

ON INNER KAN COMPLEXES IN THE CATEGORY OF DENDROIDAL SETS

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ABSTRACT. The category of dendroidal sets is an extension of that of simplicial sets, suitable for defining nerves of operads rather than just of categories. In this paper, we prove some basic properties of inner Kan complexes in the category of dendroidal sets. In particular, we extend fundamental results of Boardman and Vogt, of Cordier and Porter, and of Joyal to dendroidal sets.

1. INTRODUCTION

This paper is a companion to our paper [MW], where we introduced the category of dendroidal sets, and explained some of its applications. The main goal of the present paper is to introduce a notion of "inner Kan complex" for dendroidal sets, and to prove some of the fundamental properties of these inner Kan complexes.

Dendroidal sets provide a generalization of simplicial sets. There is an embedding $i_1 : sSet \rightarrow dSet$ along which many properties of simplicial sets can be extended to dendroidal sets. For example, the category of dendroidal sets carries a closed symmetric monoidal structure, which extends the Cartesian structure of simplicial sets (but is not itself Cartesian). This monoidal structure is closely related to the Boardman-Vogt tensor product of operads ([BV, F, D]). It plays a central role in various uses of dendroidal sets, for example in the construction of homotopy coherent nerves of operads, and in the definition of weak higher categories given in [MW].

We recall that a simplicial set is said to satisfy the Kan condition, or, to be a Kan complex, if every horn $\Lambda^i[n] \rightarrow X$ for $0 \leq i \leq n$ has a filler. Boardman and Vogt [BV] study this filler condition for $0 < i < n$, and refer to it as the restricted Kan condition. Recently, Joyal [J2] has been studying simplicial sets satisfying this condition, under the name of quasi-categories. The horns $\Lambda^i[n]$ for i different from 0 and n are called inner horns, and we shall call a simplicial set satisfying this restricted Kan condition an inner Kan complex. Note that the nerve of any category is an inner Kan complex.

In this paper, we will define inner horns and inner Kan complexes for dendroidal sets, in such a way that a simplicial set X is an inner Kan complex iff $i_1(X)$ is a dendroidal inner Kan complex. The dendroidal nerve of any operad provides an example of such a dendroidal inner Kan complex. We will prove several fundamental properties of dendroidal inner Kan complexes. Our main result is that the closed monoidal structure on dendroidal sets has the property that for any two dendroidal sets X and Y , the internal Hom $\underline{Hom}(X, Y)$ is an inner Kan complex whenever X is normal and Y is inner Kan (Theorem 9.1 below). We will also show that the homotopy coherent nerve of a topological operad is an inner Kan complex. (A more general statement for operads in monoidal model categories is given in Theorem 7.1

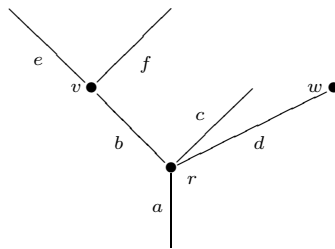
below.) These results specialise to known results for simplicial sets, and provide new proofs of these. Indeed, Cordier and Porter [CP] prove that the homotopy coherent nerve of a locally fibrant simplicial category is an inner Kan complex. And Joyal [J2] proves for any two simplicial sets X and Y that $\underline{Hom}(X, Y)$ is an inner Kan complex (quasi-category) whenever Y is (see also [N]). The latter result plays a fundamental role in Joyal’s proof of the existence of a model structure on simplicial sets in which the inner Kan complexes are the fibrant objects. We expect our result for dendroidal sets to play a similar role in establishing an analogous model structure on the category of dendroidal sets.

In [MW], we introduced a Grothendieck construction (homotopy colimit) for diagrams of dendroidal sets, necessary for our definition of weak higher categories. In Section 8 of this paper, we will prove that this dendroidal Grothendieck construction yields an inner Kan complex when applied to a diagram of inner Kan complexes. In addition, in Section 6, we will give an explicit description of the operad generated by an inner Kan complex, modelled on the one given by Boardman and Vogt [BV] in the simplicial case. We then use this description to prove that a dendroidal set satisfies the unique filler condition for inner horns iff it is the nerve of an operad.

2. THE CATEGORY OF DENDROIDAL SETS

The notion of dendroidal set was introduced in [MW]. We briefly recall the relevant definitions here.

To begin with, we introduce a category Ω whose objects are finite rooted trees. If we think of a tree as a graph, and call a vertex unary if it has only one edge attached to it, then our trees T are equipped with a distinguished unary vertex o called the output, and a set of unary vertices I (not containing o) called the set of inputs. When drawing such a tree, it is common to orient the tree "towards the output" drawn at the bottom, and delete the designated output and input vertices from the picture. Thus, in the tree T ,



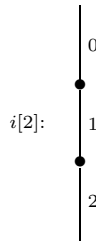
the output vertex at the edge a has been deleted, as have the input vertices at e, f, c . This tree T now has three (remaining) vertices, three *input edges* attached to the three deleted input vertices, and one *output edge* (attached to the deleted output vertex). These input and output edges are called *outer edges* (the output edge is also called the *root*, while the input edges are also called *leaves*), the others (b and d in the picture) are called *inner edges*. From now on, we will not mention the input and output vertices anymore, and "vertex" will always refer to a remaining vertex.

Attached to each such vertex in the tree, there will be one designated edge pointing towards the root; the other edges attached to this vertex are called the

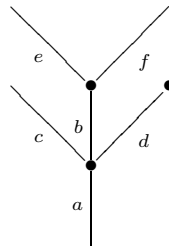
input edges of that vertex, and their number is called the *valence* of the vertex. So in the tree T pictured above, the vertex r has valence three and the vertex w has valence zero. The tree with just one edge is now drawn as



and referred to as η , or sometimes as η_e if we want to name its edge e . The linear tree, with one input edge and one output edge and n vertices, is denoted $i[n]$. It has $n + 1$ edges which we usually number from input to output as $0, 1, \dots, n$. Here is a picture of $i[2]$:



Each tree T defines a coloured operad (see [BM2] for a definition of coloured operads) which we denoted $\Omega(T)$ in [MW]. The colours of this operad are the edges of the tree, and the operations are *generated* by the vertices of the tree. A planar representation of the tree gives a specific set of generators. For example, for the tree T pictured above, $\Omega(T)$ has six colours, a, b, \dots, f . A choice of generating operations is $r \in \Omega(T)(b, c, d; a)$, $w \in \Omega(T)(-; d)$ and $v \in \Omega(T)(e, f; b)$. The other operations are units such as $1_b \in \Omega(T)(b; b)$, compositions such as $r \circ_1 v \in \Omega(T)(e, f, c, d; a)$, and permutations such as $r \cdot \tau \in \Omega(T)(c, b, d; a)$. Note that the *same* tree T can be given a different planar structure, e.g.



which defines the *same* operad $\Omega(T)$ but suggests a different choice of generators ($r \cdot \tau$ rather than r).

The category Ω is now defined as the category having these trees T (with designated output and inputs) as objects, and as arrows $T \rightarrow T'$ the maps of coloured operads $\Omega(T) \rightarrow \Omega(T')$. (Note that every such map sends colours to colours, i.e., edges of T to edges of T' , and is in fact completely determined by this).

The category of *dendroidal sets* is the category of functors $X : \Omega^{op} \rightarrow Set$ and natural transformations between them. We will denote this category by $dSet$.

For a dendroidal set X and a tree T , we will usually write X_T for $X(T)$, and call an element of the set X_T a *dendrex* of X of "shape" T .

The linear trees $i[n]$ for $n \geq 0$ define a functor (a full embedding)

$$i : \Delta \rightarrow \Omega$$

from the standard simplicial category, and hence by composition a functor

$$i^* : dSet \rightarrow sSet,$$

from dendroidal sets to the category $sSet$ of simplicial sets.

By Kan extension, this functor has both a left and a right adjoint, denoted $i_!$ and $i_* : sSet \rightarrow dSet$, respectively. The functor $i_!$ is "extension by zero"; for a simplicial set X ,

$$i_!(X)_T = \begin{cases} X_n, & \text{if } T \cong i[n] \text{ for some } n \geq 0 \\ \phi, & \text{otherwise} \end{cases}$$

This defines a full embedding $i_! : sSet \rightarrow dSet$, from simplicial sets into dendroidal sets.

Each coloured operad \mathcal{P} defines a dendroidal set $N_d(\mathcal{P})$, its *dendroidal nerve*, by

$$N_d(\mathcal{P})_T = Hom(\Omega(T), \mathcal{P}),$$

Hom denoting the set of arrows in the category of operads. If \mathcal{P} is itself an operad of the form $\Omega(S)$, given by an object S of Ω , then $N_d(\mathcal{P})$ is the representable dendroidal set given by S , which we will denote by $\Omega[S]$; i.e.,

$$N_d\Omega(S) = \Omega[S]$$

by definition. The functor N_d from operads to dendroidal sets is fully faithful, and has a left adjoint which we will denote by

$$\tau_d : dSet \rightarrow Operad.$$

For a dendroidal set X , we refer to $\tau_d(X)$ as *the operad generated by X* .

We also recall from [MW] that the Cartesian structure on $sSet$ extends to a (non-Cartesian) closed symmetric monoidal structure \otimes on $dSet$. This structure is completely determined by the identity

$$\Omega[S] \otimes \Omega[T] = N_d(\Omega(S) \otimes_{BV} \Omega(T))$$

where \otimes_{BV} denotes the Boardman-Vogt tensor product of (coloured) operads; see [BV]. The corresponding internal Hom is then determined by the Yoneda lemma, as

$$\underline{Hom}(X, Y)_T = Hom_{dSet}(\Omega[T] \otimes X, Y).$$

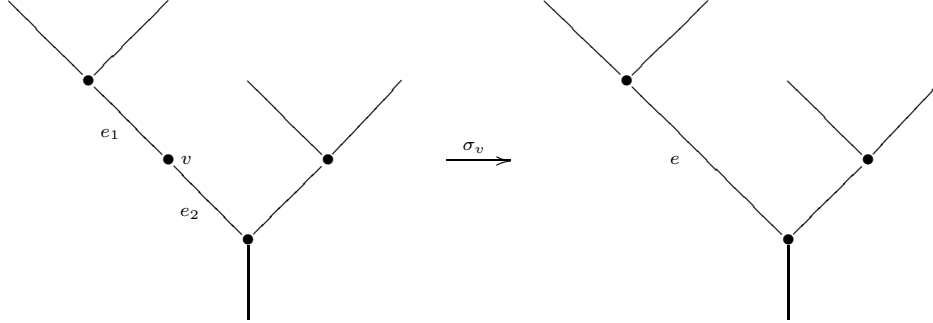
We will come back to the monoidal structure in more detail in Section 9.

3. FACES AND DEGENERACIES

Exactly as for Δ , the maps in Ω are generated by special kinds of maps.

(i) Given a tree T and a vertex $v \in T$ of valence 1, there is a tree T' , obtained from T by deleting the vertex v and merging the two edges e_1 and e_2 on either side of v into one new edge e . There is an operad map, i.e. an arrow $\sigma_v : T \rightarrow T'$ in Ω ,

which sends v to the unit 1_e . For example:

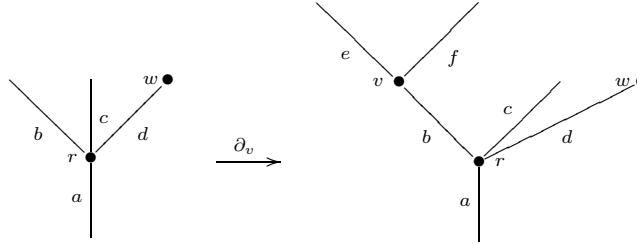


An arrow in Ω of this kind will be called a *degeneracy (map)*.

(ii) Given a tree T , and a vertex v in T with exactly one inner edge attached to it, one can obtain a new tree T/v by deleting v and all the external edges attached to it. The operad $\Omega(T/v)$ associated to T/v is simply a suboperad of the one associated to T , and this inclusion of operads defines an arrow in Ω denoted

$$\partial_v : T/v \rightarrow T.$$

An arrow in Ω of this kind is called an *outer face (map)*. For example

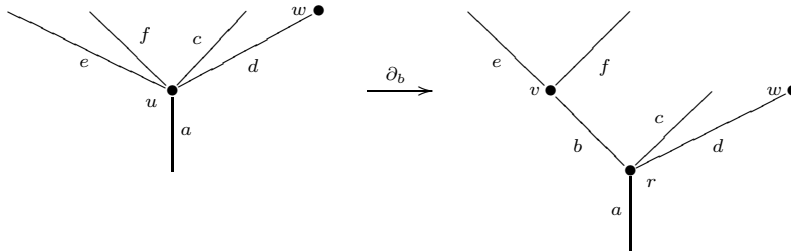


Moreover, for any tree T with exactly one vertex v , each edge e of T (necessarily outer), there is an *outer face map*

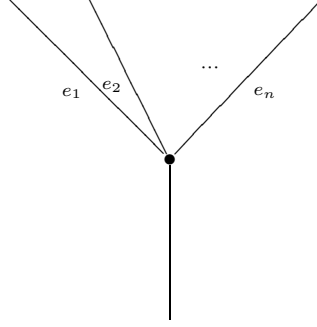
$$e : \eta \rightarrow T$$

sending the unique edge of η to e .

(iii) Given a tree T and an inner edge e in T , one can obtain a new tree T/e by contracting the edge e . There is a canonical map of operads $\partial_e : \Omega(T/e) \rightarrow \Omega(T)$ which sends the new vertex in T/e (obtained by merging the two vertices attached to e) into the appropriate composition of these two vertices in $\Omega(T)$. An arrow $\partial_e : T/e \rightarrow T$ in Ω of this kind is called an *inner face (map)*. For example



(iv) Given two trees T and T' , any isomorphism $T \rightarrow T'$ of trees, sending inputs to inputs and output to output, of course defines an isomorphism of operads $\Omega(T) \rightarrow \Omega(T')$, and hence is an isomorphism $T \xrightarrow{\cong} T'$ in Ω . For example, if C_n denotes the corolla with just one vertex, n inputs, and one output, then we might name its input edges e_1, \dots, e_n



Any permutation $\varphi \in \Sigma_n$ defines an automorphism of C_n in Ω .

For a tree T , let its *degree* $|T|$ be the number of vertices in T . Then degeneracy maps decrease degree by 1, face maps (outer or inner) increase degree by 1, and isomorphisms preserve degree. Any map $T \xrightarrow{f} T'$ in Ω can be written as $f = \delta\varphi\sigma$, where δ is a composition of (inner or outer) faces, φ is an isomorphism, and σ is a composition of degeneracies. (This composition is unique up to isomorphism).

4. SKELETAL FILTRATION

As for any presheaf category, any dendroidal set X is a colimit of representables, of the form

$$X = \varinjlim \Omega[T]$$

(see [M]). We wish to refine this a little, in a way similar to the skeletal filtration for simplicial sets. To this end, call a dendrex $x \in X_T$ of shape T *degenerate* if there is a surjective map $T \xrightarrow{\alpha} T'$ in Ω (a composition of degeneracies) such that $x = \alpha^*(x')$ for some $x' \in X_{T'}$. Here α should not be an isomorphism of course, so that T' has strictly fewer vertices than T and α is a non-empty composition of degeneracies.

Given a dendroidal set X we denote by $Sk_n(X)$ the sub dendroidal set of X generated by all non-degenerate dendrices $x \in X_T$ where $|T| \leq n$. An arbitrary dendroidal set X is clearly the colimit (union) of the sequence

$$Sk_0(X) \subseteq Sk_1(X) \subseteq Sk_2(X) \subseteq \dots \tag{1}$$

We call this the skeletal filtration of X . This filtration extends the skeletal filtration for simplicial sets in the precise sense that for any dendroidal set X and any simplicial set S , there are canonical isomorphisms

$$i^* Sk_n(X) = Sk_n(i^* X)$$

and

$$i_! Sk_n(S) = Sk_n(i_! S).$$

Consider now the following diagram:

$$\begin{array}{ccc} \coprod_{x,T} \partial\Omega[T] & \longrightarrow & Sk_n(X) \\ \downarrow & & \downarrow \\ \coprod_{x,T} \Omega[T] & \longrightarrow & Sk_{n+1}(X) \end{array}$$

where the sum is taken over isomorphism classes of pairs (x, T) in the category of elements of X where T is a tree with n vertices and $x \in X_T$ is non-degenerate, and $\partial\Omega[T]$ is the boundary of $\Omega[T]$, i.e., the union of its faces. We call the skeletal filtration of X *normal* if this square is a pushout for each $n > 0$.

Following Cisinski [C] we call a dendroidal set *normal* if for each non-degenerate dendrex $x \in X_T$, the only isomorphism fixing x is the identity. Cisinski (loc. cit.) proves that the normal dendroidal sets are precisely those whose skeletal filtrations are normal.

Example 4.1. If X is a simplicial set then $i_1(X)$ admits a normal skeletal filtration and in fact that skeletal filtration is isomorphic to the usual skeletal filtration of X . If \mathcal{P} is a Σ -free operad then $N_d(\mathcal{P})$ is normal. In particular if \mathcal{P} is the symmetrization of a planar operad then $N_d(\mathcal{P})$ is normal.

5. INNER KAN COMPLEXES

We begin by introducing inner horns. For a tree T , each face map $\partial : T' \rightarrow T$ defines a monomorphism $\Omega[T'] \rightarrow \Omega[T]$ between (representable) dendroidal sets. The union (pushout) of these subobjects is the boundary of $\Omega[T]$, denoted

$$\partial\Omega[T] \twoheadrightarrow \Omega[T],$$

as above. If e is an inner edge of T , then the union of all the faces *except*

$$\partial_e : T/e \twoheadrightarrow T$$

defines a subobject of the boundary, denoted

$$\Lambda^e[T] \twoheadrightarrow \Omega[T],$$

and called the *inner horn* associated to e (and to T). This terminology and notation extends the one

$$\Lambda^k[n] \twoheadrightarrow \partial\Delta[n] \twoheadrightarrow \Delta[n]$$

for simplicial sets, in the sense that

$$i_1(\Lambda^k[n]) = \Lambda^k[i[n]]$$

$$i_1(\partial\Delta[n]) = \partial\Omega[i[n]]$$

as subobjects of $i_1(\Delta[n]) = \Omega[i[n]]$.

A dendroidal set K is said to be a (dendroidal) *inner Kan complex* if, for any tree T and any inner edge e in T , the map

$$K_T = Hom(\Omega[T], K) \rightarrow Hom(\Lambda^e[T], K)$$

is a surjection of sets. It is called a *strict* inner Kan complex if this map is a bijection (for any T and e as above). For example, we will see (Proposition 5.3 below) that the dendroidal nerve of an operad is always a strict inner Kan complex. This terminology is analogous to the one introduced by Boardman and Vogt, who say a simplicial set X satisfies the restricted Kan condition if, for any $0 < k < n$, the map $Hom(\Delta[n], X) \rightarrow Hom(\Delta^k[n], X)$ is a surjection ([BV] Definition 4.8, page 102). In more recent work ([J1, J2]) Joyal develops the general theory of simplicial sets satisfying the restricted Kan condition. Joyal uses the terminology quasi-categories for such simplicial sets so as to stress the analogy with category theory. In fact a simplicial set X is a quasi-category iff $i_1(X)$ is a dendroidal inner Kan complex, and for any dendroidal inner Kan complex K , the restriction $i^*(K)$ is a quasi-category in the sense of Joyal.

Let us call a map $u : U \rightarrow V$ of dendroidal sets an *anodyne extension* if it can be obtained from the set of inner horn inclusions by coproducts, pushouts, compositions, and retracts (cf, [GZ], p. 60). Then obviously, the surjectivity property for inner Kan complexes extends to anodyne extensions, in the sense that the map of sets

$$u^* : Hom(V, K) \rightarrow Hom(U, K),$$

given by composition with u , is again surjective. Similarly, the map u^* is a bijection for any strict inner Kan complex.

For a tree T let $I(T)$ be the set of inner edges of T . For a non-empty subset $A \subseteq I(T)$ let $\Lambda^A[T]$ be the union of all faces of $\Omega[T]$ except those obtained by contracting an edge from A . Note that if $A = \{e\}$ then $\Lambda^A[T] = \Lambda^e[T]$.

Lemma 5.1. *For any non-empty $A \subseteq I(T)$ the inclusion $\Lambda^A[T] \rightarrow \Omega[T]$ is anodyne.*

Proof. By induction on $n = |A|$. If $n = 1$ then $\Lambda^A[T] \rightarrow \Omega[T]$ is an inner horn inclusion, thus anodyne. Assume the proposition holds for $n < k$ and suppose $|A| = k$. Choose an arbitrary $e \in A$ and put $B = A \setminus \{e\}$. The map $\Lambda^A[T] \rightarrow \Omega[T]$ factors as

$$\begin{array}{ccc} \Lambda^A[T] & \longrightarrow & \Lambda^B[T] \\ & \searrow & \downarrow \\ & & \Omega[T] \end{array}$$

The vertical map is anodyne by the induction hypothesis and it therefore suffices to prove that $\Lambda^A[T] \rightarrow \Lambda^B[T]$ is anodyne. The following diagram expresses that map as a pushout

$$\begin{array}{ccc} \Lambda^B[T/e] & \longrightarrow & \Lambda^A[T] \\ \downarrow & & \downarrow \\ \Omega[T/e] & \longrightarrow & \Lambda^B[T] \end{array}$$

and since the map $\Lambda^B[T/e] \rightarrow \Omega[T/e]$ is anodyne (by the induction hypothesis), the proof is complete. \square

We denote by $\Lambda^I[T]$ the dendroidal set $\Lambda^A[T]$ where $A = I(T)$, that is $\Lambda^I[T]$ is the union of all outer faces of $\Omega[T]$. By the above proposition the inclusion $\Lambda^I[T] \rightarrow \Omega[T]$ is anodyne.

We now consider grafting of trees. For two trees T and S , and a leaf l of T , let $T \circ_l S$ be the tree obtained by grafting S onto T by identifying l with the root (output edge) of S . Then there are obvious inclusions $\Omega[S] \rightarrow \Omega[T \circ_l S]$ and $\Omega[T] \rightarrow \Omega[T \circ_l S]$, the pushout (union) of which we denote by $\Omega[T] \cup_l \Omega[S] \rightarrow \Omega[T \circ_l S]$.

Lemma 5.2. (*Grafting*) *For any two trees T and S and any leaf l of T , the inclusion $\Omega[T] \cup_l \Omega[S] \rightarrow \Omega[T \circ_l S]$ is anodyne.*

Proof. Let us write $R = T \circ_l S$. The case where $T = \eta$ or $S = \eta$ is trivial, we therefore assume that this is not the case. We proceed by induction on $n = |T| + |S|$, the sum of the degrees of T and S . The cases $n = 0$ or $n = 1$ are taken care of by our assumption that $T \neq \eta \neq S$. For the case $n = 2$ the same assumption implies that the inclusion $\Omega[T] \cup_l \Omega[S] \rightarrow \Omega[R]$ is an inner horn inclusion. In any case it is anodyne. Assume then that the result holds for $2 \leq n < k$ and suppose $|T| + |S| = k$.

Recall that $\Lambda^I[R]$ is the union of all the outer faces of $\Omega[R]$. First notice that $\Omega[T] \cup_l \Omega[S] \rightarrow \Omega[R]$ factors as

$$\begin{array}{ccc} \Omega[T] \cup_l \Omega[S] & \longrightarrow & \Lambda^I[R] \\ & \searrow & \downarrow \\ & & \Omega[R] \end{array}$$

and the vertical arrow is anodyne by a previous result. We now show that

$$\Omega[T] \cup_l \Omega[S] \rightarrow \Lambda^I[R]$$

is anodyne by exhibiting it as a pushout of an anodyne extension. Recall ([MW]) that an external cluster is a vertex v with the property that one of the edges adjacent to it is inner while all the other edges adjacent to it are outer. Let $Cl(T)$ (resp. $Cl(S)$) be the set of all external clusters in T (resp. S) which do not contain l (resp. the root of S). For each $C \in Cl(T)$ the face of $\Omega[R]$ corresponding to C is isomorphic to $\Omega[(T/C) \circ_l S]$ and the map $\Omega[T/C] \cup_l \Omega[S] \rightarrow \Omega[(T/C) \circ_l S]$ is anodyne by the induction hypothesis. Similarly for every $C \in Cl(S)$ the face of $\Omega[R]$ that corresponds to C is isomorphic to $\Omega[T \circ_l (S/C)]$ and the map $\Omega[T] \cup_l \Omega[S/C] \rightarrow \Omega[T \circ_l (S/C)]$ is anodyne by the induction hypothesis. The following diagram is a pushout

$$\begin{array}{ccc} \coprod_{C \in Cl(T)} (\Omega[T/C] \cup_l \Omega[S]) \amalg \coprod_{C \in Cl(S)} (\Omega[T] \cup_l \Omega[S/C]) & \longrightarrow & \Omega[T] \cup_l \Omega[S] \\ \downarrow & & \downarrow \\ \coprod_{C \in Cl(T)} (\Omega[(T/C) \circ_l S]) \amalg \coprod_{C \in Cl(S)} (\Omega[T \circ_l (S/C)]) & \longrightarrow & \Lambda^I[R] \end{array}$$

where the map on the left is the coproduct of all of the anodyne extensions just mentioned. Since anodyne extensions are closed under coproducts, it follows that the map on the left of the pushout is anodyne and thus also the one on the right, which is what we set out to prove. This concludes the proof. \square

We end this section with two remarks on strict inner Kan complexes.

Proposition 5.3. *The dendroidal nerve of any operad is a strict inner Kan complex.*

Proof. Let \mathcal{P} be an operad. A dendrex $x \in N_d(\mathcal{P})_T$ is a map $x : \Omega[T] \rightarrow N_d(\mathcal{P})$ which is a map of operads $\Omega(T) \rightarrow \mathcal{P}$. If we choose a planar representative for T then $\Omega(T)$ is specifically given in terms of generators and is a free operad. It follows that x is equivalent to a labeling of the (planar representative) T as follows. The edges are labeled by colours of \mathcal{P} and the vertices are coloured by operations in \mathcal{P} where the input of such an operation is the tuple of labels of the incoming edges to the vertex and the output is the label of the outgoing edge from the vertex. Any inner horn $\Lambda^\epsilon[T] \rightarrow N_d(\mathcal{P})$ is easily seen to be equivalent to such a labeling of the tree T and thus determines a unique filler. \square

Proposition 5.4. *Any strict inner Kan complex is 2-coskeletal.*

Proof. Let X be a strict inner Kan complex. Let Y be any dendroidal set and assume $f : Sk_2 Y \rightarrow Sk_2 X$ is given. We first show that f can be extended to a dendroidal map $\hat{f} : Y \rightarrow X$. Suppose f was extended to a map $f_k : Sk_k Y \rightarrow Sk_k X$ for $k \geq 2$. Let $y \in Sk_{k+1}(Y)$ be a non-degenerate dendrex and assume $y \notin Sk_k(Y)$. So $y \in Y_T$ and T has exactly $k+1$ vertices. Choose an inner horn $\Lambda^\alpha[T]$ (such an inner horn exist since $k \geq 2$). The set $\{\beta^* y\}_{\beta \neq \alpha}$ where β runs over all faces of T , defines a horn $\Lambda^\alpha[T] \rightarrow Y$. Since this horn factors through the k -skeleton of Y we obtain, by applying f_k , a horn $\Lambda^\alpha[T] \rightarrow X$ in X given by $\{f(\beta^* y)\}_{\beta \neq \alpha}$. Let $f_{k+1}(y) \in X_T$ be the unique filler of that horn. By construction we have for each $\beta \neq \alpha$ that

$$\beta^* f_{k+1}(y) = f(\beta^* y)$$

it thus remains to show the same for α . The dendrices $f(\alpha^* y)$ and $\alpha^* f_{k+1}(y)$ both have the same boundary and they are both of shape S where S has k vertices. Since $k \geq 2$, S has an inner face, but then it follows that both $f(\alpha^* y)$ and $\alpha^* f_{k+1}(y)$ are fillers for the same inner horn in X which proves that they are equal. By repeating the process for all dendrices in $Sk_{k+1}(Y)$ it follows that f_k can be extended to $f_{k+1} : Sk_{k+1}(Y) \rightarrow Sk_{k+1}(X)$. This holds for all $k \geq 2$ which implies that f can be extended to $\hat{f} : Y \rightarrow X$. To show uniqueness of \hat{f} assume that g is another extension of f . Suppose it has been shown that \hat{f} and g agree on all dendrices of shape T where T has at most k vertices, and let $y \in X_S$ be a dendrex of shape S where S has $k+1$ vertices. But then the dendrices $\hat{f}(y)$ and $g(y)$ are dendrices in X that have the same boundary. Since $k \geq 2$ it follows that these dendrices are both fillers for the same inner horn and so are the same. This proves that $\hat{f} = g$. \square

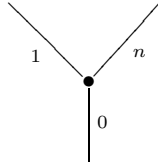
6. THE OPERAD GENERATED BY AN INNER KAN COMPLEX

We recall that $\tau_d : dSet \rightarrow Operad$ denotes the left adjoint to the dendroidal nerve functor N_d . In this section, we will give a more explicit description of the operad $\tau_d(X)$ in the case where X is an inner Kan complex. This description extends the one in [BV] of the category generated by a simplicial set satisfying the restricted Kan condition. It will lead to a proof of the following converse of Proposition 5.3.

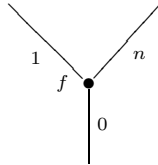
Theorem 6.1. *For any strict inner Kan complex X , the canonical map $X \rightarrow N_d(\tau_d(X))$ is an isomorphism.*

Proposition 5.3 and Theorem 6.1 together state that a dendroidal set is a strict inner Kan complex iff it is the nerve of an operad.

Consider an inner Kan complex X . For the description of $\tau_d(X)$, we first fix some notation. For each $n \geq 0$ let C_n be the n -corolla:

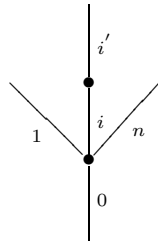


and for each $0 \leq i \leq n$ recall that $i : \eta \rightarrow C_n$ denotes the obvious (outer face) map in Ω that sends the unique edge of η to the edge i in C_n . An element $f \in X_{C_n}$ will be denoted by

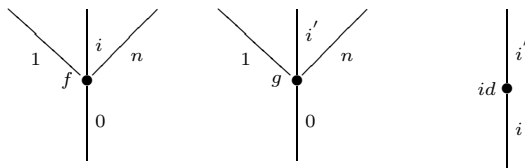


If C'_n is another n -corolla together with an isomorphism $\alpha : C'_n \rightarrow C_n$ then we will usually write f again instead of $\alpha^*(f)$. We will use this convention quite often in the coming definitions and constructions, and in each case there will be an obvious choice for the isomorphism α given by the planar representation of the trees at question, which will usually not be mentioned.

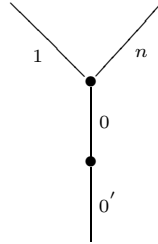
Definition 6.2. Let X be an inner Kan complex and let $f, g \in X_{C_n}$, $n \geq 0$. For $1 \leq i \leq n$ we say that f is homotopic to g along the edge i , and write $f \sim_i g$, if there is a dendrex H of shape



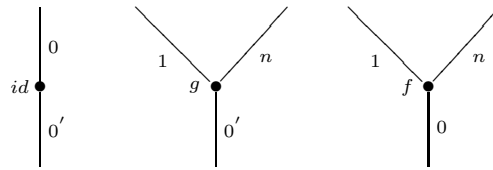
whose three faces are:



where the third one denotes a degeneracy. Similarly we will say that f is homotopic to g along the edge 0 and write $f \sim_0 g$ if there is a dendrex of shape



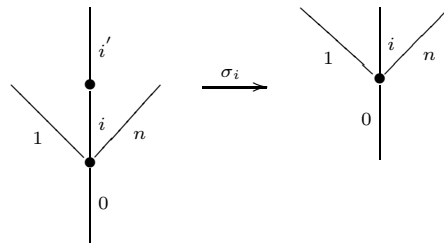
whose three faces are:



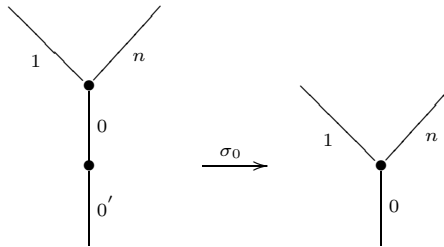
When $f \sim_i g$ for some $0 \leq i \leq n$ we will refer to the corresponding H as a *homotopy* from f to g along i and will sometimes write $H : f \sim_i g$.

Proposition 6.3. *Let X be an inner Kan complex. For each $0 \leq i \leq n$ the relation \sim_i on the set X_{C_n} is an equivalence relation.*

Proof. First we prove reflexivity. For $1 \leq i \leq n$ let

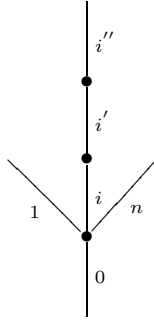


and for $i = 0$ let



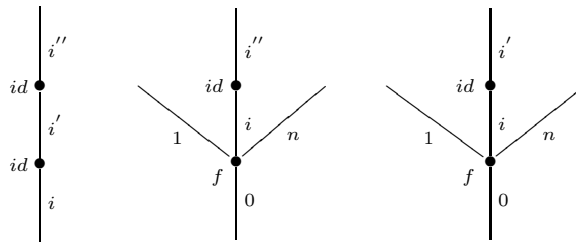
be the obvious degeneracies. It then follows that for any $f \in X_{C_n}$ the dendrex $\sigma_i^*(f)$ is a homotopy from f to f , thus $f \sim_i f$.

To prove symmetry assume $f \sim_i g$ for some $1 \leq i \leq n$ and let H_{fg} be a homotopy from f to g along i . Consider the tree T :

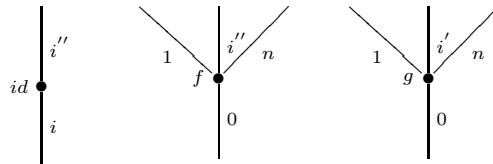


For the inner horn $\Lambda^i[T]$, corresponding to the edge i in the tree above, we now describe a map $\Lambda^i[T] \rightarrow X$. Such a map is given by specifying three dendrices in X of certain shapes such that their faces match in a suitable way. We describe this map by explicitly writing the mentioned dendrices and their faces:

H_i H_f H_{fg}

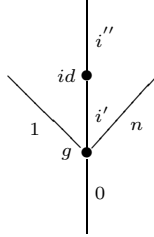


with inner faces of these dendrices:

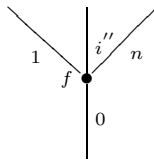


where H_i is a double degeneracy of i , H_f is a homotopy from f to f (along the branch i) and H_{fg} is the given homotopy from f to g . It is easily checked that the faces indeed match so that we have a horn in X . Let x be a filler for that horn and

consider $H_{gf} = \partial_i^*(x)$. This dendrex can be pictured as

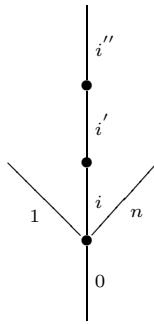


with inner face:



and is thus a homotopy from g to f along i , so that $g \sim_i f$. For $i = 0$ a similar proof works.

To prove transitivity let $f \sim_i g$ and $g \sim_i h$ for $1 \leq i \leq n$. Let H_{fg} be a homotopy from f to g and let H_{gh} be a homotopy from g to h . We again consider the tree T :

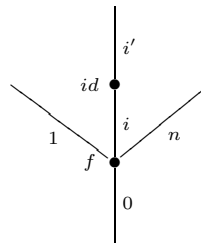
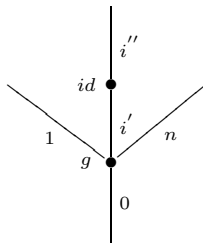
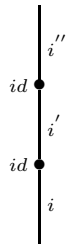


This time we look at $\Lambda^{i'}[T]$. The following describes a map $\Lambda^{i'}[T] \rightarrow X$ in X :

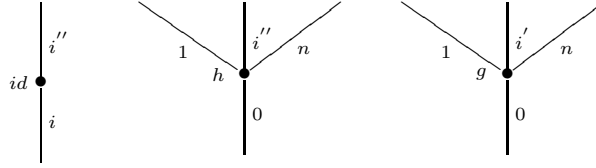
H_i

H_{gh}

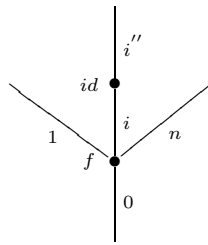
H_{fg}



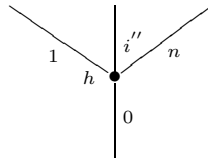
with inner faces being:



Let x be a filler for that horn and let $H_{fh} = \partial_i^*(x)$. This dendrex can be pictured as follows:



with inner face:

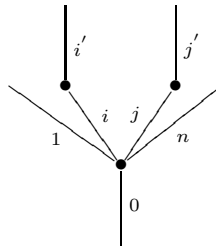


and is thus a homotopy from f to h so that $f \sim_i h$. The proof for $i = 0$ is similar. \square

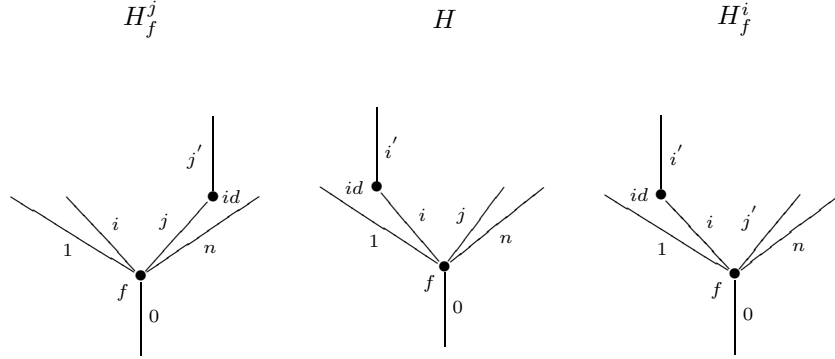
Lemma 6.4. *Let X be an inner Kan complex. The relations \sim_0, \dots, \sim_n on X_{C_n} are all equal.*

Remark 6.5. On the basis of this lemma, we will later just write $f \sim g$ instead of $f \sim_i g$.

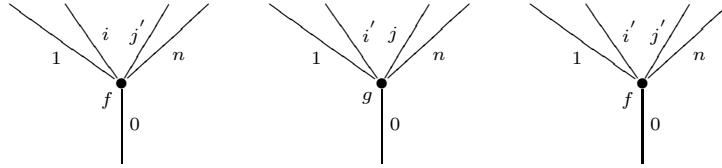
Proof. Suppose $H : f \sim_i g$ for $1 \leq i \leq n$ and let $1 \leq i < j \leq n$. We consider the tree T :



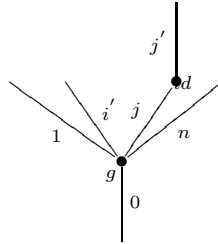
and the inner horn $\Lambda^i[T]$. The following then describes a map $\Lambda^i[T] \rightarrow X$:



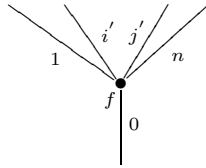
where $H_f^j : f \sim_j f$ and $H_f^i : f \sim_i f$. The inner faces of the three dendrices are



Let x be a filler for this horn, then $\partial_i^*(x)$ is the following dendrex



with inner face:



and is thus a homotopy from g to f along the j -th branch. Thus $g \sim_j f$ and so $f \sim_j g$ as well. The other cases to be considered follow in a similar way. \square

Given an inner Kan complex X and vertices $x_1, \dots, x_n, x \in X_\eta$, let us write

$$X(x_1, \dots, x_n; x) \subseteq X(C_n)$$

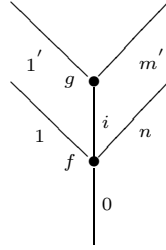
for the set of dendrices x of shape C_n with $0^*(x) = x$ and $i^*(x) = x_i$ for $i = 0, \dots, n$. Here $i : \eta \rightarrow C_n$ denotes the map in Ω sending the unique edge of η to the one of C_n with name i . The equivalence relation \sim on $X(C_n)$ given by the preceding lemma defines a quotient of $X(C_n)$ which we will denote by

$$Ho(X)(x_1, \dots, x_n; x) = X(x_1, \dots, x_n; x) / \sim .$$

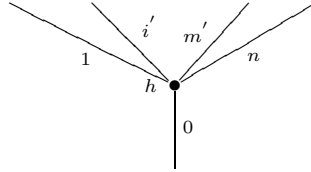
This defines a coloured collection $Ho(X)$, and a canonical quotient map of collections $Sk_1(X) \rightarrow Ho(X)$. We will now proceed to prove the following.

Proposition 6.6. *There is a unique structure of a (symmetric, coloured) operad on $Ho(X)$ for which the map of collections $Sk_1(X) \rightarrow Ho(X)$ extends to a map of dendroidal sets $X \rightarrow N_d(Ho(X))$. The latter map is an isomorphism whenever X is a strict inner Kan complex.*

To prepare for the proof of this proposition, we begin by defining the composition operations \circ_i of the operad $Ho(X)$. Let X be an inner Kan complex and let $f \in X_{C_n}$ and $g \in X_{C_m}$ be two dendrices in X . We will say that a dendrex $h \in X_{C_{n+m-1}}$ is a \circ_i -composition of f and g if there is a dendrex γ in X as follows:



with inner face

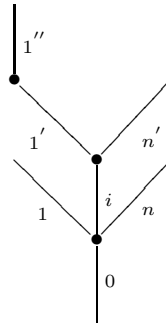


We will denote this situation by $h \sim f \circ_i g$ and call γ a *witness* for the composition.

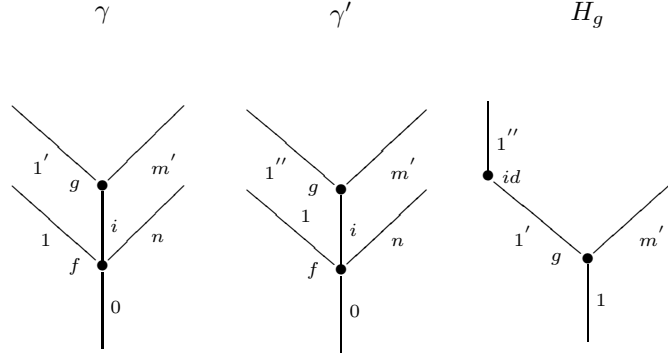
Remark 6.7. Notice that for $1 \leq i \leq n$ we have by definition that $H : f \sim_i g$ iff H is a witness for the composition $g \sim f \circ_i id$. Similarly for $i = 0$ we have that $H : f \sim_0 g$ iff H is a witness for the composition $g \sim id \circ f$.

Lemma 6.8. *In an inner Kan complex X , if $h \sim f \circ_i g$ and $h' \sim f \circ_i g$ then $h \sim h'$.*

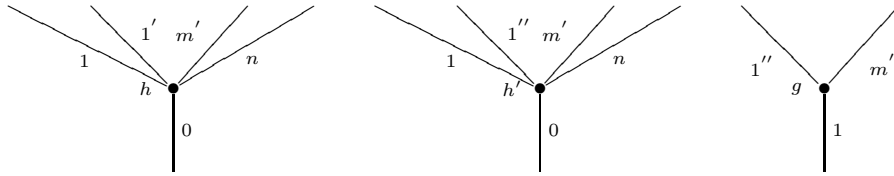
Proof. Let γ be a witness for the composition $h \sim f \circ_i g$ and γ' one for the composition $h' \sim f \circ_i g$. We consider the tree T :



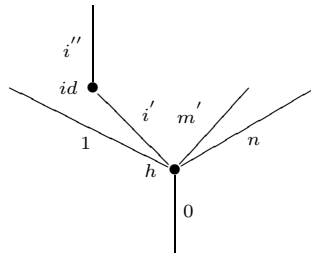
and the inner horn $\Lambda^i[T]$. Let $H_g : g \sim_i g$ and consider the following map $\Lambda^i[T] \rightarrow X$



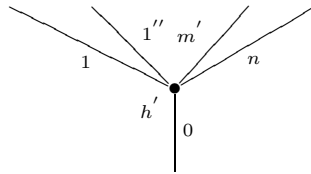
with inner faces



Let x be a filler for this horn. The face $\partial_i^* x$ is then the dendrex



whose inner face is

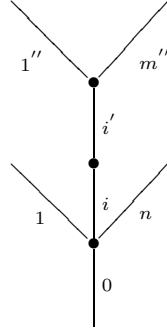


which proves that $h \sim h'$. □

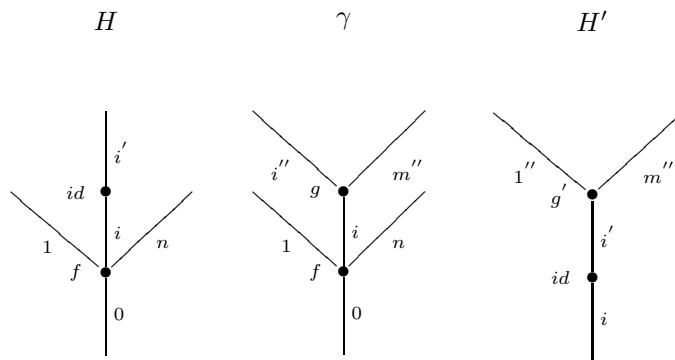
Lemma 6.9. *In an inner Kan complex X , let $f \sim f'$ and $g \sim g'$. If $h \sim f \circ_i g$ and $h' \sim f' \circ_i g'$ then $h \sim h'$.*

Proof. Let H be a homotopy from f to f' along the edge i , H' a homotopy from g' to g along the root, and γ a witness for the composition $h \sim f \circ_i g$. We now

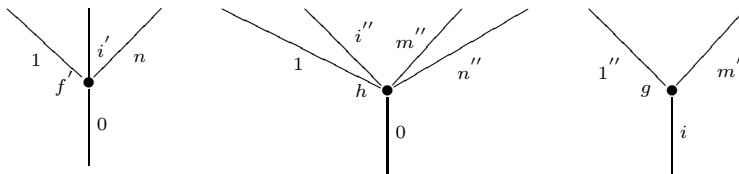
consider the tree T :



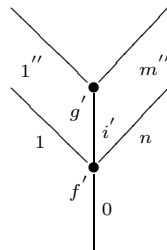
and the inner horn $\Lambda^i[T]$. The following is then a map $\Lambda^i[T] \rightarrow X$ in X :



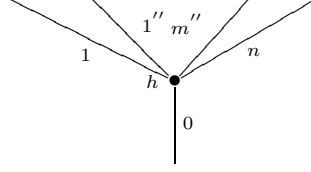
with inner faces:



The missing face of a filler for this horn is then:



with inner face



which proves that $h \sim f' \circ_i g'$, and thus by the previous result also that $h \sim h'$. \square

We now proceed to prove Proposition 5.6:

Proof. (of Proposition 5.6) Lemma 5.8 implies that for

$$[f] \in Ho(X)(x_1, \dots, x_n; x)$$

and

$$[g] \in Ho(X)(y_1, \dots, y_m; x_i)$$

the assignment

$$[f] \circ_i [g] = [f \circ_i g]$$

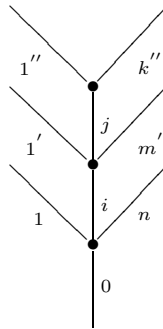
is well-defined. This provides the \circ_i operations in the operad $Ho(X)$. The Σ_n actions are defines as follows. Given a permutation $\sigma \in \Sigma_n$ let $\sigma : C_n \rightarrow C_n$ be the obvious induced map in Ω . The map $\sigma^* : X_{C_n} \rightarrow X_{C_n}$ restricts to a function

$$\sigma^* : X(x_1, \dots, x_n; x) \rightarrow X(x_{\sigma(1)}, \dots, x_{\sigma(n)}; x)$$

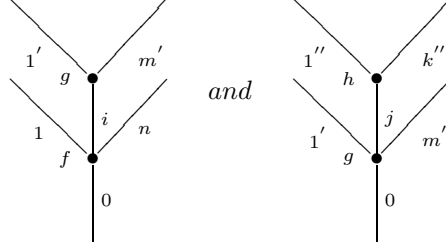
and it is trivial to verify that this map respects the homotopy relation. We thus obtain a map

$$\sigma^* Ho(X)(x_1, \dots, x_n; x) \rightarrow Ho(X)(x_{\sigma(1)}, \dots, x_{\sigma(n)}; x).$$

We now need to show that these structure maps make the coloured collection $Ho(X)$ into an operad. The verification is simple and we exemplify it by proving associativity. Let $[f] \in Ho(X)(x_1, \dots, x_n; x)$, $[g] \in Ho(X)(y_1, \dots, y_m; x_i)$ and $[h] \in Ho(X)(z_1, \dots, z_k; y_m)$. We need to prove that $[f] \circ ([g] \circ [h]) = ([f] \circ [g]) \circ [h]$ (with the obvious choice for subscripts on the \circ) which is the same as showing that $f \circ (g \circ h) \sim (f \circ g) \circ h$ for any choice for compositions $\psi \sim g \circ h$ and $\varphi \sim f \circ g$. Consider the tree T given by



and consider the anodyne extension $\Lambda^I[T] \rightarrow \Omega[T]$, cf. Lemma 5.1. The two given compositions $\psi \sim g \circ h$ and $\varphi \sim f \circ g$ define a map $\Lambda^I[T] \rightarrow X$ depicted by



whose inner faces are respectively ψ and φ . Let $x \in X_T$ be a dendrex extending this map. Let $c : C_m \rightarrow T$ be the map obtained by contracting both i and j and $\rho = c^*x$. It now follows that ∂_i^*x is a witness for the composition $\rho \sim \psi \circ h$ and ∂_j^*x is a witness for the composition $\rho \sim f \circ \varphi$. That proves associativity. The other axioms for an operad follow in a similar manner.

Next, let us show that the quotient map $q : Sk_1(X) \rightarrow Ho(X)$ extends to a map $q : X \rightarrow N_d(Ho(X))$ of dendroidal sets. Since we already know that $N_d(X)$ is 2-coskeletal, it suffices to give its values for dendrices $x \in X_T$ where T is a tree with two vertices. Let e be the inner edge of this tree. Then $\Lambda^e[T] \twoheadrightarrow \Omega[T] \xrightarrow{x} X$ factors through $Sk_1(X)$, so its composition $\Lambda^e[T] \rightarrow N_d(Ho(X))$ with q has a unique extension (Proposition 5.3), which we take to be $q(x) : \Omega[T] \rightarrow N_d(Ho(X))$. This defines $q : Sk_2(X) \rightarrow Sk_2(N_d(Ho(X)))$, and hence all of $q : X \rightarrow N_d(Ho(X))$ by 2-coskeletality, as said.

Finally, when X is itself a strict inner Kan complex, then the homotopy relation is the identity relation, so $Sk_1(X) \rightarrow Ho(X)$ is the identity map. Since X and $N_d(Ho(X))$ are now both strict inner Kan complexes, the extension $q : X \rightarrow N_d(Ho(X))$ is an isomorphism. \square

The following Proposition, together with Proposition 6.6, now provide the proof of Theorem 6.1.

Proposition 6.10. *For any inner Kan complex X , the natural map $\tau_d(X) \rightarrow Ho(X)$ is an isomorphism of operads.*

Proof. It suffices to prove that the map $q : X \rightarrow N_d(Ho(X))$ of Proposition 6.6 has the universal property of the unit of the adjunction. This means that for any operad \mathcal{P} and any map $\varphi : X \rightarrow N_d(\mathcal{P})$, there is a unique map of operads $\psi : Ho(X) \rightarrow \mathcal{P}$ for which $N_d(\psi)q = \varphi$. But φ induces a map $Ho(X) \rightarrow Ho(N_d(\mathcal{P}))$ for which

$$\begin{array}{ccc} Sk_1(X) & \xrightarrow{\varphi} & Sk_1 N_d(\mathcal{P}) \\ \downarrow q_X & & \downarrow q_{\mathcal{P}} \\ Ho(X) & \xrightarrow{Ho(\varphi)} & Ho(N_d(\mathcal{P})) \end{array}$$

commutes, and $Ho(N_d(\mathcal{P})) = \mathcal{P}$ while $q_{\mathcal{P}}$ is the identity as we have seen in (the proof of) Proposition 6.6. So $Ho(\varphi)$ in fact defines a map $\psi : Ho(X) \rightarrow \mathcal{P}$ of collections. It is easily seen that ψ is a map of operads. It is unique because q_X is surjective. \square

7. HOMOTOPY COHERENT NERVES OF OPERADS

In this section, we assume \mathcal{E} is a monoidal model category with a cofibrant unit I . We also assume that \mathcal{E} is equipped with an *interval* in the sense of [BM1]. Such an interval is given by maps

$$I \begin{array}{c} \xrightarrow{0} \\ \rightrightarrows \\ \xrightarrow{1} \end{array} H \xrightarrow{\epsilon} I$$

and

$$H \otimes H \xrightarrow{\vee} H$$

satisfying certain conditions. In particular, H is an interval in Quillen’s sense ([Q]), so 0 and 1 together define a cofibration $I \amalg I \rightarrow H$, and ϵ is a weak equivalence. Such an interval H allows one to construct for each (coloured) operad \mathcal{P} in \mathcal{E} a ”Boardman-Vogt” resolution $W_H(\mathcal{P}) \rightarrow \mathcal{P}$. Each operad in Set can be viewed as an operad in \mathcal{E} (via the functor $Set \rightarrow \mathcal{E}$ which preserves coproducts and sends the one-point set to I), and hence has such a Boardman-Vogt resolution. When we apply this to the operads $\Omega(T)$, we obtain the *homotopy coherent dendroidal nerve* $hcN_d(\mathcal{P})$ of any operad \mathcal{P} in \mathcal{E} , as the dendroidal set given by

$$hcN_d(\mathcal{P})_T = Hom(W_H(\Omega(T)), \mathcal{P})$$

where the Hom is that of operads in \mathcal{E} . See [MW] for a more detailed description and examples. Our goal here is to prove the following result.

Theorem 7.1. *Let \mathcal{P} be an operad in \mathcal{E} , with the property that for each sequence $c_1, \dots, c_n; c$ of colours of \mathcal{P} , the object $\mathcal{P}(c_1, \dots, c_n; c)$ is fibrant. Then $hcN_d(\mathcal{P})$ is an inner Kan complex.*

Remark 7.2. As explained in [MW], our construction of the dendroidal homotopy coherent nerve specializes to that of the homotopy coherent nerve of an \mathcal{E} -enriched category, and for the case where \mathcal{E} is the category of topological spaces or simplicial sets, one recovers the classical definition ([CP]). In particular, as a special case of Theorem 7.1, one obtains that for an \mathcal{E} -enriched category with fibrant Hom objects (in other words, for a locally fibrant \mathcal{E} -enriched category), its homotopy coherent nerve is a quasi-category in the sense of Joyal. This result was proved, for the case where \mathcal{E} is simplicial sets, by Cordier and Porter in [CP].

Before embarking on the proof of Theorem 7.1, we need to be a bit more explicit about the operads of the form $W_H\Omega(T)$ involved in the definition of the homotopy coherent nerve. Recall first of all the functor

$$Symm : Operad(\mathcal{E})_\pi \rightarrow Operad(\mathcal{E})$$

which is left adjoint to the forgetful functor from symmetric operads to non-symmetric (i.e., planar) ones. If T is an object in Ω and \bar{T} is a chosen planar representative of T , then \bar{T} naturally describes a planar operad $\Omega(\bar{T})$ for which $\Omega(T) = Symm(\Omega(\bar{T}))$. Since the W -construction commutes with symmetrization (as one readily verifies), it follows that

$$W_H(T) = Symm(W_H\Omega(\bar{T})).$$

This latter operad $W_H\Omega(\bar{T})$ is easily described explicitly. The colours of

$$W_H(\Omega(\bar{T}))$$

are the colours of $\Omega(\bar{T})$, i.e., the edges of T . By a *signature*, we mean a sequence $e_1, \dots, e_n; e_0$ of edges. Given a signature $\sigma = (e_1, \dots, e_n; e_0)$, we have that $W_H(\Omega(\bar{T}))(\sigma) = 0$ whenever $\Omega(\bar{T})(\sigma) = \phi$. And if $\Omega(\bar{T})(\sigma) \neq \phi$, there is a subtree T_σ of T (and a corresponding planar subtree \bar{T}_σ of \bar{T}) whose leaves are e_1, \dots, e_n , and whose root is e_0 . Then

$$W_H\Omega(\bar{T})(e_1, \dots, e_n; e_0) = \bigotimes_{f \in i(\sigma)} H,$$

where $i(\sigma)$ is the set of *inner* edges of T_σ (or of \bar{T}_σ). (This last tensor product is to be thought of as the "space" of assignments of lengths to inner edges in \bar{T}_σ ; it is the unit if $i(\sigma)$ is empty.)

Remark 7.3. The composition operations in the operad $W_H\Omega(\bar{T})$ are given in terms of the \circ_i -operations as follows. For signatures $\sigma = (e_1, \dots, e_n; e_0)$ and $\rho = (f_1, \dots, f_m; e_i)$, the composition map

$$\begin{array}{ccc} \Omega(\bar{T})(e_1, \dots, e_n; e_0) \otimes \Omega(\bar{T})(f_1, \dots, f_m; e_i) & & \\ \downarrow \circ_i & & (1) \\ \Omega(\bar{T})(e_1, \dots, e_{i-1}, f_1, \dots, f_m, e_{i+1}, \dots, e_n; e_0) & & \end{array}$$

is the following one. The trees \bar{T}_σ and \bar{T}_ρ can be grafted along e_i to form $\bar{T}_\sigma \circ_{e_i} \bar{T}_\rho$, again a planar subtree of \bar{T} . In fact

$$\bar{T}_\sigma \circ_{e_i} \bar{T}_\rho = \bar{T}_{\sigma \circ_i \rho}$$

where $\sigma \circ_i \rho$ is the signature $(e_1, \dots, e_{i-1}, f_1, \dots, f_m, e_{i+1}, \dots, e_n; e_0)$, and for the sets of inner edges we have

$$i(\sigma \circ_i \rho) = i(\sigma) \cup i(\rho) \cup \{e_i\}.$$

The composition map in (1) now is the map

$$\begin{array}{ccc} H^{\otimes i(\sigma)} \otimes H^{\otimes i(\rho)} & \xrightarrow{\quad} & H^{\otimes i(\sigma \circ_i \rho)} \\ \downarrow \cong & & \downarrow \cong \\ H^{\otimes i(\sigma) \cup i(\rho)} \otimes I & \xrightarrow{id \otimes 1} & H^{\otimes i(\sigma) \cup i(\rho)} \otimes H \end{array}$$

where $1 : I \rightarrow H$ is one of the "endpoints" of the interval H , as above.

This description of the operad $W_H\Omega(\bar{T})$ is functorial in the planar tree T . In particular, we note that for an inner edge e of T , the tree T/e inherits a planar structure \bar{T}/e from \bar{T} , and $W_H\Omega(\bar{T}/e) \rightarrow W_H\Omega(\bar{T})$ is the natural map assigning length 0 to the edge e whenever it occurs (in a subtree given by a signature).

Proof. (Of Theorem 7.1) Consider a tree T and an inner edge e in T . We want to solve the extension problem

$$\begin{array}{ccc} \Lambda^e[T] & \xrightarrow{\varphi} & hcN_d(\mathcal{P}) \\ \downarrow & \nearrow & \\ \Omega[T] & & \end{array}$$

Fix a planar representative \bar{T} of T . Then the desired map $\psi : \Omega[T] \rightarrow hcN_d(\mathcal{P})$ corresponds to a map of planar operads

$$\hat{\psi} : W_H\Omega(\bar{T}) \rightarrow \mathcal{P}.$$

Each face S of T inherits a planar structure \bar{S} from \bar{T} , and the given map $\varphi : \Lambda^e[T] \rightarrow hcN_d(\mathcal{P})$ corresponds to a map of operads in \mathcal{E} ,

$$\hat{\varphi} : W_H(\Lambda^e[T]) \rightarrow \mathcal{P},$$

where $W_H(\Lambda^e[T])$ denotes the colimit of operads in \mathcal{E} ,

$$W_H(\Lambda^e[T]) = \text{colim } W(\Omega(\bar{S})) \quad (2)$$

over all but one of the faces of T . In other words, φ corresponds to a compatible family of maps

$$\hat{\varphi}_S : W_H(\Omega(\bar{S})) \rightarrow \mathcal{P}.$$

Let us now show the existence of an operad map $\hat{\psi}$ extending the $\hat{\varphi}_S$ for all faces $S \neq T/e$. First, the colours of $\Omega(\bar{T})$ are the same as those of the colimit in (2), so we already have a map $\psi_0 = \varphi_0$ on colours:

$$\psi_0 : (\text{Edges of } T) \rightarrow (\text{Colours of } \mathcal{P}).$$

Next, if $\sigma = (e_1, \dots, e_n; e_0)$ is a signature of T for which $W_H(\Omega(\bar{T})) \neq \phi$, and if $T_\sigma \subseteq T$ is not all of T , then T_σ is contained in an outer face S of T . So $W_H(\Omega(\bar{T}))(\sigma) = W_H(\Omega(\bar{T}_\sigma))(\sigma) = W_H(\Omega(\bar{S}))(\sigma)$, and we already have a map

$$\hat{\varphi}_S(\sigma) : W_H(\Omega(\bar{T}))(\sigma) \rightarrow \mathcal{P}(\sigma),$$

given by $\hat{\varphi}_S : W_H(\Omega(\bar{S})) \rightarrow \mathcal{P}$. Thus, the only part of the operad map $\hat{\psi} : W_H(\Omega(\bar{T})) \rightarrow \mathcal{P}$ not determined by φ is the one for the signature τ where $T_\tau = T$; i.e., $\tau = (e_1, \dots, e_n; e_0)$ where e_1, \dots, e_n are all the input edges of \bar{T} (in the planar order) and e_0 is the output edge. For this signature, $\hat{\psi}(\tau)$ is to be a map

$$\hat{\psi} : W_H(\Omega(\bar{T}))(\tau) = H^{\otimes i(\tau)} \rightarrow \mathcal{P}(\tau)$$

which (i) is compatible with the $\hat{\psi}(\sigma) = \hat{\varphi}_S(\sigma)$ for other signatures σ ; and (ii) together with these $\hat{\psi}(\sigma)$ respects operad composition. The first condition determines $\hat{\psi}(\tau)$ on the subobject of $H^{\otimes i(\tau)}$ which is given by a value 0 on one of the tensor-factors marked by an edge e_i other than the given e . The second condition determines $\hat{\psi}(\tau)$ on the subobject of $H^{\otimes i(\tau)}$ which is given by a value 1 on one of the factors. Thus, if we write 1 for the map $I \xrightarrow{1} H$ and $\partial H \xrightarrow{\quad} H$

for the map $I \amalg I \rightarrow H$, and define $\partial H^{\otimes k} \xrightarrow{\quad} H^{\otimes k}$ by the Leibniz rule (i.e., $\partial(A \otimes B) = \partial(A) \otimes B \cup A \otimes \partial(B)$), then the problem of finding $\hat{\psi}(\tau)$ comes down to an extension problem of the form

$$\begin{array}{ccc} \partial(H^{\otimes i(\sigma)-\{e\}} \otimes H) \cup H^{\otimes i(\sigma)-\{e\}} \otimes I & \longrightarrow & \mathcal{P}(\tau) \\ \downarrow & & \uparrow \hat{\psi}(\sigma) \\ H^{\otimes i(\sigma)-\{e\}} \otimes H & \xrightarrow{\cong} & H^{\otimes i(\sigma)} \end{array}$$

This extension problem has a solution, because $\mathcal{P}(\tau)$ is fibrant by assumption, and because the left hand map is a trivial cofibration (by repeated use of the push-out

product axiom for monoidal model categories). This concludes the proof of the theorem. \square

8. GROTHENDIECK CONSTRUCTION FOR DENDROIDAL SETS

Let \mathbb{S} be a Cartesian category. A functor $X : \mathbb{S}^{op} \rightarrow dSet$ is called a *diagram* of dendroidal sets. In [MW] a construction was given of the dendroidal set $\int_{\mathbb{S}} X$. This construction was then applied to the specific diagram of dendroidal sets $X : Set^{op} \rightarrow dSet$, where for a set A , $X(A)$ was the dendroidal set of weak n -categories having A as set of objects. The dendroidal set $\int_{\mathbb{S}} X$ was defined to be the dendroidal set of weak n -categories. Our aim in this section is to prove that for a given diagram of dendroidal sets $X : \mathbb{S}^{op} \rightarrow dSet$, if each $X(S)$ is an inner Kan complex then $\int_{\mathbb{S}} X$ is also an inner Kan complex. For the convenience of the reader we repeat here the definition of $\int_{\mathbb{S}} X$.

It will be convenient to consider dendroidal collections. A dendroidal collection is a collection of sets $X = \{X_T\}_{T \in \Omega}$. Each dendroidal set has an obvious underlying dendroidal collection. A map of dendroidal collections $X \rightarrow Y$ is a collection of functions $\{X_T \rightarrow Y_T\}_{T \in \Omega}$. Given a Cartesian category \mathbb{S} , consider the dendroidal nerve $N_d(\mathbb{S})$ where \mathbb{S} is regarded as an operad via the Cartesian structure. There is a natural way of associating an object of \mathbb{S} with each dendrex of $N_d(\mathbb{S})$. For a tree T in Ω , let $leaves(T)$ be the set of leaves of T , and for a leaf l , write $l : \eta \rightarrow T$ also for the map sending the unique edge in η to l in T . Then, since \mathbb{S} is assumed to have finite products, each dendrex $t \in N_d(\mathbb{S})_T$ defines an object

$$in(t) = \prod_{l \in leaves(T)} l^*(t)$$

in \mathbb{S} . Notice that if $\alpha : S \rightarrow T$ is a composition of face maps, then by using the canonical symmetries and the projections in \mathbb{S} there is a canonical arrow $in(\alpha) : in(t) \rightarrow in(\alpha^*t)$ for any $t \in X_T$.

Definition 8.1. Let $X : \mathbb{S}^{op} \rightarrow dSet$ be a diagram of dendroidal sets. The dendroidal set $\int_{\mathbb{S}} X$ is defined as follows. A dendrex $\Omega[T] \rightarrow \int_{\mathbb{S}} X$ is a pair (t, x) such that $t \in N_d(\mathbb{S})_T$ and x is a map of dendroidal collections

$$x : \Omega[T] \rightarrow \prod_{S \in ob(\mathbb{S})} X(S)$$

satisfying the following conditions. For each $r \in \Omega[T]_R$ (that is an arrow $r : R \rightarrow T$), we demand that $x(r) \in X(in(r^*t))$. Furthermore we demand the following compatibility conditions to hold. For any $r \in \Omega[T]_R$ and any map $\alpha : U \twoheadrightarrow R$ in Ω

$$\alpha^*(x(r)) = X(in(\alpha))x(\alpha^*(r)).$$

Theorem 8.2. Let $X : \mathbb{S}^{op} \rightarrow dSet$ be a diagram of dendroidal sets. If for any $S \in ob(\mathbb{S})$ the dendroidal set $X(S)$ is a (strict) inner Kan complex then so is $\int_{\mathbb{S}} X$.

Proof. Let T be a tree and e an inner edge. We consider the extension problem

$$\begin{array}{ccc} \Lambda^e[T] & \longrightarrow & \int_{\mathbb{S}} X \\ \downarrow & \nearrow & \\ \Omega[T] & & \end{array}$$

The horn $\Lambda^e[T] \rightarrow \int_{\mathbb{S}} X$ is given by a compatible collection $\{(r, x_R) : \Omega[R] \rightarrow \int_{\mathbb{S}} X\}_{R \neq T/e}$. We wish to construct a dendrex $(t, x_T) : \Omega[T] \rightarrow \int_{\mathbb{S}} X$ extending this family. First notice that the collection $\{r\}_{R \neq T/e}$ is an inner horn $\Lambda^e[T] \rightarrow N_d(\mathbb{S})$ (actually this horn is obtained by composing with the obvious projection $\int_{\mathbb{S}} X \rightarrow N_d(\mathbb{S})$ sending a dendrex (t, x) to t). We already know $N_d(\mathbb{S})$ to be an inner Kan complex (actually a strict inner Kan complex) and thus there is a (unique) filler $t \in N_d(\mathbb{S})_T$ for the horn $\{r\}_{R \neq T/e}$. We now wish to define a map of dendroidal collections $x_T : \Omega[T] \rightarrow \coprod_{S \in \text{ob}(\mathbb{S})} X(S)$ that will extend the given maps x_R for $R \neq T/e$. This condition already determines the value of x_T for any dendrex $r : U \rightarrow T$ other than $id : T \rightarrow T$ and $\alpha : T/e \rightarrow T$, since for each such r , the tree U factors through one of the faces $R \neq T/e$. To determine $x_T(id_T)$ and $x_T(\alpha)$ consider the family $\{y_R = x_R(id : R \rightarrow R)\}_{R \neq T/e}$. By definition we have that $y_R \in X(in(r))_R$. For each such R let $\alpha_R : R \rightarrow T$ be the corresponding face map in Ω . Since $\alpha^*t = r$ we obtain the map $in(\alpha_R) : in(r) \rightarrow in(t)$. We can now pull back the collection $\{y_R\}_{R \neq T/e}$ using $X(in(\alpha_R))$ to obtain a collection $\{z_R = X(in(\alpha_R))(y_R)\}_{R \neq T/e}$. This collection is now a horn $\Lambda^e[T] \rightarrow X(in(T))$ (this follows from the compatibility conditions in the definition of $\int_{\mathbb{S}} X$). Since $X(in(t))$ is inner Kan there is a filler $u \in X(in(t))_T$ for that horn. We now define $x_T(id : T \rightarrow T) = u$ and $x_T(\alpha : T/e \rightarrow T) = \alpha^*(u)$. Notice that since e is inner we have that $in(t) = in(\alpha)$ and thus the image of these dendrices are in the correct dendroidal set, namely $X(in(t))$. It follows from our construction that this makes (t, x_T) a dendrex $\Omega[T] \rightarrow \int_{\mathbb{S}} X$ which extends the given horn. This concludes the proof. \square

9. THE EXPONENTIAL PROPERTY

Our aim in this section is to prove the following theorem concerning the closed monoidal structure of dendroidal sets.

Theorem 9.1. *Let K and X be dendroidal sets, and assume X is normal. If K is a (strict) inner Kan complex, then so is $\underline{Hom}_{dSet}(X, K)$.*

The internal Hom here is defined by the universal property, giving a bijective correspondence between maps $Y \otimes X \rightarrow K$ and $Y \rightarrow \underline{Hom}(X, K)$ for any dendroidal set Y , and natural in Y . We recall from Section 2 that \otimes is defined in terms of the Boardman-Vogt tensor product of operads. We remind the reader that for two (coloured) operads \mathcal{P} and \mathcal{Q} with respective sets of colours C and D , this tensor product operad $\mathcal{P} \otimes_{BV} \mathcal{Q}$ has the product $C \times D$ as its set of colours, and is described in terms of generators and relations as follows. The operations in $\mathcal{P} \otimes_{BV} \mathcal{Q}$ are generated by the operations

$$p \otimes d \in \mathcal{P} \otimes_{BV} \mathcal{Q}((c_1, d), \dots, (c_n, d); (c, d))$$

for any $p \in \mathcal{P}(c_1, \dots, c_n; c)$ and any $d \in D$, and

$$c \otimes q \in \mathcal{P} \otimes_{BV} \mathcal{Q}((c, d_1), \dots, (c, d_m); (c, d))$$

for any $q \in \mathcal{Q}(d_1, \dots, d_m; d)$ and any $c \in C$. The *relations* between these state, first of all, that for fixed $c \in C$ and $d \in D$, the maps $p \mapsto p \otimes d$ and $c \mapsto c \otimes q$ are maps of operads. Secondly, there is an *interchange law* stating that, for p and q as above, the composition $p \otimes d(c \otimes q, \dots, c \otimes q)$ in

$$\mathcal{P} \otimes_{BV} \mathcal{Q}((c_1, d_1), \dots, (c_1, d_m), \dots, (c_n, d_1), \dots, (c_n, d_m); (c, d))$$

and $c \otimes q(p \otimes d, \dots, p \otimes d)$ in

$$\mathcal{P} \otimes_{BV} \mathcal{Q}((c_1, d_1), \dots, (c_n, d_1), \dots, (c_1, d_m), \dots, (c_n, d_m); (c, d))$$

are mapped to each other by the obvious permutation $\tau \in \Sigma_{n \times m}$ which puts the two sequences of input colours in the same order. The tensor product of dendroidal sets is then uniquely determined (up to isomorphism) by the fact that it preserves colimits in each variable separately, together with the identity

$$\Omega[S] \otimes \Omega[T] = N_d(\Omega(S) \otimes_{BV} \Omega(T))$$

stated in Section 2, which gives the tensor product of two representable dendroidal sets.

First of all, let us prove that Theorem 9.1 follows by a standard argument from the following proposition.

Proposition 9.2. *For any two objects S and T of Ω , and any inner edge e in S , the map*

$$\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \twoheadrightarrow \Omega[S] \otimes \Omega[T]$$

is an anodyne extension.

In the proposition above, the union is that of subobjects of the codomain, which is the same as the pushout over the intersection $\Lambda^e[S] \otimes \partial\Omega[T]$.

Proof. (of Theorem 9.1 from Proposition 9.2) The theorem states that for any tree S and any inner edge $e \in S$, any map of dendroidal sets

$$\varphi : \Lambda^e[S] \otimes X \rightarrow K$$

extends to some map (uniquely in the strict case)

$$\psi : \Omega[S] \otimes X \rightarrow K.$$

By writing X as the union of its skeleta,

$$X = \varinjlim Sk_n(X)$$

as in Section 4, and using the fact that the skeletal filtration is normal, we can build this extension ψ by induction on n . For $n = 0$, $Sk_0(X)$ is a sum of copies of $\Omega[\eta]$, the unit for the tensor product, so obviously the restriction $\varphi_0 : \Lambda^e[S] \otimes Sk_0(X) \rightarrow K$ extends to a map

$$\psi_0 : \Omega[S] \otimes Sk_0(X) \rightarrow K.$$

Suppose now that we have found an extension $\psi_n : \Omega[S] \otimes Sk_n(X) \rightarrow K$ of the restriction $\varphi_n : \Lambda^e[S] \otimes Sk_n(X) \rightarrow K$. Consider the following diagram:

$$\begin{array}{ccccc}
 \coprod \Lambda^e[S] \otimes \partial\Omega[T] & \longrightarrow & \coprod \Lambda^e[S] \otimes \Omega[T] & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & \Lambda^e[S] \otimes Sk_n(X) & \longrightarrow & \Lambda^e[S] \otimes Sk_{n+1}(X) \\
 & & \downarrow & & \downarrow \\
 \coprod \Omega[S] \otimes \partial\Omega[T] & \longrightarrow & \coprod \Omega[S] \otimes \Omega[T] & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & \Omega[S] \otimes Sk_n(X) & \longrightarrow & \Omega[S] \otimes Sk_{n+1}(X)
 \end{array}$$

In this diagram, the top and bottom faces are pushouts given by the normal skeletal filtration of X . Now inscribe the pushouts U and V in the back and front face, fitting into a square

$$\begin{array}{ccc}
 U & \longrightarrow & \coprod \Omega[S] \otimes \Omega[T] \\
 \downarrow & & \downarrow \\
 V & \longrightarrow & \Omega[S] \otimes Sk_{n+1}(X)
 \end{array}$$

The maps $\psi_n : \Omega[S] \otimes Sk_n(X) \rightarrow K$ and $\varphi_{n+1} : \Lambda^e[S] \otimes Sk_{n+1}(X) \rightarrow K$ together define a map $V \rightarrow K$. So, to find ψ_{n+1} , it suffices to prove that

$$V \twoheadrightarrow \Omega[S] \otimes Sk_{n+1}(X)$$

is anodyne. But, by a diagram chase argument, the square above is a pushout, so in fact, it suffices to prove that $U \twoheadrightarrow \coprod \Omega[S] \otimes \Omega[T]$ is anodyne. The latter map is a sum of copies of anodyne extensions as in the statement of the proposition. \square

Corollary 9.3. *The monoidal structure on the category of coloured operads given by the Boardman-Vogt tensor product is closed (see [MW]). It is related to the closed monoidal structure on dendroidal sets by two natural isomorphisms*

$$\tau_d(N_d\mathcal{P} \otimes N_d\mathcal{Q}) = \mathcal{P} \otimes_{BV} \mathcal{Q}$$

and

$$N_d(\underline{Hom}(\mathcal{Q}, \mathcal{R})) = \underline{Hom}(N_d\mathcal{Q}, N_d\mathcal{R})$$

for any operads \mathcal{P}, \mathcal{Q} and \mathcal{R} .

Proof. The first isomorphism was proved in [MW]. The second isomorphism follows from the first one together with (the strict version of) Theorem 9.1, Theorem 6.1, and the fact that N_d is fully faithful. \square

In the rest of this section, we will be concerned with the proof of Proposition 9.2, and we fix S, T and e as in the statement of the proposition from now on. Our strategy will be as follows. First, let us write

$$A_0 \subseteq \Omega[S] \otimes \Omega[T]$$

for the dendroidal set given by the image of $\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T]$. We are going to construct a sequence of dendroidal subsets

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_N = \Omega[S] \otimes \Omega[T]$$

such that each inclusion is an anodyne extension. This will be done by writing $\Omega[S] \otimes \Omega[T]$ as a union of representables, as follows. We will explicitly describe a sequence of trees

$$T_1, T_2, \dots, T_N$$

together with canonical monomorphisms (all called)

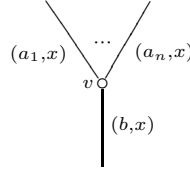
$$m : \Omega[T_i] \hookrightarrow \Omega[S] \otimes \Omega[T],$$

and we will write $m(T_i) \subseteq \Omega[S] \otimes \Omega[T]$ for the dendroidal subset given by the image of this monomorphism. We will then define

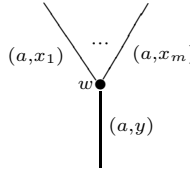
$$A_{i+1} = A_i \cup m(T_{i+1}) \quad (i = 0, \dots, N-1)$$

and prove that each $A_i \hookrightarrow A_{i+1}$ thus constructed is anodyne. For the rest of this section, we will fix planar structures on the trees S and T . These will then induce a natural planar structure on each of the trees T_i , and avoid unnecessary discussion involving automorphisms in the category Ω .

To define the T_i , let us think of the vertices of S as *white* (drawn \circ) and those of T as *black* (drawn \bullet). The edges of T_i are (labelled by) pairs (a, x) where a is an edge of S and x one of T . We refer to a as the *S-colour* of this edge (a, x) , and to x as its *T-colour*. There are two kinds of vertices in T_i (corresponding to the generators for $\Omega[S] \otimes \Omega[T]$ coming from vertices of S or of T). There are *white* vertices in T_i labelled



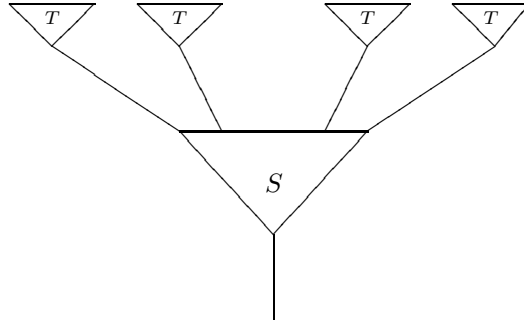
where v is a vertex in S with input edges a_1, \dots, a_n and output edge b , while x is an edge of T ; and there are *black* vertices in T_i labelled



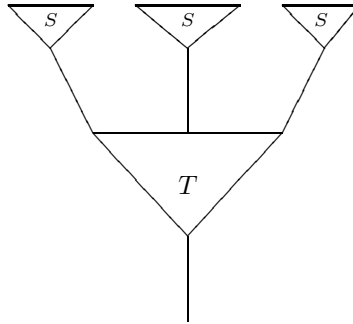
where w is a vertex in T with input edges x_1, \dots, x_m and output edge y , while a is an edge in S . Moreover, each such tree T_i is *maximal*, in the sense that its output (root) edge is labelled (r_S, r_T) where r_S and r_T are the roots of S and T , and its input edges are labelled by all pairs (a, x) where a is an input edge of S and x one of T .

All the possible such trees T_i come in a natural (partial) order. The minimal tree T_1 in the poset is the one obtained by stacking a copy of the black tree T on top of each of the input edges of the white tree S . Or, more precisely, on the bottom of

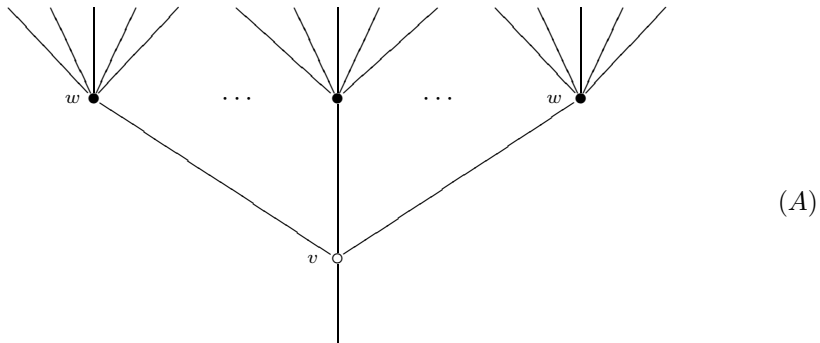
T_1 there is a copy $S \otimes r_T$ of the tree S all whose edges are renamed (a, r_T) where r_T is the output edge at the root of T . For each input edge b of S , a copy of T is grafted on the edge (b, r) of $S \otimes r$, with edges x in T renamed (b, x) . The maximal tree T_N in the poset is the similar tree with copies of the white tree S grafted on each of the input edges of the black tree. Pictorially T_1 looks like



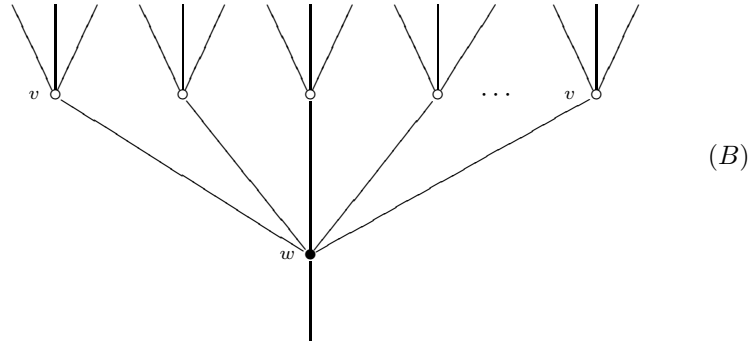
and T_N looks like



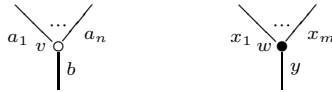
The intermediate trees T_k ($1 < k < N$) are obtained by letting the black vertices in T_1 slowly percolate in all possible ways towards the root of the tree. Each T_k is obtained from an earlier T_l by replacing a configuration



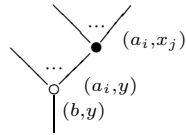
in T_l by



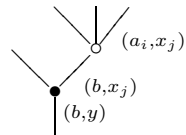
in T_k . More explicitly, if v and w are vertices in S and T ,



then the edges in (A) are named

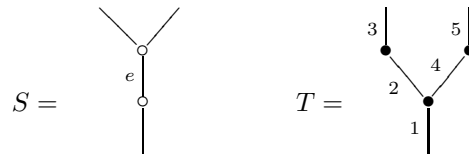


and those in (B) are named

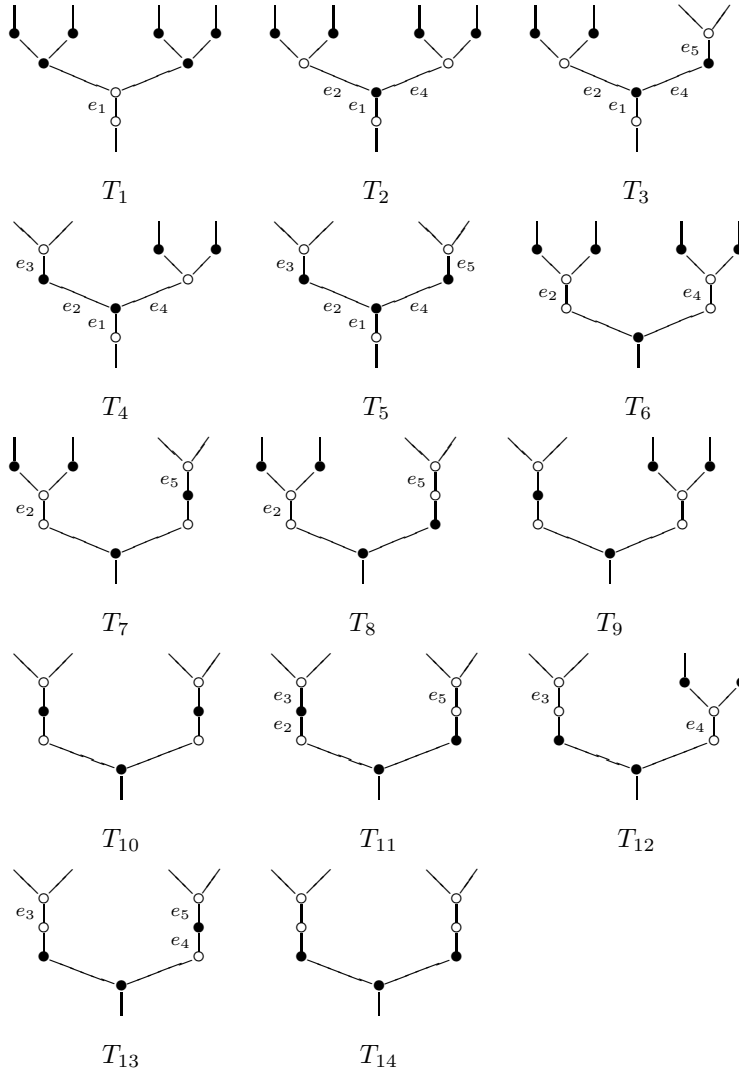


We will refer to these trees T_i as the *percolation schemes* for S and T , and if T_k is obtained from T_l by replacing (A) by (B), then we will say that T_l is obtained by a *single percolation step*.

Example 9.4. Many of the typical phenomena that we will encounter already occur for the following two trees S and T ; here, we have singled out one particular edge e in S , we've numbered the edges of T as $1, \dots, 5$, and denoted the colour (e, i) in T_i by e_i .

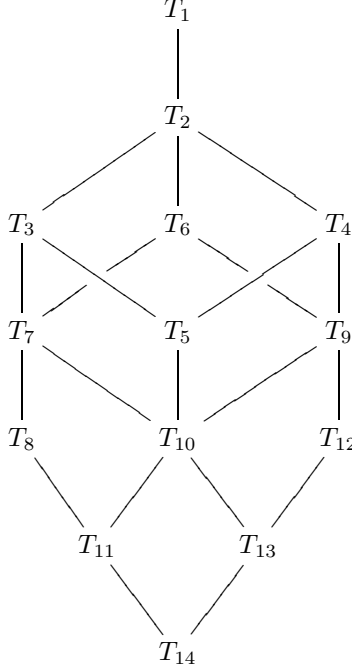


There are 14 percolation schemes T_1, \dots, T_{14} in this case. Here is the complete list of them:



As claimed, there is a partial order on the percolation schemes T_1, \dots, T_N for $S \otimes T$, in which T_1 (copies of T on top of S) is the minimal element and T_N (copies of S on top of T) the maximal one. The partial order is given by defining $T \leq T'$ whenever the percolation scheme T' can be obtained from the percolation scheme T by a sequence of percolations. For example, the poset structure on the percolation

trees above is:



The planar structures of S and T provide a way to refine this partial order by a linear order. It is not important exactly how this is done, but we shall from now on assume that the percolation schemes for S and T are ordered T_1, \dots, T_N where T_i comes before T_j only if $T_i \leq T_j$ in the partial order.

Lemma 9.5. (and notation) *Each percolation scheme T_i is equipped with a canonical monomorphism*

$$m : \Omega[T_i] \twoheadrightarrow \Omega[S] \otimes \Omega[T] .$$

The dendroidal subset given by the image of this monomorphism will be denoted

$$m(T_i) \subseteq \Omega[S] \otimes \Omega[T].$$

Proof. The vertices of the dendroidal set $\Omega[T_i]$ are the edges of the tree T_i . The map m is completely determined by asking it to map an edge named (a, x) in T_i to the vertex with the same name in $\Omega[S] \otimes \Omega[T]$. This map is a monomorphism. In fact, any map

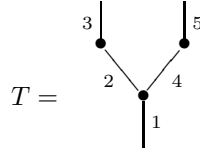
$$\Omega[R] \rightarrow X,$$

from a representable dendroidal set to an arbitrary one, is a monomorphism as soon as the map $\Omega[R]_\eta \rightarrow X_\eta$ on vertices is. \square

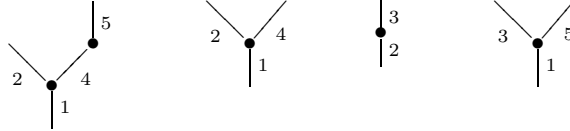
Before we continue, we need to introduce a bit of terminology for trees, i.e., for objects of Ω . Let R be such a tree. A map $R' \rightarrow R$ which is a composition of basic face maps (maps of type (ii) or (iii) in Section 3) will also be referred to as a *face* of R , just like for simplicial sets. If it is a composition of *inner* faces (resp. *outer* faces), the map $R' \twoheadrightarrow R$ will be called an *inner face* (resp. *outer face*) of R . A *top face* of R is an outer face map $\partial_v : R' \rightarrow R$ where R' is obtained by deleting a

top vertex from R . An *initial segment* $R' \twoheadrightarrow R$ is a composition of top faces (it is a special kind of outer face of R). If v is the vertex above the root of R and e is an input edge of v , then R contains a subtree R' whose root is e . We'll refer to an inclusion of this kind as a *bottom* face of R (it is again a special kind of outer face). In all these cases, we'll often leave the monomorphism $R' \twoheadrightarrow R$ implicit, and apply the same terminology not only to the map $R' \twoheadrightarrow R$ but also to the tree R' .

For example, for the tree T constructed above



The following sub-trees are examples of, respectively, a top face, an initial segment, a bottom face, and an inner face:



Remark 9.6. We observe the following simple properties, which we will repeatedly use in the proofs of the lemmas below. In stating these properties and below, we denote by $m(R)$ the image of the composition of the inclusion $\Omega[R] \twoheadrightarrow \Omega[T_i]$ given by a subtree (a face) R of T_i and the canonical monomorphism

$$m : \Omega[T_i] \twoheadrightarrow \Omega[S] \otimes \Omega[T] .$$

(i) Let R be a subtree of T_i . If $m(R) \subseteq A_0$ then R misses a T -colour, or an S -colour other than e , or a stump of either S or T . Here, a stump is a top vertex of valence zero (i.e., without input edges). We say that R "misses" such a stump $v \in S$, for example, if $m(R) \subseteq \partial_v[S] \otimes \Omega[T]$. The tree R is a sub-tree of T_i , where edges are coloured by pairs (a, x) , where a is an S -colour and x a T -colour. By saying that R "misses" a T -colour y , we mean that none of the colours (a, x) occurring in R has $x = y$ as second coordinate. "Missing an S -colour" is interpreted similarly.

(ii) This implies in particular that for any bottom face $R \twoheadrightarrow T_i$ of any percolation scheme T_i the dendroidal set $m(R)$ is contained in A_0 , because it must miss either the root colour r_S (in case the root of T_i is white), or the root colour r_T (in case the root of T_i is black), and $r_S \neq e$ because e is assumed inner.

(iii) If F, G are faces of T_i then F is a face of G iff $m(F) \subseteq m(G)$. (This is clear from the fact that the map from $\Omega[R]$ onto its image $m(R)$ is an isomorphism of dendroidal sets.)

(iv) Let Q and R be initial segments of T_i , and let F be an inner face of Q . If $m(F) \subseteq m(R)$ then also $m(Q) \subseteq m(R)$ (and hence Q is a face of R , by (iii)). In fact, let $Inn(Q)$ denote the set of all inner edges of Q and $Q/Inn(Q) \twoheadrightarrow Q$ the inner

face of Q given by contracting all these. Then if $m(Q/\text{Inn}(Q)) \subseteq m(R/\text{Inn}(R))$, it follows by comparing labels of input edges of Q and R that Q is a face of R .

These remarks prepare the ground for the following lemma. Recall that $A_k = A_0 \cup m(T_1) \cup \dots \cup m(T_k)$, where $m(T_i)$ is the image in $\Omega[S] \otimes \Omega[T]$ of the dendroidal set $\Omega[T_i]$.

Lemma 9.7. *Let R, Q_1, \dots, Q_p be a family of initial segments in T_{k+1} and write $B = m(Q_1) \cup \dots \cup m(Q_p) \subseteq \Omega[S] \otimes \Omega[T]$. Suppose*

(i) *For every top face F of R , $m(F) \subseteq A_k \cup B$.*

(ii) *There exists an edge ξ in R such that for every inner face $F \twoheadrightarrow R$, if $m(F)$ is not contained in $A_k \cup B$ then neither is $m(F/(\xi))$.*

Then the inclusion $A_k \cup B \twoheadrightarrow A_k \cup B \cup m(R)$ is anodyne.

We call ξ a characteristic edge of R with respect to Q_1, \dots, Q_p .

Proof. If $m(R) \subseteq A_k \cup B$ there is nothing to prove. If not, then by (ii), $m(R/(\xi))$ is not contained in $A_k \cup B$. Let

$$\xi = \xi_0, \xi_1, \dots, \xi_n$$

be all the inner edges in R such that the dendroidal set $m(R/(\xi_i))$ is not contained in $A_k \cup B$. For a sub-sequence $\xi_{i_1}, \dots, \xi_{i_p}$ of these ξ_0, \dots, ξ_n , we have the dendroidal subset of $\Omega[S] \otimes \Omega[T]$,

$$m(R/(\xi_{i_1}, \dots, \xi_{i_p})), \quad (1)$$

obtained by contracting each of $\xi_{i_1}, \dots, \xi_{i_p}$ and composing with $m : \Omega[T_{k+1}] \rightarrow \Omega[S] \otimes \Omega[T]$. We are going to consider a sequence of anodyne extensions

$$A_k \cup B = B_0 \twoheadrightarrow B_1 \twoheadrightarrow \dots \twoheadrightarrow B_{2^n} = A_k \cup B \cup m(R)$$

by considering images of faces of $\Omega[R]$ of this type (1).

Consider first

$$R_{(0)} = m(R/(\xi_1, \dots, \xi_n)).$$

If $m(R_{(0)})$ is contained in $A_k \cup B$, let $B_1 = B_0 = A_k \cup B$. Otherwise, let B_1 be the pushout

$$\begin{array}{ccc} m(\Lambda^{\xi_0} R_{(0)}) & \longrightarrow & B_0 \\ \downarrow & & \downarrow \\ m(R_{(0)}) & \longrightarrow & B_1 \end{array}$$

Notice that $m(\Lambda^{\xi_0} R_{(0)})$ is indeed contained in $B_0 = A_k \cup B$. For, any outer face F of $R_{(0)}$ is a face of an outer face G of R

$$\begin{array}{ccc} F & \longrightarrow & R/(\xi_1, \dots, \xi_n) = R_{(0)} \\ \downarrow & & \downarrow \\ G & \longrightarrow & R \end{array}$$

if G is a top face, then $m(G) \subseteq A_k \cup B$ by assumption (i); and if G is a bottom face, it already factors through $A_0 \subseteq A_k$ (cf Remark 9.6 (ii) before the lemma). On the other hand, if $F \subseteq R_{(0)}$ is an inner face of $R_{(0)}$ given by contracting an edge ζ in $R/(\xi_1, \dots, \xi_n)$, then F is a face of $R/(\zeta)$. So if $m(F) \not\subseteq B_0$ then $m(R/(\zeta))$ wouldn't

be contained in B_0 either, and hence ζ must be one of ξ_0, \dots, ξ_n . But ξ_1, \dots, ξ_n are no longer edges in $R/(\xi_1, \dots, \xi_n)$, so ζ must be ξ_0 . This shows that for any inner face F of $R_{(0)}$ other than $R_{(0)}/(\xi_0)$, the dendroidal set $m(F)$ is contained in B_0 , as claimed.

Next, consider all sub-sequences $(\xi_1, \dots, \hat{\xi}_i, \dots, \xi_n)$, and the faces

$$R_{(i)} = R/(\xi_1, \dots, \hat{\xi}_i, \dots, \xi_n) \quad i = 1, \dots, n$$

We will define

$$B_2, \dots, B_{n+1}$$

by considering these $R_{(1)}, \dots, R_{(n)}$. Suppose B_1, \dots, B_i have been defined. Consider $R_{(i)}$ to form B_{i+1} . If its image $m(R_{(i)})$ is contained in B_i , let $B_{i+1} = B_i$. Otherwise, $m(R_{(i)}) \rightarrow \Omega[S] \otimes \Omega[T]$ does not factor through B_i , and a fortiori doesn't factor through $A_k \cup B = B_0$ either. So by assumption (ii), we have that $m(R_{(i)}/(\xi_0)) \not\subseteq A_k \cup B$. But then $m(R_{(i)}/(\xi_0))$ is not contained in B_i either, because by Remark 9.6(iv), if $m(R_{(i)}/(\xi_0))$ would be contained in one of $m(R_{(0)}), \dots, m(R_{(i-1)})$, then $R_{(i)}/(\xi_0)$ would be a face of one of $R_{(0)}, \dots, R_{(i-1)}$, which is obviously not the case. On the other hand, ξ_0 is the *only* edge of $R_{(i)}$ for which $m(R_{(i)}/(\xi_0))$ is not contained in B_i (indeed, the only other candidate would be ξ_i , but $R_{(i)}/\xi_i = R_{(0)}$ and $m(R_{(0)}) \subseteq B_1$). So, we can form the pushout

$$\begin{array}{ccc} m(\Lambda^{\xi_0} R_{(i)}) & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ m(R_{(i)}) & \longrightarrow & B_{i+1} \end{array}$$

Next, consider for each $i < j$ the tree

$$R_{(ij)} = R/(\xi_1, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_n),$$

and order these lexicographically, say as

$$R_1^2, \dots, R_u^2 \quad (u = \binom{n}{2}).$$

We are going to form anodyne extensions of B_{n+1} by using these trees,

$$B_{n+1} \twoheadrightarrow B_{n+2} \twoheadrightarrow \dots \twoheadrightarrow B_{n+1+u},$$

treating R_p^2 in the step to form $B_{n+p} \twoheadrightarrow B_{n+p+1}$ (for each $p = 1, \dots, u$). Suppose B_{n+p} has been formed, and consider $R_p^2 = R_{(ij)}$ say. If $m(R_p^2) \subseteq B_{n+p}$ then let $B_{n+p+1} = B_{n+p}$. If not, then surely $m(R_p^2) \not\subseteq A_k \cup B$, so by assumption (ii) $m(R_p^2/(\xi_0)) = m(R/(\xi_0, \xi_1, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_n))$ is not contained in $A_k \cup B$. On the other hand, Remark 9.6(iv) implies that $m(R_p^2/(\xi_0))$ cannot be contained in any of $m(R_1), \dots, m(R_n), m(R_1^2), \dots, m(R_{p-1}^2)$ either. So $m(R_p^2/(\xi_0))$ is not contained in B_{n+p} . As before, ξ_0 is the *only* inner edge ζ for which $m(R_p^2/(\zeta))$ is not contained in B_{n+p} . So we can form the pushout

$$\begin{array}{ccc} m(\Lambda^{\xi_0} (R_{(p)}^2)) & \longrightarrow & B_{n+p} \\ \downarrow & & \downarrow \\ m(R_{(p)}^2) & \longrightarrow & B_{n+p+1} \end{array}$$

Next consider for each $i_1 < i_2 < i_3$ the tree

$$R_{(i_1 i_2 i_3)} = R/(\xi_1, \dots, \hat{\xi}_{i_1}, \dots, \hat{\xi}_{i_2}, \dots, \hat{\xi}_{i_3}, \dots, \xi_n)$$

and adjoin the pushout along

$$m(\Lambda^{\xi_0} R_{(i_1 i_2 i_3)}) \twoheadrightarrow m(R_{(i_1 i_2 i_3)})$$

if necessary. Continuing in this way for all $l = 0, 1, \dots, n - 1$ and all sub-sequences $i_1 < \dots < i_l$ and corresponding trees

$$R/(\xi_1, \dots, \hat{\xi}_{i_1}, \dots, \hat{\xi}_{i_l}, \dots, \xi_n),$$

we end up with a sequence of anodyne extensions

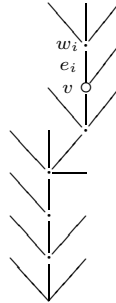
$$B_1 \twoheadrightarrow \dots \twoheadrightarrow B_q$$

where $q = \sum_{l=0}^{n-1} \binom{n}{l} = 2^n - 1$, and where $m(R/(\xi_i))$ is contained in B_q for each $i = 1, \dots, n$. In the final step, and exactly as before, we let $B_{2^n} = B_{2^n-1}$ if $m(R) \subseteq B_{2^n-1}$; and if not, we form the pushout

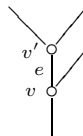
$$\begin{array}{ccc} m(\Lambda^{\xi_0}(R)) & \longrightarrow & B_{2^n-1} \\ \downarrow & & \downarrow \\ m(R) & \longrightarrow & B_{2^n} \end{array}$$

Then B_{2^n} is the pushout of $A_0 \cup B$ and $m(R)$ over $(A_0 \cup B) \cap m(R)$ (because every face F of R for which $m(F)$ is contained in $A_0 \cup B$ occurs in some corner of the pushouts taken in the construction of the B_i). This proves the lemma. \square

Consider the tree T_{k+1} , and look at all lowest occurrences of the S -colour e (Recall e is the fixed edge in S , occurring in the statement of Proposition 9.2). More precisely, let $e_i = (e, x_i)$ for $i = 1, \dots, t$ be all the edges in T_{k+1} whose S -colour is e , while the S -colour of the edge immediately below it isn't. This means that (e, x_i) is an edge having a white vertex at its bottom. Let β_i be the branch in T_{k+1} from the root to and including this edge e_i . Each such β_i is an initial segment in T_{k+1} , to which we will refer as the *spine* through e_i . For example, this is a picture of a spine in T_{k+1} ,

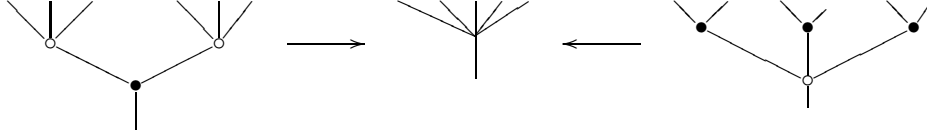


corresponding to the edge e in S



Lemma 9.8. *Let R, Q_1, \dots, Q_p be initial segments in T_{k+1} , as in the preceding lemma, and suppose condition (i) of that lemma is satisfied. Then for any spine β_i contained in R , the edge $e_i \in \beta_i$ is characteristic for R with respect to Q_1, \dots, Q_p .*

Proof. We have to check condition (ii) of Lemma 9.7. So, suppose F is an inner face of R , and suppose $m(F/(e_i))$ is contained in $A_k \cup B = A_0 \cup m(T_1) \cup \dots \cup m(T_k) \cup m(Q_1) \cup \dots \cup m(Q_p) \subseteq m(T_{k+1})$. Since $m(F/(e_i))$ is isomorphic to the representable dendroidal set $\Omega[F/(e_i)]$, it must be contained in one of the dendroidal sets constituting this union. But, if $m(F/(e_i))$ is contained in A_0 , then by Remark 9.6 (ii) $m(F)$ is also contained in A_0 . (the only colour occurring in F but possibly not in $F/(e_i)$ is the S -colour e). And, if $m(F/(e_i))$ is contained in $m(T_j)$ for some $j \leq k$, then there must be a tensor product relation applying to the image of $F/(e_i)$, which allows a black vertex to move up so as to get into an earlier T_j , as in:



where the left tree is in T_{k+1} , the right tree is in T_j , and the middle one in $F/(e_i)$.

But then the same relation must apply to the image of F , because the edge e_i , having a white vertex at its root, cannot contribute to this relation. Finally, if $m(F/(e_i))$ is contained in $m(Q_l)$ for some $l \leq p$, then by Remark 9.6 (iv), we have $m(R) \subseteq m(Q_l)$. So a fortiori, $m(F)$ is contained in $m(Q_l)$. This proves the lemma. \square

Recall that our aim is to prove for $A_k = A_0 \cup m(T_1) \cup \dots \cup m(T_k)$ that each inclusion

$$A_k \twoheadrightarrow A_{k+1}$$

is anodyne. Consider the tree T_{k+1} , and let β_1, \dots, β_t be all the spines contained in it. We shall prove by induction that $A_k \twoheadrightarrow A_k \cup m(R_1) \cup \dots \cup m(R_q)$ is anodyne, for any family R_1, \dots, R_q of initial segments each of which contains at least one such spine. The induction will be on the number of such initial segments as well as on their size. When applied to the maximal initial segment T_{k+1} itself, this will show that $A_k \twoheadrightarrow A_k \cup m(T_{k+1}) = A_{k+1}$ is anodyne, as claimed. The precise form of induction is given by the following lemma.

Lemma 9.9. *Fix l with $0 \leq l \leq t$. Let Q_1, \dots, Q_p be a family of initial segments in T_{k+1} , each containing at least one and at most l spines. Let R_1, \dots, R_q be initial segments, each of which contains $l+1$ spines. Then the inclusion*

$$A_k \twoheadrightarrow A_k \cup B \cup C$$

for $B = m(Q_1) \cup \dots \cup m(Q_p)$ and $C = m(R_1) \cup \dots \cup m(R_q)$, is anodyne.

Proof. We can measure the size of each of the initial segments R_j by counting the number of vertices in R_j which are not on one of the $l+1$ spines. If this number is not bigger than u , we say that R_j has size at most u , and write $size(R_j) \leq u$. Let $\Lambda(l, u)$ be the assertion that the lemma holds for l , for any families $\{Q_i\}$ and $\{R_j\}$ where the R_j all have $size(R_j) \leq u$. We will prove $\Lambda(l, u)$ by induction, first on l and then on u .

Case $l = 0$: This is the case where there are no Q 's, i.e., $p = 0$. For $l = 0$, first consider the case where $u = 0$ also. Then each R_i is itself a spine, say β_i , with top inner edge e_i running from a copy of v to a copy w_i of w . We will prove that each of the inclusions

$$A_k \cup m(R_1) \cup \cdots \cup m(R_{i-1}) \hookrightarrow A_k \cup m(R_1) \cup \cdots \cup m(R_i)$$

for $i = 0, \dots, q$, is anodyne. If $R_i = \beta_i$ coincides with one of the earlier spines R_j , $j < i$, then there is nothing to prove. If R_i is a different spine, then its outer top face is contained in A_0 because it misses the vertex v' which is above e in S . So condition (i) of Lemma 9.7 is satisfied, where R_i, R_1, \dots, R_{i-1} take the role of R, Q_1, \dots, Q_p in that lemma. By Lemma 9.8, the edge $e_i \in R_i$ is characteristic so Lemma 9.7 gives that $A_k \cup m(R_1) \cup \cdots \cup m(R_{i-1}) \hookrightarrow A_k \cup m(R_1) \cup \cdots \cup m(R_i)$ is anodyne, as claimed. The composition of these inclusions will then be anodyne also, which proves the statement $\Lambda(0, 0)$.

Suppose now that $\Lambda(0, u)$ has been proved, and consider families R_1, \dots, R_q of initial segments which are each of size not bigger than $u + 1$. Suppose that among these, $R_1, \dots, R_{q'}$ actually have size not bigger than u , while $R_{q'+1}, \dots, R_q$ have size $u + 1$. We shall prove that

$$A_k \hookrightarrow A_k \cup m(R_1) \cup \cdots \cup m(R_q)$$

is anodyne, by induction on the number $r = q - q'$ of initial segments that have size $u + 1$. If $r = 0$, this holds by $\Lambda(0, u)$. Suppose we have proved this for *any* family with not more than r initial segments of size $u + 1$, and consider such a family R_1, \dots, R_q where $q - q' = r + 1$. Write β_q for the spine contained in R_q (there is only one such because we are still in the case $l = 0$). For a top outer face $\partial_x(R_q)$ of R_q , either $x \neq w_q$ so that $\partial_x(R_q)$ still contains β_q but has size at most u , or $x = w_q$ so that $m(\partial_x(R_i))$ is contained in A_0 because it misses the vertex v' immediately above e in S . Thus, if we let

$$P = m(R_1) \cup \cdots \cup m(R_{q-1}) \cup \bigcup_x m(\partial_x(R_q))$$

where x ranges over all the top vertices in R_q , then by the fact that $\Lambda(0, u + 1)$ is assumed to hold for $r = (q - 1) - q'$,

$$A_k \hookrightarrow A_k \cup P \tag{1}$$

is anodyne. To prove that $A_k \cup P \hookrightarrow A_k \cup P \cup m(R_q) = A_k \cup m(R_1) \cup \cdots \cup m(R_q)$ is anodyne as well, we can now apply Lemma 9.7. Indeed, the family of initial segments containing P is made to contain the images of all the top faces of R_q , and $e_q \in R_q$ is characteristic by Lemma 9.8. This proves that $A_k \cup P \hookrightarrow A_k \cup P \cup m(R_q)$ is anodyne, as claimed. When composed with (1), we find that $A_k \hookrightarrow A_k \cup m(R_1) \cup \cdots \cup m(R_q)$ is anodyne. This proves $\Lambda(0, u + 1)$ and completes the inductive proof of $\Lambda(l, u)$ for $l = 0$ and all u .

Suppose now that we have proved $\Lambda(l', u)$ for all $l' \leq l$ and all u . We will now prove $\Lambda(l + 1, u)$ by induction on u .

Case $u = 0$: This is the assertion that for any given initial segments

$$Q_1, \dots, Q_p, R_1, \dots, R_q$$

of T_{k+1} , where the Q_j contain at most l spines while each R_i is made up out of exactly $l + 1$ spines (and no other vertices), the inclusion

$$A_k \twoheadrightarrow A_k \cup m(Q_1) \cup \cdots \cup m(Q_p) \cup m(R_1) \cup \cdots \cup m(R_q) \quad (2)$$

is anodyne. We shall prove by induction on q that this holds for all p . For $q = 0$, the conclusion follows by the inductive assumption that $\Lambda(l, u)$ holds. Suppose the assertion holds for $q - 1$, and consider R_q . Each top vertex of R_q lies at the end of a spine, so $\partial^x(R_q)$ contains at most l spines. Let

$$D = m(Q_1) \cup \cdots \cup m(Q_p) \cup \bigcup_x \partial_x(R_q)$$

where x ranges over the top vertices of R_q . Then, by the assumption for $q - 1$,

$$A_k \twoheadrightarrow A_k \cup D \cup m(R_1) \cup \cdots \cup m(R_{q-1}) \quad (3)$$

is anodyne. To prove that (2) is anodyne, it then suffices to apply Lemma 9.7, and show that R_q has a characteristic edge with respect to the family of initial segments containing the union $D \cup m(R_1) \cup \cdots \cup m(R_{q-1})$ in (3). But by Lemma 9.8, any top edge e_q of R_q is characteristic. This proves $\Lambda(l + 1, u)$, for $u = 0$.

Case $u + 1$: Suppose now $\Lambda(l + 1, u)$ holds. To prove $\Lambda(l + 1, u + 1)$, consider families

$$Q_1, \cdots, Q_p, R_1, \cdots, R_{q'}, R_{q'+1}, \cdots, R_q \quad (4)$$

of initial segments in T_{k+1} , where the Q_i contain at most l spines, the R_i contain exactly $l + 1$ spines, the $R_1, \cdots, R_{q'}$ are of size not more than u , and $R_{q'+1}, \cdots, R_q$ are of size exactly $u + 1$. We will show by induction on the last number $r = q - q'$ that for any such family, the inclusion

$$A_k \twoheadrightarrow A_k \cup m(Q_1) \cup \cdots \cup m(Q_p) \cup m(R_1) \cup \cdots \cup m(R_q) \quad (5)$$

is anodyne. For $r = q - q' = 0$ there is nothing to prove, because this is the case covered by $\Lambda(l + 1, u)$. Suppose we have proved that (5) is anodyne for *any* family (4) with $q - q' \leq r$, and consider such a family with $q - q' = r + 1$. The initial segment R_q has two kinds of top outer faces, namely the $\partial_x(R_q)$ which remove the top of a spine, and the $\partial_x(R_q)$ where x does not lie on a spine. Outer faces of the first kind contain l spines only, and outer faces of the second kind are of size not more than u . Let

$$D = m(Q_1) \cup \cdots \cup m(Q_p) \cup \bigcup_x m(\partial_x R_q)$$

where x ranges over the top vertices of R_q which are on a spine. Let

$$E = m(R_1) \cup \cdots \cup m(R_{q'}) \cup \bigcup_x m(\partial_x(R_q))$$

where x ranges over the top vertices of R_q which are not on a spine. Then, by the assumption that $\Lambda(l + 1, u + 1)$ has been established for families (4) where $q - q' \leq r$, we see that

$$A_k \twoheadrightarrow A_k \cup D \cup E \cup R_{q'+1} \cup \cdots \cup R_{q-1} \quad (6)$$

is anodyne. The union $D \cup E$ is made to contain all the images $\partial_x(R_q)$ of top faces $\partial_x(R_q)$ of R_q , and by Lemma 9.8, any edge e_q on the top of a spine β_q in R_q is

characteristic with respect to the family of initial segments making up the union on the right-hand-side of (6). So by Lemma 9.7, the map

$$A_k \cup D \cup E \cup R_{q'+1} \cup \cdots \cup R_{q-1} \twoheadrightarrow A_k \cup D \cup E \cup R_{q'+1} \cup \cdots \cup R_q$$

is anodyne. When composed with (6), this gives (5), and proves the case $u + 1$.

This established $\Lambda(l + 1, u + 1)$ and completes, for $l + 1$, the induction on u , thus completing the proof. \square

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