories of presheaves and sheaves on a site, and in the gluing category or Freyd cover. (We also have an explicit description in the case of Hyland's realizability topos, which will be presented in [17].) These explicit calculations can be formalized in a weak predicative metatheory, and lead to the result that if $\mathcal{E}$ is any suitably filtered pretopos with dependent products and W-types, then so is the category of internal sheaves on a site in $\mathcal{E}$ (Remark 5.9).

Our paper is organized as follows. In Section 2 we review some standard definitions concerning pretoposes and dependent products. In Section 3 we present the categorical definition of the W-construction, and in Section 4 we prove some of its basic functoriality properties; e.g., that it turns coequalizers into equalizers. In Section 5, a construction is presented which to each map between (pre)sheaves of sets associates a sheaf of wellfounded trees, and it is proved that this is in fact the W-type in the category (pre)sheaves of sets (Theorem 5.6). In Section 6, we discuss the W-construction for the Freyd cover. Finally, in Section 7 it is shown how these categorical constructions are not only analogous to but explicitly related to Martin-Löf type theory.

2 Pretoposes and dependent products

In this preliminary section we review some familiar definitions concerning the basic structures we shall work with. Recall that a structure sufficient to interpret first order intuitionistic logic (with sums and quotients) is that of a *Heyting pretopos*. For the convenience of the reader, we provide the definition below. But first we give the categorical formulation of equivalence relations and their quotients.

Definition 2.1 Let $\mathcal{C}$ be a category with finite limits. An *equivalence relation* on an object X of $\mathcal{C}$ is a subobject $\langle \partial_0, \partial_1 \rangle : R \rightarrowtail X \times X$ with the property that for any object Y of $\mathcal{C}$, the set defined by

$$\{(\partial_0 k, \partial_1 k) \mid k : Y \longrightarrow R\}$$

is an equivalence relation (in the usual sense) on $\mathrm{Hom}(Y, X)$.

Definition 2.2 A diagram

$$R \underset{g}{\overset{f}{\rightrightarrows}} X \xrightarrow{h} Y \tag{1}$$

is *exact,* if it is a coequalizer diagram and $R \underset{g}{\overset{f}{\rightrightarrows}} X$ is the kernel pair of $X \xrightarrow{h} Y$ (i.e. $R \cong X \times_Y X$ as subobjects of $X \times X$).

Definition 2.3 A category $\mathcal{C}$ is a *pretopos* if it satisfies the following conditions P1-4.

(P1) $\mathcal{C}$ has all finite limits.

(P2) $\mathcal{C}$ has finite sums, and these are disjoint and stable.

(Disjointness means that for a finite sum $Y = Y_1 + \cdots + Y_n$, the pullback $Y_i \times_Y Y_j$ isomorphic to the initial object 0, whenever $i \neq j$. Moreover stability means that for any family $\{f_i : Y_i \to X \mid i = 1, \ldots, n\}$, $n \geq 0$, and any arrow $X' \to X$, the canonical map $\Sigma(X' \times_X Y_i) \to X' \times_X \Sigma Y_i$ is an isomorphism.)

(P3) For any equivalence relation $R \rightrightarrows X$ there exists some arrow $X \to Y$ for which $R \rightrightarrows X \longrightarrow Y$ is exact.

(P4) If $R \rightrightarrows X \longrightarrow Y$ is exact, then for any arrow $Z \to Y$ the diagram

$$Z \times_Y R \rightrightarrows Z \times_Y X \longrightarrow Z \times_Y Y = Z$$

is again exact.

Remark 2.4 Often an extra axiom is assumed in the definition of a pretopos: for any epi $X \to Y$ there exists $R \rightrightarrows X$ such that $R \rightrightarrows X \longrightarrow Y$ is exact. However, A. Carboni has pointed out to us that this axiom is a consequence of (P1–4). (See also Freyd and Scedrov [5, p. 111].) Note that from this extra axiom and (P4), it immediately follows that in a pretopos the pullback of an epi along any map is an epi.

Definition 2.5 Let $\mathcal{C}$ be a pretopos. Let $\mathrm{Sub}_{\mathcal{C}}(X)$ be the partial order of subobjects of X in $\mathcal{C}$. For any $\mathcal{C}$-morphism $f : X \to Y$ there is a pullback map $f^{-1} : \mathrm{Sub}_{\mathcal{C}}(Y) \to \mathrm{Sub}_{\mathcal{C}}(X)$. The pretopos $\mathcal{C}$ is said to be *Heyting* if every such pullback map f^{-1} has a right adjoint

$$\forall_f : \mathrm{Sub}_{\mathcal{C}}(X) \to \mathrm{Sub}_{\mathcal{C}}(Y).$$

We remark that for a Heyting pretopos $\mathcal{E}$, the slice category $\mathcal{E}/X$ is again a Heyting pretopos (for any object in X in $\mathcal{E}$). Moreover, for any map $\alpha : Y \to X$ in $\mathcal{E}$, the pullback (or "substitution") functor $\alpha^* : \mathcal{E}/X \to \mathcal{E}/Y$ preserves the Heyting pretopos structure.

As pointed out above, the "internal logic" of Heyting pretoposes is exactly first order intuitionistic logic (with sums and quotients). We will often exploit this fact and describe constructions in a Heyting pretopos $\mathcal{E}$ by logical or set-theoretic notation. For example, for arrows $X \stackrel{f}{\leftarrow} Y \stackrel{g}{\to} Z \stackrel{h}{\leftarrow} A$, the image along f of the pullback $Y \times_Z A$ could be denoted

$$\{x \in X \mid \exists y \in Y\, \exists a \in A\, f(y) = x \,\&\, g(y) = h(a)\}.$$

It is well-known that the substitution functor $\alpha^* : \mathcal{E}/X \to \mathcal{E}/Y$ given by pullback along $\alpha : Y \to X$ always has a left adjoint. This left adjoint is usually denoted

$$\Sigma_\alpha : \mathcal{E}/Y \longrightarrow \mathcal{E}/X$$

and described by composition with α. A pretopos $\mathcal{E}$ is said to have *dependent products* if each substitution functor also has a right adjoint

$$\Pi_\alpha : \mathcal{E}/Y \longrightarrow \mathcal{E}/X.$$

This is certainly the case if $\mathcal{E}$ is the category of sets, where for a map $A \stackrel{u}{\to} Y$, the right adjoint $\Pi_\alpha u$ is the set over X with fiber

$$(\Pi_\alpha u)_x = \prod_{y \in \alpha^{-1}(x)} A_y \tag{2}$$

(here $A_y = u^{-1}(y)$ is the fiber of u). More generally, any elementary topos $\mathcal{E}$ has dependent products. For a pretopos with dependent products, we will often describe these informally using set-theoretic notation, such as (2) or a variant thereof.

Remark 2.6 For a pullback square

$$\begin{array}{ccc} Y' & \xrightarrow{\gamma} & Y \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ X' & \xrightarrow[\beta]{} & X \end{array}$$

the left adjoints Σ_α and $\Sigma_{\alpha'}$ satisfy a "Beck–Chevalley" identity $\beta^*\Sigma_\alpha = \Sigma_{\alpha'}\gamma^*$ (where $=$ denotes canonical isomorphism). It follows by taking right adjoints that the Π-functors satisfy the identity

$$\alpha^*\Pi_\beta = \Pi_\gamma \alpha'^*.$$

In other words, substitution preserves dependent products.

We also recall that for arrows $Z \stackrel{\beta}{\longrightarrow} Y \stackrel{\alpha}{\longrightarrow} X$ in a pretopos $\mathcal{E}$ with dependent products (in fact, in any category with pullbacks and dependent products), the operations Σ_β and Π_α satisfy a distributivity law of the form $\Pi_\alpha\Sigma_\beta A = \Sigma\Pi A'$ for any map $A \to Z$. In set-theoretic notation this is the familiar identity

$$\prod_{y \in \alpha^{-1}(x)} \sum_{z \in \beta^{-1}(y)} A_z = \sum_\tau \prod_{y \in \alpha^{-1}(x)} A_{\tau(y)} \tag{3}$$

where τ ranges over functions $\alpha^{-1}(x) \to Z$ with $\beta\tau = 1$. Categorically, it can be written as

$$\Pi_\alpha\Sigma_\beta A = \Sigma_\pi\Pi_{\alpha'}\mathrm{ev}^*(A),$$

where we use the notation $\Pi_\alpha\beta = (P \xrightarrow{\pi} X)$ and $\mathrm{ev} : P \times_X Y \to Z$ for the evaluation, $\alpha' : P \times_X Y \to P$ for the projection. In the context of the propositions-as-types interpretation [15], Martin-Löf refers to the distributivity law (3) as the axiom of choice.

Remark 2.7 A pretopos $\mathcal{E}$ has dependent products if, and only if, each slice $\mathcal{E}/X$ is cartesian closed (i.e. $\mathcal{E}$ is locally cartesian closed). In particular, if $\mathcal{E}$ has dependent products it is a Heyting pretopos. Indeed, for subobjects $A, B \subseteq X$, the implication $(A \Rightarrow B) \subseteq X$ is the exponential of $(B \rightarrowtail X)$ and $(A \rightarrowtail X)$ in $\mathcal{E}/X$. For $A \subseteq X$ and $f : X \to Y$, the universal quantifier is $\forall_f(A) = \Pi_f(A \to X)$.

Remark 2.8 A pretopos $\mathcal{E}$ with dependent products and a natural numbers object has all finite colimits. In fact, since $\mathcal{E}$ already has coproducts, and coequalizers of equivalence relations, it is enough to be able to define the transitive, symmetric closure R^* of a relation $R \subseteq X^2$. In the internal logic of $\mathcal{E}$ this closure can be expressed by letting $R^*(x, y)$ be

$$(\exists n \in N)\,(\exists h \in X^N)\,[h(0) = x \,\&\, h(n) = y \,\&\, \\ (\forall i < n)\,(R(h(i), h(i+1)) \text{ or } R(h(i+1), h(i)))].$$

Remark 2.9 We shall mainly be interested in non-boolean categories, since any boolean pretopos $\mathcal{E}$ with dependent products is a topos. (A subobject classifier for $\mathcal{E}$ is given by $t : 1 \to 1 + 1$.)

2.10 Projectives. The following notions will be needed in Section 4. Let $\mathcal{E}$ be a pretopos with dependent products. Recall that an object P in $\mathcal{E}$ is called *projective* if $\mathrm{Hom}_{\mathcal{E}}(P, -)$ preserves epimorphisms; in other words, for any epi $e : Y \to X$ and any $\alpha : P \to X$ there is a $\beta : P \to Y$ with $e\beta = \alpha$. Using the axioms of a pretopos one can show that P is projective exactly when every epi $e : X \twoheadrightarrow P$ has a section, i.e. there is some $s : P \to X$ with $es = 1_P$. The object P is said to be *internally projective* if the internal hom functor, i.e. the exponential functor $(-)^P : \mathcal{E} \to \mathcal{E}$, preserves epis. This means that, in the internal logic of $\mathcal{E}$, the axiom of choice is valid for quantifier combinations of the form $\forall p \in P\, \exists y(\cdots)$. For this reason, one also calls an internally projective object P in $\mathcal{E}$ a "choice object", and an internally projective object $B \to A$ of $\mathcal{E}/A$ a "choice map".

3 Wellfounded trees

A W-type is a direct generalization of the free term algebra from finite arities to arbitrary arities (specified by a signature), and is thus an algebra of possibly infinite wellfounded trees. In this section we study W-types in a pretopos $\mathcal{E}$ with dependent products, although many definitions also make sense in any locally cartesian closed category.

3.1 Algebras. Let $T : \mathcal{E} \to \mathcal{E}$ be any endofunctor. Recall that a T-*algebra* is an object X of $\mathcal{E}$ equipped with a map $\mu : TX \to X$. A map between two such algebras

$$h : (X, \mu) \longrightarrow (Y, \nu)$$

is a map $h : X \to Y$ in $\mathcal{E}$ which preserves the operations in the sense that $\nu \circ Th = h \circ \mu$. This defines a category of T-algebras $\mathrm{Alg}_T(\mathcal{E})$. The *free* T-algebra is an initial object of $\mathrm{Alg}_T(\mathcal{E})$. (It need not exist.) A result of Lambek asserts that for the free algebra (W, η), the structure map $\eta : TW \to W$ is an isomorphism [11].

3.2 Algebras for polynomial endofunctors. Any map $f : B \to A$ in a pretopos $\mathcal{E}$ with dependent products defines a "polynomial" endofunctor P_f, by

$$P_f(X) = \sum_{a \in A} X^{B_a},$$

where $B_a = f^{-1}(a)$ is the fiber of f over a, as before. More explicitly, $P_f(X)$ is the total space of the exponential

$$(X \times A \xrightarrow{\pi_2} A)^{(B \xrightarrow{f} A)}$$

in $\mathcal{E}/A$. A P_f-algebra X should be thought of as an object X together with, for each $a \in A$, an $f^{-1}(a)$-ary operation

$$\mu_a : X^{f^{-1}(a)} \longrightarrow X.$$

Remark 3.3 Although we will not use the following in this paper, we would like to point out that, more generally, any map

$$\begin{array}{ccc} B & \xrightarrow{\ f\ } & A \\ & \searrow \quad \swarrow & \\ & I & \end{array}$$

in $\mathcal{E}/I$ gives rise to a "family of polynomial functors"

$$P_f^{(I)} : \mathcal{E} \to \mathcal{E}/I$$

defined in the obvious way: $P_f^{(I)}(X)_i = P_{f_i}(X)$ where $f_i : B_i \to A_i$. If $P : \mathcal{E} \to \mathcal{E}/I$ is a family of polynomial functors, then for any $\alpha : I \to J$, so are $\Sigma_\alpha \circ P$, and, by the distributivity law of Remark 2.6, also $\Pi_\alpha \circ P$. It is easy to see that the families of polynomial functors form the smallest class of functors $\mathcal{E} \to \mathcal{E}/I$ (for varying I) closed under Σ and Π in this sense, and containing the pullback functor $\mathcal{E} \to \mathcal{E}/I$ for each object I. This remark gives a generalization of a result of Dybjer [4].

Definition 3.4 The initial algebra of a polynomial functor P_f, if it exists, is called the *(extensional) W-type* for the map f and is denoted

$$\mathrm{W}(f).$$

The map f is called the *branching data* or the *signature* of the W-type.

We will show below that these W-types are preserved by "slicing" $\mathcal{E} \to \mathcal{E}/I$. Hence we can use the set-theoretic notation (i.e. the internal language of $\mathcal{E}$) to describe properties of W-types. Thus, since $\mathrm{W}(f)$ is a P_f-algebra, it has for each $a \in A$, an operation $\mathrm{W}(f)^{f^{-1}(a)} \longrightarrow \mathrm{W}(f)$, which we denote by $\sup_a(-)$ or $\sup(a,-)$. The freeness of $\mathrm{W}(f)$ can then be expressed by the fact that for any other P_f-algebra (X, μ) there is a unique map $\varphi : \mathrm{W}(f) \to X$ with the property that

$$\varphi(\sup_a(t)) = \mu_a(\varphi \circ t) \tag{4}$$

for any $a \in A$ and any $t : f^{-1}(a) \to \mathrm{W}(f)$. We think of φ as defined "by induction": if φ has already been defined on the values of t, then φ is defined on $\sup_a(t)$ by (4). Note that Lambek's result states in this case that every $x \in W(f)$ is of the form $\sup_a t$ for unique $a \in A$ and $t : f^{-1}(a) \to W(f)$.

Notice also that by initiality of $W(f)$, any subalgebra $R \subseteq W(f)$ coincides with $W(f)$. We will use this in the internal logic of $\mathcal{E}$ as an induction principle, stating that if

$$(\forall a \in A)(\forall t : f^{-1}(a) \to W(f))[(\forall b \in f^{-1}(a))t(b) \in R \to \sup_a(t) \in R]$$

then

$$(\forall x \in W(f))\, x \in R.$$

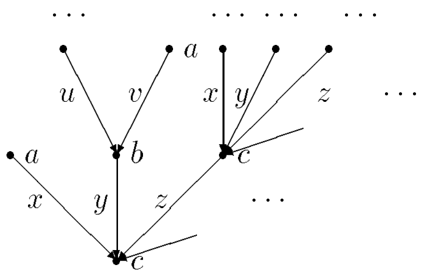
The principle implies a useful double induction principle for $Q \subseteq W(f) \times W(f)$: To prove $Q = W(f) \times W(f)$ it suffices to show that for all relevant a, a', t, t':

$$(\forall b \in f^{-1}(a))\,(\forall b' \in f^{-1}(a'))\,(t(b), t'(b')) \in Q \Rightarrow (\sup_a(t), \sup_{a'}(t')) \in Q.$$

(This is seen by considering the subobject $R = \{x \in W(f) : (\forall y \in W(f))\,(x, y) \in Q\}$.)

Examples 3.5 (a) In **Sets**, $\mathrm{W}(f)$ exists and can be described explicitly as the set of wellfounded trees with nodes labelled by elements a of A, and edges into a node labelled a enumerated by the elements of $f^{-1}(a)$.

$f^{-1}(a) = \emptyset$
$f^{-1}(b) = \{u, v\}$
$f^{-1}(c) = \{x, y, z, \ldots\}$
$\ldots$



For example, if f is $\{1\} \hookrightarrow \{0, 1\}$, then $\mathrm{W}(f) \cong \mathbb{N}$. And if f is $|\cdot| : \{-1, 1\} \longrightarrow \{0, 1\}$, then $\mathrm{W}(f)$ is the set of finite binary trees.

(b) The *Brouwer ordinals* are built from a tree T_b whose branching at each node is indexed by a singleton or the natural numbers. Define a function $g : \omega \longrightarrow \{0, 1, 2\}$ by letting $g(x) = 1 + \min(x, 1)$. Then $T_b = \mathrm{W}(g)$.

Led by the description in Example 3.5.(a), it is easy to see that a tree can be coded as a set of finite sequences of elements from $A + B$. By using the higher order logic of an elementary topos we can then define the set of wellfounded such trees. This leads to the following proposition. Here and below, we always assume an elementary topos to have a natural numbers object.

Proposition 3.6 *W-types exist in any elementary topos.*

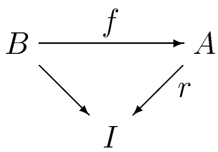
Definition 3.7 A pretopos $\mathcal{E}$ with dependent products is said to *have W-types* if for any map $f : B \longrightarrow A$ in $\mathcal{E}$, the free algebra $\mathrm{W}(f)$ exists.

In order to be able to use W-types in the internal logic and to exploit the corresponding induction principles, one needs the existence of W-types in all the slices $\mathcal{E}/I$ and the preservation of W-types by the pullback $\mathcal{E}/I \longrightarrow \mathcal{E}/J$ along any map $J \longrightarrow I$. The following proposition, due to A. Simpson, establishes this.

Proposition 3.8 *If $\mathcal{E}$ is a pretopos with dependent products and all W-types, then $\mathcal{E}/X$ has all W-types, for every $X \in \mathcal{E}$. Moreover, for any map $J \longrightarrow I$, the change-of-base functor $\mathcal{E}/I \longrightarrow \mathcal{E}/J$ preserves W-types.*

Proof. In this proof we will make use of the covariant functoriality of W-types, as described in the first few lines of Section 4.1 below.

Consider a map $B \longrightarrow A$ over I,



and write $f_i : B_i \to A_i$ for the fiber over $i \in I$. One can construct the W-type in the slice $\mathcal{E}/I$

$$W_I(f) = \sum_{i \in I} W(f_i)$$

from the "global" W-type $W(f)$, as follows. Construct the equalizer

$$\mathrm{W}_I(f) \xrightarrow{\ \varepsilon\ } \mathrm{W}(f) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathrm{W}(f \times I) \tag{5}$$

where α is the map induced by the pullback

$$\begin{array}{ccc} B & \xrightarrow{(1, rf)} & B \times I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times I} \\ A & \xrightarrow[(1, r)]{} & A \times I \end{array}$$

i.e. $\alpha = (1, r)_!$ in the notation of Section 4.1, and β is informally constructed as

$$\beta(t) = (1, k(t))_!(t),$$

where $k(t)$ is the map $A \to I$ which is constant with value $r(\rho(t))$, and $\rho(t) \in A$ is the label of the root of t. Since we cannot (yet) use the internal language to reason about the universal property of W-types, we must construct β explicitly. Let $R = I^A$. Then define a map $\psi_X : P_f(X^R) \times R \to P_{f \times I}(X)$ by letting

$$((a, t), g) \mapsto ((a, g(a)), u \mapsto t(\pi_1(u))(g)).$$

If $\eta = \eta_{f \times I} : P_{f \times I}(W(f \times I)) \to W(f \times I)$ denotes the canonical isomorphism, we have

$$\eta\psi_{W(f \times I)} : P_f(W(f \times I)^R) \times R \to W(f \times I).$$

Thus the transpose $\widehat{\eta\psi} : P_f(W(f \times I)^R) \to W(f \times I)^R$ yields a P_f-algebra. Let $\hat{h} : W(f) \to W(f \times I)^R$ be the universal map for this algebra, and we have informally

$$h(t, g) = (1, g)_!(t).$$

Then $\beta = h \circ (1, \hat{k})$ where $k = r\rho\pi_1 : W(f) \times A \to I$.

It is clear that $W_I(f) = \sum_{i \in I} W(f_i)$ "set-theoretically", i.e. it is the collection of trees whose branching type is constant in I.

To check the universal property, notice that for any $P_f^{(I)}$-algebra $X \to I$ in the slice $\mathcal{E}/I$, i.e. any family $(X_i : i \in I)$, X_i a P_{f_i}-algebra, we can construct a P_f-algebra $\tilde{X}$ in $\mathcal{E}$, as follows

$$\tilde{X} = \sum_{t \in W(f)} X_{r(\rho(t))}^{[\![t \in W_I(f)]\!]}.$$

Here $[\![t \in W_I(f)]\!]$ is the "truth-value", obtained by pulling back $\varepsilon : W_I(f) \to W(f)$ along $t : 1 \to W(f)$. The structure of a P_f-algebra on $\tilde{X}$ is obtained as accordingly: given $a \in A$ and $\varepsilon : f^{-1}(a) \to \tilde{X}$, define $x = \sup_a(\varphi) \in \tilde{X}$ as follows. First, by projection $\tilde{X} \to W(f)$, φ induces a map

$$\varphi_0 : B_a = f^{-1}(a) \to W(f)$$

and this gives a tree

$$t_0 = \sup_a(\varphi_0) \in W(f).$$

If $t_0 \in W_I(f)$, then for any $b \in B_a$, $\varphi_0(b)$ is a tree with label of the root over the same i as a, namely $i = r(a) = r\rho(t_0)$. So each $\varphi(b)$ sits in the summand $X_{r(\rho(\varphi_0(b)))} = X_i$, and we can define

$$\sup_a(\varphi) : * \in [\![t_0 \in W_I(f)]\!] \mapsto \sup_a^{(i)}(\varphi) \in X_i$$

where $\sup^{(i)}$ is the operation of X_i.

Notice that we have a pullback

$$\begin{array}{ccc} W_I(f) \times_I X & \longrightarrow & \tilde{X} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ W_f(I) & \xrightarrow[\varepsilon]{} & W(f) \end{array}$$

Thus X is embedded in a P_f-algebra $(\tilde{X}, \sup)$. Let $H : W(f) \to \tilde{X}$ be the unique homomorphism. Then H is a section of $\pi : \tilde{X} \to W(f)$, and so pulls back to a section

$$H' : W_I(f) \to W_I(f) \times_I X$$

of π_1 (cf. the previous diagram). Then $\pi_2 \circ H' : W_I(f) \to X$ is the required homomorphism in $\mathcal{E}/I$.

For the uniqueness of this map $\pi_2 \circ H' : W_I(f) \to X$ it suffices to observe that any homomorphism $\lambda : W_I(f) \to X$ over I extends to a homomorphism

$$\tilde{\lambda} : W(f) \to \tilde{X}$$

by $\tilde{\lambda} = \langle t, * \mapsto \lambda(t) \rangle$ (if $* \in [\![t \in W_I(f)]\!]$), and use the universal propety of $W(f)$.

This proves that for any map in $\mathcal{E}/I$,

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ & \searrow \quad \swarrow & \\ & I & \end{array}$$

the W-type $W_I(f)$ exists in $\mathcal{E}/I$.

Using the explicit construction of $W_I(f)$, it is now easy to check that for any map $u : J \to I$, the pullback functor $u^* : \mathcal{E}/I \to \mathcal{E}/J$ preserves this construction i.e.

$$u^*(W_I(f)) = W_J(u^* f),$$

as required. ∎

4 Functorial properties of W

In this section, we assume that all initial algebras $\mathrm{W}(f)$ involved in the discussion exist. We begin by discussing some elementary functorial properties of these free algebras.

4.1 Covariant character of W

The construction of the free algebra $\mathrm{W}(f)$ is covariant along pullbacks, in the sense that any pullback diagram

$$\begin{array}{ccc} B' & \xrightarrow{\beta} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ A' & \xrightarrow[\alpha]{} & A \end{array} \tag{6}$$

induces a map $\alpha_! : \mathrm{W}(f') \to \mathrm{W}(f)$. Indeed, since the fiber $f'^{-1}(a')$ is isomorphic to the fiber $f^{-1}(\alpha(a'))$, there is, for every object X, an obvious map $P_{f'}(X) \to P_f(X)$, natural in X. This makes every P_f-algebra into a $P_{f'}$-algebra. Applying this to $W(f)$, the initiality of $W(f')$ gives a map

$$\alpha_! : W(f') \to W(f).$$

In the internal language, we can think of $\alpha_!$ as defined inductively by

$$\alpha_!(\sup_{a'}(t)) = \sup_{\alpha(a')}(\alpha_! \circ t \circ \beta_{a'}^{-1}),$$

where $a' \in A'$, $t : (f')^{-1}(a') \to \mathrm{W}(f')$ and $\beta_{a'} : (f')^{-1}(a') \xrightarrow{\sim} f^{-1}(\alpha(a'))$ is the restriction of β to the fibers. The functoriality is covariant, $\alpha_! \circ \alpha'_! = (\alpha \circ \alpha')_!$.

Example 4.1 Let $B = \{-1, 1\}$, $A = \{0, 1\}$, and $f(x) = |x|$; let $B' = B \times B$, $A' = \{-1, 0, 1\}$ and $f' = \pi_1$. Moreover $\alpha(x) = |x|$ and $\beta = \pi_2$. This forms a pullback as in (6). Then both $\mathrm{W}(f)$ and $\mathrm{W}(f')$ are sets of binary trees, but in the latter case the nodes can have one of two different labels. The map $\alpha_!$ removes this labelling.

Suppose α above is epi, so that by pullback we obtain, using the properties of a pretopos, a diagram

$$\begin{array}{ccccc} B'' & \underset{\rho_2}{\overset{\rho_1}{\rightrightarrows}} & B' & \xrightarrow{\beta} & B \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ A'' & \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} & A' & \xrightarrow{\alpha} & A \end{array} \qquad A'' = A' \times_A A'$$

whose rows are coequalizers. For such a diagram, the W-construction has the following property.

Proposition 4.2 *If α is epi and f' is internally projective as an object of $\mathcal{E}/A'$ then the diagram*

$$\mathrm{W}(f'') \underset{(\pi_2)_!}{\overset{(\pi_1)_!}{\rightrightarrows}} \mathrm{W}(f') \xrightarrow{\alpha_!} \mathrm{W}(f)$$

is a coequalizer, and in particular $\alpha_!$ is epi.

Proof. Construct the coequalizer

$$\mathrm{W}(f'') \underset{(\pi_2)_!}{\overset{(\pi_1)_!}{\rightrightarrows}} \mathrm{W}(f') \xrightarrow{\psi} Q \tag{7}$$

in $\mathcal{E}$. We will prove that Q has the universal property required of the initial P_f-algebra $W(f)$, for which ψ satisfies the same identity that defines $\alpha_!$ inductively. This will prove the lemma.

First, we claim that diagram (7) is *exact*, i.e. that

$$\langle \pi_{1!}, \pi_{2!} \rangle : W(f'') \to W(f') \times W(f') \tag{8}$$

is an equivalence relation. This involves checking "by induction" that the map in (8) is monic, and defines a reflexive, symmetric and transitive relation.

To see that (8) is monic, take $x = \sup_{(a_1,a_2)}(s)$ and $\overline{x} = \sup_{(\overline{a}_1,\overline{a}_2)}(\overline{s})$ in $W(f'')$ such that $\pi_{1!}(s) = \pi_{1!}(\overline{s})$ and $\pi_{2!}(s) = \pi_{2!}(\overline{s})$. Thus

$$\sup_{a_i}(\pi_{i!} \circ s \circ \rho_i^{-1}) = \sup_{\overline{a}_i}(\pi_{i!} \circ \overline{s} \circ \rho_i^{-1}) \qquad (i = 1, 2),$$

and hence

$$a_i = \overline{a}_i, \quad \pi_{i!} \circ s \circ \rho_i^{-1} = \pi_{i!} \circ \overline{s} \circ \rho_i^{-1} \qquad (i = 1, 2).$$

If we now assume for our inductive hypothesis that (8) is monic on predecessors of x and $\overline{x}$, we find for $r = s \circ \rho_1^{-1} \circ \beta^{-1} = s \circ \rho_2^{-1} \circ \beta^{-1}$ and the similarly defined $\overline{r}$ that $r = \overline{r}$. This shows that $s = \overline{s}$, hence $x = \overline{x}$, and completes the proof that (8) is mono.

To see that (8) is reflexive, notice that for the diagonal

$$\begin{array}{ccc} B' & \xrightarrow{\varepsilon} & B'' \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f''} \\ A' & \xrightarrow[\delta]{} & A'' \end{array}$$

we have that $\pi_{1!}\delta_! = \pi_{2!}\delta_! = 1$. Symmetry is proved similarly, using the twist map $\tau : A'' \to A''$ defined by $\pi_1\tau = \pi_2$, $\pi_2\tau = \pi_1$. Finally, we prove that (8) is a transitive relation. Choose x and y in $W(f'')$ such that $\pi_{2!}(x) = \pi_{1!}(y)$. We need to find a $z \in W(f'')$, necessarily unique, such that $\pi_{1!}(z) = \pi_{1!}(x)$ and $\pi_{2!}(z) = \pi_{2!}(y)$. To do this, we proceed by induction and assume this property holds for all predecessors of x and y. Write

$$x = \sup_{(a_1,a_2)}(s), \quad y = \sup_{(a_3,a_4)}(t).$$

Then by the assumption that $\pi_{2!}(x) = \pi_{1!}(y)$, we find

$$\sup_{a_2}(\pi_{2!} \circ s \circ (\rho_2)^{-1}_{a_1,a_2}) = \sup_{a_3}(\pi_{1!} \circ t \circ (\rho_1)^{-1}_{a_3,a_4}),$$

whence $a_2 = a_3$ and $\pi_{2!} \circ s \circ \rho_2^{-1} = \pi_{1!} \circ t \circ \rho_1^{-1}$. By induction hypothesis, there is a function $r : (f')^{-1}(a_2) \longrightarrow W(f'')$ so that

$$\pi_{1!} \circ r = \pi_{1!} \circ s \circ \rho_2^{-1}, \quad \pi_{2!} \circ r = \pi_{2!} \circ t \circ \rho_1^{-1}.$$

Let

$$\begin{aligned} u &= r \circ \beta_{a_2}^{-1} \circ \beta_{a_1} \circ (\rho_1)_{(a_1,a_4)} : (f'')^{-1}(a_1, a_4) \longrightarrow W(f''), \\ z &= \sup_{(a_1,a_4)}(u). \end{aligned}$$

Then

$$\begin{aligned} \pi_{1!}(z) &= \sup_{a_1}(\pi_{1!} \circ u \circ (\rho_1)^{-1}_{a_1,a_4}) \\ &= \sup_{a_1}(\pi_{1!} \circ r \circ \beta_{a_2}^{-1} \circ \beta_{a_1}) \\ &= \sup_{a_1}(\pi_{1!} \circ s \circ (\rho_2)^{-1}_{(a_1,a_2)} \circ \beta_{a_2}^{-1} \circ \beta_{a_1}) \\ &= \sup_{a_1}(\pi_{1!} \circ s \circ (\rho_1)^{-1}_{(a_1,a_2)}) \\ &= \pi_{1!}(x), \end{aligned}$$

and similarly $\pi_{2!}(z) = \pi_{2!}(y)$. This proves transitivity of the relation (8), and hence the asserted exactness of the diagram (7).

Next, we show that Q has a canonical P_f-algebra structure. For $a \in A$ and $t : f^{-1}(a) \longrightarrow Q$ we define $\sup_a(t) \in Q$ as follows. First, since α is epi, there is an $a_1 \in A'$ with $\alpha(a_1) = a$; and since ψ is epi while f is internally projective, there is an $s_1 : f^{-1}(a) \longrightarrow W(f')$ with $\psi s_1 = t$. Define

$$\sup_a(t) = \psi(\sup_{a_1}(s_1 \circ \beta_{a_1})). \tag{9}$$

We need to prove that this definition is independent of the choice of a_1 and s_1. But if a_2 and s_2 also satisfy $\alpha(a_2) = a$ and $\psi s_2 = t$, then $(a_1, a_2) \in A''$, and $(s_1, s_2) : f^{-1}(a) \longrightarrow W(f') \times W(f')$ maps into $W(f'')$ by exactness of (7). So

$$r = (s_1, s_2) \circ \beta_{a_1} \circ (\rho_1)_{a_1,a_2} = (s_1, s_2) \circ \beta_{a_2} \circ (\rho_2)_{a_1,a_2}$$

defines a map $r : (f'')^{-1}(a_1, a_2) \longrightarrow W(f'')$ with $\pi_{1!}(r) = \sup_{\mathrm{a}_1}(s_1 \circ \beta_{a_1})$ and $\pi_{2!}(r) = \sup_{\mathrm{a}_2}(s_2 \circ \beta_{a_2})$. Thus

$$\psi(\sup_{a_1}(s_1 \circ \beta_{a_1})) = \psi(\sup_{a_2}(s_2 \circ \beta_{a_2})),$$

showing that the definition (9) of $\sup_a(t)$ does not depend on the choices involved.

It is also clear from this independence that the map $\psi : W(f') \longrightarrow Q$ satisfies the identity

$$\psi(\sup_{a_1}(s_1)) = \sup_{\alpha(a_1)}(\psi \circ s_1 \circ \beta_{a_1}^{-1}),$$

similar to the defining identity for $\alpha_!$.

Finally, for the universal property, suppose (X, σ) is a P_f-algebra, with operations

$$\sigma_a : X^{f^{-1}(a)} \longrightarrow X \qquad (a \in A).$$

By composition with $\beta_{a'}^{-1} : f^{-1}(\alpha(a')) \longrightarrow (f')^{-1}(a')$, one can define operations

$$\tau_{a'} = \sigma_{\alpha(a')} \circ X^{\beta_{a'}^{-1}} : X^{(f')^{-1}(a')} \longrightarrow X,$$

giving X the structure of a $P_{f'}$-algebra (X, τ). By the universal property of $W(f')$, there is a unique map $\varphi : W(f') \longrightarrow X$ with the property that

$$\varphi(\sup_{a'}(t)) = \tau_{a'}(\varphi \circ t),$$

for $a' \in A'$ and $t : (f')^{-1}(a') \longrightarrow W(f')$. We claim that $\varphi \circ \pi_{1!} = \varphi \circ \pi_{2!}$. To see this, take $x \in W(f'')$ and write $x = \sup_{(a_1,a_2)}(t)$, where $a_1, a_2 \in A'$ with $\alpha(a_1) = a = \alpha(a_2)$ and $t : (f'')^{-1}(a_1, a_2) \longrightarrow W(f'')$. Suppose for the induction that

$$\varphi(\pi_1)_!(tb) = \varphi(\pi_2)_!(tb), \tag{10}$$

for each $b \in (f'')^{-1}(a_1, a_2)$. Then

$$\begin{aligned}
\varphi\pi_{1!}(x) &= \varphi \sup_{a_1}(\pi_{1!} \circ t \circ \rho_1^{-1}{}_{(a_1,a_2)}) \\
&= \tau_{a_1}(\varphi \circ \pi_{1!} \circ t \circ \rho_1^{-1}{}_{(a_1,a_2)}) \\
&= \sigma_a(\varphi \circ \pi_{1!} \circ t \circ \rho_1^{-1}{}_{(a_1,a_2)} \circ \beta_{a_1}^{-1}) \\
&= \sigma_a((\varphi\pi_{1!}t) \circ (\beta\rho_1)_{(a_1,a_2)}^{-1})
\end{aligned}$$

and similarly for $\varphi\pi_{2!}(x)$. Since $\varphi\pi_{1!}t = \varphi\pi_{2!}t$ by induction hypothesis, and $\beta\rho_1 = \beta\rho_2$, we see that $\varphi\pi_{1!}(x) = \varphi\pi_{2!}(x)$. This shows that $\varphi\pi_{1!} = \varphi\pi_{2!}$, as claimed.

It follows that φ factors uniquely through the coequalizer ψ, say as $\varphi = \overline{\varphi} \circ \psi$, for $\overline{\varphi} : Q \longrightarrow X$. To conclude the proof, we check that $\overline{\varphi}$ is an algebra map. For

$a \in A$ and $t : f^{-1}(a) \longrightarrow Q$, and a choice of a_1 and s_1 as for (9) above,

$$\begin{array}{rcl} \overline{\varphi}(\sup_a(t)) & = & \overline{\varphi}\psi \sup_{a_1}(s_1 \circ \beta_{a_1}) \\ & = & \varphi(\sup_{a_1}(s_1 \circ \beta_{a_1})) \\ & = & \tau_{a_1}(\varphi \circ s_1 \circ \beta_{a_1}) \\ & = & \sigma_a(\varphi \circ s_1) \\ & = & \sigma_a(\overline{\varphi} \circ \psi \circ s_1) \\ & = & \sigma_a(\overline{\varphi} \circ t), \end{array}$$

as required. ∎

4.2 Contravariant character of W

The construction of the free algebra $\mathrm{W}(f)$ is contravariant in the sense that a commutative triangle

$$\begin{array}{ccc} C & \xrightarrow{\ \pi\ } & B \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & A & \end{array}$$

induces a map $\pi^* : \mathrm{W}(f) \longrightarrow \mathrm{W}(g)$. Informally, π^* is defined "inductively" by the identity

$$\pi^*(\sup_a t) = \sup_a(\pi^* \circ (t \circ \pi_a)) \tag{11}$$

where $t : f^{-1}(a) \longrightarrow \mathrm{W}(f)$ and $\pi_a : g^{-1}(a) \longrightarrow f^{-1}(a)$ is the restriction of π to the fiber. We have contravariant functoriality $\pi^* \circ \rho^* = (\rho \circ \pi)^*$, as is easily checked.

Example 4.3 Let $f : \{1\} \hookrightarrow \{0,1\}$, and let $g : \omega \longrightarrow \{0,1\}$ be given by $g(x) = 1$. Then if $\pi : \omega \longrightarrow \{1\}$, π^* defines an embedding of $\mathrm{W}(f)$, the natural numbers, into $\mathrm{W}(g)$. Here $\pi^*(n)$ is the full ω-branching tree of depth n.

If the map π is epi, then by the pretopos axioms there is a coequalizer diagram in $\mathcal{E}/A$,

$$\begin{array}{ccccc} C \times_B C & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & C & \xrightarrow{\ \pi\ } & B \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle f} & \\ & & A & & \end{array} \tag{12}$$

which induces maps

$$\mathrm{W}(f) \xrightarrow{\ \pi^*\ } \mathrm{W}(g) \overset{\pi_1^*}{\underset{\pi_2^*}{\rightrightarrows}} \mathrm{W}(h) \tag{13}$$

Proposition 4.4 *For any epimorphism π, the diagram (13) is an equalizer, and in particular π^* is mono.*

Proof. To begin with, we prove that the map π^* is mono, by showing "inductively" that

$$\pi^*(x) = \pi^*(y) \Rightarrow x = y, \tag{14}$$

for all $x, y \in W(f)$. To this end, take any x and y in $W(f)$, and write

$$x = \sup_a(t), \quad y = \sup_{a'}(t')$$

where $t : f^{-1}(a) \longrightarrow W(f)$ and $t' : f^{-1}(a') \longrightarrow W(f)$. Assume that (14) holds for the predecessors of x and y, i.e.

$$\pi^*(tb) = \pi^*(t'b') \Rightarrow tb = t'b', \tag{15}$$

for any $b \in f^{-1}(a)$ and $b' \in f^{-1}(a')$.

To prove (14), suppose now that $\pi^*(x) = \pi^*(y)$. By definition of π, this means that

$$\sup_a(\pi^* t\pi) = \sup_{a'}(\pi^* t'\pi),$$

in $W(g)$. In particular, $a = a'$ and $\pi^* t\pi(c) = \pi^* t'\pi(c)$ for all $c \in g^{-1}(a)$. By (15), we find $t\pi(c) = t'\pi(c)$ for any $c \in g^{-1}(a)$. Thus $t = t'$ since $\pi : g^{-1}(a) \longrightarrow f^{-1}(a)$ is surjective. This shows that $a = a'$ and $t = t'$, so $x = y$. Thus π^* is mono, as claimed.

To complete the proof of the proposition, we now show for any $x \in W(g)$ that

$$\pi_1^*(x) = \pi_2^*(x) \Rightarrow (\exists y \in W(f))\, \pi^*(y) = x, \tag{16}$$

again by induction on x. Write $x = \sup_a(t)$, $a \in A$, $t : g^{-1}(a) \longrightarrow W(g)$, and assume that (16) holds for each $t(b)$; i.e.

$$\pi_1^*(tb) = \pi_2^*(tb) \Rightarrow (\exists y \in W(f))\, \pi^*(y) = tb. \tag{17}$$

Notice that this y is necessarily unique if it exists. Thus, by function comprehension we derive from (17) that

$$\pi_1^* \circ t = \pi_2^* \circ t \Rightarrow (\exists! s : g^{-1}(a) \longrightarrow W(f))\, \pi^* \circ s = t. \tag{18}$$

To prove (16) we now suppose $\pi_1^*(x) = \pi_2^*(x)$, i.e.,

$$\sup_a(\pi_1^* \circ t \circ \pi_1) = \sup_a(\pi_2^* \circ t \circ \pi_2)$$

in $W(h)$. In particular,

$$\pi_1^* \circ t \circ \pi_1 = \pi_2^* \circ t \circ \pi_2. \tag{19}$$

By precomposing (19) with the diagonal $\delta : g^{-1}(a) \longrightarrow h^{-1}(a)$ and postcomposing it with δ^* we find that

$$\pi_1^* \circ t = \pi_2^* \circ t, \quad t \circ \pi_1 = t \circ \pi_2. \tag{20}$$

The first identity gives by (18) that $t = \pi^* \circ s$ for a unique $s : g^{-1}(a) \longrightarrow W(f)$, while the second one in (20) gives together with the monicity of π^*, and the universal property of the coequalizer (12), that $s = r \circ \pi$ for a unique $r : f^{-1}(a) \longrightarrow W(g)$. Together, these give $t = \pi^* \circ r \circ \pi$, where

$$\begin{aligned} x &= \sup_a(t) \\ &= \sup_a(\pi^* \circ r \circ \pi) \\ &= \pi^*(\sup_a(r)), \end{aligned}$$

which proves (16) if we take $y = \sup_a(r)$. This completes the proof. ∎

5 Sheaves of wellfounded trees

In Section 3 we observed that W-types exist in any elementary topos (always assumed to have a natural numbers object). This applies in particular to the category $\mathrm{Psh}(\mathbb{C})$ of presheaves of sets on a small category $\mathbb{C}$, and to the category $\mathrm{Sh}(\mathbb{C})$ of sheaves for a given Grothendieck topology on $\mathbb{C}$. More generally, if $\mathbb{C}$ is any internal category or site in an elementary topos $\mathcal{E}$, the internal presheaves and sheaves form toposes, $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$ and $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C})$, and hence have all W-types.

We begin by recalling the construction of dependent products of presheaves. For the moment, let $\mathcal{E}$ be the category of sets, and let $\mathbb{C}$ be a small category, i.e. a category in $\mathcal{E}$. A presheaf $\mathcal{P}$ on $\mathbb{C}$ is a functor $\mathbb{C}^{\mathrm{op}} \longrightarrow \mathcal{E}$. Thus $\mathcal{P}$ is given by a set $\mathcal{P}(C)$ for each object $C \in \mathbb{C}$, and a "restriction operation" $\alpha^* : \mathcal{P}(C) \longrightarrow \mathcal{P}(D)$ for each arrow $\alpha : D \longrightarrow C$ in $\mathcal{E}$. We usually write $x \cdot \alpha$ rather than $\alpha^*(x)$. We will also use the notation

$$|\mathcal{P}| = \{(x, C) \ : \ C \in \mathbb{C}, x \in \mathcal{P}(C)\}$$

for the "underlying set" of $\mathcal{P}$.

A map between presheaves $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is a natural transformation, i.e. a family of maps $f_C : \mathcal{P}(C) \longrightarrow \mathcal{Q}(C)$, $C \in \mathbb{C}$, which commutes with restrictions. In this way we obtain a category $\mathrm{Psh}(\mathbb{C})$ of presheaves on $\mathbb{C}$.

This definition, in fact, still makes sense when **Sets** is replaced by any category $\mathcal{E}$ with finite limits [13, p. 242], thus giving a category $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$ of internal presheaves in $\mathcal{E}$. The following proposition is well-known.

Proposition 5.1 *If $\mathcal{E}$ is a pretopos with dependent products, then so is* $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$.

One way to see that $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$ has dependent products if $\mathcal{E}$ does, is to write down the explicit description of dependent products for the case where $\mathcal{E}$ is **Sets**, and observe that this description makes sense in (the internal language of) any locally cartesian closed category $\mathcal{E}$. Since we need this explicit description later anyway, we give it now.

5.2 Dependent products of presheaves. For maps of presheaves $g : \mathcal{W} \to \mathcal{B}$ and $f : \mathcal{B} \to \mathcal{A}$, the presheaf $\mathcal{P} = \Pi_f\, g$ is described as follows. Elements of $\mathcal{P}(C)$ are pairs (a, t), where $a \in \mathcal{A}(C)$, and t is a map assigning to each $\beta : D \to C$ in $\mathbb{C}$ and each $b \in \mathcal{B}(D)$, with $f(b) = a \cdot \beta$, an element $t(\beta, b) \in \mathcal{W}(D)$. This map t is required to satisfy the identities

$$\begin{aligned} g(t(\beta, b)) &= b \\ t(\beta\gamma, b \cdot \gamma) &= t(\beta, b) \cdot \gamma \end{aligned}$$

for any $E \xrightarrow{\gamma} D$ and β, b as above. Thus, if we write

$$\mathcal{B}_a(D) = \{(\beta, b) \mid \beta : D \to C, b \in \mathcal{B}(D), f(b) = a \cdot \beta\}$$

then t is a map of presheaves $t : \mathcal{B}_a \to \mathcal{W}$ with $gt = \pi_2 : \mathcal{B}_a \to \mathcal{B}$. For $C' \xrightarrow{\alpha} C$, the restriction $\mathcal{P}(C) \to \mathcal{P}(C')$ is defined by

$$(a, t) \cdot \alpha = (a \cdot \alpha, t \cdot \alpha)$$

where

$$(t \cdot \alpha)(\beta, b) = t(\alpha\beta, b).$$

Note that, like any dependent product, $\mathcal{P}$ is equipped with an evident projection map $\mathcal{P} \to \mathcal{A}$, and an "evaluation" map $\mathcal{P} \times_{\mathcal{A}} \mathcal{B} \to \mathcal{W}$ given by

$$(a, t, b) \mapsto t(1, b).$$

5.3 Polynomial functors. The previous remark yields in particular an explicit description of the polynomial functor $\mathcal{P}_f$ associated to a map $f : \mathcal{B} \to \mathcal{A}$ between presheaves. For any presheaf $\mathcal{W}$ and the associated presheaf $\mathcal{P}_f(\mathcal{W})$, the set $\mathcal{P}_f(\mathcal{W})(C)$ consists of pairs (a, t) where $a \in \mathcal{A}(C)$ and t is a map of presheaves $\mathcal{B}_a \to \mathcal{W}$.

5.4 W-presheaves. We now construct the W-type $\mathcal{W}(f)$ associated to a map of presheaves $f : \mathcal{B} \to \mathcal{A}$. To begin with, we consider the set $\mathcal{S}$ of wellfounded trees with nodes labelled by pairs $(a, C) \in |\mathcal{A}|$, and branches into such a node labelled by the set $|\mathcal{B}_a|$, i.e. by pairs

$$(\beta, b) \qquad (\beta : D \to C, b \in \mathcal{B}(D), f(b) = a \cdot \beta).$$

This set $\mathcal{S}$ can be constructed as the W-type in **Sets** for the evident map $\dot{\bigcup}\{|\mathcal{B}_a| : (a, C) \in |\mathcal{A}|\} \to |\mathcal{A}|$.

Thus any $T \in \mathcal{S}$ is of the form

$$T = \sup_{(a,C)} t \tag{21}$$

where $(a, C) \in |\mathcal{A}|$ and $t : |\mathcal{B}_a| \to \mathcal{S}$. For such a tree T, we will write $C = \rho(T)$ for the object of $\mathbb{C}$ occurring in the label of its root. Write $\mathcal{S}(C)$ for the collection of trees $T \in \mathcal{S}$ with $\rho(T) = C$. Then $\mathcal{S}$ has the structure of a presheaf, with restriction along $\alpha : C' \to C$ given by

$$T \cdot \alpha = \sup_{(a \cdot \alpha, C')} \alpha^*(t) \tag{22}$$

where $\alpha^*(t) : |\mathcal{B}_{a \cdot \alpha}| \to \mathcal{S}$ is given by

$$\alpha^*(t)(\beta, b) = t(\alpha\beta, b),$$

for any $\beta : D \to C'$ and any $b \in \mathcal{B}(D)$ with $f(b) = a \cdot (\alpha\beta)$.

Now we define two hereditary properties of trees. This means that we are defining predicates by transfinite recursion, and require more than just the universal property of W-types; see Remark 5.9. Let us call a tree T as in (21) *composable* if for any $(\beta, b) \in |\mathcal{B}_a|$, the tree $t(\beta, b)$ is composable and moreover $\rho(t(\beta, b)) = \text{dom}(\beta)$. Furthermore, let us call a composable tree as in (21) *natural* if for any $(\beta : D \to C, b) \in \mathcal{B}_a$, the tree $t(\beta, b)$ is natural, and moreover for any $\gamma : E \to D$,

$$t(\beta, b) \cdot \gamma = t(\beta\gamma, b \cdot \gamma). \tag{23}$$

Lemma 5.5 *If T is natural then so is $T \cdot \alpha$, for any arrow $\alpha : C' \to C$.*

Proof. Let $T = \sup_{(a,C)} t$ as in (21). Clearly $T \cdot \alpha$ is composable whenever T is. To see that it is also natural, it suffices to check (23) for $\alpha^*(t)$, assuming it holds for t. To this end, take $(\beta, b) \in \mathcal{B}_{a \cdot \alpha}$ where $\beta : D \to C'$, and write $t(\alpha\beta, b) = \sup_{(a', D)}(s)$. Then for any $\gamma : E \to D$, the definition of $\alpha^*(t)$ and the naturality of t give

$$\alpha^*(t)(\beta\gamma, b \cdot \gamma) = t(\alpha\beta\gamma, b \cdot \gamma) = \sup_{(\alpha \cdot \gamma, E)} \gamma^*(s),$$

and the right hand side is exactly $\alpha^*(t)(\beta, b) \cdot \gamma$ by the definition (22) of the restriction operation on trees. ∎

Let us write $\mathcal{W}(C) \subseteq \mathcal{S}(C)$ for the collection of natural trees rooted in C. The lemma shows that $\mathcal{W}$ is a subpresheaf of $\mathcal{S}$. It also shows that a natural tree $T \in \mathcal{W}(C)$ is uniquely of the form (21) for a *natural transformation* $t : \mathcal{B}_a \to \mathcal{W}$ into this presheaf.

Theorem 5.6 *The presheaf $\mathcal{W}$ carries a canonical operation*

$$S : \mathcal{P}_f(\mathcal{W}) \longrightarrow \mathcal{W}$$

which makes it into the free $\mathcal{P}_f$-algebra in the category of presheaves.

Proof. Using the notation of 5.3, the operation S is defined on a pair (a, t) where $a \in \mathcal{A}(C)$ and $t : \mathcal{B}_a \longrightarrow \mathcal{W}$, simply by

$$S_C(a, t) = \sup_{(a,C)} t.$$

Here $\sup_{(a,C)}(t)$ is the tree obtained by applying the sup operation of the set $\mathcal{S}$. Note that $S_C(a, t)$ is a natural tree, by the remark just preceding the statement of the theorem. Furthermore, S is evidently natural in a and t. To verify the universal property, let $(\mathcal{X}, \sigma)$ be any $\mathcal{P}_f$-algebra. Define a map $\varphi : \mathcal{W} \longrightarrow \mathcal{X}$ by induction on natural trees:

$$\varphi(\sup_{(a,C)} t) = \sigma_C(a, \varphi \circ t) \tag{24}$$

where $C \in \mathcal{C}$, $a \in \mathcal{C}$, $t : \mathcal{B}_a \longrightarrow \mathcal{W}$ as above, and $\varphi \circ t$ is the composite which makes sense because φ is assumed to be already defined on the values of t. It is readily checked that φ is a natural transformation $\mathcal{W} \longrightarrow \mathcal{X}$, and is the unique one satisfying (24). ∎

Everything we have said so far extends immediately to sheaves. Here we use a definition of Grothendieck topology in terms of bases only, so that it makes sense for any internal category $\mathbb{C}$ in a pretopos $\mathcal{E}$. More explicitly, such a Grothendieck topology is given by a collection of covering families $\{C_i \longrightarrow C\}$ satisfying the stability condition of [13, Ch.III, Ex.3].

It is wellknown that for an internal category $\mathbb{C}$ equipped with a Grothendieck topology in a pretopos $\mathcal{E}$ with dependent products, the category $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C})$ of internal sheaves is again a pretopos with dependent products, analogous to Proposition 5.1. In fact, the dependent products are those of $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$, because if the presheaves $\mathcal{W}$, $\mathcal{B}$ and $\mathcal{A}$ as in 5.2 are sheaves, then so is the presheaf $\mathcal{P}$ constructed there.

Proposition 5.7 *If the presheaves $\mathcal{A}$ and $\mathcal{B}$ are sheaves, then so is the presheaf $\mathcal{W}$ of wellfounded trees constructed as in 5.4.*

Proof. We prove by induction on trees that compatible families of trees have a unique amalgamation. To this end, consider a cover $\{\alpha_i : C_i \longrightarrow C\}$, and trees $T_i \in \mathcal{W}(C_i)$ so that for any two arrows $C_i \xleftarrow{\beta} B \xrightarrow{\gamma} C_j$ with $\alpha_i \beta = \alpha_j \gamma$ we have $T_i \cdot \beta = T_j \cdot \gamma$. Write

$$T_i = \sup_{(a_i, C_i)} t_i.$$

Then the a_i form a compatible family of elements in the sheaf $\mathcal{A}$, so they can be glued to a unique $a \in \mathcal{A}(C)$ with $a \cdot \alpha_i = a_i$. We now wish to glue the functions t_i to a function t for which

$$T = \sup_{(a,C)} t$$

is an amalgamation of the trees T_i. To define t, take any $\beta : D \longrightarrow C$ and $b \in \mathcal{B}(D)$ with $f(b) = a \cdot \beta$. By the stability condition on the Grothendieck topology, there exists a cover $\{\gamma_j : D_j \longrightarrow D\}$ of D such that each $\beta \circ \gamma_j$ factors through some C_i,

$$\forall j \, \exists i \, \exists \delta \, (\beta \circ \gamma_j = \alpha_i \circ \delta). \tag{25}$$

Thus for each j we have a tree

$$S_j = t_i(\delta, b \cdot \gamma_j) \in \mathcal{W}(D_j);$$

here, for a given j, the index i and the arrow δ are as in (25), and S_j does not depend on the choice of i and δ by compatibility of the family $\{t_i\}$, as one readily checks. We claim that these trees S_j form a compatible family for the cover $\{\gamma_j : D_j \longrightarrow D\}$. Indeed, for two indices j and j' and a choice of i, δ and i', δ', we have for any arrows $D_j \xleftarrow{\varepsilon} E \xrightarrow{\varepsilon'} D_{j'}$ with $\gamma_j \varepsilon = \gamma_{j'} \varepsilon'$ that

$$\begin{aligned} S_j \cdot \varepsilon &= t_i(\delta, b \cdot \gamma_j) \cdot \varepsilon \\ &= t_i(\delta\varepsilon, b \cdot \gamma_j \varepsilon) \\ &= t_{i'}(\delta'\varepsilon', b \cdot \gamma_{j'} \varepsilon') \\ &= t_{i'}(\delta', b \cdot \gamma_{j'}) \cdot \varepsilon' \\ &= S_{j'} \cdot \varepsilon'. \end{aligned}$$

Here the first and last identities follows by definition, the second and fourth are the naturality of t and t', and the third follows since t_i is compatible with $t_{i'}$ while $\alpha_i \delta \varepsilon = \alpha_{i'} \delta' \varepsilon'$ and $\gamma_j \varepsilon = \gamma_{j'} \varepsilon'$. By induction hypothesis, the S_j now glue to a unique tree $S \in \mathcal{W}(D)$, and we define $t(\beta, b)$ to be this S. This completes the definition of the function t and hence of the tree $T \in \mathcal{W}(C)$. We leave it to the reader to check that T is indeed a natural tree, and that it is the unique one satisfying $T \cdot \alpha_i = T_i$. ∎

Remark 5.8 For the case where the site $\mathbb{C}$ is a (sufficiently) complete Boolean algebra with the usual topology in which suprema cover, a sheaf of Brouwer ordinals was introduced in [2]. This sheaf can in fact be seen to be a special instance of the general construction given in 5.4.

Remark 5.9 In the proof of Proposition 5.7, we have used a definition by "transfinite induction" to define $\varphi : \mathcal{W} \longrightarrow \mathcal{X}$, which uses more than just the universal property of W-types $\mathcal{S}$ in **Sets**, which lay at the basis of the construction of the presheaf $\mathcal{W}$. In fact, in terms of the category of sets and the object $\mathcal{S}$, we defined by transfinite induction a relation

$$R \subseteq \mathcal{S} \times |\mathcal{X}|,$$

by $(\sup_{(a,C)}(t), x) \in R$ iff $x \in \mathcal{X}(C)$, $\sup_{(a,C)}(t)$ is a natural tree, $x = \sigma(a, r)$ for some $a \in \mathcal{A}(C)$ and $r : \mathcal{B}_a \to \mathcal{W}$, and $(t(\beta, b), r(\beta, b)) \in R$ for all $(\beta, b) \in \mathcal{B}_a$. This relation R is then the "graph" of the map of presheaves $\varphi : \mathcal{W} \to \mathcal{X}$. A similar definition of subobjects by "transfinite recursion" already occurred in the construction of W-presheaves in 5.4. This argument does not go through, in general, when we replace the category of sets by an arbitrary pretopos $\mathcal{E}$ with dependent products and W-types.

In [16], we will introduce a predicative analogue of the notion of elementary topos, for which these arguments can indeed be formalized in the internal language. Such a "predicative topos" is called a stratified pseudo-topos in [16]. Any stratified pseudo-topos is in particular a pretopos with dependent products and W-types. Moreover, it is proved in [16] that if $\mathbb{C}$ is an internal site in a *stratified pseudo-topos* $\mathcal{E}$*,* then the categories $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$ and $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C})$ of internal presheaves and sheaves, respectively, are again stratified pseudo-toposes.

6 Gluing

Let $\mathcal{C}$ be a category with finite limits. From $\mathcal{C}$ we can construct a new category $\check{\mathcal{C}}$ whose objects are triples (S, C, α) where $\alpha : S \to \Gamma C = \mathrm{Hom}(1, C)$ is a function from a set S to the set of global sections of an object $C \in \mathcal{C}$. In other words, an object of $\check{\mathcal{C}}$ is given by an object $C \in \mathcal{C}$ together with an indexed family of arrows $\{\alpha(s) : 1 \to C\}_{s \in S}$. Arrows $(S, C, \alpha) \to (T, D, \beta)$ are pairs (f, u) where $f : S \to T$ is a function between sets and $u : C \to D$ is an arrow in $\mathcal{C}$ such that $\beta(fs) = u \circ \alpha(s)$ for any $s \in S$. Thus, $\check{\mathcal{C}}$ is the *comma category*

$$\mathbf{Sets}/\Gamma$$

associated to the left exact functor $\Gamma : \mathcal{C} \to \mathbf{Sets}$. This construction of $\check{\mathcal{C}}$ out of $\mathcal{C}$ is wellknown, in particular for the case of an elementary topos $\mathcal{C}$ where it is often referred to as the Freyd cover of $\mathcal{C}$ (cf. [12]), and used to prove existence and disjunction properties of intuitionistic theories. A syntactic version of this construction has been given for a type theory with dependent sums and products by Smith [19]. The following proposition belongs to the folklore:

Proposition 6.1 *(i) If $\mathcal{C}$ is a pretopos then so is $\check{\mathcal{C}}$.*

(ii) If $\mathcal{C}$ has dependent products then so does $\check{\mathcal{C}}$. Moreover, the forgetful functor $\check{\mathcal{C}} \to \mathcal{C}$ preserves the pretopos constructions as well as the dependent products.

Proof. We present some details of the proof, since the explicit constructions will be used later, but we leave the verification of the relevant universal properties to the reader.

(i) The product of two objects $(\alpha : S \to \Gamma C)$ and $(\beta : T \to \Gamma D)$ of $\check{\mathcal{C}}$ is

$$S \times T \xrightarrow{\alpha\times\beta} \Gamma C \times \Gamma D \xrightarrow{\sim} \Gamma(C \times D),$$

and the coproduct is

$$S + T \xrightarrow{\alpha+\beta} \Gamma C + \Gamma D \xrightarrow{\text{can}} \Gamma(C + D),$$

where "can" is the canonical map. Other limits and colimits in $\check{\mathcal{C}}$ are constructed similarly from those of $\mathcal{C}$.

(ii) We first describe the exponential

$$(\beta : T \to \Gamma D)^{(\alpha : S \longrightarrow \Gamma C)}$$

as $(F \xrightarrow{\pi_1} \Gamma(D^C))$. Here D^C is the exponential in $\mathcal{C}$, and F is the set of morphisms $(f, u) : (\alpha : S \to \Gamma C) \to (\beta : T \to \Gamma D)$ in $\check{\mathcal{C}}$; the map $\pi_2 : F \to \Gamma(D^C)$ is the projection.

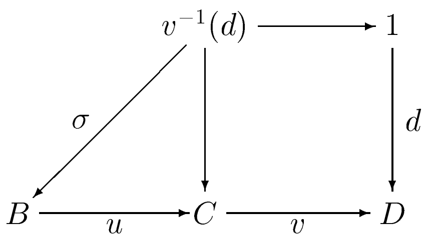
The construction of dependent products is similar: For maps

$$(R \xrightarrow{\alpha} \Gamma B) \xrightarrow{(f,u)} (S \xrightarrow{\beta} \Gamma C) \xrightarrow{(g,v)} (T \xrightarrow{\gamma} \Gamma D)$$

in $\check{\mathcal{C}}$ we construct $\Pi_{(g,v)}(f, u)$ as the object

$$P \longrightarrow \Gamma(\Pi_v u),$$

as follows. First, $\Pi_v u$ is the dependent product in $\mathcal{C}$, so that $\Gamma(\Pi_v u)$ is the set of pairs (d, σ) where $d : 1 \to D$ and $\sigma : v^{-1}(d) = C \times_D 1 \to B$ is a section of u:



The set P consists of triples (t, ρ, σ) where $(t, \rho) \in \Pi_g f$ (i.e. $t \in T$ and $\rho : g^{-1}(t) \to R$ is a section of f), while for $d = \gamma(t)$, the pair (d, σ) is an element of the set $\Gamma(\Pi_v u)$ described above, and moreover

$$(\sigma \circ \beta)(s) = (\alpha \circ \rho)(s), \qquad (s \in g^{-1}(t)). \blacksquare$$

Remark 6.2 For a map in $\check{\mathcal{C}}$, say

$$(f,u) : (T \xrightarrow{\beta} \Gamma D) \longrightarrow (S \xrightarrow{\alpha} \Gamma C),$$

we obtain an explicit description of the polynomial functor $P_{(f,u)} : \check{\mathcal{C}} \to \check{\mathcal{C}}$ from the last part of the proof: For an object $(X \xrightarrow{\xi} \Gamma W)$, the value $P_{(f,u)}(X \xrightarrow{\xi} \Gamma W)$ is the object

$$Y \xrightarrow{\pi} \Gamma(P_u(W)),$$

where $P_u(W) \in \mathcal{C}$ is the value of the polynomial functor P_u at W, and Y is the set of triples (s, σ, ζ) where $s \in S$, $\sigma : f^{-1}(s) \to X$ (i.e. $(s,\sigma) \in P_f(X)$), and $\zeta : u^{-1}(\alpha(s)) \to W$ is a map in $\mathcal{C}$ such that

$$(\xi \circ \sigma)(t) = (\zeta \circ \beta)(t) \qquad (t \in f^{-1}(s)).$$

The map $Y \xrightarrow{\pi} \Gamma(P_u(W))$ sends the triple (s, σ, ζ) to $(\alpha(s), \zeta)$.

Theorem 6.3 *Let $\mathcal{C}$ be a pretopos with dependent products. If $\mathcal{C}$ has W-types then so does $\check{\mathcal{C}}$, and the forgetful functor $\check{\mathcal{C}} \to \mathcal{C}$ preserves them.*

Proof. Consider a map

$$(R \xrightarrow{\beta} \Gamma B) \xrightarrow{(f,u)} (S \xrightarrow{\alpha} \Gamma A)$$

in $\check{\mathcal{C}}$. We will construct $W(f,u)$ as an object of the form

$$\langle \cdot \rangle : Q \longrightarrow \Gamma(W(u)),$$

where $W(u)$ is the W-type in $\mathcal{C}$ which exists by hypothesis. Before describing the set Q, we consider an auxiliary set Q_0 of wellfounded trees. The nodes of Q_0 are labelled by pairs (s, ζ) where $s \in S$ and $\zeta : u^{-1}(\alpha s) \to W(u)$; the branches of Q_0 into such a node are indexed by the elements $t \in f^{-1}(s)$.

For each node (s, ζ) in Q_0 one has a map

$$\mathrm{sup}_{\alpha(s)}(\zeta) : 1 \to W(u),$$

i.e. an element of $\Gamma(W(u))$. In particular, for each wellfounded tree $T \in Q_0$ one obtains a global section $\langle T \rangle \in \Gamma W(u)$, defined by

$$\langle T \rangle = \mathrm{sup}_{\alpha(s_0)}(\zeta_0),$$

where (s_0, ζ_0) is the label at the root of T. Call a tree $T \in Q_0$ *coherent* if for any node in T with label (s, ζ), and any branch labelled $t \in f^{-1}(s)$ from this node, the tree T_t above this branch has the property that

$$\langle T_t \rangle = \zeta \circ \beta(t) : 1 \xrightarrow{\beta(t)} u^{-1}(\alpha s) \xrightarrow{\zeta} W(u).$$

Coherence can be defined "by induction", by stating that for $s_0 \in S$, $\zeta_0 : u^{-1}(\alpha s_0) \to W(u)$, and $\varphi : f^{-1}(s_0) \to Q_0$, the tree

$$T = \sup_{(s_0,\zeta_0)}(\varphi)$$

is coherent iff for each $t \in f^{-1}(s)$ the tree $\varphi(t)$ is coherent, and moreover $\langle \varphi(t) \rangle = \zeta_0 \circ \beta(t) \in \Gamma(W(u))$. We now let $Q \subseteq Q_0$ be the set of all coherent trees, thus completing the definition of the object

$$W(f,u) = (Q \xrightarrow{\langle\,\rangle} \Gamma W(u))$$

in $\check{\mathcal{C}}$.

To describe the "operations"

$$\sup : P_{(f,u)}(W(f,u)) \longrightarrow W(f,u) \tag{26}$$

we use the explicit description of Remark 6.2. Thus, we write

$$P_{(f,u)}(W(f,u)) = (\overline{Y} \xrightarrow{\overline{\pi}} \Gamma P_u(W(u))),$$

where the elements of the set $\overline{Y}$ are triples $s \in S$, $\sigma : f^{-1}(s) \to Q$ and $\zeta : u^{-1}(\alpha s) \to W(u)$ such that

$$\langle \sigma(t) \rangle = \zeta \circ \beta(t) \qquad (t \in f^{-1}(s)). \tag{27}$$

Now the $\mathcal{C}$-component of the sup-map (26) is the $\sup : P_u(W(u)) \to W(u)$ in $\mathcal{C}$, of course, while the **Sets**-component is the map $\overline{Y} \to Q$ sending a triple (s,σ,ζ) to the tree $\sup_{(s,\zeta)}(\sigma) \in Q_0$. This "sup" is the sup-operation of Q_0, and the tree belongs to $Q \subseteq Q_0$, i.e. is coherent, precisely because of the identity (27).

In order to verify the universal property, suppose $(X \xrightarrow{\xi} \Gamma W)$ is any other object with "operations"

$$(K,\theta) : P_{(f,u)}(X \xrightarrow{\xi} \Gamma W) \longrightarrow (X \xrightarrow{\xi} \Gamma W).$$

Thus $\theta : P_u(W) \to W$ in $\mathcal{C}$ and $K : Y \to X$ where $Y \to \Gamma(P_u(W))$ is the map π as in Remark 6.2, and moreover $\xi \circ K = \Gamma(\theta) \circ \pi$. By the universal property of $W(u)$ there is a unique map

$$v : W(u) \longrightarrow W$$

in $\mathcal{C}$ which commutes with the operations, i.e. $\theta \circ P_u(v) = v \circ \sup$. We can complete v to a map in $\check{\mathcal{C}}$,

$$(g,v) : (Q \xrightarrow{\langle\cdot\rangle} \Gamma W(u)) \longrightarrow (X \xrightarrow{\xi} \Gamma W),$$

by defining $g : Q \to X$ inductively on coherent trees: if $T = \sup_{(s,\zeta)}(\sigma)$ is such a tree where $(s, \sigma, \zeta) \in \overline{Y}$, and $g\sigma(t)$ has been defined for all $t \in f^{-1}(s)$, define

$$g(T) = K(s, g \circ \sigma, v \circ \zeta). \tag{28}$$

Then (g, v) is indeed a map in $\check{\mathcal{C}}$, because

$$\begin{aligned} \xi(g(T)) &= \xi(K(s, g\sigma, v\zeta)) \\ &= \Gamma(\theta) \circ \pi(s, g\sigma, v\zeta) \\ &= \Gamma(\theta)(\alpha(s), v\zeta) \\ &= \Gamma(\theta)P_u(v)(\alpha(s), \zeta) \\ &= \Gamma(v) \circ \sup(\alpha(s), \zeta) \\ &= \Gamma(v)\langle T \rangle. \end{aligned}$$

Moreover, (g, v) commutes with the operations. Indeed, unravelling the definition, we see that this means that v does, and that g satisfies (28). ■

Remark 6.4 In the proof of this theorem, we have used the W-type $W(u)$ in $\mathcal{C}$, and we have cut out the coherent part Q of the W-type Q_0 in **Sets**. Moreover, in the verification of the universal property in $\check{\mathcal{C}}$, we have defined a map g on Q rather than on Q_0. Thus, we have used more than just the universal W-type property of Q is **Sets**. This situation is completely analogous to the one for presheaves discussed in Remark 5.9, and leads to the similar conclusion that W-types still exist in $\check{\mathcal{C}}$ when **Sets** is replaced by a stratified pseudo-topos.

Remark 6.5 It is known from topos theory [20] that the gluing construction $\check{\mathcal{C}}$ is a special case of the construction of the category $\mathrm{Coalg}_G(\mathcal{F})$ of coalgebras for a left exact comonad G on $\mathcal{F}$. It is likely that if $\mathcal{F}$ is a stratified pseudo-topos and G is a monad respecting a filtration of $\mathcal{F}$, then $\mathrm{Coalg}_G(\mathcal{F})$ is again a stratified pseudo-topos, but we have not checked this.

7 Relation to type theory

In Martin-Löf type theory [15] the category of sets, **Sets**, is most naturally defined to be the category of types (or presets) with equivalence relations $X = (\overline{X}, =_X)$ and functions preserving these equivalences. We refer to [8] for a more detailed treatment. The basic type theory of Martin-Löf, $\mathbf{ML}_0$, consists of rules for Σ- and Π-types, disjoint sum-type $(+)$, natural numbers $\mathbb{N}$, the canonical finite sets $\mathbb{N}_k = \{0, \ldots, k-1\}$, the intensional identity type and the boolean type $\mathbb{B}(x)$ $(x \in \mathbb{N}_2)$ such that $\mathbb{B}(0) = \mathbb{N}_0$ (empty set) and $\mathbb{B}(1) = \mathbb{N}_1$. (This is not a minimal axiomatization.) It is well-known that in $\mathbf{ML}_0$ the category **Sets** is locally cartesian closed. Using the particular axiomatization (P1-4) of pretoposes it is easy to obtain

Proposition 7.1 *In* $\mathbf{ML}_0$ *the category* **Sets** *is a pretopos with dependent products.*

Proof. The axioms (P1) and (P2) are straightforward to check, where (P2) requires the disjoint sum construction and the boolean type of the theory.

To verify (P3) suppose that $R \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} X$ is an equivalence relation in **Sets**. Define the following relation on $\overline{X}$, where $X = (\overline{X}, =_X)$,

$$a =^* b \iff_{\text{def}} (\exists r \in R)\,[\partial_0(r) =_X a \,\&\, \partial_1(r) =_X b].$$

It is easily checked that this is an equivalence relation. Let $Y = (\overline{X}, =^*)$ and define $i : X \to Y$ by $i(x) = x$. Now the usual argument that

$$R \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} X \xrightarrow{\;i\;} Y \tag{29}$$

is exact can be formalized in $\mathbf{ML}_0$.

To check the axiom (P4) one utilizes the fact that a coequalizing map in an exact diagram is surjective. We leave the straightforward details to the reader. ∎

We now consider an extension $\mathbf{ML}_{<\omega}$ of $\mathbf{ML}_0$ where an infinite, cumulative sequence of universes $\mathbb{U}_n, \mathbb{T}_n(\cdot)$, $n < \omega$, is assumed. To be cumulative means that for each type A there is some n and some $a \in \mathbb{U}_n$ such that $A = \mathbb{T}_n(a)$. Note that n is an external index. Moreover we have a function $\mathsf{t}_n : \mathbb{U}_n \to \mathbb{U}_{n+1}$ and a constant $\mathsf{u}_n \in \mathbb{U}_{n+1}$ with

$$\mathbb{T}_{n+1}(\mathsf{t}_n(a)) = \mathbb{T}_n(a), \qquad \mathbb{T}_{n+1}(\mathsf{u}_n) = \mathbb{U}_n.$$

This type theory is defined in [15]. For each $n < \omega$, let $\mathbf{Sets}_n$ be the full subcategory of **Sets** where the objects are sets $A = (\overline{A}, =_A)$ with $\overline{A} = \mathbb{T}_n(a)$, for some $a \in \mathbb{U}_n$, and $x =_A y$ is of the form $\mathbb{T}_n(e(x,y))$ for some $e \in (\mathbb{T}_n(a \times a) \to \mathbb{U}_n)$. Clearly $\mathbf{Sets}_0 \subseteq \mathbf{Sets}_1 \subseteq \mathbf{Sets}_2 \subseteq \cdots$.

Existence of W-sets. In type theory, the W-type is, as other types, defined by giving natural deduction style rules, thus specifying introduction rules that tell us how new elements are formed in the type, and elimination rules describing how functions are defined on the type. For any family of types $B(x)$ $(x \in A)$, we can form a W-type: $W = (\mathbb{W}x \in A)B(x)$, with the following rules:

$$\text{(intro.)} \quad \frac{a \in A \qquad f \in B(a) \to W}{\mathsf{sup}(a, f) \in W}$$

$$\text{(elim.)}\quad \frac{c \in W \qquad \begin{array}{c}(x \in A, f \in B(x) \to W, z \in (\Pi u \in B(x))C(f(u)))\\ \vdots \\ d(x,f,z) \in C(\mathsf{sup}(x,f))\end{array}}{\mathsf{R}_W(c,d) \in C(c)}$$

Moreover we have the computation rule

$$\mathsf{R}_W(\mathsf{sup}(a,g),d) = d(a,g,\lambda u \in B(a).\mathsf{R}_W(g(u),d)).$$

We extend the theory $\mathbf{ML}_{<\omega}$ by W-types. This means more precisely that the rules for W-types are included, and that every universe U_n, T_n is closed under formation of W-types. The extension is denoted $\mathbf{ML}_{<\omega}\mathbf{W}$. In [16] we will show that the category **Sets** in this extended theory is a stratified pseudo-topos (cf. Remark 5.9), here we restrict our attention to W-types.

Theorem 7.2 *In* $\mathbf{ML}_{<\omega}\mathbf{W}$*, the category* **Sets** *has W-types.*

Proof. Let $\varphi : B \to A$ be a map in $\mathbf{Sets}_n$. Then the inverse image $\varphi^{-1}(a) = B_a$ is given by $B_a = (\overline{B_a}, =_{B_a})$ where $\overline{B_a} = (\Sigma b \in \overline{B})\,[\varphi(b) =_A a]$, and $(b,p) =_{B_a} (b',p')$ iff $b =_B b'$. If $q : a =_A a'$ is a proof object, then for every $(b,p) \in \overline{B_a}$ there is a p' with $(b,p') \in \overline{B_{a'}}$. Since the equality on $B_{a'}$ ignores the second component, this defines a function $\beta_{aa'q} : B_a \to B_{a'}$ which does not depend on q. The relevant endofunctor for φ is constructed as follows. Let $P(X)$ be the set with

$$\overline{P(X)} = (\Sigma a \in \overline{A})\,\overline{X^{B_a}}$$

and where $(a,f) =_{P(X)} (a',f')$ iff there exists $p : a =_A a'$ with f and $f' \circ \beta_{aa'p}$ equal in X^{B_a}. For maps $h : X \to Y$ let $P(h)((a,f)) = (a, h \circ f)$.

Let W^0 be the W-type $(\mathbb{W}a \in \overline{A})\overline{B_a}$ with the partial equivalence relation $\simeq_W$ inductively defined by: $\mathsf{sup}(a,f) \simeq_W \mathsf{sup}(a',f')$ iff for some $p : a' =_I a$ and for all $x \in B_a, x' \in B_{a'}$:

$$x =_{B_a} \beta_{a'ap}(x') \Longrightarrow f(x) \simeq_W f'(x').$$

This relation can be realized as a propositional function $W^0 \times W^0 \to \mathbb{U}_n$. It follows by an inductive argument that $=_W$ is symmetric and transitive. Let $\overline{W} = (\Sigma x \in W^0)\,[x \simeq_W x]$, and $(x,p) =_W (x',p')$ iff $x \simeq_W x'$. This makes $W = (\overline{W}, =_W)$ into a set in $\mathbf{Sets}_n$.

Define $\sigma : T(W) \to W$ by

$$\sigma((a,f)) = (\mathsf{sup}(a, \pi_1 \circ f), p),$$

where π_1 is the first projection and p is a proof that $\mathsf{sup}(a,\pi_1 \circ f) =_W \mathsf{sup}(a,\pi_1 \circ f)$. The latter follows from the assumption that f is extensional.

We show that (W, σ) is an initial P-algebra. Let X be any set and let $h : PX \to X$ be a function. We shall find (a unique) $r : W \to X$ such that

$$\begin{array}{ccc} PW & \xrightarrow{\sigma} & W \\ {\scriptstyle Pr}\downarrow & & \downarrow{\scriptstyle r} \\ PX & \xrightarrow{h} & X \end{array} \tag{30}$$

commutes, i.e. $r \circ \sigma = h \circ Pr$. But this is equivalent to the condition

$$r(\sigma(a, f)) = h(a, r \circ f) \qquad (a \in A, f : B_a \to W).$$

Writing $\sigma(a, f) = (\mathsf{sup}(a, g), p)$, $g = \pi_1 \circ f$, $q = \pi_2 \circ f$ this is

$$r(\mathsf{sup}(a, g), p) = h(a, \lambda x.r(g(x), q(x))).$$

Such an r can be defined by W^0-recursion. Namely, put $r(u, p) = r_0(u)(p)$ where $r_0 \in (\Pi u \in W^0)[u \simeq_W u \to A]$ and let

$$r_0(\mathsf{sup}(a, g)) = H(a, g, \lambda x.r_0(g(x))),$$

where $H(a, g, f) = (\lambda p \in \mathsf{sup}(a, g) \simeq_W \mathsf{sup}(a, g))\, h(a, \lambda x \in B_a.f(x)(t(p, x)))$, and where $t(p, x)$ is the proof of $g(x) \simeq_W g(x)$. The latter can easily be extracted from p.

By W^0-induction it follows that r is extensional and makes the diagram (30) commute. The uniqueness of r is clear, since only the first component of W determines the equality. ∎

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