

# On the Connes-Kreimer construction of Hopf Algebras

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*Abstract:* We give a universal construction of families of Hopf  $\mathbb{P}$ -algebras for any Hopf operad  $\mathbb{P}$ . As a special case, we recover the Connes-Kreimer Hopf algebra of rooted trees.

*Keywords:* Hopf operad, Hopf algebra, Hochschild cohomology.

In [K], [CK] a Hopf algebra  $H$  of rooted trees is discussed. This algebra originates in problems of renormalisation [K] and is closely related to the Hopf algebra introduced in [CM] in the context of cyclic homology and foliations. The algebra  $H$  is the polynomial algebra on countably many indeterminates  $T$ , one for each finite rooted tree  $T$ . Its comultiplication is given by the formula

$$\Delta(T) = 1 \otimes T + T \otimes 1 + \sum_c F_c \otimes R_c,$$

see [CK]. Here  $c$  ranges over all “cuts” (prunings) of the tree  $T$ . Such cuts are assumed non-empty, and to contain at most one edge on each branch.  $R_c$  is the part of the tree which remains after having performed the pruning, and  $F_c$  is the product of subtrees which have fallen on the ground. In [CK] it is proved that this comultiplication indeed makes  $H$  into a Hopf algebra. Furthermore,  $H$  is equipped with a linear endomorphism  $\lambda$ , which is a universal cocycle for a suitably defined Hochschild cohomology of Hopf algebras.

The first aim of this note is to show that all these properties can in fact be deduced from a more basic universal property of  $H$ . Namely,  $H$  is the initial object in the category of (commutative unitary) algebras equipped with a linear endomorphism. Having realized that this is the case, it becomes clear that  $H$  is in fact equipped with a large family of Hopf algebra structures, all making the endomorphism  $\lambda$  into a universal cocycle for the corresponding Hochschild cohomology. For example, for any two complex numbers  $q_1$  and  $q_2$ , there is a coproduct on  $H$ , uniquely determined by the identity

$$\Delta(\lambda(T)) = \sum q_1^{|T_{(1)}|} \cdot T_{(1)} \otimes \lambda(T_{(2)}) + \lambda(T_{(1)}) \otimes q_2^{|T_{(2)}|} \cdot T_{(2)},$$

where  $|T|$  denotes the number of nodes in the tree  $T$ . For  $q_1 = 1$  and  $q_2 = 0$  one recovers the Hopf algebra structure of [CK].

The second aim is to describe how this construction applies more generally to “algebras” for any operad  $\mathbb{P}$  on an additive category, as soon as one has a well-behaved tensor product of algebras. More precisely, we will show that if  $\mathbb{P}$  is a “Hopf operad” on a symmetric monoidal additive category, then the initial object in the category of  $\mathbb{P}$ -algebras equipped with a “linear” endomorphism is naturally equipped with a family of natural Hopf  $\mathbb{P}$ -algebra structures. The algebra of rooted trees then becomes the extreme instance of this construction where the operad  $\mathbb{P}$  is the unit object in each degree.

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## 1 Operads and algebras.

**1.1 The underlying category.** In this preliminary section we will consider operads on a category  $\mathcal{C}$ . We will assume that  $\mathcal{C}$  is a symmetric monoidal additive category, with countable sums and quotients of actions by finite groups on objects of  $\mathcal{C}$ . (In most cases,  $\mathcal{C}$  will be closed under all small colimits.) As an example, the reader may wish to keep the category of vector spaces over a field  $k$  in mind in what follows. We will write  $k$  for the unit object of  $\mathcal{C}$ , and  $a, l, r$  for the associativity and unit isomorphisms. The symmetry will be denoted by  $c$ , with components  $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ . We will assume that  $\otimes$  is an additive functor in each variable separately. Often, the isomorphisms  $a, l, r$  will be suppressed from the notation, and we identify  $k \otimes X$  with  $X$ , and  $X \otimes (Y \otimes Z)$  with  $(X \otimes Y) \otimes Z$ , etc. This is justified, on the basis of Mac Lane’s coherence theorem. See [CWM] for details.

**1.2 Operads.** ([M], [KM], [GK], ...) We will consider operads  $\mathbb{P}$  on such a category  $\mathcal{C}$ , and write  $\mathbb{P}(n)$  for the object (of  $\mathcal{C}$ ) of  $n$ -ary operations. We will always assume that our operads have a distinguished “unit element”  $u : k \rightarrow \mathbb{P}(0)$ . We will *not* assume that this map is an isomorphism, i.e. that  $\mathbb{P}$  is unitary in the sense of [KM]. Many operads are unitary, but the constructions of 1.3 lead us out of unitary operads. Note that the unit  $u : k \rightarrow \mathbb{P}(0)$  provides us with a unit  $u_A : k \rightarrow A$  in any  $\mathbb{P}$ -algebra  $A$ .

The functor underlying the monad on  $\mathcal{C}$  whose algebras are  $\mathbb{P}$ -algebras will be denoted by  $F_{\mathbb{P}} : \mathcal{C} \rightarrow \mathcal{C}$ ; so for any object  $V$  in  $\mathcal{C}$ ,

$$F_{\mathbb{P}}(V) = \coprod_{n \geq 0} \mathbb{P}(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

This object  $F_{\mathbb{P}}(V)$  is the free  $\mathbb{P}$ -algebra generated by  $V$ .

**1.3 Two constructions.** (i) If  $\mathbb{P}$  is an operad on  $\mathcal{C}$  and  $G$  is an object of  $\mathcal{C}$ , there is an operad  $\mathbb{P}_G$  whose algebras are  $\mathbb{P}$ -algebras equipped with a map from  $G$ . Thus,  $\mathbb{P}_G$  is obtained from  $\mathbb{P}$  by adding  $G$  to the space  $\mathbb{P}(0)$  of “constants” (nullary operations). Explicitly,

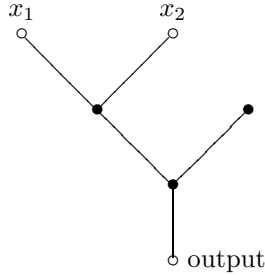
$$\mathbb{P}_G(n) = \coprod_{p \geq 0} \mathbb{P}(n+p) \otimes_{\Sigma_p} G^{\otimes p}.$$

Note that the initial  $\mathbb{P}_G$ -algebra  $\mathbb{P}_G(0)$  is the free  $\mathbb{P}$ -algebra  $F_{\mathbb{P}}(G)$  on  $G$ .

(ii) Let  $\mathbb{P}$  be an operad on  $\mathcal{C}$ . A  $\mathbb{P}[t]$ -algebra is a pair  $(A, \alpha)$  where  $A$  is a  $\mathbb{P}$ -algebra and  $\alpha : A \rightarrow A$  is a map in  $\mathcal{C}$ . (We will often refer to maps in  $\mathcal{C}$  as “linear maps”, to contrast them with  $\mathbb{P}$ -algebra homomorphisms.) A map between  $\mathbb{P}[t]$ -algebras  $(A, \alpha) \rightarrow (B, \beta)$  is a map of  $\mathbb{P}$ -algebras  $f : A \rightarrow B$  such that  $\beta f = f \alpha$ . This defines a category of  $\mathbb{P}[t]$ -algebras. This category is the category of algebras for an operad, again denoted  $\mathbb{P}[t]$ . It is the operad obtained

by freely adjoining a unary operation “ $t$ ” to  $\mathbb{P}$ . It is not difficult to give an explicit description of  $\mathbb{P}[t]$  in terms of trees, analogous to constructions in [GK]. We will not need such an explicit description.

**1.4 Example.** Let  $\mathcal{C}$  be the category of vector spaces over a field  $k$ , and let  $\mathbb{P}$  be the operad  $\mathbb{P}(n) = k$ . Its algebras are commutative unitary  $k$ -algebras, and the monad  $F_{\mathbb{P}}$  associated to  $\mathbb{P}$  is the symmetric algebra functor. The associated operad  $\mathbb{P}[t]$  can be described as follows. The space  $\mathbb{P}[t](n)$  is the vector space on rooted finite trees  $\mathbb{T}$ , with one “*output node*”, the root, and  $n$  “*input nodes*”, labelled by  $x_1, \dots, x_n$ . The *inner nodes* represent application of the new unary operation  $t$ . For example, the tree



represents the binary operation  $t(t(x_1 \cdot x_2) \cdot t(1))$ . The tree consisting of just the output vertex represents the element (nullary operation) 1. We will refer to the algebra  $\mathbb{P}[t](0)$  as the algebra of *finite rooted trees*. It can be identified with the Connes-Kreimer algebra  $H$  mentioned in the introduction. (There is a slight difference in notation, in that we have merged a product of trees into one tree with a new output node added to it.)

## 2 Hopf operads.

**2.1 Coalgebras.** Let  $\mathcal{C}$  be a category as in 1.1. A *coalgebra*  $\underline{X} = (X, \varepsilon, \Delta)$  is an object  $X$  of  $\mathcal{C}$  equipped with a coassociative comultiplication  $\Delta : X \rightarrow X \otimes X$ , and a counit  $\varepsilon : X \rightarrow k$  for this comultiplication. The associated category  $\text{Coalg}(\mathcal{C})$  is again a (symmetric) monoidal category, with the usual tensor product ( $\underline{X} \otimes \underline{Y}$  is  $X \otimes Y$  with as comultiplication the composition of  $\Delta_X \otimes \Delta_Y : X \otimes Y \rightarrow (X \otimes X) \otimes (Y \otimes Y)$  and the symmetry  $X \otimes c \otimes Y : (X \otimes X) \otimes (Y \otimes Y) \rightarrow (X \otimes Y) \otimes (X \otimes Y)$ ).

**2.2 Hopf operads.** A *Hopf operad* on  $\mathcal{C}$  is an operad  $\mathbb{P}$  on  $\mathcal{C}$  equipped with additional structure making it an operad on  $\text{Coalg}(\mathcal{C})$ . Thus, each  $\mathbb{P}(n)$  has the structure of a coalgebra,

$$k \xleftarrow{\varepsilon} \mathbb{P}(n) \xrightarrow{\Delta} \mathbb{P}(n) \otimes \mathbb{P}(n), \quad (1)$$

this structure is  $\Sigma_n$ -invariant, and the structure maps of the operad  $\mathbb{P}(n) \otimes \mathbb{P}(k_1) \otimes \dots \otimes \mathbb{P}(k_n) \rightarrow \mathbb{P}(k_1 + \dots + k_n)$  are coalgebra maps. The notion of a Hopf operad has been introduced in [GJ]. (But beware that their coalgebras are not necessarily counital.) I will sometimes write  $\underline{\mathbb{P}}$  for this operad on  $\text{Coalg}(\mathcal{C})$ , as opposed to the operad  $\mathbb{P}$  on  $\mathcal{C}$ . The Hopf operad  $\mathbb{P}$  is *cocommutative* if each of the coalgebras  $\mathbb{P}(n)$  is.

If  $\mathbb{P}$  is a Hopf operad, then the tensor product  $A \otimes B$  of two  $\mathbb{P}$ -algebras  $A$  and  $B$  is again a  $\mathbb{P}$ -algebra, by the maps

$$\begin{aligned} \mathbb{P}(n) \otimes (A \otimes B)^{\otimes n} &\xrightarrow{\Delta \otimes \text{id}} \mathbb{P}(n) \otimes \mathbb{P}(n) \otimes (A \otimes B)^{\otimes n} \xrightarrow{c} \\ &(\mathbb{P}(n) \otimes A^{\otimes n}) \otimes (\mathbb{P}(n) \otimes B^{\otimes n}) \longrightarrow A \otimes B. \end{aligned}$$

Moreover, the counits  $\varepsilon : \mathbb{P}(n) \rightarrow k$  in (1) make  $k$  into a  $\mathbb{P}$ -algebra, which is a unit for this tensor product of  $k$ -algebras. Thus, the category of  $\mathbb{P}$ -algebras is again a monoidal category (symmetric if  $\mathbb{P}$  is cocommutative). A coalgebra in this category of  $\mathbb{P}$ -algebras is the same thing as a  $\underline{\mathbb{P}}$ -algebra in the category  $\text{Coalg}(\mathcal{C})$  of coalgebras, and (as in [GJ]) will be referred to as a *Hopf  $\mathbb{P}$ -algebra*.

**2.3 Example.** The free  $\mathbb{P}$ -algebra  $F_{\mathbb{P}}(G)$  on an object  $G$  has a canonical Hopf  $\mathbb{P}$ -algebra structure, cocommutative if  $\mathbb{P}$  is. Indeed, since  $F_{\mathbb{P}}(G)$  is *free*, the maps  $0 : G \rightarrow k$  and  $\text{id} \otimes 1 + 1 \otimes \text{id} : G \rightarrow F_{\mathbb{P}}(G) \otimes F_{\mathbb{P}}(G)$  into  $\mathbb{P}$ -algebras extend uniquely to  $\mathbb{P}$ -algebra maps

$$k \xleftarrow{\varepsilon} F_{\mathbb{P}}(G) \xrightarrow{\Delta} F_{\mathbb{P}}(G) \otimes F_{\mathbb{P}}(G),$$

and one easily checks that this provides the claimed structure.

### 3 The Connes-Kreimer construction.

Let  $\mathbb{P}$  be a Hopf operad on a category  $\mathcal{C}$  as before, and let  $\mathbb{P}[t]$  be the associated operad whose algebras are  $\mathbb{P}$ -algebras equipped with a “linear” endomorphism. We now present a general construction of Hopf  $\mathbb{P}$ -algebras, of which the Connes-Kreimer Hopf algebra is a special case.

**3.1 The initial  $\mathbb{P}[t]$ -algebra.** Let  $(H, \lambda)$  denote the initial  $\mathbb{P}[t]$ -algebra, i.e.  $(H, \lambda) = \mathbb{P}[t](0)$ . Thus  $H$  is a  $\mathbb{P}$ -algebra,  $\lambda : H \rightarrow H$  is a linear map (i.e. just an arrow in  $\mathcal{C}$ ), and these have the following universal property: For any  $\mathbb{P}$ -algebra  $A$  and any linear map  $\alpha : A \rightarrow A$ , there is a *unique*  $\mathbb{P}$ -algebra map  $\varphi : H \rightarrow A$  such that  $\alpha\varphi = \varphi\lambda$ .

**3.2 Lemma.** *There is a unique augmentation  $\varepsilon : H \rightarrow k$  with  $\lambda\varepsilon = 0$ .*

*Proof:* Apply the universal property to the  $\mathbb{P}$ -algebra  $k$  with the zero endomorphism.  $\square$

Next, let  $\sigma_1, \sigma_2 : H \rightarrow H$  be two linear maps. Let

$$(\sigma_1, \sigma_2) = \sigma_1 \otimes \lambda + \lambda \otimes \sigma_2 : H \otimes H \rightarrow H \otimes H.$$

This gives  $H \otimes H$  the structure of a  $\mathbb{P}[t]$ -algebra. So there is a unique  $\mathbb{P}$ -algebra map

$$\Delta = \Delta_{\sigma_1, \sigma_2} : H \rightarrow H \otimes H$$

such that  $(\sigma_1, \sigma_2) \circ \Delta = \Delta \circ \lambda$ .

**3.3 Lemma.** (i) If  $\varepsilon\sigma_i = \varepsilon$  for  $i = 1, 2$  then  $\varepsilon : H \rightarrow k$  is a counit for  $\Delta$ .  
(ii) If, in addition,  $\Delta\sigma_i = (\sigma_i \otimes \sigma_i)\Delta$  for  $i = 1, 2$  then  $\Delta$  is coassociative.

*Proof:* (i) Consider the maps

$$(H, \lambda) \xrightarrow{\Delta} (H \otimes H, (\sigma_1, \sigma_2)) \xrightleftharpoons[\varepsilon \otimes \text{id}]{\text{id} \otimes \varepsilon} (H, \lambda),$$

where on the right the isomorphisms  $H \otimes k = H = k \otimes H$  have been suppressed. By initiality of  $H$ , it is enough to prove that  $\text{id} \otimes \varepsilon$  and  $\varepsilon \otimes \text{id}$  are  $\mathbb{P}[t]$ -homomorphisms. This is indeed the case, since

$$\begin{aligned} (\text{id} \otimes \varepsilon)(\sigma_1, \sigma_2) &= (\text{id} \otimes \varepsilon)(\sigma_1 \otimes \lambda + \lambda \otimes \sigma_2) && \text{(definition)} \\ &= \sigma_1 \otimes \varepsilon\lambda + \lambda \otimes \varepsilon\sigma_2 \\ &= \lambda \otimes \varepsilon\sigma_2 && (\varepsilon\lambda = 0) \\ &= \lambda \otimes \varepsilon && \text{(assumption)} \\ &= \lambda \circ (\text{id} \otimes \varepsilon), \end{aligned}$$

and similarly  $(\varepsilon \otimes \text{id})(\sigma_1, \sigma_2) = \lambda \circ (\varepsilon \otimes \text{id})$ .

(ii) Consider the map  $\nu : H \otimes H \otimes H \rightarrow H \otimes H \otimes H$ ,

$$\nu = \lambda \otimes \sigma_2 \otimes \sigma_2 + \sigma_1 \otimes \lambda \otimes \sigma_2 + \sigma_1 \otimes \sigma_1 \otimes \lambda.$$

This makes  $H^{\otimes 3}$  into a  $\mathbb{P}[t]$ -algebra, so there is a unique  $\mathbb{P}[t]$ -homomorphism  $(H, \lambda) \rightarrow (H^{\otimes 3}, \nu)$ . It thus suffices to show that  $(\text{id} \otimes \Delta)\Delta$  and  $(\Delta \otimes \text{id})\Delta$  both are. For the first,

$$\begin{aligned} (\text{id} \otimes \Delta)\Delta\lambda &= (\text{id} \otimes \Delta)(\sigma_1 \otimes \lambda + \lambda \otimes \sigma_2)\Delta \\ &= (\sigma_1 \otimes \Delta\lambda + \lambda \otimes \Delta\sigma_2)\Delta \\ &= (\sigma_1 \otimes \sigma_1 \otimes \lambda + \sigma_1 \otimes \lambda \otimes \sigma_2 + \lambda \otimes \sigma_2 \otimes \sigma_2)(\text{id} \otimes \Delta)\Delta \\ &= \nu(\text{id} \otimes \Delta)\Delta. \end{aligned}$$

The calculation for  $(\Delta \otimes \text{id})\Delta$  is similar. □

The preceding lemmas prove:

**3.4 Theorem.** *The initial  $\mathbb{P}[t]$ -algebra  $(H, \lambda)$  has a natural family of Hopf  $\mathbb{P}$ -algebra structures, parametrized by pairs  $\sigma_1, \sigma_2 : H \rightarrow H$  satisfying the conditions of Lemma 3.3.*

**3.5 Example.** The conditions of Lemma 3.3 are always satisfied if one takes  $\sigma_i$  to be the identity  $H \rightarrow H$  or the composition of the counit  $\varepsilon : H \rightarrow k$  and the unit  $u : k \rightarrow H$ , or any convex combination  $\alpha \cdot \text{id} + \beta \cdot u\varepsilon : H \rightarrow H$  of these two (for  $\alpha, \beta : k \rightarrow k$  with  $\alpha + \beta = \text{id}$ ). This provides many different Hopf  $\mathbb{P}$ -algebra structures on  $H$ .

**3.6 Example.** Consider again the case of the commutative unitary algebra operad of 1.4. Then  $H$  is the algebra of finite rooted trees  $T$ . Note that  $\varepsilon(T) = 0$  as soon as  $T$  has at least one inner node. Write  $|T|$  for the number of inner nodes of  $T$ . Now let  $q_1, q_2 \in k$  be any two numbers, and let

$$\sigma_i = q_i^{|T|} \cdot T, \quad \text{for } i = 1, 2$$

Then  $\sigma_1$  and  $\sigma_2$  satisfy the condition of Lemma 3.3. Thus for any two  $q_1, q_2 \in k$ , the algebra  $H$  has a Hopf algebra structure, with the usual counit, and with comultiplication completely determined by the identity

$$\Delta\lambda(T) = \sum q_1^{|T_{(1)}|} T_{(1)} \otimes \lambda(T_{(2)}) + \lambda(T_{(1)}) \otimes q_2^{|T_{(2)}|} \cdot T_{(2)}$$

where we write  $\Delta(T) = \sum T_{(1)} \otimes T_{(2)}$  as usual [S]. For the values  $q_1 = 1$  and  $q_2 = 0$  one finds  $\sigma_1 = \text{id}$  and  $\sigma_2 = \varepsilon$ , and one recovers the Hopf algebra structure of [CK].

**3.7 Remark.** The results and examples in this section have been stated for the initial  $\mathbb{P}[t]$ -algebra  $(H, \lambda) = \mathbb{P}[t](0)$ . Similar facts hold for the free  $\mathbb{P}[t]$ -algebra generated by any object  $G$  of  $\mathcal{C}$ . Writing  $(H[G], \lambda)$  for this algebra and  $j : G \rightarrow H[G]$  for the universal map from  $G$ , one defines  $\Delta : H[G] \rightarrow H[G] \otimes H[G]$  from  $\sigma_1$  and  $\sigma_2$  as the unique map of  $\mathbb{P}[t]$ -algebras satisfying  $\Delta\lambda = (\sigma_1 \otimes \lambda + \lambda \otimes \sigma_2)\Delta$  as before and extending the map  $u \otimes j + j \otimes u : G \rightarrow H[G] \otimes H[G]$  (where  $u : k \rightarrow H[G]$  is the unit). However, rather than doing the calculation again, this can be seen as a formal consequence of the statements made for the initial algebra, because the free  $\mathbb{P}[t]$ -algebra on  $G$  is the initial  $\mathbb{P}_G[t]$ -algebra (cf. 1.3.(i)), and  $\mathbb{P}_G$  is a Hopf operad whenever  $\mathbb{P}$  is.

## 4 Hochschild cohomology.

In [CK] it is proved that for the Connes-Kreimer algebra  $(H, \lambda)$  (cf. Example 3.6), the map  $\lambda$  is a universal 1-cocycle for Hochschild cohomology. In this section, we show that this result extends to our more general construction.

Recall the definition of the Hochschild cohomology groups  $H^*(A, M)$  for any algebra  $A$  and any bimodule  $M$ , from the complex with maps  $A^{\otimes n} \rightarrow M$  as cochains (see e.g. [L, formula (1.5.1.1)]). Turning around all the arrows in a diagrammatic form of this definition, one obtains a cohomology  $H^*(E, C)$  of a coalgebra  $C$  with coefficients in a bicomodule  $E$ , as the cohomology of the complex  $C^n(E, C) = \text{Hom}_C(E, C^{\otimes n})$ . Explicitly, this is the cohomology of the simplicial abelian group with the face maps  $d_i : C^{n-1}(E, C) \rightarrow C^n(E, C)$  defined for  $\varphi : E \rightarrow C^{\otimes(n-1)}$  by

$$d_i(\varphi) = \begin{cases} E \xrightarrow{l} C \otimes E \xrightarrow{C \otimes \varphi} C \otimes C^{\otimes n-1} = C^{\otimes n} & (i = 0) \\ E \xrightarrow{\varphi} C^{\otimes n-1} \xrightarrow{\Delta^{(i)}} C^{\otimes n} & (0 < i < n) \\ E \xrightarrow{r} E \otimes C \xrightarrow{\varphi \otimes C} C^{\otimes n} & (i = n). \end{cases}$$

Here  $l$  and  $r$  are the left and right coactions, and  $\Delta^{(i)} = C^{\otimes(i-1)} \otimes \Delta \otimes C^{\otimes(n-i-1)}$ . Note that this cohomology  $H^*(E, C)$  is *contravariant* in  $E$  and *covariant* in  $C$ .

In particular, given “linear” maps  $\sigma_1, \sigma_2 : C \rightarrow C$ , we can view  $C$  itself as a  $C$ -bimodule  ${}_{\sigma_1}C_{\sigma_2}$ , with left action  $C \xrightarrow{\Delta} C \otimes C \xrightarrow{\sigma_1 \otimes C} C \otimes C$  and right action  $C \xrightarrow{\Delta} C \otimes C \xrightarrow{C \otimes \sigma_2} C \otimes C$ . We denote the corresponding cohomology by

$$HH_{\sigma_1, \sigma_2}^*(C). \quad (2)$$

A map  $\varphi : C \rightarrow C$  is a 1-cocycle for this cohomology precisely when

$$\Delta \circ \varphi = (\sigma_1 \otimes \varphi + \varphi \otimes \sigma_2)\Delta. \quad (3)$$

Now let us go back to the context of a Hopf operad  $\mathbb{P}$  on our underlying category  $\mathcal{C}$ .

**4.1 Natural twisting functions.** Call  $\sigma$  a natural twisting function if  $\sigma$  assigns to each Hopf  $\mathbb{P}$ -algebra  $C$  a linear endomorphism  $\sigma = \sigma^{(C)} : C \rightarrow C$ , which is natural for morphisms of augmented  $\mathbb{P}$ -algebras (i.e. if  $f : C \rightarrow D$  is such a morphism then  $f \circ \sigma^{(C)} = \sigma^{(D)} \circ f$ ), and has the property that  $\sigma^{(k)}$  is the identity. Note that this implies that  $\varepsilon \circ \sigma^{(C)} = \varepsilon$ . For example, the identity  $C \rightarrow C$  and the composition  $C \xrightarrow{\varepsilon} k \xrightarrow{u} C$  of the augmentation and the unit are natural twisting functions, as is any convex combination  $\alpha \cdot \text{id} + \beta \cdot u\varepsilon : C \rightarrow C$  of these two (for  $\alpha, \beta : k \rightarrow k$  with  $\alpha + \beta = \text{id}$ ).

Now let  $(H, \lambda)$  be the initial  $\mathbb{P}[t]$ -algebra, and let  $\sigma_1 = \sigma_1^{(H)}, \sigma_2 = \sigma_2^{(H)} : H \rightarrow H$  be the components of two natural twisting functions. Suppose that  $\sigma_1$  and  $\sigma_2$  define a Hopf  $\mathbb{P}$ -algebra structure  $(H, \Delta, \varepsilon)$  on  $H$ , by Theorem 3.4. Observe that the defining equation  $(\sigma_1, \sigma_2)\Delta = \Delta\lambda$  for the coproduct states precisely that  $\lambda$  is a 1-cocycle for  $HH_{\sigma_1, \sigma_2}^*(H)$ . The following theorem is now a consequence of the universal property (3.1) of  $(H, \lambda)$ .

**4.2 Theorem.** *The map  $\lambda$  is the universal 1-cocycle. More explicitly, if  $B$  is a Hopf  $\mathbb{P}$ -algebra and  $\gamma$  is a 1-cocycle in the complex defining  $HH_{\sigma_1, \sigma_2}^*(B)$ , there is a unique Hopf  $\mathbb{P}$ -algebra map  $c_\gamma : H \rightarrow B$  such that  $c_\gamma \circ \lambda = \gamma \circ c_\gamma$ .*

*Proof:* By the universal property of  $H$  and  $\lambda$ , there is a unique  $\mathbb{P}$ -algebra map  $c = c_\gamma : H \rightarrow B$  such that  $\gamma c = c\lambda$ . It suffices to show that  $c$  is a coalgebra map. First, we show that  $c$  is a map of augmented algebras, i.e.  $\varepsilon \circ c = \varepsilon$ . By initiality of  $(H, \lambda)$ , it suffices to show that the composite  $(H, \lambda) \xrightarrow{c} (B, \gamma) \xrightarrow{\varepsilon} (k, 0)$  is a map of  $\mathbb{P}[t]$ -algebras; in other words, that  $\varepsilon\gamma = 0$ . To prove this, apply  $\varepsilon \otimes \varepsilon$  to the cocycle condition  $\Delta\gamma = (\sigma_1 \otimes \gamma + \gamma \otimes \sigma_2)\Delta$ . Using that  $(\varepsilon \otimes \varepsilon)\Delta = \varepsilon$ , and  $\varepsilon\sigma_i = \varepsilon$  (as observed above), this yields  $\varepsilon\gamma = (\varepsilon \otimes \varepsilon\gamma + \varepsilon\gamma \otimes \varepsilon)\Delta = \varepsilon\gamma + \varepsilon\gamma$ . Thus  $\varepsilon\gamma = 0$ , as desired.

Next, we show that the map  $c$  preserves coproducts. Observe that, by initiality of  $(H, \lambda)$ , the square

$$\begin{array}{ccc} (H, \lambda) & \xrightarrow{\Delta} & (H \otimes H, \sigma_1^{(H)} \otimes \lambda + \lambda \otimes \sigma_2^{(H)}) \\ c \downarrow & & \downarrow c \otimes c \\ (B, \gamma) & \xrightarrow{\Delta} & (B \otimes B, \sigma_1^{(B)} \otimes \gamma + \gamma \otimes \sigma_2^{(B)}) \end{array}$$

necessarily commutes as soon as all four maps are  $\mathbb{P}[t]$ -algebra homomorphisms. The map  $c \otimes c$  is the only one for which this still has to be shown. But, we have just proved that  $c$  is a map of augmented  $\mathbb{P}$ -algebras, so  $c \circ \sigma_i^{(H)} = \sigma_i^{(B)} \circ c$  by naturality. Since also  $c\lambda = \gamma c$ , the map  $c \otimes c$  is indeed a map of  $\mathbb{P}[t]$ -algebras. This completes the proof of the theorem.  $\square$

## 5 Remarks on functoriality.

We continue to work in the context of Hopf operads on a category  $\mathcal{C}$  as in 1.1.

**5.1 Adjoint functors.** Let  $\varphi : \mathbb{Q} \rightarrow \mathbb{P}$  be a map of Hopf operads. Then  $\varphi$  induces functors  $\varphi^* : (\mathbb{P}\text{-algebras}) \rightarrow (\mathbb{Q}\text{-algebras})$  and  $\overline{\varphi}^* : (\text{Hopf } \mathbb{P}\text{-algebras}) \rightarrow (\text{Hopf } \mathbb{Q}\text{-algebras})$ . Also,  $\varphi$  gives a functor  $\varphi^* : (\mathbb{P}[t]\text{-algebras}) \rightarrow (\mathbb{Q}[t]\text{-algebras})$ , by  $\varphi^*(B, \beta) = (\varphi^*(B), \beta)$ . If the relevant coequalizers exists in  $\mathcal{C}$  then the first functor  $\varphi^*$  has a left adjoint  $\varphi_! : (\mathbb{Q}\text{-algebras}) \rightarrow (\mathbb{P}\text{-algebras})$ , see e.g. [GJ]. Note that  $\varphi^*(k) = k$  and that the (first) functor  $\varphi^*$  commutes with tensor products of algebras. Hence by adjointness, there are canonical maps of  $\mathbb{P}$ -algebras  $\varphi_!(k) \rightarrow k$  and  $\varphi_!(A \otimes B) \rightarrow \varphi_!(A) \otimes \varphi_!(B)$ . Using these maps, one obtains a lifting of  $\varphi_!$  to a left adjoint  $\overline{\varphi}_! : (\text{Hopf-}\mathbb{P}\text{-algebras}) \rightarrow (\text{Hopf-}\mathbb{Q}\text{-algebras})$  for  $\overline{\varphi}^*$ .

Now let  $(H, \lambda)$  be the initial  $\mathbb{P}[t]$ -algebra and  $(K, \mu)$  the one for  $\mathbb{Q}$ . Let  $j_0 : (K, \mu) \rightarrow (\varphi^*(H), \lambda)$  be the unique map of  $\mathbb{Q}[t]$ -algebras, and note that this is a map of augmented  $\mathbb{Q}$ -algebras. Let  $j : \varphi_!(K) \rightarrow H$  be the adjoint map; this is a map of augmented  $\mathbb{P}$ -algebras. Next, consider natural twisting functions  $\sigma_1, \sigma_2$  on  $\mathbb{Q}$ -algebras. These also induce  $\sigma_i : H \rightarrow H$  on any  $\mathbb{P}$ -algebra  $H$ , by  $\sigma_i = \sigma_i^{(\varphi^*(H))}$ .

**5.2 Proposition.** *Suppose  $\sigma_1$  and  $\sigma_2$  satisfy the conditions of Theorem 3.4 so as to make  $H$  and  $K$  into Hopf  $\mathbb{P}$ - (respectively  $\mathbb{Q}$ -) algebras. Then  $j_0 : K \rightarrow \varphi^*(H)$  and  $j : \varphi_!(K) \rightarrow H$  are maps of Hopf  $\mathbb{P}$ - (resp.  $\mathbb{Q}$ -) algebras.*

*Proof:* The second assertion for  $j$  follows from the first for  $j_0$  by adjointness. To see that the map  $j_0$  preserves the coproduct, simply apply initiality of  $(K, \mu)$  to the square

$$\begin{array}{ccc} (K, \mu) & \xrightarrow{\Delta} & (K \otimes K, \sigma_1^{(K)} \otimes \mu + \mu \otimes \sigma_2^{(K)}) \\ j_0 \downarrow & & \downarrow j_0 \otimes j_0 \\ (\varphi^*(H), \lambda) & \xrightarrow{\Delta} & (\varphi^*(H) \otimes \varphi^*(H), \sigma_1^{(H)} \otimes \lambda + \lambda \otimes \sigma_2^{(H)}), \end{array}$$

exactly as in the proof of Theorem 4.2.  $\square$

**5.3 The operad  $\mathbb{B}$ .** A *pointed object* is an object  $X$  of  $\mathcal{C}$  equipped with a “basepoint”  $u : k \rightarrow X$ . We call  $X$  *well-pointed* if  $X$  is equipped with an augmentation  $\varepsilon : X \rightarrow k$  with  $\varepsilon u = \text{id}$ . Such an object splits as  $X = k \oplus \tilde{X}$  where  $\tilde{X} = \text{Ker}(\varepsilon)$ . Let  $\mathbb{B}$  be the operad whose algebras are pointed objects. If  $\mathbb{P}$  is any (Hopf) operad then the unit of  $\mathbb{P}$  gives a map of operads  $u : \mathbb{B} \rightarrow \mathbb{P}$ . We consider the left adjoint  $u_!$  of the induced functor  $u^* : (\mathbb{P}\text{-algebras}) \rightarrow (\mathbb{B}\text{-algebras})$ .

**5.4 Lemma.** *If  $X$  is well-pointed then  $u_!(X) = F_{\mathbb{P}}(\tilde{X})$ , the free  $\mathbb{P}$ -algebra on  $\tilde{X}$ .*

*Proof:* Let  $k \xrightarrow{u} X \xrightarrow{\varepsilon} k$  be a well-pointed object. Let  $w : X \rightarrow F_{\mathbb{P}}(\tilde{X}) = F(\tilde{X})$  be the map  $k \oplus \tilde{X} \rightarrow F(\tilde{X})$  obtained from the unit  $u_{F(\tilde{X})} : k \rightarrow F(\tilde{X})$  of this free algebra together with the canonical map  $\mu : \tilde{X} \rightarrow F(\tilde{X})$ . We claim that  $w$  is the universal base-point preserving map from  $X$  into a  $\mathbb{P}$ -algebra. Indeed, suppose  $f : X \rightarrow A$  is any map into the underlying object  $A$  of a  $\mathbb{P}$ -algebra  $\underline{A}$ , with  $f \circ u = u_{\underline{A}}$ . Since  $F(\tilde{X})$  is the free algebra, the restriction  $f \upharpoonright \tilde{X} : \tilde{X} \rightarrow A$



extends uniquely to a  $\mathbb{P}$ -algebra map  $\underline{f} : F(\tilde{X}) \rightarrow \underline{A}$ . It is easy to check that  $\underline{f} \circ w = f$  for this map  $\underline{f}$ .  $\square$

Now let  $(A, \alpha)$  be the initial  $\mathbb{B}[t]$ -algebra, and  $(H, \lambda)$  the initial  $\mathbb{P}[t]$ -algebra as before. Let  $\sigma_1, \sigma_2$  be natural twisting functions on  $\mathbb{B}$ -algebras. Suppose  $\sigma_1^{(A)}, \sigma_2^{(A)} : A \rightarrow A$  define a Hopf algebra structure on  $A$ , and  $\sigma_1^{(H)}, \sigma_2^{(H)} : H \rightarrow H$  one on  $H$ , by Theorem 3.4.

**5.5 Proposition.** *There is a canonical retraction*

$$u_!(A) \begin{array}{c} \xrightarrow{j} \\ \xleftarrow{r} \end{array} H, \quad r \circ j = \text{id},$$

where  $j$  is a map of Hopf  $\mathbb{P}$ -algebras and  $r$  one of augmented  $\mathbb{P}$ -algebras.

*Proof:* The map  $j : u_!(A) \rightarrow H$  is the one of Proposition 5.2. The map  $r : H \rightarrow u_!(A)$  is the unique map  $(H, \lambda) \rightarrow (u_!(A), \bar{\alpha})$  of  $\mathbb{P}[t]$ -algebras, for the map  $\bar{\alpha}$  defined as follows. Since  $A$  has an augmentation  $\varepsilon$  with  $\varepsilon\alpha = 0$  (Lemma 3.2), we can write  $A = k \oplus \tilde{A}$  where  $\alpha$  maps  $A$  into  $\tilde{A}$ . Also, the free  $\mathbb{P}$ -algebra  $u_!(A) = F_{\mathbb{P}}(\tilde{A})$ , briefly  $F(\tilde{A})$ , is augmented, hence splits as  $u_!(A) = k \oplus F(\tilde{A})^{\sim}$ . Now define  $\bar{\alpha}$  on these two summands separately: on  $k$  it is the composition

$$k \xrightarrow{u} A \xrightarrow{\alpha} \tilde{A} \rightarrow F(\tilde{A})$$

and on the other summand it is the map

$$F(\tilde{A})^{\sim} \subseteq F(\tilde{A}) \xrightarrow{F(\tilde{\alpha})} F(\tilde{A})$$

where  $\tilde{\alpha} : \tilde{A} \rightarrow \tilde{A}$  is the restriction of  $\alpha$ . Note that the map  $\bar{\alpha}$  thus defined satisfies the identities

$$\bar{\alpha}w = w\alpha, \quad \varepsilon\bar{\alpha} = 0,$$

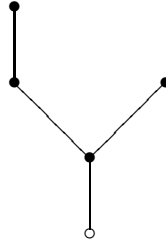
where  $w : A \rightarrow u_!(A)$  is the universal map as in the proof of the previous lemma.

We claim that  $r \circ j = \text{id}$ . By adjointness, it suffices to show  $rjw = w$  as maps of pointed objects. Now  $w\alpha = \bar{\alpha}w$  as we have seen. Also,  $j : u_!(A) \rightarrow H$  is obtained from  $j_0 : A \rightarrow u^*(H)$  by adjointness, hence  $jw = j_0$ . Thus  $(rjw)\alpha = rj_0\alpha = r\lambda j_0 = \bar{\alpha}rj_0 = \bar{\alpha}(rjw)$ . This shows that  $w$  and  $rjw$  are both maps of  $\mathbb{B}[t]$ -algebras on  $(A, \alpha)$ , hence equal by initiality.

It remains to observe that  $r$  respects the augmentation. Since  $r : (H, \lambda) \rightarrow (u_!(A), \bar{\alpha})$  and  $\varepsilon : (u_!(A), \bar{\alpha}) \rightarrow (k, 0)$  are both maps of  $\mathbb{P}[t]$ -algebras, so is the composite  $\varepsilon r$ . So  $\varepsilon r = \varepsilon$  by initiality of  $(H, \lambda)$ . This shows that  $r$  preserves the augmentation, and completes the proof.  $\square$

**5.6 Example.** Let  $(H, \lambda)$  be the Connes-Kreimer Hopf algebra of Example 3.6. For the same twisting functions  $\sigma_1 = \text{id}$  and  $\sigma_2 = u\varepsilon$ , the initial  $\mathbb{B}[t]$ -algebra  $(A, \alpha)$  is the vector space with basis  $x_0, x_1, x_2, \dots$ , where  $x_0$  is the base point and  $\alpha(x_n) = x_{n+1}$ . Thus  $u_!(A)$  is the algebra  $k[x_1, x_2, \dots]$ , where we identify  $x_0$  with  $1 \in u_!(A)$ . The Hopf algebra structure is given by  $\Delta(x_n) = \sum_{i=0}^n x_i \otimes x_{n-i}$ . The embedding  $j$  identifies  $u_!(A)$  with the subalgebra of “linear trees” of  $H$

(considered also in [CK]), and  $x_n$  with  $\lambda^n(1) \in H$ . The retraction  $r : H \rightarrow u_1(A)$  sends a tree  $T$  to the product of all the maximal branches through  $T$ . For example, the tree



representing  $\lambda(\lambda^2(1) \cdot \lambda(1))$  is sent to  $x_3 \cdot x_1$ . Note that  $r$  does not commute with coproducts.

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