ON THE DEVELOPABILITY OF LIE SUBALGEBROIDS

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Abstract. In this paper, the Almeida-Molino obstruction to developability of transversely complete foliations is extended to Lie groupoids.

Introduction

In this paper we continue our study – begun in [13] – of some of the most basic properties of Lie subgroups and subalgebras, in the wider context of Lie groupoids and Lie algebroids. These objects occur naturally in many contexts, such as Poisson geometry, group actions, quantization and foliation theory [2, 3, 4, 6, 10, 16]. One of the main features which makes the basic theory of Lie algebroids and Lie groupoids so much more involved than that of Lie algebras and groups is that Lie algebroids may not be integrable. In other words, any Lie groupoid has a Lie algebroid as its infinitesimal part, but not every Lie algebroid arises in this way.

An obstruction to integrability was first observed by Almeida and Molino, in the context of foliations on compact manifolds. Recall that a foliation on a manifold M is called developable if its lift to the universal covering space $\tilde{M}$ of M is given by the fibers of a submersion into another manifold. Given any transversely complete foliation $\mathcal{F}$ of a compact manifold M, Almeida and Molino [1] discovered an associated Lie algebroid $\mathfrak{b}(M,\mathcal{F})$, and proved that

(1) $\mathcal{F}$ is developable if and only if $\mathfrak{b}(M,\mathcal{F})$ is integrable.

One way to construct a developable foliation is as the kernel of a Maurer-Cartan form on M with coefficients in a Lie algebra. One can view the Almeida-Molino result as stating that *any* transversely complete foliation on a compact manifold is the kernel of a Maurer-Cartan form with coefficients in a Lie *algebroid*, and that this Lie algebroid is integrable if and only if the foliation is developable.

Our goal is to extend this result to Lie groupoids, in the following way: Consider a Lie groupoid G over a manifold M, and a subalgebroid $\mathfrak{h}$ of the Lie algebroid $\mathfrak{g} = \mathfrak{L}(G)$ of G. We say that $\mathfrak{h}$ is *developable* if it can be integrated to a closed subgroupoid of the universal covering groupoid $\tilde{G}$ of G. Under conditions of source-compactness and transverse completeness (as in the Almeida-Molino case) we will construct another Lie algebroid $\mathfrak{b}(G,\mathfrak{h})$, and prove an equivalence

(2) $\mathfrak{h}$ is developable if and only if $\mathfrak{b}(G,\mathfrak{h})$ is integrable.

(see Theorem 1.8 for a precise statement). This is formally similar to the Almeida-Molino result (1), and in fact includes the latter. To explain the relation, recall that any manifold M defines a pair groupoid $G = M \times M$, whose universal covering groupoid $\tilde{G}$ is the fundamental groupoid $\Pi(M)$ given by homotopy classes of paths in M. Any transversely complete foliation $\mathcal{F}$ on M can be viewed as a subalgebroid of the Lie algebroid of G, and we will show (Corollary 3.4) that $\mathcal{F}$ is a developable

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foliation if and only if it can be integrated by a closed subgroupoid of $\Pi(M)$. This shows that (1) is a special case of (2).

In Section 1, we discuss Lie algebroid valued Maurer-Cartan forms, and state our main result (Theorem 1.8). For its proof, we observe that for a Lie groupoid G over a manifold M, any subalgebroid $\mathfrak{h}$ of the Lie algebroid $\mathfrak{L}(G)$ of G defines a foliation $\mathcal{F}(\mathfrak{h})$ of G, whose leaves are contained in the fibers of the source map $\mathrm{s}\colon G \to M$. In Section 2 we will show that if the foliation $\mathcal{F}(\mathfrak{h})$ is locally transversely parallelizable and the fibers of s are compact, then $\mathcal{F}(\mathfrak{h})$ makes $\mathrm{s}\colon G \to M$ into a locally trivial bundle of transversely complete foliations. This fact is independent of the groupoid structure on G, and we will prove it in the general context of a submersion $s : N \to M$ with suitably foliated fibers, see Theorem 2.3. Next, we will give various characterizations of the fiberwise developability of a locally trivial bundle $N \to M$ of foliated manifolds, and prove that this property is equivalent to the integrability of a specific algebroid (Theorem 3.9). Finally, we will apply our results on bundles of foliated manifold to groupoids, and prove Theorem 1.8.

1. Developable subgroupoids and Maurer-Cartan forms

We begin by introducing the notion of *developability* for subalgebroids, which is motivated by the well-known notion of developability for foliations.

For a Lie groupoid G over a manifold M, we shall denote by $\mathrm{s} : G \to M$ the *source* map of G, and by $\mathrm{t} : G \to M$ the *target* map of G. Recall that G is said to be *source-connected*, if the fibers of the source map of G are connected, and *source-simply connected* if the fibers of the source map of G are simply connected. In this paper, we shall assume that all Lie groupoid are source-connected.

Let G be a (source-connected) Lie groupoid over a manifold M. Denote by $\mathfrak{g} = \mathfrak{L}(G)$ the Lie algebroid associated to G. We denote by $\tilde{G}$ the source-simply connected covering groupoid of G which has the same Lie algebroid as G.

We shall consider subalgebroids of $\mathfrak{g}$ over the same base M. Recall from [13] that such a subalgebroid $\mathfrak{h}$ of $\mathfrak{g}$ corresponds to a right-invariant foliation $\mathcal{F}(\mathfrak{h})$ of G which refines the foliation $\mathcal{F}(\mathrm{s})$ of G given by the fibers of the source map, $\mathcal{F}(\mathfrak{h}) \subset \mathcal{F}(\mathrm{s})$. Any (injectively immersed source-connected) Lie subgroupoid H of G over the same base M gives rise to a Lie subalgebroid $\mathfrak{h}$ of $\mathfrak{g}$, and in turn the leaves of $\mathcal{F}(\mathfrak{h})$ are precisely the right cosets of H in G. In this case we write $\mathcal{F}(H) = \mathcal{F}(\mathfrak{h})$, and we say that H integrates $\mathfrak{h}$, or more precisely, that the inclusion $H \to G$ integrates the inclusion $\mathfrak{h} \to \mathfrak{g}$. We denote by G/H the space of leaves of the associated foliation $\mathcal{F}(H)$.

We say that a subalgebroid $\mathfrak{h}$ of $\mathfrak{g}$ is *developable* if $\mathfrak{h}$ can be integrated to a closed subgroupoid of the source-simply connected cover $\tilde{G}$ of G. A subgroupoid H of G is *developable* if the associated Lie subalgebroid of $\mathfrak{g}$ is.

Recall that a foliation $\mathcal{F}$ of a manifold N is *simple* if it is given by the components of the fibers of a submersion into a Hausdorff manifold. It is called *strictly simple* if there exists a smooth structure of a Hausdorff manifold on the space of leaves $N/\mathcal{F}$ such that the quotient map $N \to N/\mathcal{F}$ is a submersion. The results of [13] provide the following equivalent formulation of developability:

Proposition 1.1. *Let G be an source-connected Lie groupoid with Lie algebroid $\mathfrak{g}$, and let $\mathfrak{h}$ be a subalgebroid of $\mathfrak{g}$. The following two statements are equivalent:*

(i) *The Lie algebroid $\mathfrak{h}$ is developable.*
(ii) *The pull-back of the foliation $\mathcal{F}(\mathfrak{h})$ to the source-simply connected cover of G is strictly simple.*

Proof. This equivalence follows directly from [13, Proposition 3.1], [13, Proposition 3.2 (ii)] and the fact that pull-back of the foliation $\mathcal{F}(\mathfrak{h})$ to the source-simply connected cover $\tilde{G}$ of G is the foliation of $\tilde{G}$ given by the same subalgebroid $\mathfrak{h} \subset \mathfrak{g} = \mathfrak{L}(\tilde{G})$. □

Example 1.2. Recall that a foliation $\mathcal{F}$ of a manifold N is called *developable* if its pull-back to the universal covering space of the manifold is strictly simple. Such a foliation $\mathcal{F}$ can be viewed as a subalgebroid of the Lie algebroid $T(M)$. The Lie algebroid $T(M)$ is integrated by the pair groupoid $M \times M$, but also by the source-simply connected fundamental groupoid $\Pi(M)$ of M. By Corollary 3.4 below, the foliation $\mathcal{F}$ is developable if and only if $\mathcal{F}$, viewed as a subalgebroid of $T(M)$, is integrable by a closed subgroupoid of $\Pi(M)$. In this way, our definition of developability of subalgebroids extends the usual one for foliations.

An important class of examples are the subalgebroids which arise as kernels of Maurer-Cartan forms, which we now discuss.

Let $\mathfrak{g}$ be a Lie algebroid over M. Let $\mathfrak{k}$ be a finite dimensional Lie algebra. Then $\mathfrak{k}$ pulls back to an algebroid $\mathfrak{k}_M = \mathfrak{k} \times M$ over M, with zero anchor. Suppose that we are given a (left) action ∇ of $\mathfrak{g}$ on $\mathfrak{k}_M$ along the identity map of M. Recall from [9, 11] that such an action assigns to each section $X \in \Gamma(\mathfrak{g})$ a derivation ∇_X on the Lie algebra $C^\infty(M, \mathfrak{k})$ of smooth $\mathfrak{k}$-valued functions, $C^\infty(M)$-linear and flat in X,

$$\nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X \ ,$$

which moreover satisfies the Leibniz law

$$\nabla_X(f\alpha) = f\nabla_X(\alpha) + \mathrm{an}(X)(f)\alpha$$

for any $f \in C^\infty(M)$ and any $\alpha \in C^\infty(M, \mathfrak{k})$.

A *Maurer-Cartan form* on $\mathfrak{g}$ with coefficients in $\mathfrak{k}$ is a map $\omega \colon \mathfrak{g} \to \mathfrak{k}_M$ of vector bundles over M satisfying the Maurer-Cartan equation

$$d\omega + \frac{1}{2}[\omega, \omega] = 0 \ .$$

Here d is the differential of the Lie algebroid cohomology of $\mathfrak{g}$ with coefficients in $\mathfrak{k}$,

$$2d\omega(X, Y) = \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega([X, Y]) \ ,$$

while the bracket on maps $\omega, \lambda \colon \mathfrak{g} \to \mathfrak{k}_M$ of vector bundles over M is given by

$$2[\omega, \lambda](X, Y) = [\omega(X), \lambda(Y)] - [\omega(Y), \lambda(X)] \ .$$

Example 1.3. Let $\mathfrak{k}$ be a finite dimensional Lie algebra, and let $\mathfrak{g}$ be a Lie algebroid over M. The *trivial action* ∇^{triv} of $\mathfrak{g}$ on $\mathfrak{k} \times M = \mathfrak{k}_M$, along the identity map of M, is given by the derivative along the anchor,

$$\nabla^{\mathrm{triv}}_X(\alpha) = \mathrm{an}(X)(\alpha) \ .$$

For $\mathfrak{g} = T(M)$, a Maurer-Cartan form on $T(M)$ with values in $\mathfrak{k}$ with respect to the trivial action is a usual Maurer-Cartan form on the manifold M.

Lemma 1.4. *Let $\mathfrak{g}$ be a Lie algebroid over M, let $\mathfrak{k}$ be a finite dimensional Lie algebra, and suppose that $\mathfrak{k}_M$ is equipped with an action ∇ of $\mathfrak{g}$ along the identity map. For a map $\omega \colon \mathfrak{g} \to \mathfrak{k}_M$ of vector bundles over M, the following conditions are equivalent:*

(i) *ω is a Maurer-Cartan form on $\mathfrak{g}$ with respect to the action ∇.*
(ii) *$\omega([X, Y]) = [\omega(X), \omega(Y)] + \nabla_X(\omega(Y)) - \nabla_Y(\omega(X))$ for any $X, Y \in \Gamma(\mathfrak{g})$.*
(iii) *$(\mathrm{id}, \omega) \colon \mathfrak{g} \to \mathfrak{g} \ltimes \mathfrak{k}_M$ is a morphism of Lie algebroids over M.*

In (iii), the semi-direct product $\mathfrak{g} \ltimes \mathfrak{k}_M$ is the bundle $\mathfrak{g} \oplus \mathfrak{k}_M$, with bracket defined by

$$[(X, \alpha), (Y, \beta)] = ([X, Y], [\alpha, \beta] + \nabla_X(\beta) - \nabla_Y(\alpha))$$

for any $X, Y \in \Gamma(\mathfrak{g})$ and $\alpha, \beta \in C^\infty(M, \mathfrak{k})$. The equivalence of conditions (i)-(iii) is a trivial calculation.

Proposition 1.5. *Let $\mathfrak{k}$ be a finite-dimensional Lie algebra, let G be a source-connected Lie groupoid with Lie algebroid $\mathfrak{g}$ acting on the Lie algebroid $\mathfrak{k}_M = \mathfrak{k} \times M$, and let $\omega \colon \mathfrak{g} \to \mathfrak{k}_M$ be a non-degenerated Maurer-Cartan form on $\mathfrak{g}$. Then* $\mathrm{Ker}(\omega)$ *is a developable subalgebroid of* $\mathfrak{g}$.

Proof. Let $\tilde{G}$ be the source-simply connected Lie groupoid covering G, and let K be the simply connected Lie group with Lie algebra $\mathfrak{k}$. Write $K_M = K \times M$ for the trivial bundle of Lie groups over M with fiber K, which integrates the Lie algebroid $\mathfrak{k}_M$. By [11], the action of $\mathfrak{g}$ on $\mathfrak{k}_M$ integrates to an action of $\tilde{G}$ on K (more precisely, an action on K_M), and we can form the semi-direct product $\tilde{G} \ltimes K_M$ over M. (Its arrows are pairs $(\tilde{g}, k)$, where $\tilde{g} \colon x \to y$ is an arrow in $\tilde{G}$ and $k \in K$; composition is given by $(\tilde{h}, l)(\tilde{g}, k) = (\tilde{h}\tilde{g}, l(\tilde{h}k))$.) By Lemma 1.4 and [11, Proposition 3.5], the morphism of Lie algebroids (id, ω) integrates to a morphism of Lie groupoids $(\mathrm{id}, \Omega) \colon \tilde{G} \to \tilde{G} \ltimes K_M$. Then $\Omega \colon \tilde{G} \to K_M$ is a 'twisted' homomorphism,

$$\Omega(\tilde{h}\tilde{g}) = \Omega(\tilde{h})(\tilde{h}\Omega(\tilde{g}))\ ,$$

whose kernel is a closed subgroupoid of $\tilde{G}$. This closed subgroupoid integrates the Lie algebroid $\mathrm{Ker}(\omega)$, hence the latter is developable. □

Let $\mathcal{F}$ be a foliation of a manifold N. Recall that a vector field Y on N is *projectable* with respect to $\mathcal{F}$ if its (local) flow preserves the foliation, or equivalently, if the Lie derivative of Y in the direction of a vector field tangent to $\mathcal{F}$ is again tangent to $\mathcal{F}$. A foliation $\mathcal{F}$ of N is *transversely complete* if any tangent vector on N can be extended to a complete projectable vector field on N (see also Remarks 2.2 below).

A subgroupoid H of a Lie groupoid G is *transversely complete* if the associated foliation $\mathcal{F}(H)$ of G is transversely complete. For example, any transitive subgroupoid H of G is automatically transversely complete [13, Proposition 3.5].

Our aim is to prove that for any transversely complete subgroupoid H of a source-compact groupoid G (i.e. G is Hausdorff and its source map is proper), its Lie algebra $\mathfrak{h}$ is the kernel of a Maurer-Cartan form. This result can be interpreted as a generalization of Molino's theorem for transversely complete foliations. As in Molino's case, it requires the Maurer-Cartan form to take values in a Lie algebroid (see Example 1.7 (2)).

Let G be a Lie groupoid over M with Lie algebroid $\mathfrak{g}$. Let $\mathfrak{k}$ be a Lie algebroid over W equipped with an action of $\mathfrak{g}$ along a submersion $f : W \to M$. Recall from [11] that this means in particular that $\mathfrak{k} \to W$ is a bundle of Lie algebroids over M, and that $\mathfrak{g}$ acts on W and $\mathfrak{k}$, via suitable maps $R \colon \Gamma(\mathfrak{g}) \to \mathfrak{X}(W)$ and ∇ assigning to each $X \in \Gamma(\mathfrak{g})$ a derivation $(\nabla_X, R(X))$ on $\mathfrak{k}$. A *Maurer-Cartan form* on $\mathfrak{g}$ with values in $\mathfrak{k}$ is a section of the projection map of Lie algebroids $\mathfrak{g} \ltimes \mathfrak{k} \to \mathfrak{g}$ over $f \colon W \to M$. Such a section is given by a map $\omega \colon \mathfrak{g} \to \mathfrak{k}$ of vector bundles over a section $\alpha \colon M \to W$ of f, with the property that

$$((\alpha, \mathrm{id}), \omega) \colon \mathfrak{g} \longrightarrow f^*(\mathfrak{g}) \oplus \mathfrak{k}$$

defines a morphism of Lie algebroids $\mathfrak{g} \to \mathfrak{g} \ltimes \mathfrak{k}$ over $\alpha \colon M \to W$.

Remark 1.6. It is of course possible to spell out the last condition in detail: First of all, compatibility with the anchor maps of $\mathfrak{g}$ and $\mathfrak{g} \ltimes \mathfrak{k}$ means that ω should satisfy the identity

$$\mathrm{an}(\omega(v)) = (d\alpha)_x(\mathrm{an}(v)) - R_{\alpha(x)}(v) \tag{3}$$

for any $x \in M$ and any $v \in \mathfrak{g}_x$; next, compatibility of the brackets can be expressed by the equation

$$\omega \circ [X, Y] = ([\omega(X), \omega(Y)] + \nabla_X(\omega(Y)) - \nabla_Y(\omega(X))) \circ \alpha\ . \tag{4}$$

Here X and Y are sections of $\mathfrak{g}$, while $\omega(X)$ and $\omega(Y)$ denote arbitrary sections of $\mathfrak{k}$ which extend $\omega \circ X : M \to \mathfrak{k}$ respectively $\omega \circ Y : M \to \mathfrak{k}$ along the embedding $\alpha : M \to W$. It follows from (3) that the right hand side of (4) is independent of the choice of these extensions.

Examples 1.7. (1) A Maurer-Cartan form with values in a Lie algebra $\mathfrak{k}$ (as discussed above) can be seen as a special case of a Maurer-Cartan form with values in the Lie algebroid $\mathfrak{k}_M = \mathfrak{k} \times M$, where the action is taken along the identity map of the base manifold M.

(2) (Molino) Let M be a compact manifold equipped with a transversely complete foliation $\mathcal{F}$. The closures of the leaves of such a foliation are the fibers of a locally trivial fiber bundle $M \to W$ of foliations [14]. There is the associated Lie algebroid $\mathfrak{b}(M, \mathcal{F})$ over W such that $\mathcal{F}$ is the kernel of a map of Lie algebroids over the projection $M \to W$ [12, 15].

$$
\begin{tikzcd}
T(M) \arrow[r] \arrow[d] & \mathfrak{b}(M,\mathcal{F}) \arrow[d] \\
M \arrow[r] & W
\end{tikzcd}
\tag{5}
$$

To see this as a Maurer-Cartan form, consider the Lie algebroid $\mathfrak{b}(M, \mathcal{F}) \times M$ over $W \times M$. This is a bundle of Lie algebroids over M, and has the canonical action of the pair groupoid $M \times M$ (just by acting on the second component only). This action differentiates to an action of $T(M)$ on $\mathfrak{b}(M, \mathcal{F}) \times M$. The morphism of algebroids in (5) induces a section of the projection $T(M) \ltimes (\mathfrak{b}(M, \mathcal{F}) \times M) \to T(M)$.

Recall from [13, Theorem 3.7] that if H is a transversely complete subgroupoid of a Hausdorff Lie groupoid G, then its closure $\bar{H}$ in G is a closed Lie subgroupoid of G, and the space of right cosets $G/\bar{H}$ has the natural structure of a Hausdorff manifold such that the projection $G \to G/\bar{H}$ is a locally trivial fiber bundle.

We can now state our main theorem, which we shall prove in Section 4.

Theorem 1.8. *Let G be an source-compact Lie groupoid with Lie algebroid $\mathfrak{g}$, and let H be a transversely complete subgroupoid of G with Lie algebroid $\mathfrak{h} \subset \mathfrak{g}$.*

(i) There exists a regular Lie algebroid $\mathfrak{b}(G, \mathfrak{h})$ over $G/\bar{H}$ such that the projection $G \to G/\bar{H}$ lifts to a natural surjective morphism of Lie algebroids $\mathcal{F}(\mathrm{s}) \to \mathfrak{b}(G, \mathfrak{h})$ with kernel $\mathcal{F}(\mathfrak{h})$.

(ii) The natural action of $\mathfrak{g}$ on $G/\bar{H}$ lifts to an action of $\mathfrak{g}$ on $\mathfrak{b}(G, \mathfrak{h})$.

(iii) The Lie algebroid $\mathfrak{h}$ is the kernel of a Maurer-Cartan form on $\mathfrak{g}$ with values in $\mathfrak{b}(G, \mathfrak{h})$.

(iv) The Lie algebroid $\mathfrak{h}$ is developable if and only if $\mathfrak{b}(G, \mathfrak{h})$ is integrable.

The assumption that G is source-compact is used to ensure that the source map of G is a locally trivial bundle of transversely complete foliations (Theorem 2.3), and can in fact be replaced by this weaker condition.

Part (ii) of Theorem 1.8 states in particular that $\mathfrak{b}(G, \mathfrak{h})$ is in fact a bundle of Lie algebroids over the base space M of G; more precisely, its anchor is annihilated by the map $T(G/\bar{H}) \to T(M)$ induced by the differential of the source map. Thus, $\mathfrak{b}(G, \mathfrak{h})$ is a bundle of Lie algebras in case the map $G/\bar{H} \to M$, induced by the source, is a diffeomorphism. The result of Douady and Lazard [7] thus gives the following corollary:

Corollary 1.9. *Let G be an source-compact Lie groupoid with Lie algebroid $\mathfrak{g}$, and let H be a transversely complete subgroupoid of G with Lie algebroid $\mathfrak{h} \subset \mathfrak{g}$. Then H is developable whenever the cosets of H are dense in the source-fibers of G.*

Example 1.10. Let $\mathfrak{l}$ be the Lie algebra of a compact connected Lie group L, and let ω be a non-singular Maurer-Cartan form with values in $\mathfrak{l}$ on a connected compact manifold M. Then ω defines a transitive (hence transversely complete) subgroupoid H of the groupoid $M \times L \times M$ whose source-fibers are the holonomy covers of ω, see [13, Example 3.8]. If the holonomy group of ω is dense in L, then the cosets of H are dense in $M \times L$, hence H is developable by Corollary 1.9.

2. Bundles of transversely complete foliations

First, we recall some standard definitions. Let $\mathcal{F}$ be a foliation of a manifold N. Write $\mathfrak{X}(\mathcal{F})$ for the Lie algebra of vector fields on N which are tangent to $\mathcal{F}$, and let $L(N,\mathcal{F})$ be the normalizer of $\mathfrak{X}(\mathcal{F})$ in the Lie algebra $\mathfrak{X}(N)$ of vector fields on N. Vector fields in $L(N,\mathcal{F})$ are called *projectable* vector fields on $(N,\mathcal{F})$, and the quotient

$$l(N,\mathcal{F}) = L(N,\mathcal{F})/\mathfrak{X}(\mathcal{F})$$

is the Lie algebra of transverse vector fields on $(N,\mathcal{F})$. The transverse vector fields on $(N,\mathcal{F})$ can be identified with the holonomy invariant sections of the normal bundle $\nu(\mathcal{F}) = T(N)/T(\mathcal{F})$. Both $L(N,\mathcal{F})$ and $l(N,\mathcal{F})$ are modules over the algebra $\Omega^0_{\mathrm{bas}}(N,\mathcal{F})$ of basic functions on $(N,\mathcal{F})$, and the quotient projection

$$L(N,\mathcal{F}) \longrightarrow l(N,\mathcal{F}) , \qquad Y \mapsto \bar{Y} ,$$

is $\Omega^0_{\mathrm{bas}}(N,\mathcal{F})$-linear.

A foliation $\mathcal{F}$ of N is *transversely parallelizable* [5, 15] if there exists a global frame of $\nu(\mathcal{F})$ consisting of transverse vector fields on $(N,\mathcal{F})$. Such a frame of $\nu(\mathcal{F})$ is referred to as *transverse parallelism* on $(N,\mathcal{F})$.

A foliation $\mathcal{F}$ of N is *locally transversely parallelizable* if for any $y \in N$ and any tangent vector $\xi \in T_y(N)$, there exists a projectable vector field Y on $(N,\mathcal{F})$ such that $Y_y = \xi$. A foliation $\mathcal{F}$ of N is *transversely complete* [14, 15] if for any $y \in N$ and any tangent vector $\xi \in T_y(N)$ there exists a complete projectable vector field Y on $(N,\mathcal{F})$ such that $Y_y = \xi$.

Example 2.1. Consider a Maurer-Cartan form ω as in Proposition 1.5. Let $\mathfrak{k}_G$ be the trivial bundle of Lie algebras over G, and let $\mathcal{F}(\mathrm{s})$ be the foliation on G by the fibers of the source map. Then ω defines a flat $\mathcal{F}(\mathrm{s})$-partial connection on $\mathfrak{k}_G$, and hence a 'monodromy' map

$$\Pi^{\mathrm{s}}(G) \longrightarrow \mathrm{Aut}(\mathfrak{k}) .$$

If this map is trivial, i.e. if ω has trivial monodromy, then the invariant foliation corresponding to $\mathrm{Ker}(\omega)$ is transversely parallelizable. In particular, if G is source-compact and ω has finite monodromy, then there exists an source-compact covering groupoid G' of G with the same algebroid $\mathfrak{g}$, on which $\mathrm{Ker}(\omega)$ defines a transversely parallelizable foliation.

Remarks 2.2. (1) Note that a foliation $\mathcal{F}$ of N is locally transversely parallelizable if and only if the evaluation map $l(N,\mathcal{F}) \to \nu_y(N,\mathcal{F})$ is surjective for any $y \in N$. Equivalently, a foliation $\mathcal{F}$ of N is locally transversely parallelizable if for any $y \in N$ there exist projectable vector fields $Y_1,\ldots,Y_q$ such that $((\bar{Y}_1)_y,\ldots,(\bar{Y}_q)_y)$ is a basis of $\nu_y(N,\mathcal{F})$. In particular, any transversely parallelizable foliation is locally transversely parallelizable.

Any locally transversely parallelizable foliation of a compact manifold is transversely complete. Examples of transversely complete foliations also include foliations given by the fibers of locally trivial fiber bundles, and Lie foliations on compact manifolds [8].

(2) Let $\mathcal{F}$ be a foliation on N, and let $\bar{Y}_1,\ldots,\bar{Y}_q$ be transverse vector fields on $(N,\mathcal{F})$. We say that $y \in N$ is a *regular point* of $(\bar{Y}_1,\ldots,\bar{Y}_q)$ if the normal vectors

$(\bar{Y}_1)_y, \ldots, (\bar{Y}_k)_y \in \nu_y(N, \mathcal{F})$ form a basis of $\nu_y(\mathcal{F})$. The *regular set* of $(\bar{Y}_1, \ldots, \bar{Y}_q)$ is the set of all regular points of $(\bar{Y}_1, \ldots, \bar{Y}_q)$, and will be denoted by

$$\mathrm{reg}(\bar{Y}_1, \ldots, \bar{Y}_q) \subset N \ .$$

This set is open in N, and also $\mathcal{F}$-saturated because of the holonomy invariance of the transverse vector fields. We say that $(\bar{Y}_1, \ldots, \bar{Y}_q)$ is a *local transverse parallelism* on $(N, \mathcal{F})$ with regular set $\mathrm{reg}(\bar{Y}_1, \ldots, \bar{Y}_q)$.

Note that $\mathcal{F}$ is locally transversely parallelizable if and only if the regular sets of local transverse parallelisms on $(N, \mathcal{F})$ cover N. If $\mathcal{F}$ is a locally transversely parallelizable foliation of a manifold N, then any leaf L of $\mathcal{F}$ has an open $\mathcal{F}$-saturated neighbourhood $U \subset N$ on which the foliation $\mathcal{F}|_U$ is transversely parallelizable.

(3) Any locally transversely parallelizable foliation has trivial holonomy. Furthermore, any transversely complete foliation $\mathcal{F}$ of a connected manifold M is homogeneous, i.e. the group $\mathrm{Aut}(M, \mathcal{F})$ of its foliation automorphisms acts transitively on M. This is a direct consequence of the fact that a vector field is projectable if and only if its (local) flow preserves the foliation.

Any strictly simple foliation $(M, \mathcal{F})$ is simple; if the foliation is homogeneous, then these two notions coincide [12, Theorem 4.3 (vi)]. Note that simple foliations are preserved under the pull-back along a covering projection, while strictly simple are not. Since the homogeneity is also preserved under the pull-back along a covering projection, a homogeneous foliation is developable if and only if its pull-back to a covering space of the manifold is simple.

(4) Recall that a homogeneous foliation $\mathcal{F}$ of N admits an associated *basic* foliation $\mathcal{F}_{\mathrm{bas}}$, given by

$$\mathfrak{X}(\mathcal{F}_{\mathrm{bas}}) = \{X \in \mathfrak{X}(N) \,|\, X(\Omega^0_{\mathrm{bas}}(N, \mathcal{F})) = 0\} \supset \mathfrak{X}(\mathcal{F}) \ .$$

The foliation $\mathcal{F}_{\mathrm{bas}}$ is again homogeneous, and satisfies $\Omega^0_{\mathrm{bas}}(N, \mathcal{F}_{\mathrm{bas}}) = \Omega^0_{\mathrm{bas}}(N, \mathcal{F})$ and $L(N, \mathcal{F}) \subset L(N, \mathcal{F}_{\mathrm{bas}})$. Furthermore, the space of basic leaves $N/\mathcal{F}_{\mathrm{bas}}$ has a natural structure of a Hausdorff manifold such that the basic projection $\pi_{\mathrm{bas}} \colon N \to N/\mathcal{F}$ is a submersion, and $\Omega^0_{\mathrm{bas}}(N, \mathcal{F}) = C^\infty(N/\mathcal{F}_{\mathrm{bas}})$.

(5) Suppose that $\mathcal{F}$ is a transversely complete foliation of a manifold N. By Molino's structure theorem [14], the closures of the leaves of a transversely complete foliation $\mathcal{F}$ of M are the fibers of the associated basic fibration $N \to N/\mathcal{F}_{\mathrm{bas}}$, which is in fact a locally trivial fiber bundle of (Lie) foliations. In particular, this implies that if two transversal vector fields on $(N, \mathcal{F})$ agree at a point of N, they also agree along the basic leaf through that point. Furthermore, the regular set of any local transverse parallelism on $(N, \mathcal{F})$ is $\mathcal{F}_{\mathrm{bas}}$-saturated.

Let $\mathcal{F}$ be a locally transversely parallelizable foliation of a connected manifold N, and let $s : N \to M$ be a surjective submersion with connected compact $\mathcal{F}$-saturated fibers. Since the image of the homomorphism $s^* : C^\infty(M) \to C^\infty(N)$ is a subalgebra of $\Omega^0_{\mathrm{bas}}(N, \mathcal{F})$ and the fibers of s are compact, it follows that $\mathcal{F}$ is transversely complete and therefore homogeneous. It is clear that the compact fibers of s are also $\mathcal{F}_{\mathrm{bas}}$-saturated. Furthermore, the submersion s factors through the associated basic fibration $\pi_{\mathrm{bas}} : N \to N/\mathcal{F}_{\mathrm{bas}}$ as a surjective submersion $\bar{s} : N/\mathcal{F}_{\mathrm{bas}} \to M$ with compact connected fibers.

The aim of this section is to prove the following theorem, which says that any such surjective submersion $s \colon (N, \mathcal{F}) \to M$ with connected compact $\mathcal{F}$-saturated fibers is actually a locally trivial bundle of foliated manifolds.

Theorem 2.3. *Let $\mathcal{F}$ be a locally transversely parallelizable foliation of codimension q of a connected manifold N, and let $s \colon N \to M$ be a surjective submersion with connected compact $\mathcal{F}$-saturated fibers. Put $m = \dim(M)$ and $\bar{q} = \mathrm{codim}(\mathcal{F}_{\mathrm{bas}})$.*

Then for any $y \in N$ there exist vector fields $Y_1, \ldots, Y_q \in L(N, \mathcal{F}) \cap L(N, \mathcal{F}(s))$ such that

(i) *$(\bar{Y}_1, \ldots, \bar{Y}_q)$ is a local transverse parallelism on $(N, \mathcal{F})$, regular on a neighbourhood of y,*
(ii) *$ds(Y_i) = 0$ for $i = m+1, \ldots, q$, and*
(iii) *$d\pi_{\mathrm{bas}}(Y_i) = 0$ for $i = \bar{q}+1, \ldots, q$.*

In particular, the restriction of $\mathcal{F}$ to any fiber N_x of s over $x \in M$ is a transversely complete foliation, and the submersion $s : (N, \mathcal{F}) \to M$ is a locally trivial fiber bundle of foliated compact manifolds.

As we shall show, this theorem is a consequence of Proposition 2.5 below, which we formulate and prove first.

Suppose that $\phi : N \to M$ is a surjective submersion, and write $\mathcal{F}(\phi)$ for the foliation of N given by the connected components of the fibers of ϕ. Let C be a subspace of $L(N, \mathcal{F}(\phi))$ and D a subspace of $\mathfrak{X}(M)$ such that $d\phi(C) \subset D$. Then we say that the linear map $d\phi : D \to C$ has the *lifting property* if for any vector field $X \in C$ and any $\xi \in T_y(N)$ such that $d\phi(\xi) = X_{\phi(y)}$ there exists a vector field $Y \in D$ such that $Y_y = \xi$ and $d\phi(Y) = X$. Thus the lifting property of $d\phi\colon D \to C$ in particular implies that $d\phi(D) = C$.

Example 2.4. Let $\phi\colon N \to M$ be a surjective submersion, and denote by $\mathcal{F}(\phi) = \mathrm{Ker}(d\phi)$ the foliation of N with the connected components of the fibers of ϕ as leaves. Then the Lie algebra homomorphism $d\phi\colon L(N, \mathcal{F}(\phi)) \to \mathfrak{X}(M)$ has the lifting property. In particular, the foliation $\mathcal{F}(\phi)$ is locally transversely parallelizable.

To see this, take any vector field $X \in \mathfrak{X}(M)$ and any $\xi \in T_y(N)$ with $d\phi(\xi) = X_{\phi(y)}$. Note that we can choose an open cover (U^α) of N and vector fields $Y^\alpha \in L(U^\alpha, \mathcal{F}(\phi)|_{U^\alpha})$ such that $d\phi(Y^\alpha) = X|_{U^\alpha}$. Furthermore, we can assume that $y \in U^{\alpha_0}$, $Y^{\alpha_0}_y = \xi$ and $y \notin U^\alpha$ for $\alpha \neq \alpha_0$. Let (η^α) be a partition of unity subordinated to (U^α), and define

$$Y = \sum_\alpha \eta^\alpha Y^\alpha \ .$$

Then $Y_y = Y^{\alpha_0}_y = \xi$ and $d\phi(Y_z) = X_{\phi(z)}$ for any $z \in N$. Thus $Y \in L(N, \mathcal{F}(\phi))$ and $d\phi(Y) = X$.

Proposition 2.5. *Let $\mathcal{F}$ be a locally transversely parallelizable foliation of a connected manifold N, let $s : N \to M$ be a surjective submersion with connected compact $\mathcal{F}$-saturated fibers, and let $\bar{s}\colon N/\mathcal{F}_{\mathrm{bas}} \to M$ be the submersion induced by s. Then the homomorphisms of Lie algebras*

$$d\pi_{\mathrm{bas}}\colon L(N, \mathcal{F}) \cap L(N, \mathcal{F}(s)) \longrightarrow L(N/\mathcal{F}_{\mathrm{bas}}, \mathcal{F}(\bar{s})) \ ,$$

$$d\bar{s}\colon L(N/\mathcal{F}_{\mathrm{bas}}, \mathcal{F}(\bar{s})) \longrightarrow \mathfrak{X}(M)$$

and

$$ds\colon L(N, \mathcal{F}) \cap L(N, \mathcal{F}(s)) \longrightarrow \mathfrak{X}(M)$$

have the lifting property.

Proof. Example 2.4 implies that $d\bar{s} : L(N/\mathcal{F}_{\mathrm{bas}}, \mathcal{F}(\bar{s})) \to \mathfrak{X}(M)$ has the lifting property. Because $ds = d\bar{s} \circ d\pi_{\mathrm{bas}}$ and $L(N, \mathcal{F}) \subset L(N, \mathcal{F}_{\mathrm{bas}})$, we have

$$d\pi_{\mathrm{bas}}(L(N, \mathcal{F}) \cap L(N, \mathcal{F}(s))) \subset L(N/\mathcal{F}_{\mathrm{bas}}, \mathcal{F}(\bar{s})) \ ,$$

and it is sufficient to show that $d\pi_{\mathrm{bas}} : L(N, \mathcal{F}) \cap L(N, \mathcal{F}(s)) \to L(N/\mathcal{F}_{\mathrm{bas}}, \mathcal{F}(\bar{s}))$ has the lifting property.

Write $W = N/\mathcal{F}_{\mathrm{bas}}$. Take any projectable vector field $Z \in L(W, \mathcal{F}(\bar{s}))$, and choose $\xi \in T_y(N)$ such that $d\pi_{\mathrm{bas}}(\xi) = Z_{\pi_{\mathrm{bas}}(y)}$. Put $w = \pi_{\mathrm{bas}}(y)$.

First we shall show that there exists an open neighbourhood V of w in W and a vector field $Y^V \in L(\pi_{\mathrm{bas}}^{-1}(V), \mathcal{F}|_{\pi_{\mathrm{bas}}^{-1}(V)})$ such that $Y^V_y = \xi$ and $d\pi_{\mathrm{bas}}(Y^V) = Z|_V$. To see this, choose projectable vector fields $Y_1, \dots, Y_q \in L(N, \mathcal{F})$ such that y is a regular point of the associated local transverse parallelism $(\bar{Y}_1, \dots, \bar{Y}_q)$ on $(N, \mathcal{F})$. Furthermore, we can choose these projectable vector fields so that $\xi = \sum_{i=1}^q c_i (Y_i)_y$ for some constants $c_1, \dots, c_q$. Indeed, we can replace Y_1 by the vector field $Y_1 + Y'$, for a suitable vector field $Y' \in \mathfrak{X}(\mathcal{F})$, so that ξ is in the span of tangent vectors $(Y_1)_y, \dots, (Y_q)_y$.

Denote $X_i = \pi_{\mathrm{bas}}(Y_i)$, for $i = 1, \dots, q$. We can reorder $Y_1, \dots, Y_q$, so that $((X_1)_w, \dots, (X_{\bar{q}})_w)$ is a basis of $T_w(W)$. Therefore we can choose an open neighbourhood V of w in W such that $((X_1)_v, \dots, (X_{\bar{q}})_v)$ is a basis of $T_v(W)$ for any $v \in V$. By Example 2.4 there exists a vector field $Z' \in L(W, \mathcal{F}(\bar{s}))$ such that $Z'_w = \sum_{i=1}^{\bar{q}} c_i (X_i)_w$. Write

$$Z|_V - Z'|_V - \sum_{j=\bar{q}+1}^{q} c_j X_j|_V = \sum_{i=1}^{\bar{q}} h_i X_i|_V$$

and

$$Z'|_V = \sum_{i=1}^{\bar{q}} h'_i X_i|_V \ ,$$

so $h_i, h'_i \in C^\infty(V)$, $h_i(w) = 0$ and $h'_i(w) = c_i$, for $i = 1, \dots, \bar{q}$.

Let $g_i = h_i + h'_i$ for $i = 1, \dots, \bar{q}$, and put $g_i = c_i$ for $i = \bar{q}+1, \dots, q$. Thus for any $i = 1, \dots, q$ we have $g_i \in C^\infty(V)$, $g_i(w) = c_i$ and $Z|_V = \sum_{i=1}^q g_i X_i|_V$. Now we define $Y^V \in L(\pi_{\mathrm{bas}}^{-1}(V), \mathcal{F}|_{\pi_{\mathrm{bas}}^{-1}(V)})$ by

$$Y^V = \sum_{i=1}^{q} (g_i \circ \pi_{\mathrm{bas}}) Y_i|_{\pi_{\mathrm{bas}}^{-1}(V)} \ .$$

We have $Y^V_y = \sum_{i=1}^q g_i(w)(Y_i)_y = \xi$ and $d\pi_{\mathrm{bas}}(Y^V) = \sum_{i=1}^q g_i X_i|_V = Z|_V$.

From this, we can conclude that there exist an open cover (V^α) of W and projectable vector fields $Y^\alpha = Y^{V^\alpha} \in L(\pi_{\mathrm{bas}}^{-1}(V^\alpha), \mathcal{F}|_{\pi_{\mathrm{bas}}^{-1}(V^\alpha)})$ such that $d\pi_{\mathrm{bas}}(Y^\alpha) = Z|_{V^\alpha}$. Furthermore, we can assume that $w \in V^{\alpha_0}$, $Y^{\alpha_0}_y = \xi$ and $y \notin V^\alpha$ for $\alpha \neq \alpha_0$. Let (η^α) be a partition of unity subordinated to (V^α), and define

$$Y = \sum_{\alpha} (\eta^\alpha \circ \pi_{\mathrm{bas}}) Y^\alpha \ .$$

Observe that $Y \in L(N, \mathcal{F})$ because the functions $\eta^\alpha \circ \pi_{\mathrm{bas}}$ are basic, and that $Y_y = Y^{\alpha_0}_y = \xi$. Furthermore, $d\pi_{\mathrm{bas}}(Y_z) = \sum_\alpha \eta^\alpha(\pi_{\mathrm{bas}}(z)) Z_{\pi_{\mathrm{bas}}(z)}) = Z_{\pi_{\mathrm{bas}}(z)}$ for any $z \in N$. Finally, we have $Y \in L(N, \mathcal{F}(s))$ because $d\pi_{\mathrm{bas}}(Y) = Z \in L(W, \mathcal{F}(\bar{s}))$. □

Proof of Theorem 2.3. Since $T_y(\mathcal{F}) \subset T_y(\mathcal{F}_{\mathrm{bas}}) \subset \mathrm{Ker}(ds)_y \subset T_y(N)$, we can choose tangent vectors $\xi_1, \dots, \xi_q \in T_y(N)$ such that their projections to $\nu_y(\mathcal{F})$ form a basis of $\nu_y(\mathcal{F})$, $ds(\xi_{m+1}) = \ldots = ds(\xi_q) = 0$ and $d\pi_{\mathrm{bas}}(\xi_{\bar{q}+1}) = \ldots = d\pi_{\mathrm{bas}}(\xi_q) = 0$. By Proposition 2.5 we can choose vector fields $Z_1, \dots, Z_{\bar{q}} \in L(N/\mathcal{F}_{\mathrm{bas}}, \mathcal{F}(\bar{s}))$ such that

$$(Z_i)_{\pi_{\mathrm{bas}}(y)} = d\pi_{\mathrm{bas}}(\xi_i) \ , \qquad i = 1, \dots, \bar{q} \ .$$

Set

$$Z_i = 0 \ , \qquad i = \bar{q}+1, \dots, q \ .$$

Again by Proposition 2.5, we can find $Y_1, \dots, Y_q \in L(N, \mathcal{F}) \cap L(N, \mathcal{F}(s))$ such that $(Y_i)_y = \xi_i$ and $d\pi_{\mathrm{bas}}(Y_i) = Z_i$. We can use the flows of the vector fields $Y_1, \dots, Y_q$ to obtain a local trivialization of s as a bundle of foliated manifolds, while the restrictions of $Y_{m+1}, \dots, Y_q$ to a fiber $N_{s(y)}$ provide a local transverse parallelism on $(N_{s(y)}, \mathcal{F}|_{N_{s(y)}})$ for which y is a regular point. □

Theorem 2.3 motivates the following definition. Let M be a connected manifold. A *bundle of transversely complete foliations* $s\colon (N,\mathcal{F}) \to M$ with connected compact fibers over M is a manifold N, equipped with a locally transversely parallelizable foliation $\mathcal{F}$ and a surjective submersion $s : N \to M$ with connected compact $\mathcal{F}$-saturated fibers. In particular, any such bundle is a locally trivial fiber bundle of foliations.

For such a bundle $s\colon (N,\mathcal{F}) \to M$ we shall write

$$L^s(N,\mathcal{F}) = L(N,\mathcal{F}) \cap \mathfrak{X}(\mathcal{F}(s))$$

for the Lie algebra of s-vertical projectable vector fields on $(N,\mathcal{F})$, which is the normalizer of $\mathfrak{X}(\mathcal{F})$ in the Lie algebra $\mathfrak{X}(\mathcal{F}(s))$. Let

$$l^s(N,\mathcal{F}) = L^s(N,\mathcal{F})/\mathfrak{X}(\mathcal{F})$$

be the associated quotient Lie algebra of s-vertical transverse vector fields on $(N,\mathcal{F})$. It is a subalgebra of the Lie algebra $l(N,\mathcal{F})$.

The subbundle $\mathrm{Ker}(ds) \subset T(N)$ of s-vertical tangent vectors, which is the tangent bundle of the foliation $\mathcal{F}(s)$, will also be denoted by $T^s(N)$. We shall write $\nu^s(\mathcal{F}) = T^s(N)/T(\mathcal{F})$ for the corresponding subbundle of the normal bundle $\nu(\mathcal{F})$. Note that $ds : T(N) \to T(M)$ induces a map of vector bundles $\nu(\mathcal{F}) \to T(M)$ (which we denote again by ds) with kernel $\nu^s(\mathcal{F})$. A transversal vector field on $(N,\mathcal{F})$ is s-vertical if and only if it is a section of $\nu^s(\mathcal{F})$.

3. Developability of bundles of transversely complete foliations

Let $s\colon (N,\mathcal{F}) \to M$ be a locally trivial fiber bundle of foliations with connected fibers. We shall denote by $\mathcal{F}_x$ the restriction of $\mathcal{F}$ to the fiber $N_x = s^{-1}(x)$ over a point $x \in M$. We say that such a bundle of foliations $s\colon (N,\mathcal{F}) \to M$ is *developable* if the foliation $(N_x,\mathcal{F}_x)$ is developable for any $x \in M$.

Our aim in this section is to provide some characterization of this notion of developability, first in terms of the fiberwise fundamental groupoid of s (Theorem 3.3), and second in terms of the integrability of an associated Lie algebroid constructed in Subsection 3.2 (Theorem 3.9).

3.1. **The fiberwise fundamental groupoid.** Let $s : (N,\mathcal{F}) \to M$ be a locally trivial fiber bundle of foliations with connected fibers. For any open subset U of M, we shall write $N|_U = s^{-1}(U)$ and $\mathcal{F}|_U = \mathcal{F}|_{s^{-1}(U)}$. Note that $s|_U\colon (N|_U,\mathcal{F}|_U) \to U$ is again a locally trivial fiber bundle of foliations.

Denote by $\Pi^s(N)$ the monodromy groupoid of $(N,\mathcal{F}(s))$. This groupoid can be viewed as the fiberwise fundamental groupoid of the bundle s. The space $\Pi^s(N)$ is in fact a locally trivial fiber bundle over M, and the map $(\mathrm{t},\mathrm{s})\colon \Pi^s(N) \to N \times_M N$ is a covering projection. We define the pull-back foliation $\Pi^s(\mathcal{F}) = (\mathrm{t},\mathrm{s})^*(\mathcal{F}\times 0)$, which is a right invariant foliation of the groupoid $\Pi^s(N)$. If M is a one-point space, the Lie groupoid $\Pi^s(N)$ is simply the fundamental groupoid $\Pi(N)$; the corresponding foliation $\Pi^s(\mathcal{F})$ will in this case be denoted by $\Pi(\mathcal{F})$.

For a local section $\sigma\colon U \to N$ of s, defined on an open subset U of M, we define $\Pi^s_\sigma(N)$ by the following pull-back:

$$\begin{tikzcd} \Pi^s_\sigma(N) \arrow[r] \arrow[d, "\tau"'] & \Pi^s(N) \arrow[r, "\mathrm{t}"] \arrow[d, "\mathrm{s}"] & N \\ U \arrow[r, "\sigma"] & N & \end{tikzcd}$$

The elements of $\Pi^s_\sigma(N)$ are the homotopy classes of paths in s-fibers starting at a point in $\sigma(U)$. In particular, the restriction of the target map of $\Pi^s(N)$ to the fiber $\Pi^s_{\sigma(x)}(N) = \Pi^s(N)(\sigma(x),\text{-}\,)$ over a point $x \in U$ is the universal covering projection onto N_x, and the restriction of $\Pi^s(\mathcal{F})$ to $\Pi^s_{\sigma(x)}(N)$ is the pull-back of $\mathcal{F}_x$ along the covering projection $\mathrm{t}\colon \Pi^s_{\sigma(x)}(N) \to N_x$.

Since s is a locally trivial fiber bundle of foliations, it follows that the map

$$s \circ \mathrm{s} = s \circ \mathrm{t}\colon (\Pi^s(N), \Pi^s(\mathcal{F})) \longrightarrow M$$

is a locally trivial fiber bundle of foliated Lie groupoids, i.e. for any $x_0, x \in M$ there is an open neighbourhood U of x in M, and an isomorphism of Lie groupoids

$$\Pi^s(N)|_{s^{-1}(U)} \longrightarrow \Pi(N_{x_0}) \times U$$

over U, which maps the restriction of the foliation $\Pi^s(\mathcal{F})$ to the foliation $\Pi(\mathcal{F}_{x_0}) \times 0$. In particular, the map $\mathrm{s}\colon (\Pi^s(N), \Pi^s(\mathcal{F})) \to N$ is a locally trivial fiber bundle of foliations with connected fibers.

Lemma 3.1. *Let $s\colon (N, \mathcal{F}) \to M$ be a locally trivial fiber bundle of foliations with connected fibers. If $\sigma\colon U \to N$ is a local section of s, defined on an open subset U of M, then $\mathrm{t}\colon \Pi^s_\sigma(N) \to N|_U$ is a covering projection, and*

$$(\Pi^s_\sigma(N), \mathrm{t}^*(\mathcal{F}|_U)) \longrightarrow U$$

is a locally trivial fiber bundle of foliated manifolds.

Proof. Take any $x_0, x_1 \in U$, and choose an open simply connected neighbourhood V of x_1 in U such that $(N|_V, \mathcal{F}|_V)$ is a trivial bundle of foliated manifolds, with a trivialization

$$\mu = (\delta, s) : N|_V \longrightarrow N_{x_0} \times V$$

which maps the $\mathcal{F}|_V$ to the foliation $\mathcal{F}_{x_0} \times 0$. Choose $y_0 \in N_{x_0}$. Since $\mathrm{t}\colon \Pi^s_{y_0}(N) \to N_{x_0}$ is a covering projection and V is simply connected, we can lift $\delta \circ \sigma|_V$ to a map $\tau\colon V \to \Pi^s_{y_0}(N)$, so $\tau(x)\colon y_0 \to \delta(\sigma(x))$ for any $x \in V$.

Define a map $\tilde{\mu}\colon \Pi^s_{\sigma|_V}(N) \to \Pi^s_{y_0}(N) \times V$ by

$$\tilde{\mu}(\alpha) = (\delta_*(\alpha)\tau(x), x) ,$$

for any $\alpha\colon \sigma(x) \to y$ in $\Pi^s_{\sigma|_V}(N)$. This map is clearly a diffeomorphism, and the diagram

$$\begin{tikzcd}
\Pi^s_{\sigma|_V}(N) \arrow[r, "\tilde{\mu}"] \arrow[d, "\mathrm{t}"] & \Pi^s_{y_0}(N) \times V \arrow[d, "\mathrm{t}\times\mathrm{id}"] \\
N|_V \arrow[r, "\mu"] \arrow[d, "s"] & N_{x_0} \times V \arrow[dl, "\mathrm{pr}_2"] \\
V &
\end{tikzcd}$$

commutes. This shows that $\mathrm{t}\colon \Pi^s_\sigma(N) \to N|_U$ is a covering projection. Since the map μ trivializes the bundle $(N|_V, \mathcal{F}|_V) \to V$ of foliated manifolds, it follows that $(\Pi^s_\sigma(N), \mathrm{t}^*(\mathcal{F}|_U)) \to U$ is also a locally trivial fiber bundle of foliated manifolds. $\square$

We first observe that strict simplicity is a fiberwise property:

Proposition 3.2. *Let $s\colon (N, \mathcal{F}) \to M$ be a locally trivial fiber bundle of foliations with connected fibers. Then the foliation $\mathcal{F}$ is strictly simple if and only if for each $x \in M$ the foliation $\mathcal{F}_x$ is strictly simple.*

Proof. Suppose that $\mathcal{F}$ is strictly simple, thus given by a submersion $f : N \to T$ with connected fibers. Since the fibers of s are saturated, the map f induces a surjective submersion $\bar{f}\colon T \to M$. In particular, the fiber $T_x = \bar{f}^{-1}(x)$ is a closed submanifold of T. Moreover, the restriction

$$f|_{N_x}\colon N_x \longrightarrow T_x$$

is a submersion with connected fibers and defines the foliation $\mathcal{F}_x$.

Conversely, suppose that $\mathcal{F}_x$ is strictly simple for any $x \in M$. Equivalently, there is a structure of a Hausdorff manifold on the space of leaves $N_x/\mathcal{F}_x$ such that the quotient projection $N_x \to N_x/\mathcal{F}_x$ is a submersion. Since N is a locally trivial fiber bundle of foliated manifolds, $N/\mathcal{F}$ can also be given a structure of a Hausdorff manifold such that the quotient projection $N \to N/\mathcal{F}$ is a submersion. It follows that $\mathcal{F}$ is strictly simple. □

Theorem 3.3. *Let $s : (N, \mathcal{F}) \to M$ be a locally trivial fiber bundle of foliations with connected fibers. The following conditions are equivalent.*

(i) The bundle $s\colon (N, \mathcal{F}) \to M$ is developable.

(ii) The foliation $\Pi^s(\mathcal{F})$ of $\Pi^s(N)$ is strictly simple.

(iii) For any open subset U of M with a section $\sigma\colon U \to M$ of s, the restriction of the foliation $\Pi^s(\mathcal{F})$ to $\Pi^s_\sigma(N)$ is strictly simple.

Proof. (i)⇔(ii) Recall that $\mathrm{s}\colon (\Pi^s(N), \Pi^s(\mathcal{F})) \to N$ is a locally trivial fiber bundle of foliations with connected fibers. If we apply Proposition 3.2 to this bundle, we see that $\Pi^s(\mathcal{F})$ is strictly simple if and only if the restriction of $\Pi^s(\mathcal{F})$ to $\Pi^s(N)(y, \text{-})$ is strictly simple, for any $y \in N$. But

$$\mathrm{t}\colon \Pi^s(N)(y, \text{-}) \longrightarrow N_{s(y)}$$

is the universal cover of $N_{s(y)}$, and the restriction of $\Pi^s(\mathcal{F})$ to $\Pi^s(N)(y, \text{-})$ is the pull-back of $\mathcal{F}_{s(y)}$ along this covering.

(i)⇔(iii) By definition, the bundle $s\colon (N, \mathcal{F}) \to M$ is developable if and only if the pull-back of the foliation $(N_x, \mathcal{F}_x)$ to the universal cover of N_x is strictly simple, for any $x \in M$. This is true if and only if the foliation $(\Pi^s_{\sigma(x)}(N), \mathrm{t}^*(\mathcal{F}_x))$ is strictly simple, for any section $\sigma\colon U \to N$ of s, because $\Pi^s_{\sigma(x)}(N)$ is the universal cover of N_x. In turn, this holds true if and only if the foliation $(\Pi^s_\sigma(N), \mathrm{t}^*(\mathcal{F}|_U))$ is strictly simple. Indeed, to see this, apply Proposition 3.2 to the map

$$s \circ \mathrm{t}\colon (\Pi^s_\sigma(N), \mathrm{t}^*(\mathcal{F}|_U)) \longrightarrow U ,$$

which is a locally trivial fiber bundle of foliated manifolds. Since $\Pi^s_\sigma(N)$ is a covering space of $N|_U$ by Lemma 3.1, this completes the proof. □

Corollary 3.4. *For a foliation $\mathcal{F}$ on a manifold M, the following two statements are equivalent:*

(i) *The foliation $\mathcal{F}$ is developable.*
(ii) *The foliation $\mathcal{F}$, viewed as a subalgebroid of $T(M)$, is integrable by a closed subgroupoid of the fundamental groupoid $\Pi(M)$.*

Proof. The groupoid $\Pi(M)$ integrates the algebroid $T(M)$, so the subalgebroid $\mathcal{F} \subset T(M)$ correspond to an invariant foliation of $\Pi(M)$ by [13, Lemma 2.1], which is readily identified to the foliation $\Pi(\mathcal{F})$. Thus by Theorem 3.3, $\mathcal{F}$ is developable if and only if $\Pi(\mathcal{F})$ is strictly simple, and by [13, Proposition 3.2 (ii)] this is the case if and only if $\mathcal{F}$ can be integrated by a closed subgroupoid of $\Pi(M)$. □

3.2. **The basic Lie algebroid.** As the main result of this section, we shall now construct a Lie algebroid associated to a bundle $s : (N, \mathcal{F}) \to M$ of transversely complete foliations, and show that integrability of this algebroid is equivalent to developability of the bundle s.

Lemma 3.5. *Let $s\colon (N, \mathcal{F}) \to M$ be a bundle of transversely complete foliations with compact connected fibers, and let $\pi_{\mathrm{bas}} : N \to N/\mathcal{F}_{\mathrm{bas}} = W$ be the associated basic fibration. The groupoid $N \times_W N$ naturally acts linearly both on the normal bundle $\nu(\mathcal{F})$ and its subbundle $\nu^s(\mathcal{F})$ along π_{bas}.*

Proof. For any $(z,y) \in N \times_W N$ and $\xi \in \nu_z(\mathcal{F})$, put

$$\Theta(\xi, z, y) = \bar{Y}_y \;,$$

where $\bar{Y}$ is any transverse vector field on $(N, \mathcal{F})$ such that $\bar{Y}_z = \xi$. This definition is independent of the choice of $\bar{Y}$ by Remark 2.2 (5). Using local transverse parallelisms and the corresponding local trivializations of the normal bundle, it is straightforward to check that Θ is indeed a smooth linear action.

This action restricts to an action of $N \times_W N$ on $\nu^s(\mathcal{F})$ by Proposition 2.5. Indeed, for any $(z,y) \in N \times_W N$ and any $\xi \in \nu^s_z(\mathcal{F})$, we can find $\bar{Y} \in L^s(N, \mathcal{F})$ with $\bar{Y}_z = \xi$. □

Remark 3.6. Note that if y and z lie on the same leaf of $\mathcal{F}$, the corresponding linear map $\Theta(\text{-}, z, y) \colon \nu_z(\mathcal{F}) \to \nu_y(\mathcal{F})$ is simply the linear holonomy isomorphism of an arbitrary path in the leaf of $\mathcal{F}$ from z to y.

Let $s \colon (N, \mathcal{F}) \to M$ be a bundle of transversely complete foliations with compact connected fibers over a connected manifold M, and write $W = N/\mathcal{F}_{\mathrm{bas}}$. We shall denote by

$$\mathfrak{b}^s(N, \mathcal{F}) = \nu^s(\mathcal{F})/(N \times_W N)$$

the space of orbits of the natural action of the groupoid $N \times_W N$ on $\nu^s(\mathcal{F})$ (Lemma 3.5). In the next lemma, we show that the natural projection $\mathfrak{b}^s(N, \mathcal{F}) \to W$ is a vector bundle.

Lemma 3.7. *Let $s \colon (N, \mathcal{F}) \to M$ be a bundle of transversely complete foliations with compact connected fibers, and let $\pi_{\mathrm{bas}} : N \to N/\mathcal{F}_{\mathrm{bas}} = W$ be the associated basic fibration. Then the orbit space $\mathfrak{b}^s(N, \mathcal{F})$ has the structure of a vector bundle over W such that*

(i) the quotient projection $\theta \colon \nu^s(\mathcal{F}) \to \mathfrak{b}^s(N, \mathcal{F})$ is a $(N \times_W N)$-principal bundle, and

(ii) the diagram

$$\begin{tikzcd} \nu^s(\mathcal{F}) \arrow[r, "\theta"] \arrow[d] & \mathfrak{b}^s(N, \mathcal{F}) \arrow[d] \\ N \arrow[r, "\pi_{\mathrm{bas}}"] & W \end{tikzcd}$$

is a fibered product of vector bundles.

Proof. Since N is a principal $N \times_W N$-bundle over W and the map $\nu^s(\mathcal{F}) \to N$ is $N \times_W N$-equivariant, it follows that there is a structure of smooth Hausdorff manifold on $\mathfrak{b}^s(N, \mathcal{F})$ such that the quotient projection θ is a $N \times_W N$-principal bundle. Since the action on $\nu^s(\mathcal{F})$ is given by linear isomorphisms, the fibers of $\mathfrak{b}^s(N, \mathcal{F}) \to W$ have vector space structures such that the restriction $\theta_y \colon \nu^s_y(\mathcal{F}) \to \mathfrak{b}^s(N, \mathcal{F})_{\pi_{\mathrm{bas}}(y)}$ of θ is a linear isomorphism, for any $y \in N$. Indeed, if $z \in N$ is another point on the same basic leaf as y, then $\theta_y \circ \Theta(\text{-}, z, y) = \theta_z$, where Θ denotes the natural action of $N \times_W N$ on $\nu(\mathcal{F})$.

Take any local transverse parallelism $(\bar{Y}_1, \ldots, \bar{Y}_q)$ with regular set U, which is of the form given by Theorem 2.3. It gives us a local trivialization of $\mathfrak{b}^s(N, \mathcal{F})$

$$\alpha \colon \pi_{\mathrm{bas}}(U) \times \mathbb{R}^{q-m} \longrightarrow \mathfrak{b}^s(N, \mathcal{F})|_{\pi_{\mathrm{bas}}(U)}$$

over $\pi_{\mathrm{bas}}(U)$ by

$$\alpha(\pi_{\mathrm{bas}}(y), t_{m+1}, \ldots, t_q) = \theta_y \left(\sum_{j=m+1}^{q} t_j (\bar{Y}_j)_y \right) \,.$$

This is well-defined because for another point $z \in U$ with $\pi_{\mathrm{bas}}(y) = \pi_{\mathrm{bas}}(z)$ we have $\Theta((\bar{Y}_j)_z, z, y) = (\bar{Y}_j)_y$. Thus we can conclude that $\mathfrak{b}^s(N, \mathcal{F})$ is indeed a vector bundle over W. The diagram in (ii) is a pull-back because θ restricts to the isomorphism θ_y on each fiber $\nu^s_y(\mathcal{F})$. □

Proposition 3.8. *Let $s\colon (N, \mathcal{F}) \to M$ be a bundle of transversely complete foliations with compact connected fibers. The bundle $\mathfrak{b}^s(N, \mathcal{F})$ has a natural structure of a regular Lie algebroid over $N/\mathcal{F}_{\mathrm{bas}}$ such that*

(i) the quotient projection $\nu^s(\mathcal{F}) \to \mathfrak{b}^s(N, \mathcal{F})$ over the basic fibration induces an isomorphism of Lie algebras

$$\Gamma(\mathfrak{b}^s(N, \mathcal{F})) \longrightarrow l^s(N, \mathcal{F}) \ ,$$

(ii) the foliation of $N/\mathcal{F}_{\mathrm{bas}}$ corresponding to this regular algebroid is the simple foliation given by the submersion $N/\mathcal{F}_{\mathrm{bas}} \to M$ induced by s, and

(iii) the natural projection $T^s(N) \to \mathfrak{b}^s(N, \mathcal{F})$ is a surjective morphism of Lie algebroids.

Proof. The fact that the diagram in Lemma 3.7 (ii) is a fibered product implies that any section X of $\mathfrak{b}^s(N, \mathcal{F})$ induces a section $\pi^*_{\mathrm{bas}}X$ of $\nu^s(\mathcal{F})$, so we get a map

$$\pi^*_{\mathrm{bas}}\colon \Gamma(\mathfrak{b}^s(N, \mathcal{F})) \longrightarrow \Gamma(\nu^s(\mathcal{F})) \ ,$$

which is clearly $\Omega^0_{\mathrm{bas}}(N, \mathcal{F})$-linear (recall that the composition with the basic projection $\pi_{\mathrm{bas}}\colon N \to N/\mathcal{F}_{\mathrm{bas}} = W$ gives us the identification $\Omega^0_{\mathrm{bas}}(N, \mathcal{F}) = C^\infty(W)$).

Note that a section σ of the bundle $\nu^s(\mathcal{F})$ is a transverse vector field if and only if it is holonomy invariant, i.e. if and only if

$$\Theta(\sigma_z, z, y) = \sigma_y \tag{6}$$

for any two points y, z on the same leaf of $\mathcal{F}$. Furthermore, Remark 2.2 (5) implies that for a transverse vector field the condition (6) in fact automatically holds for any y, z on the same leaf of $\mathcal{F}_{\mathrm{bas}}$. Thus, by the definition of $\mathfrak{b}^s(N, \mathcal{F})$ and by Lemma 3.7, the s-vertical transverse vector fields on $(N, \mathcal{F})$ are exactly those sections of $\nu^s(\mathcal{F})$ which can be projected along π_{bas} to a section of $\mathfrak{b}^s(N, \mathcal{F})$. We can therefore conclude that

$$\pi^*_{\mathrm{bas}}\colon \Gamma(\mathfrak{b}^s(N, \mathcal{F})) \longrightarrow l^s(N, \mathcal{F})$$

is an $\Omega^0_{\mathrm{bas}}(N, \mathcal{F})$-linear isomorphism. We therefore define a Lie algebra structure on $\Gamma(\mathfrak{b}^s(N, \mathcal{F}))$ so that π^*_{bas} is also a Lie algebra isomorphism.

To give $\mathfrak{b}^s(N, \mathcal{F})$ a Lie algebroid structure, we have to define its anchor, which is a morphism of vector bundles $\mathrm{an}\colon \mathfrak{b}^s(N, \mathcal{F}) \longrightarrow T(W)$ over W. We define it by

$$\mathrm{an}(\theta_y(\zeta + T_y(\mathcal{F}))) = (d\pi_{\mathrm{bas}})_y(\zeta)$$

for any $\zeta \in T^s_y(N)$. It is straightforward to check that $\mathfrak{b}^s(N, \mathcal{F})$ is a Lie algebroid over W and that $T^s(N) \to \mathfrak{b}^s(N, \mathcal{F})$ is a surjective morphism of Lie algebroids. □

The Lie algebroid

$$\mathfrak{b}^s(N, \mathcal{F})$$

over $N/\mathcal{F}_{\mathrm{bas}}$ will be called the *basic Lie algebroid* associated to the bundle of transversally complete foliations $s\colon (N, \mathcal{F}) \to M$.

Theorem 3.9. *Suppose that $s\colon (N, \mathcal{F}) \to M$ is a bundle of transversely complete foliations with compact connected fibers. Then $s\colon (N, \mathcal{F}) \to M$ is developable if and only if the associated basic Lie algebroid $\mathfrak{b}^s(N, \mathcal{F})$ is integrable.*

Proof. Recall from Theorem 2.3 that the restriction $\mathcal{F}_x$ of $\mathcal{F}$ to the fiber N_x over $x \in M$ is a transversely complete (and therefore homogeneous) foliation. Theorem 2.3 also implies $(\mathcal{F}_x)_{\mathrm{bas}} = \mathcal{F}_{\mathrm{bas}}|_{N_x}$, so the fiber $W_x = \bar{s}^{-1}(x)$ over x is the space of leaves $N_x/(\mathcal{F}_x)_{\mathrm{bas}}$.

Since $\mathfrak{b}^s(N, \mathcal{F})$ is a locally trivial bundle of transitive Lie algebroids over M, its integrability is equivalent to the integrability of each fiber. The restriction of $\mathfrak{b}^s(N, \mathcal{F})$ to the fiber W_x is exactly the basic algebroid $\mathfrak{b}(N_x, \mathcal{F}_x)$ of the transversely complete foliation $\mathcal{F}_x$ of the compact fiber N_x (see also [15]). On the other hand, developability of $s\colon (N, \mathcal{F}) \to M$ is also given fiberwise. Therefore we may assume without loss of generality that M is a one-point space. In this case, the statement follows from the Almeida-Molino theorem for transversely complete foliations of compact manifolds [15] (see also [12]). □

4. Proof of the main theorem

In this final section, we return to the setting of Section 1, and use the results of Sections 2 and 3 to prove Theorem 1.8. In the proof of this theorem, we use the following simple observations on the universal covering groupoid $\tilde{G}$ and the fiberwise fundamental groupoid $\Pi^{\mathrm{s}}(G)$.

Lemma 4.1. *Let G be a source-connected Lie groupoid with Lie algebroid $\mathfrak{g}$.*

(i) There is a natural action of G on $\Pi^{\mathrm{s}}(G)$ which differentiates to an action ∇ by $\mathfrak{g}$ on the Lie algebroid $T^{\mathrm{s}}(G) \to G$ along the source map $\mathrm{s}\colon G \to M$. For any $X \in \Gamma(\mathfrak{g})$, the corresponding derivation $\nabla(X) = (\nabla_X, R(X))$ is given by

$$R(X) = \tilde{X}^{-1}$$

and

$$\nabla_X(Y) = [R(X), Y]$$

for any $Y \in \mathfrak{X}^{\mathrm{s}}(G)$, where $\tilde{X}^{-1}$ the image of the right invariant extension $\tilde{X}$ of X along the inverse map of G.

(ii) The universal covering groupoid $\tilde{G}$ of G embed into $G \ltimes \Pi^{\mathrm{s}}(G)$ as the full subgroupoid on the space of units $\mathrm{u}\colon M \to G$, and this inclusion differentiates to the diagonal embedding $\mathfrak{g} \to \mathfrak{g} \ltimes T^{\mathrm{s}}(G)$ of Lie algebroids.

Proof. Note that $\Pi^{\mathrm{s}}(G)$ is a bundle of groupoids over M whose fiber over $x \in M$ is the fundamental groupoid $\Pi(\mathrm{s}^{-1}(x))$ of $\mathrm{s}^{-1}(x)$. The natural left action of an arrow $g\colon x \to y$ of G is the map of fundamental groupoids $\Pi(\mathrm{s}^{-1}(x)) \to \Pi(\mathrm{s}^{-1}(y))$ induced by the right translation $R_{g^{-1}}$. The semi-direct product $G \ltimes \Pi^{\mathrm{s}}(G)$ can be described explicitly as the groupoid over G whose arrows $g \to g'$ are pairs (h, α), where $h\colon \mathrm{s}(g) \to \mathrm{s}(g')$ is an arrow in G and $\alpha\colon g \to g'h$ is an arrow in $\Pi^{\mathrm{s}}(G)$. From this description it is clear that the restriction of $G \ltimes \Pi^{\mathrm{s}}(G)$ to the units is precisely $\tilde{G}$. The rest follows by straightforward inspection. □

Proof of Theorem 1.8. Before proving (i)-(iv), let us observe that since the source map of G has compact fibers and the foliation $\mathcal{F}(\mathfrak{h}) \subset \mathcal{F}(\mathrm{s})$ is transversely complete, the map $\mathrm{s} : (G, \mathcal{F}(\mathfrak{h})) \to M$ is a bundle of transversely complete foliations with connected compact fibers. Moreover, the foliations $\mathcal{F}(\mathfrak{h})$ and $\mathcal{F}(\mathrm{s})$, as well as the basic foliation $\mathcal{F}(\mathfrak{h})_{\mathrm{bas}}$ associated to $\mathcal{F}(\mathfrak{h})$, are all right invariant.

The closure $\bar{H}$ of H in G is a closed subgroupoid of G which corresponds to the basic foliation $\mathcal{F}(\mathfrak{h})_{\mathrm{bas}}$, and the space of cosets $G/\bar{H}$ is the associated space of basic leaves [13, Theorem 3.7]. Furthermore, the quotient projection $\pi = \pi_{\mathrm{bas}}\colon G \to G/\bar{H}$ is a locally trivial fiber bundle.

(i) We take $\mathfrak{b}(G, \mathfrak{h})$ to be the basic Lie algebroid $\mathfrak{b}^{\mathrm{s}}(G, \mathcal{F}(\mathfrak{h}))$ associated to the bundle $\mathrm{s} : (G, \mathcal{F}(\mathfrak{h})) \to M$. Proposition 3.8 (iii) implies that we have a natural surjective morphism of Lie algebroids $\omega\colon T^{\mathrm{s}}(G) \to \mathfrak{b}(G,\mathfrak{h})$ with kernel $\mathcal{F}(\mathfrak{h})$.

$$\begin{tikzcd} T^{\mathrm{s}}(G) \arrow[r, "\omega"] \arrow[d] & \mathfrak{b}(G,\mathfrak{h}) \arrow[d] \\ G \arrow[r, "\pi"] & G/\bar{H} \end{tikzcd} \tag{7}$$

(ii) Note that the construction of $\mathfrak{b}(G,\mathfrak{h})$ in Section 3 is invariant under the right G-action, so G acts on the Lie algebroids $T^{\mathrm{s}}(G)$ and $\mathfrak{b}(G,\mathfrak{h})$, and the map ω is invariant under this action. This action, formally inverted to a left action, differentiates to an action of $\mathfrak{g}$ on $\mathfrak{b}(G,\mathfrak{h})$.

(iii) We claim that the restriction of ω to $\mathfrak{g} \subset T^{\mathrm{s}}(G)$ provides a Maurer-Cartan form with kernel $\mathcal{F}(\mathfrak{h})$. By the action of $\mathfrak{g}$ on $\mathfrak{b}(G,\mathfrak{h})$, diagram (7) gives us a map of Lie algebroids over G,

$$\mathfrak{g} \ltimes \omega\colon \mathfrak{g} \ltimes T^{\mathrm{s}}(G) \longrightarrow \mathfrak{g} \ltimes \mathfrak{b}(G,\mathfrak{h}) \;,$$

with kernel $\mathfrak{g} \ltimes \mathcal{F}(\mathfrak{h})$. Precomposing this map with the natural diagonal section $\mathfrak{g} \to \mathfrak{g} \ltimes T^{\mathrm{s}}(G)$ over the unit section $\mathrm{u}\colon M \to G$, Lemma 4.1 provides a morphism of Lie algebroids

$$\mathfrak{g} \longrightarrow \mathfrak{g} \ltimes \mathfrak{b}(G,\mathfrak{h})$$

over $\pi \circ \mathrm{u}\colon M \to G/\bar{H}$, which gives us the required Maurer-Cartan form.

(iv) The foliation $\mathcal{F}(\mathfrak{h})$ of G by cosets of H is locally transversely parallelizable, and by Theorem 2.3 the source map $\mathrm{s}\colon (G,\mathcal{F}(\mathfrak{h})) \to M$ is a locally trivial bundle of transversely complete foliations with compact connected fibers. By Theorem 3.9 the basic Lie algebroid $\mathfrak{b}(G,\mathfrak{h})$ is integrable if and only if the bundle $\mathrm{s}\colon (G,\mathcal{F}(\mathfrak{h})) \to M$ is developable, and by Theorem 3.3 this is true if and only if the corresponding foliation $\Pi^{\mathrm{s}}_{\mathrm{u}}(\mathcal{F}(\mathfrak{h}))$ of $\tilde{G} = \Pi^{\mathrm{s}}_{\mathrm{u}}(G)$ is strictly simple (here u is the unit section of the source map). By Proposition 1.1 this is equivalent to developability of the Lie algebroid $\mathfrak{h}$. □

Remark 4.2. The vector bundle $\mathfrak{b}(G,\mathfrak{h})$ is a locally trivial bundle of Lie algebroids over M, the fiber over $x \in M$ being a transitive algebroid over $\mathrm{s}^{-1}(x) \subset G/\bar{H}$. If $\mathfrak{b}(G,\mathfrak{h})$ is integrable, write B for its source-simply connected integral. Then B is a locally trivial bundle of transitive Lie groupoids over M. It follows that B is Hausdorff. The action of G on $\mathfrak{b}(G,\mathfrak{h})$ integrates to an action of G on B (see [11]), and the Maurer-Cartan form integrates to a twisted homomorphism $F : \tilde{G} \to B$, whose kernel is the closed subgroupoid integrating $\mathfrak{h}$.

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