ON THE INTEGRABILITY OF LIE SUBALGEBROIDS

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Abstract. Let G be a Lie groupoid with Lie algebroid $\mathfrak{g}$. It is known that, unlike in the case of Lie groups, not every subalgebroid of $\mathfrak{g}$ can be integrated by a subgroupoid of G. In this paper we study conditions on the invariant foliation defined by a given subalgebroid under which such an integration is possible. We also consider the problem of integrability by closed subgroupoids, and we give conditions under which the closure of a subgroupoid is again a subgroupoid.

Introduction

The basic theory of Lie groups and Lie algebras, contained in any course on the subject, establishes an almost perfect correspondence between finite-dimensional Lie algebras and Lie groups. In particular, any abstract Lie algebra can be 'integrated' to a Lie group, and this group is uniquely determined when one requires it to be simply connected. Morphisms from a simply connected Lie group to another one are in bijective correspondence with morphisms between their Lie algebras. If $\mathfrak{g}$ is the Lie algebra of a Lie group G, subalgebras of $\mathfrak{g}$ correspond to immersed subgroups of G. The closure of such an immersed Lie subgroup is again one, and an immersed subgroup is embedded if and only if it is closed.

The basic Lie theory becomes much more involved (and interesting) in the wider context where one allows *local* symmetries, i.e. in the context of so-called Lie groupoids and their algebroids. These objects arise naturally in the theory of actions of Lie groups or algebras on manifolds, in foliation theory, in Poisson geometry, in areas related to mathematical physics such as the theory of (gerbes on) orbifolds and quantization theory, and in many other situations [2, 3, 5, 7, 8, 14, 15, 22]. Like for groups, any Lie groupoid G has an infinitesimal counterpart, its Lie algebroid $\mathfrak{g} = \mathfrak{L}(G)$. Again, morphisms of Lie groupoids correspond bijectively to morphisms between their Lie algebroids under suitable connectivity assumptions [16, 17], but this is about as far as the simple picture of groups extends to groupoids. The main difference lies in the fact that not every Lie algebroid is integrable, i.e. arises as the Lie algebroid of a Lie groupoid. This was first noticed by Almeida and Molino [1, 21], who proved that the developability of a certain kind of foliation was obstructed by the integrability of a naturally associated algebroid, a result which immediately led to natural counterexamples to integrability. Around the same time, Mackenzie developed a cohomological obstruction to integrability of Lie groupoids [15]. Since then, our understanding of integrability of algebroids has advanced considerably, and we refer the reader to [8] and the references therein for a recent state of the art.

This note and its sequel [19] are concerned with the integrability problem for subalgebroids. To explain our results, consider a Lie groupoid G with Lie algebroid $\mathfrak{g}$. It is known that any subalgebroid $\mathfrak{h} \subset \mathfrak{g}$ of such an integrable Lie algebroid $\mathfrak{g}$ is again integrable, by a Lie groupoid H which admits an immersion $H \to G$.

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However, unlike the simple case of Lie groups and algebras, there may not exist a *subgroupoid* of G integrating $\mathfrak{h}$, or more precisely, there may not exist an *injective* such immersion $H \to G$ integrating $\mathfrak{h} \subset \mathfrak{g}$. And even if there is such an injective immersion, the closure of its image may not be a Lie subgroupoid of G. In this paper, we will use foliation theory to investigate under which conditions results like in the case of Lie groups hold in the context of Lie groupoids. The starting point is that any subalgebroid $\mathfrak{h} \subset \mathfrak{g}$ gives rise to a right invariant foliation on G. We prove that $\mathfrak{h} \subset \mathfrak{g}$ can be integrated to an injective immersion $H \to G$ if and only if this foliation has trivial holonomy (a condition automatically fulfilled if G is a group). We prove also that if the invariant foliation associated to $\mathfrak{h}$ is transversely complete (again, automatically true in the group case), then $\mathfrak{h}$ integrates to an injective submersion $\iota\colon H \to G$, with the property that the closure of its image is a Lie subgroupoid $\bar{H} \subset G$. Moreover, the image of ι itself is closed if and only if ι is an embedding.

These results will be used in a sequel [19] to this paper, where we will consider the question when a given Lie subalgebroid $\mathfrak{h}$ of $\mathfrak{g}$ can be integrated to a *closed* subgroupoid of a possibly larger groupoid $\tilde{G}$ integrating $\mathfrak{g}$. Under some assumptions, we will prove that this is the case if and only if a specific Lie algebroid, naturally associated to $\mathfrak{h}$ and G, is integrable. This result extends the classical Almeida-Molino theorem referred to above.

1. Preliminaries

For the convenience of the reader, and to fix the notation, we begin by summarizing some basic definitions. For detailed exposition and many examples, the reader might wish to consult one of the books [3, 15, 18, 21] and references cited there.

1.1. **Lie algebroids.** Let M be a smooth manifold. We recall that a *Lie algebroid* over M is a (real) vector bundle $\mathfrak{g}$ over M, equipped with a vector bundle map $\mathrm{an}\colon \mathfrak{g} \to T(M)$ (called the *anchor*) and a Lie algebra structure on the vector space $\Gamma(\mathfrak{g})$ of sections of $\mathfrak{g}$. This structure has to satisfy two axioms: On sections, the map $\mathrm{an}: \Gamma(\mathfrak{g}) \to \mathfrak{X}(M)$ should be a homomorphism of Lie algebras; furthermore, the Leibniz rule

$$[X, fY] = f[X,Y] + \mathrm{an}(X)(f)Y$$

should hold for any $X, Y \in \Gamma(\mathfrak{g})$ and any $f \in C^\infty(M)$. A *morphism* $\mathfrak{g} \to \mathfrak{h}$ of Lie algebroids over M is a map of vector bundles which preserves the structure. (There is also a more involved notion of morphism between algebroids over different manifolds, see [11].) Such Lie algebroids arise in many contexts: Any foliated manifold $(M, \mathcal{F})$ can be viewed as a Lie algebroid, whose anchor is the inclusion $\mathcal{F} \to T(M)$. Any Poisson manifold P can be viewed as a Lie algebroid, whose anchor $T^*(P) \to T(P)$ is defined in terms of the Poisson bracket on the functions by $\mathrm{an}(df)(g) = -\{f, g\}$. Any infinitesimal action of a finite dimensional Lie algebra $\mathfrak{g}$ on a manifold M defines a Lie algebroid $\mathfrak{g} \ltimes M$, which is the trivial vector bundle $\mathfrak{g} \times M \to M$ with the anchor given by the infinitesimal action.

1.2. **Lie groupoids and integrability.** A Lie groupoid G over a manifold M will be denoted

$$G \underset{\mathrm{t}}{\overset{\mathrm{s}}{\rightrightarrows}} M \xrightarrow{\mathrm{u}} G \;\; ,$$

where s and t are the source and target maps, and u maps each point $x \in M$ to the unit arrow 1_x. The multiplication or composition of two arrows $g : x \to y$ and $h : y \to z$ is denoted by $hg : x \to z$. The maps s and t are required to be submersions. We will generally assume that M and the fibers of s (and hence also of t) are Hausdorff manifolds, but examples force us to allow for the possibility that G is non-Hausdorff.

For such a groupoid G, the bundle $T^{\mathrm{s}}(G)$ of source-vertical tangent vectors pulls back along $\mathrm{u}\colon M \to G$ to a vector bundle $\mathrm{u}^*(T^{\mathrm{s}}(G))$ over M, which has the structure of a Lie algebroid. This algebroid is called the Lie algebroid of the groupoid G, and denoted by $\mathfrak{L}(G)$, or sometimes simply by $\mathfrak{g}$. Lie algebroids which arise in this way are called *integrable*.

A Lie groupoid G is called *source-connected* if $\mathrm{s}\colon G \to M$ has connected fibers. Every Lie groupoid G contains an open subgroupoid whose source-fibers are the connected components of the source-fibers of G containing the units. This groupoid has the same Lie algebroid as G. For this reason, all Lie groupoids considered in this paper will be assumed source-connected.

For any Lie groupoid G there exists a unique (up to unique isomorphism) cover $p : \tilde{G} \to G$ by a Lie groupoid $\tilde{G}$ whose source-fibers are simply connected (we say that $\tilde{G}$ is *source-simply connected*). The source-fiber of $\tilde{G}$ is the universal cover of the corresponding source-fiber of G, and $\tilde{G}$ again determines the same Lie algebroid as G, i.e. the induced morphism of Lie algebroids $\mathfrak{L}(p)\colon \mathfrak{L}(\tilde{G}) \to \mathfrak{L}(G)$ is an isomorphism.

As an example, and for later reference, we mention that any foliation $\mathcal{F}$ of M, viewed as a Lie algebroid, is always integrable. It is the Lie algebroid of the *holonomy groupoid* $\mathrm{Hol}(M, \mathcal{F})$ of the foliation, and also of its source-simply connected cover, which is known as the *monodromy groupoid* $\mathrm{Mon}(M, \mathcal{F})$ of the foliation. The source-fibers of the latter groupoid are the universal covers of the leaves of $\mathcal{F}$. (For details, see [18].)

2. Subalgebroids and invariant foliations

Throughout this section G denotes a fixed Lie groupoid over M, and $\mathfrak{g}$ denotes its Lie algebroid. Recall that all Lie groupoids considered are assumed source-connected.

A *Lie subalgebroid* of $\mathfrak{g}$ is a vector subbundle $\mathfrak{h} \subset \mathfrak{g}$ for which $\Gamma(\mathfrak{h}) \subset \Gamma(\mathfrak{g})$ is a Lie subalgebra. This makes $\mathfrak{h}$ into a Lie algebroid over M, and the inclusion $\mathfrak{h} \to \mathfrak{g}$ is a morphism of Lie algebroids. (In this paper, we shall only consider subalgebroids over the same base.) We recall from [17] that any subalgebroid of an integrable Lie algebroid is itself integrable, and in particular $\mathfrak{h}$ integrates to a Lie groupoid H, while the morphism $\mathfrak{h} \to \mathfrak{g}$ integrates to an immersion $H \to G$ of Lie groupoids. Our first goal is to describe the range of immersed groupoids which arise as integrals of the given subalgebroid $\mathfrak{h}$.

Let $\mathcal{F}(\mathrm{s}) = T^{\mathrm{s}}(G)$ be the foliation of G by the fibers of the source map. There is a canonical isomorphism

$$\mathrm{t}^*(\mathfrak{g}) \xrightarrow{\cong} \mathcal{F}(\mathrm{s})$$

of vector bundles over G, and $\mathfrak{h} \subset \mathfrak{g}$ defines another foliation $\mathcal{F}(\mathfrak{h}) \cong \mathrm{t}^*(\mathfrak{h})$ which refines $\mathcal{F}(\mathrm{s})$, i.e. $\mathcal{F}(\mathfrak{h}) \subset \mathcal{F}(\mathrm{s})$. The groupoid G acts on itself by right multiplication along the source map, and $\mathcal{F}(\mathfrak{h})$ is invariant under this action. Conversely, any such right invariant foliation $\mathcal{F} \subset \mathcal{F}(\mathrm{s})$ is of the form $\mathcal{F}(\mathfrak{h})$ for a unique subalgebroid $\mathfrak{h}$, defined in terms of $\mathcal{F}$ and the unit section $\mathrm{u} : M \to G$ by $\mathfrak{h} = \mathrm{u}^*(\mathcal{F})$. For later reference, we record this correspondence.

Lemma 2.1. *Let G be a Lie groupoid with Lie algebroid $\mathfrak{g}$. Subalgebroids of $\mathfrak{g}$ correspond bijectively to right invariant foliations $\mathcal{F} \subset \mathcal{F}(\mathrm{s})$ of G.*

By the invariance of such a foliation $\mathcal{F}(\mathfrak{h})$, the Lie groupoid G is expected to act naturally on structures associated to $\mathcal{F}(\mathfrak{h})$. In particular:

Lemma 2.2. *Let G be a Lie groupoid with Lie algebroid $\mathfrak{g}$, and let $\mathfrak{h}$ be a subalgebroid of $\mathfrak{g}$. Then G naturally acts on the monodromy and holonomy groupoids of the associated right invariant foliation $\mathcal{F}(\mathfrak{h})$ of G.*

Proof. The groupoids $\mathrm{Hol}(G,\mathcal{F}(\mathfrak{h}))$ and $\mathrm{Mon}(G,\mathcal{F}(\mathfrak{h}))$ are groupoids over the space G, and this space carries and action of the groupoid G by right multiplication. An arrow in $\mathrm{Mon}(G,\mathcal{F}(\mathfrak{h}))$ is the homotopy class $[\alpha]$ of a path α inside a leaf of $\mathcal{F}(\mathfrak{h})$, which is contained in a source-fiber $\mathrm{s}^{-1}(x)$ of G. An arrow $g : y \to x$ of G acts on $[\alpha]$ in the obvious way by $[\alpha]g = [\alpha g]$, where $(\alpha g)(t) = \alpha(t)g$ for any $t \in [0,1]$. This is a path in a leaf of $\mathcal{F}(\mathfrak{h})$ because $\mathcal{F}(\mathfrak{h})$ is right invariant, and this action is obviously well-defined on homotopy classes. This describes the action of G on $\mathrm{Mon}(G,\mathcal{F}(\mathfrak{h}))$.

We only need to check that this action also descends to an action on $\mathrm{Hol}(G,\mathcal{F}(\mathfrak{h}))$. To see this, choose a local bisection $\sigma\colon U \to G$, defined on a neighbourhood U of y, with $\sigma(y) = g$. Now the action by elements of $\sigma(U)$ gives us a diffeomorphism $\mathrm{s}^{-1}(\mathrm{t}(\sigma(U))) \to \mathrm{s}^{-1}(U)$ which preserves the foliation $\mathcal{F}(\mathfrak{h})$. This in particular implies that if α is a loop with trivial holonomy, then so is αg. Thus the action of G on $\mathrm{Mon}(G,\mathcal{F}(\mathfrak{h}))$ descends to an action on $\mathrm{Hol}(G,\mathcal{F}(\mathfrak{h}))$ for which the quotient map $\mathrm{Mon}(G,\mathcal{F}(\mathfrak{h})) \to \mathrm{Hol}(G,\mathcal{F}(\mathfrak{h}))$ is G-equivariant. □

Since the G-action on itself is principal, so are the G-actions on $\mathrm{Mon}(G,\mathcal{F}(\mathfrak{h}))$ and $\mathrm{Hol}(G,\mathcal{F}(\mathfrak{h}))$ [18, Lemma 5.35]. Therefore [17, Lemma 2.1] we can form the quotient Lie groupoids

$$H_{\max} = \mathrm{Mon}(G,\mathcal{F}(\mathfrak{h}))/G$$

and

$$H_{\min} = \mathrm{Hol}(G,\mathcal{F}(\mathfrak{h}))/G$$

over M.

Theorem 2.3. *Let G be a Lie groupoid with source-simply connected cover $\tilde{G}$ and with Lie algebroid $\mathfrak{g}$, and let $\mathfrak{h}$ be a subalgebroid of $\mathfrak{g}$.*

(i) The Lie groupoids $H_{\max}$ and $H_{\min}$ defined above both integrate the subalgebroid $\mathfrak{h}$, and fit into a natural commutative square

$$\begin{tikzcd} H_{\max} \arrow[r] \arrow[d] & H_{\min} \arrow[d] \\ \tilde{G} \arrow[r] & G \end{tikzcd}$$

where both vertical maps are immersions integrating the inclusion $\mathfrak{h} \to \mathfrak{g}$.

(ii) Let $\iota\colon H \to G$ be any immersion of Lie groupoids over M which integrates the inclusion $\mathfrak{h} \to \mathfrak{g}$. Then there are natural maps

$$H_{\max} \longrightarrow H \longrightarrow H_{\min}$$

of Lie groupoids over M which both integrate the identity on $\mathfrak{h}$, and ι factors through $H \to H_{\min}$ as the canonical immersion $H_{\min} \to G$.

Remark 2.4. Motivated by this theorem, the Lie groupoid $H_{\min}$ will be referred as the *minimal integral* of $\mathfrak{h}$ over G. Analogously, the Lie groupoid $H_{\max}$ will be called the *maximal integral* of $\mathfrak{h}$. Note, however, that $H_{\max}$ is in fact the source-simply connected integral $\tilde{H}$ of $\mathfrak{h}$. In particular, it is independent of the choice of G (up to isomorphism). Since we are assuming that H is source-connected, the maps in (ii) are surjective submersions. Moreover, they restrict to covering projections on source-fibers.

Proof of Theorem 2.3. (i) That $H_{\max}$ integrates $\mathfrak{h}$ follows easily from the observation [17, Remark on p. 573] that the quotient map $\mathrm{Mon}(G, \mathcal{F}(\mathfrak{h})) \to H_{\max}$ induces an isomorphism of source-fibers. Exactly the same argument applies to the map $\mathrm{Hol}(G, \mathcal{F}(\mathfrak{h})) \to H_{\min}$. Next, since $\mathcal{F}(\mathfrak{h})$ refines the simple foliation $\mathcal{F}(\mathrm{s})$, there is a natural square

$$\begin{tikzcd}
\mathrm{Mon}(G, \mathcal{F}(\mathfrak{h})) \arrow[r] \arrow[d] & \mathrm{Hol}(G, \mathcal{F}(\mathfrak{h})) \arrow[d] \\
\mathrm{Mon}(G, \mathcal{F}(\mathrm{s})) \arrow[r] & \mathrm{Hol}(G, \mathcal{F}(\mathrm{s}))
\end{tikzcd}$$

of Lie groupoids over G and G-equivariant morphisms between them. Factoring out the principal G-action, we obtain the diagram of Lie groupoids over M as in the statement.

(ii) Suppose that $\iota\colon H \to G$ is an immersion which integrates $\mathfrak{h} \to \mathfrak{g}$. Consider the Lie groupoid $\hat{H}$ over G whose arrows $g \to g'$ are arrows h of H with $\iota(h)g = g'$. Then $\hat{H}$ integrates $\mathcal{F}(\mathfrak{h})$ and carries an obvious principal G-action for which $\hat{H}/G = H$. By [9, Proposition 1] there are maps

$$\mathrm{Mon}(G, \mathcal{F}(\mathfrak{h})) \longrightarrow \hat{H} \longrightarrow \mathrm{Hol}(G, \mathcal{F}(\mathfrak{h}))$$

of foliation groupoids over G which all integrate the same foliation $\mathcal{F}(\mathfrak{h})$. Factoring out the G-action, we obtain the desired maps of Lie groupoids over M. □

Corollary 2.5. *Let G be a Lie groupoid with Lie algebroid $\mathfrak{g}$, and let $\mathfrak{h}$ be a Lie subalgebroid of $\mathfrak{g}$ with minimal integral $H_{\min}$ over G. Then the following conditions are equivalent:*

(i) *The foliation $\mathcal{F}(\mathfrak{h})$ has trivial holonomy.*
(ii) *The canonical immersion $H_{\min} \to G$ is injective.*
(iii) *The inclusion $\mathfrak{h} \to \mathfrak{g}$ can be integrated to an injective immersion $H \to G$.*

Proof. Note first that $\mathcal{F}(\mathfrak{h})$ has trivial holonomy if and only if the map

$$\mathrm{Hol}(G, \mathcal{F}(\mathfrak{h})) \longrightarrow \mathrm{Hol}(G, \mathcal{F}(\mathrm{s}))$$

is injective. Since the G-actions on $\mathrm{Hol}(G, \mathcal{F}(\mathfrak{h}))$ and on $\mathrm{Hol}(G, \mathcal{F}(\mathrm{s}))$ are principal, this is equivalent to injectivity of the map $H_{\min} \to G$, which integrates the inclusion $\mathfrak{h} \to \mathfrak{g}$ by Theorem 2.3 (i). This shows equivalence between (i) and (ii).

Since (ii) is clearly stronger than (iii), we only need to show that (iii) implies (ii). Indeed, if $H \to G$ is any injective immersion integrating the inclusion $\mathfrak{h} \to \mathfrak{g}$, we can assume that H is source-connected, and we can use Theorem 2.3 (ii) to obtain a map $H \to H_{\min}$ which integrates the identity on $\mathfrak{h}$. Furthermore, since the composition

$$H \longrightarrow H_{\min} \longrightarrow G$$

is injective by assumption, and since $H \to \mathrm{Hol}(G, \mathcal{F}(\mathfrak{h}))$ is a covering projection on each source-fiber [9], the map $H_{\min} \to G$ is injective as well. □

Examples 2.6. (1) Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra of right invariant vector fields. Then any subalgebra $\mathfrak{h} \subset \mathfrak{g}$ determines a right invariant foliation $\mathcal{F}(\mathfrak{h})$, whose tangent space at $g \in G$ is $\mathcal{F}(\mathfrak{h})_g = dR_g(\mathfrak{h}) \subset T_g(G)$. The subalgebra $\mathfrak{h}$ is integrated by a connected Lie group H which occurs as the leaf of $\mathcal{F}(\mathfrak{h})$ through $1 \in G$, so obviously H is injectively immersed in G. By Corollary 2.5 the foliation $\mathcal{F}(\mathfrak{h})$ has trivial holonomy, a fact which is well-known in this case.

(2) Let $\mathcal{F}$ be a foliation of a manifold M. We can view $\mathcal{F}$ as a subalgebroid of $T(M)$, which is the Lie algebroid of the pair groupoid $M \times M$ over M. The source-simply connected cover of the pair groupoid is the fundamental groupoid $\Pi(M)$ of M. The foliation $\mathcal{F}$ can be integrated by an injectively immersed groupoid

$H \to M \times M$ if and only if it has trivial holonomy, and by an injectively immersed groupoid $H' \to \Pi(M)$ if and only if the pull-back of $\mathcal{F}$ to the universal cover $\tilde{M} \to M$ has trivial holonomy.

(3) Consider the torus $T^2 = S^1 \times S^1$, which is a Lie group with commutative Lie algebra $\mathbb{R}^2$. Any $\theta \in \mathbb{R}P^1$ is a one-dimensional subspace of $\mathbb{R}^2$ and hence a Lie subalgebra, so by (i) it determines a foliation $\mathcal{F}(\theta)$ of T^2 with trivial holonomy (it is the Kronecker foliation of the torus with slope θ). There is a countable dense subset A of $\mathbb{R}P^1$ such that $\mathcal{F}(\theta)$ has only compact leaves for $\theta \in A$ and only non-compact dense leaves for $\theta \notin A$.

These foliations together define a foliation $\mathcal{F}$ of the trivial bundle of Lie groups $T^2 \times \mathbb{R}P^1$ over $\mathbb{R}P^1$. If we view this bundle as a Lie groupoid over $\mathbb{R}P^1$, then the leaves of $\mathcal{F}$ are contained in the source fibers and $\mathcal{F}$ is right invariant. Therefore it determines a subalgebroid $\mathfrak{h}$ of the Lie algebroid $\mathfrak{g} = \mathbb{R}^2 \times \mathbb{R}P^1$ associated to the groupoid $T^2 \times \mathbb{R}P^1$. This subalgebroid is not integrable by a groupoid injectively immersed in $T^2 \times \mathbb{R}P^1$. Indeed, the local Reeb stability theorem implies that the holonomy of $\mathcal{F}$ is not trivial.

On the other hand, note that $\mathfrak{g}$ is also integrable by the source-simply connected groupoid $\mathbb{R}^2 \times \mathbb{R}P^1$, and that the subalgebroid $\mathfrak{h}$ is integrable by a groupoid injectively immersed in $\mathbb{R}^2 \times \mathbb{R}P^1$.

3. Subgroupoids of Lie groupoids

Let G be a fixed Lie groupoid over a connected manifold M, and write $\mathfrak{g}$ for the Lie algebroid of G. Recall that all Lie groupoids considered are assumed source-connected.

A *Lie subgroupoid* of G is another Lie groupoid H over M, equipped with an injective immersion $\iota\colon H \to G$, which is also a homomorphism of Lie groupoids over M. We sometimes say for emphasis that H is an *immersed* subgroupoid of G. In case $\iota\colon H \to G$ is an embedding, we say that H is an *embedded subgroupoid* of G. A *closed subgroupoid* is a subgroupoid H for which $\iota\colon H \to G$ is a closed embedding. (In this paper, we shall only consider subgroupoids over the same base.)

An immersed (source-connected) subgroupoid H of G is completely determined by its Lie algebroid $\mathfrak{h}$, which is a subalgebroid of $\mathfrak{g}$. By Corollary 2.5, the corresponding foliation $\mathcal{F}(H) = \mathcal{F}(\mathfrak{h})$ of G has trivial holonomy. Conversely, we say that a Lie subalgebroid $\mathfrak{h}$ of $\mathfrak{g}$ is integrable by a subgroupoid H of G if the associated injective immersion $\iota\colon H \to G$ integrates the inclusion $\mathfrak{h} \to \mathfrak{g}$. If such an integrating subgroupoid exists, it is unique. We can therefore rephrase Corollary 2.5 as follows.

Proposition 3.1. *Let G be a Lie groupoid with Lie algebroid $\mathfrak{g}$. A Lie subalgebroid $\mathfrak{h} \subset \mathfrak{g}$ can be integrated by a Lie subgroupoid of G if and only if the foliation $\mathcal{F}(\mathfrak{h})$ has trivial holonomy.*

For an immersed subgroupoid $\iota\colon H \to G$ and an arrow $g\colon x \to y$ in G, the (right) *coset* Hg is the immersed submanifold of G given by

$$H(y, \text{-}) \longrightarrow G\,, \qquad h \mapsto \iota(h)g\,.$$

These cosets are exactly the leaves of the associated foliation $\mathcal{F}(H)$ given by the Lie subalgebroid $\mathfrak{h} \subset \mathfrak{g}$ corresponding to H. We will write $G/H = G/\mathcal{F}(H)$ for the space of these right cosets, with the quotient topology. Thus, G/H is the quotient of G obtained by identifying two arrows g and g' if and only if $\mathrm{s}(g) = \mathrm{s}(g')$ and $g'g^{-1}$ belongs to (the image of) H. The map $\mathrm{s}\colon G \to M$ factors as a map $G/H \to M$, and the right action of G on itself by multiplication induces a right action on G/H along this map $G/H \to M$.

Recall that a foliation $\mathcal{F}$ of a manifold M is *simple* if it is given by the components of the fibers of a submersion into a Hausdorff manifold. It is called *weakly simple* if there exists a smooth structure of a possibly non-Hausdorff manifold on $M/\mathcal{F}$ such that the quotient map $M \to M/\mathcal{F}$ is a submersion. A foliation $\mathcal{F}$ of M is *strictly simple* if it is weakly simple and the space of leaves $M/\mathcal{F}$ is Hausdorff. In particular, any strictly simple foliation is simple, and any simple foliation is weakly simple. If the group of diffeomorphisms of $(M, \mathcal{F})$ acts transitively on M, then these three notions coincide [18, Theorem 4.3 (vi)].

Proposition 3.2. *Let H be a subgroupoid of a Lie groupoid G, and let $\mathcal{F}(H)$ be the associated foliation of G.*

(i) The subgroupoid H is embedded in G if and only if the foliation $\mathcal{F}(H)$ is weakly simple.

(ii) The subgroupoid H is closed in G if and only if the foliation $\mathcal{F}(H)$ is strictly simple.

Remark 3.3. In other words, the Lie subgroupoid H is embedded in G if and only if there is a structure of a possibly non-Hausdorff manifold on the space of cosets G/H such that the projection $G \to G/H$ is a submersion. If this is the case, then H is closed if and only if G/H is Hausdorff.

Proof of Proposition 3.2. (i) Suppose that H is embedded, and consider the equivalence relation R on G defining G/H. This relation is the image of the map $H \times_M G \to G \times G$ sending a pair (h, g) with $\mathrm{s}(h) = \mathrm{t}(g)$ to (hg, g). Since H is embedded in G, the product $H \times G$ is also embedded in $G \times G$. But $H \times_M G$ is a closed submanifold of $H \times G$, therefore it follows that it is also embedded in $G \times G$. Thus R is an embedded submanifold of $G \times G$. Both the projection $H \times_M G \to G$ and the composition $H \times_M G \to G$ are submersions: the first because it is a pull-back of the submersion $\mathrm{s}\colon H \to M$, the other because it is isomorphic to the first. By the Godement criterion [23] it follows that G/H is a (possibly non-Hausdorff) manifold such that $G \to G/H$ is a submersion.

Conversely, suppose that the foliation $\mathcal{F}(H)$ is weakly simple, so G/H has a structure of a possibly non-Hausdorff manifold such that the quotient projection $f\colon G \to G/H$ is a submersion. Since $\mathcal{F}(H)$ refines the foliation of G by the source-fibers, the source map of G factors through f as a submersion $\bar{\mathrm{s}}\colon G/H \to M$. The unit section $\mathrm{u}\colon M \to G$ induces a section (hence an embedding) $f \circ \mathrm{u}\colon M \to G/H$ of $\bar{\mathrm{s}}$. Since the groupoid H fits into the pull-back square

$$\begin{tikzcd} H \arrow[r, "\mathrm{s}"] \arrow[d] & M \arrow[d, "f\circ \mathrm{u}"] \\ G \arrow[r, "f"] & G/H \end{tikzcd}$$

it follows that H is embedded in G.

(ii) This follows directly from (i) and the fact that H is closed if and only if G/H is Hausdorff. $\square$

Recall that a vector field Y on a manifold M is called *projectable* with respect to a foliation $\mathcal{F}$ of M if its local flow preserves the foliation; or equivalently, if the Lie derivative of Y along any vector field tangent to $\mathcal{F}$ is again tangent to $\mathcal{F}$. Following Molino [20, 21], a foliation $\mathcal{F}$ of M is *transversely complete* if any tangent vector on M can be extended to a complete projectable vector field on M. By Molino's structure theorem [20], the closures of the leaves of a transversely complete foliation $\mathcal{F}$ of M are the fibers of a submersion $M \to W$, which is in fact a locally trivial fiber bundle of (Lie) foliations. We shall use (only) this property of transversely complete foliations in this paper. For our purpose it is also relevant to note that any transversely complete foliation has trivial holonomy. Examples of transversely complete foliations include foliations given by the fibers of locally trivial fiber bundles, transversely parallelizable foliations on compact manifolds [6, 21] and Lie foliations on compact manifolds [10].

An (immersed) subgroupoid H of a Hausdorff Lie groupoid G is said to be *transversely complete* if the foliation $\mathcal{F}(H)$ of G by cosets of H is transversely complete. Recall that a right invariant foliation $\mathcal{F} \subset \mathcal{F}(\mathrm{s})$ of G has trivial holonomy whenever it is transversely complete. Hence Lemma 2.1 and Proposition 3.1 imply that any such foliation $\mathcal{F}$ is the foliation $\mathcal{F}(H)$ associated to a transversely complete subgroupoid H of G.

Examples 3.4. (1) Suppose that G is a connected Lie group, thus a Lie groupoid over a one-point space $M = \{\mathrm{pt}\}$. Then any connected subgroup $H \subset G$ is transversely complete.

(2) Let M be a manifold equipped with a foliation $\mathcal{F}$. Let G be the pair groupoid $M\times M$. Then $\mathcal{F}$ can be viewed as a subalgebroid $\mathfrak{h}$ of the Lie algebroid $\mathfrak{g} = T(M)$ of $M\times M$, and the corresponding foliation $\mathcal{F}(\mathfrak{h})$ is $\mathcal{F}\times 0 \subset T(M)\times T(M) = T(M\times M)$. If $\mathcal{F}$ has trivial holonomy, then $\mathrm{Hol}(M,\mathcal{F})$ is an immersed subgroupoid of $M \times M$, which is transversely complete whenever $\mathcal{F}$ is. Thus, our terminology extends the usual one for foliations.

Recall that a Lie groupoid G over M is *transitive* if the map $(\mathrm{t},\mathrm{s})\colon G \to M \times M$ is a surjective submersion [15, 18]. Any transitive Lie groupoid is automatically Hausdorff.

Proposition 3.5. *Any transitive Lie subgroupoid of any Lie groupoid is transversely complete.*

Proof. Let H be a transitive subgroupoid of a Lie groupoid G over M. First we show that in this case G is transitive as well. To see this, recall from [18, Proposition 5.14] that a Lie groupoid over M is transitive if and only if the restriction of the target map to a source-fiber is a surjective submersion onto M. Take any arrow $g\colon x \to y$ in G. Transitivity of H implies that there exists an arrow $h\colon x \to y$ in H, and that the derivative $(d\mathrm{t})_h\colon T_h(H(x, \text{-})) \to T_y(M)$ is surjective. Since the right translation

$$R_{h^{-1}g}\colon H(x, \text{-}) \longrightarrow G(x, \text{-})$$

preserves the target and maps h to g, it follows that $(d\mathrm{t})_g\colon T_h(G(x, \text{-})) \to T_y(M)$ is surjective as well. Thus we proved that G is transitive, and in particular Hausdorff.

We will now show that the foliation $\mathcal{F}(H)$ of G associated to H is transversely complete. Transitivity of G implies that any $x_0 \in M$ has a neighbourhood U and a section $\sigma\colon U \to G$ of the source map such that $\mathrm{t}(\sigma(x)) = x_0$ for any $x \in U$. The right translation by this section provides a local trivialization $\mathrm{s}^{-1}(x_0)\times U \to \mathrm{s}^{-1}(U)$ of the source map, which also respects the foliation $\mathcal{F}(H)$ because $\mathcal{F}(H)$ is right invariant. From this it follows that it is enough to show that the restriction of $\mathcal{F}(H)$ to a source-fiber is transversely complete. Thus, for any arrow $g\colon x_0 \to y$ of G and any $u \in T_g(G(x_0, \text{-}))$ we need to find a complete source-vertical projectable vector field on $G(x_0, \text{-})$ with value u at g. Because the isomorphism $R_g\colon \mathrm{s}^{-1}(y) \to \mathrm{s}^{-1}(x_0)$ given by right translation preserves the foliation and sends 1_y into g, we can assume without loss of generality that $g = 1_{x_0}$.

Since H is transitive, the restriction of the target map to $\mathrm{s}^{-1}(x_0)$ is a surjective submersion onto M, and hence we can write

$$u = v + w$$

for a vector v tangent to H and a vector w with $d\mathrm{t}(w) = 0$. We can extend v to a section of the Lie algebroid $\mathfrak{h}$ of H, and we can assume that this section has compact support. This section can be uniquely extended to a right invariant source-vertical vector field X on G, which is tangent to $\mathcal{F}(H)$ and has value v at 1_{x_0}. The vector field X is complete because its anchor is [13, p. 264]. Next, we can uniquely extend w to a left invariant vector field Y on $G(x_0, \text{-})$. The flow of this vector field is given by the family of right translations $R_{\exp(tw)}$, $t \in \mathbb{R}$, where $\exp(tw)$ is the one-parameter subgroup of the Lie group G_{x_0} corresponding to w. In particular, this flow is globally defined and preserves the right invariant foliation $\mathcal{F}(H)$, so Y is complete and projectable. Furthermore, the vector field Y commutes with the right invariant vector field X. This means that $X + Y$ is a complete projectable vector field on $G(x_0, \text{-})$ and has value u at 1_{x_0}. $\square$

Example 3.6. Unlike the case where G is a Lie group, an embedded Lie subgroupoid of a Lie groupoid is not necessarily closed, and its closure may not be a Lie subgroupoid. Indeed, Proposition 3.2 implies that the holonomy groupoid of a simple, but not strictly simple, foliation $\mathcal{F}$ of a manifold M is an embedded subgroupoid of the pair groupoid $M \times M$ which is not closed. If we take $M = \mathbb{R}^2 \setminus (\{0\} \times [0, \infty))$ and if $\mathcal{F}$ is the simple foliation of M given by the second projection, the holonomy groupoid of $(M, \mathcal{F})$ is embedded in $M \times M$, but its closure is not a Lie subgroupoid.

The following theorem shows that transversely complete Lie subgroupoids behave very much like subgroups of Lie groups.

Theorem 3.7. *Let G be a Hausdorff Lie groupoid, and let H be a transversely complete Lie subgroupoid of G.*

(i) The closure $\bar{H}$ of H in G is an embedded Lie subgroupoid of G.

(ii) If H itself is embedded in G, then it is closed, and $G \to G/H$ is a locally trivial fiber bundle.

Proof. (i) By the Molino structure theorem referred to above, the closures of the leaves of $\mathcal{F}(H)$ are the fibers of a locally trivial fiber bundle $\pi : G \to W$. Let $\bar{\mathrm{s}} \colon W \to M$ be the submersion induced by s, with the section $\bar{\mathrm{u}} = \pi \circ \mathrm{u}$. Consider the following diagram:

$$\begin{tikzcd}
 & P \arrow[r] \arrow[d, "j"] & M \arrow[d, "\bar{\mathrm{u}}"] \\
H \arrow[ur] \arrow[r, "\iota"] & G \arrow[r, "\pi"] \arrow[d, "\mathrm{s}"] & W \arrow[dl, "\bar{\mathrm{s}}"] \\
 & M &
\end{tikzcd}$$

Here P denotes the pull-back $\pi^{-1}(\bar{\mathrm{u}}(M))$, so $j \colon P \to G$ is again a closed embedding. Note that a point $g \colon x \to y$ of G belongs to P precisely when $g \in \bar{L}_{1_x}$, the closure of the leaf of $\mathcal{F}(H)$ through the unit 1_x. It follows that P is a closed subgroupoid of G. (If $g \in \bar{L}_{1_x}$ and $h \colon y \to z$ belongs to $\bar{L}_{1_y}$, then also $hg \in \bar{L}_g$ by right invariance of the foliation. Hence $hg \in \bar{L}_{1_x}$ because $\bar{L}_g = \bar{L}_{1_x}$.)

Also, the immersion $\iota \colon H \to G$ evidently factors through P. Finally, $H \to P$ is dense, because the restriction of $H \to P$ to the source-fiber over a point $x \in M$ is the inclusion $L_{1_x} \to \bar{L}_{1_x}$, hence dense. In particular we have $P = \bar{H}$.

(ii) Note that the immersion of each leaf of $\mathcal{F}(H)$ into the corresponding fiber of π is an embedding as well as dense. Thus the leaves of $\mathcal{F}(H)$ are equal to the fibers of π, and $H = P$ in the diagram above. It is clear that $G \to G/H$ is a locally trivial bundle in this case. $\square$

Example 3.8. Let G be a simply connected Lie group, and let $\mathfrak{g}$ be the associated Lie algebra of right invariant vector fields on G. Suppose that $\mathcal{F}$ is a Lie foliation on a manifold M, given by the kernel of a Maurer-Cartan form

$$\omega \in \Omega^1(M, \mathfrak{g})$$

with values in a $\mathfrak{g}$ [10, 21]. This means that ω is non-singular and has trivial formal curvature, $d\omega + \frac{1}{2}[\omega, \omega] = 0$. The foliation $\mathcal{F}$ is transversely parallelizable [6, 21]; in fact, a vector field Y on M is projectable with respect to $\mathcal{F}$ if and only if the function $\omega(Y)$ is basic, i.e. constant along the leaves of $\mathcal{F}$.

The form $-\omega$ can be extended uniquely to a flat connection form η on the trivial principal G-bundle $M \times G$ over M. (Here we take $-\omega$ instead of ω because we use the Lie algebra of right invariant vector fields on G instead the usual left invariant ones.) The kernel of η therefore defines a foliation $\mathcal{G}$ of $M \times G$ which is G-invariant and has trivial holonomy. Any leaf of $\mathcal{G}$ is a covering space of M with respect to the projection to M, and the lift of $\mathcal{F}$ to such a leaf is a strictly simple foliation given by the fibers of the projection to G. Any vector tangent to the foliation $\mathcal{G}$ can be extended to a vector field tangent to $\mathcal{G}$ of the form $(X, \omega(X))$, where X is a (projectable) vector field on M for which $\omega(X)$ is a constant function (and its value viewed as a right invariant vector field on G).

Consider the Lie groupoid

$$M \times G \times M$$

over M, i.e. the groupoid induced by G along the map $M \to \{\mathrm{pt}\}$. The arrows $x \to y$ in $M \times G \times M$ are the triples (y, g, x), where $g \in G$. The foliation $\mathcal{G} \times 0$ of $M \times G \times M$ is right invariant because $\mathcal{G}$ is right G-invariant. Thus $\mathcal{G} \times 0$ is the foliation $\mathcal{F}(\mathfrak{h})$ of a subalgebroid $\mathfrak{h}$ of the Lie algebroid of $M \times G \times M$,

$$\mathcal{G} \times 0 = \mathcal{F}(\mathfrak{h}) \; .$$

Furthermore, this foliation has trivial holonomy because $\mathcal{G}$ has trivial holonomy, therefore it defines a subgroupoid H of the Lie groupoid $M \times G \times M$ which integrates the Lie subalgebroid $\mathfrak{h}$.

Denote by $\Pi(M)$ the fundamental groupoid of M, and define a homomorphism of Lie groupoids

$$\phi \colon \Pi(M) \longrightarrow M \times G \times M$$

over M by path-lifting inside the leaves of $\mathcal{G}$: For any homotopy class $[\sigma]$ of a path σ in M from x to y, let $\phi([\sigma])$ be the unique lift of σ which starts at $(x, 1, x)$ and lies inside a leaf of $\mathcal{G} \times 0$. The image of ϕ is precisely the groupoid H. Using this path-lifting construction, we see that an arrow (y, g, x) of $M \times G \times M$ belongs to H if and only if the points (y, g) and $(x, 1)$ of $M \times G$ belong to the same leaf of $\mathcal{G}$. Notice that H is a transitive groupoid and therefore transversely complete by Proposition 3.5. Its isotropy group H_x at $x \in M$ is the holonomy group of the Maurer-Cartan form ω.

By Theorem 3.7 (i) we know that the closure $\bar{H}$ of H in $M \times G \times M$ is an embedded subgroupoid of $M \times G \times M$. It is possible to give the following explicit description of $\bar{H}$ in terms of the closures K_x of H_x inside G (such a description in fact applies to any transitive subgroupoid): An arrow $(y, g, x) \colon x \to y$ in $M \times G \times M$ belongs to $\bar{H}$ if and only if it can be factored as $g = hk$ for some $(y, h, x) \in H$ and $k \in K_x$. In other words, the groupoid H acts by conjugation on the bundle K of groups K_x over M, and $\bar{H}$ is constructed as the twisted product of H and K. This example relates to the Cartier construction of the groupoid closure [4] in the context of Galois theory for differential equations.

Remark 3.9. Note that, more generally, any connection on a principal G-bundle P over M gives us a transitive Lie subalgebroid $\mathfrak{h}$ of the gauge algebroid associated to P. Indeed, as explained in [12, Section II.7], the connection defines a G-invariant foliation on P, as the smallest involutive subbundle of $T(P)$ generated by the horizontal vector fields. The leaves of this foliation of P are exactly the holonomy bundles corresponding to the connection. This G-invariant foliation on P uniquely determines a right invariant foliation $\mathcal{F}(\mathfrak{h}) \subset \mathcal{F}(\mathrm{s})$ on the gauge groupoid Gauge(P) of the bundle. The subalgebroid $\mathfrak{h}$ is the one corresponding to the foliation $\mathcal{F}(\mathfrak{h})$ by Lemma 2.1. Since the foliation $\mathcal{F}(\mathfrak{h})$ has trivial holonomy, Corollary 2.5 implies that it is the algebroid of a (transitive) Lie subgroupoid of Gauge(P). This Lie groupoid H is isomorphic to the gauge groupoid of a holonomy bundle, and its isotropy group is the holonomy group of the connection. The Ambrose-Singer Theorem [12, Section II.8] gives an explicit description of the kernel of the anchor map of the subalgebroid $\mathfrak{h}$. By Theorem 3.7 and Proposition 3.5, the closure of the subgroupoid H is again a Lie subgroupoid, which can be described in terms of the holonomy group of the connection. For related discussions, we refer the reader to the treatment of holonomy in [15].

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