ON INNER KAN COMPLEXES IN THE CATEGORY OF DENDROIDAL SETS

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ABSTRACT. The category of dendroidal sets is an extension of that of simplicial sets, suitable for defining nerves of operads rather than just of categories. In this paper, we prove some basic properties of inner Kan complexes in the category of dendroidal sets. In particular, we extend fundamental results of Boardman and Vogt, of Cordier and Porter, and of Joyal to dendroidal sets.

1. INTRODUCTION

This paper is a companion to our paper [MW], where we introduced the category of dendroidal sets, and explained some of its applications. The main goal of the present paper is to introduce a notion of ”inner Kan complex” for dendroidal sets, and to prove some of the fundamental properties of these inner Kan complexes.

Dendroidal sets provide a generalization of simplicial sets. There is an embedding $i_1 : sSet \to dSet$ along which many properties of simplicial sets can be extended to dendroidal sets. For example, the category of dendroidal sets carries a closed symmetric monoidal structure, which extends the Cartesian structure of simplicial sets (but is not itself Cartesian). This monoidal structure is closely related to the Boardman-Vogt tensor product of operads ([BV], [D]). It plays a central role in various uses of dendroidal sets, for example in the construction of homotopy coherent nerves of operads, and in the definition of weak higher categories given in [MW].

We recall that a simplicial set is said to satisfy the Kan condition, or, to be a Kan complex, if every horn $\Lambda^i[n] \to X$ for $0 \leq i \leq n$ has a filler. Boardman and Vogt ([BV]) study this filler condition for $0 < i < n$, and refer to it as the restricted Kan condition. Recently, Joyal ([D]) has been studying simplicial sets satisfying this condition, under the name of quasi-categories. The horns $\Lambda^i[n]$ for $i$ different from 0 and $n$ are called inner horns, and we shall call a simplicial set satisfying this restricted Kan condition an inner Kan complex. Note that the nerve of any category is an inner Kan complex.

In this paper, we will define inner horns and inner Kan complexes for dendroidal sets, in such a way that a simplicial set $X$ is an inner Kan complex iff $i_1(X)$ is a dendroidal inner Kan complex. The dendroidal nerve of any operad provides an example of such a dendroidal inner Kan complex. We will prove several fundamental properties of dendroidal inner Kan complexes. Our main result is that the closed monoidal structure on dendroidal sets has the property that for any two dendroidal sets $X$ and $Y$, the internal Hom $\text{Hom}(X,Y)$ is an inner Kan complex whenever $X$ is normal and $Y$ is inner Kan (Theorem 9.1 below). We will also show that the homotopy coherent nerve of a topological operad is an inner Kan complex. (A more general statement for operads in monoidal model categories is given in Theorem 7.1.)
These results specialise to known results for simplicial sets, and provide new proofs of these. Indeed, Cordier and Porter \cite{CP} prove that the homotopy coherent nerve of a locally fibrant simplicial category is an inner Kan complex. And Joyal \cite{J2} proves for any two simplicial sets \( X \) and \( Y \) that \( \text{Hom}(X,Y) \) is an inner Kan complex (quasi-category) whenever \( Y \) is (see also \cite{N}). The latter result plays a fundamental role in Joyal’s proof of the existence of a model structure on simplicial sets in which the inner Kan complexes are the fibrant objects. We expect our result for dendroidal sets to play a similar role in establishing an analogous model structure on the category of dendroidal sets.

In \cite{MW}, we introduced a Grothendieck construction (homotopy colimit) for diagrams of dendroidal sets, necessary for our definition of weak higher categories. In Section 8 of this paper, we will prove that this dendroidal Grothendieck construction yields an inner Kan complex when applied to a diagram of inner Kan complexes. In addition, in Section 6, we will give an explicit description of the operad generated by an inner Kan complex, modelled on the one given by Boardman and Vogt \cite{BV} in the simplicial case. We then use this description to prove that a dendroidal set satisfies the unique filler condition for inner horns iff it is the nerve of an operad.

2. The category of dendroidal sets

The notion of dendroidal set was introduced in \cite{MW}. We briefly recall the relevant definitions here.

To begin with, we introduce a category \( \Omega \) whose objects are finite rooted trees. If we think of a tree as a graph, and call a vertex unary if it has only one edge attached to it, then our trees \( T \) are equipped with a distinguished unary vertex \( o \) called the output, and a set of unary vertices \( I \) (not containing \( o \)) called the set of inputs. When drawing such a tree, it is common to orient the tree "towards the output" drawn at the bottom, and delete the designated output and input vertices from the picture. Thus, in the tree \( T \),

```
\begin{tikzpicture}
  \node (a) at (0,0) [circle,draw] {a};
  \node (b) at (1.5,0) [circle,draw] {b};
  \node (c) at (3,1.5) [circle,draw] {c};
  \node (d) at (4.5,0) [circle,draw] {d};
  \node (e) at (1.5,3) [circle,draw] {e};
  \node (f) at (3,1.5) [circle,draw] {f};
  \node (g) at (4.5,1.5) [circle,draw] {g};
  \draw (a) edge (b);
  \draw (b) edge (c);
  \draw (b) edge (d);
  \draw (a) edge (e);
  \draw (b) edge (f);
  \draw (b) edge (g);
\end{tikzpicture}
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the output vertex at the edge \( a \) has been deleted, as have the input vertices at \( e, f, c \). This tree \( T \) now has three (remaining) vertices, three input edges attached to the three deleted input vertices, and one output edge (attached to the deleted output vertex). These input and output edges are called outer edges (the output edge is also called the root, while the input edges are also called leaves), the others (\( b \) and \( d \) in the picture) are called inner edges. From now on, we will not mention the input and output vertices anymore, and "vertex" will always refer to a remaining vertex.

Attached to each such vertex in the tree, there will be one designated edge pointing towards the root; the other edges attached to this vertex are called the
input edges of that vertex, and their number is called the valence of the vertex. So in the tree $T$ pictured above, the vertex $r$ has valence three and the vertex $w$ has valence zero. The tree with just one edge is now drawn as

$$i[2]:$$

and referred to as $\eta$, or sometimes as $\eta_e$ if we want to name its edge $e$. The linear tree, with one input edge and one output edge and $n$ vertices, is denoted $i[n]$. It has $n + 1$ edges which we usually number from input to output as $0, 1, \cdots, n$. Here is a picture of $i[2]$: 

Each tree $T$ defines a coloured operad (see [BM2] for a definition of coloured operads) which we denoted $\Omega(T)$ in [MW]. The colours of this operad are the edges of the tree, and the operations are generated by the vertices of the tree. A planar representation of the tree gives a specific set of generators. For example, for the tree $T$ pictured above, $\Omega(T)$ has six colours, $a, b, \cdots, f$. A choice of generating operations is $r \in \Omega(T)(b, c, d; a)$, $w \in \Omega(T)(-; d)$ and $v \in \Omega(T)(e, f; b)$. The other operations are units such as $1_b \in \Omega(T)(b; b)$, compositions such as $r \circ v \in \Omega(T)(e, f, c, d; a)$, and permutations such as $r \cdot \tau \in \Omega(T)(c, b, d; a)$. Note that the same tree $T$ can be given a different planar structure, e.g.

which defines the same operad $\Omega(T)$ but suggests a different choice of generators ($r \cdot \tau$ rather than $r$).

The category $\Omega$ is now defined as the category having these trees $T$ (with designated output and inputs) as objects, and as arrows $T \to T'$ the maps of coloured operads $\Omega(T) \to \Omega(T')$. (Note that every such map sends colours to colours, i.e., edges of $T$ to edges of $T'$, and is in fact completely determined by this).

The category of dendroidal sets is the category of functors $X : \Omega^{op} \to \text{Set}$ and natural transformations between them. We will denote this category by $d\text{Set}$.

For a dendroidal set $X$ and a tree $T$, we will usually write $X_T$ for $X(T)$, and call an element of the set $X_T$ a dendrex of $X$ of "shape" $T$.

The linear trees $i[n]$ for $n \geq 0$ define a functor (a full embedding)

$$i : \Delta \to \Omega$$
from the standard simplicial category, and hence by composition a functor

\[ i^* : dSet \to sSet, \]

from dendroidal sets to the category \( sSet \) of simplicial sets.

By Kan extension, this functor has both a left and a right adjoint, denoted \( i_l \) and \( i_r : sSet \to dSet \), respectively. The functor \( i_l \) is "extension by zero"; for a simplicial set \( X \),

\[ i_l(X)_T = \begin{cases} X_n, & \text{if } T \cong \Delta[n] \text{ for some } n \geq 0 \\ \phi, & \text{otherwise} \end{cases} \]

This defines a full embedding \( i_l : sSet \to dSet \), from simplicial sets into dendroidal sets.

Each coloured operad \( \mathcal{P} \) defines a dendroidal set \( N_d(\mathcal{P}) \), its \textit{dendroidal nerve}, by

\[ N_d(\mathcal{P})_T = \text{Hom}(\Omega(T), \mathcal{P}), \]

\textit{Hom} denoting the set of arrows in the category of operads. If \( \mathcal{P} \) is itself an operad of the form \( \Omega(S) \), given by an object \( S \) of \( \Omega \), then \( N_d(\mathcal{P}) \) is the representable dendroidal set given by \( S \), which we will denote by \( \Omega[S] \); i.e.,

\[ N_d\Omega(S) = \Omega[S] \]

by definition. The functor \( N_d \) from operads to dendroidal sets is fully faithful, and has a left adjoint which we will denote by

\[ \tau_d : dSet \to \text{Operad}. \]

For a dendroidal set \( X \), we refer to \( \tau_d(X) \) as \textit{the operad generated by} \( X \).

We also recall from [MW] that the Cartesian structure on \( sSet \) extends to a (non-Cartesian) closed symmetric monoidal structure \( \otimes \) on \( dSet \). This structure is completely determined by the identity

\[ \Omega[S] \otimes \Omega[T] = N_d(\Omega(S) \otimes_{BV} \Omega(T)) \]

where \( \otimes_{BV} \) denotes the Boardman-Vogt tensor product of (coloured) operads; see [BV]. The corresponding internal \textit{Hom} is then determined by the Yoneda lemma, as

\[ \text{Hom}(X, Y)_T = \text{Hom}_{dSet}(\Omega[T] \otimes X, Y). \]

We will come back to the monoidal structure in more detail in Section 9.

3. \textbf{Faces and Degeneracies}

Exactly as for \( \Delta \), the maps in \( \Omega \) are generated by special kinds of maps.

(i) Given a tree \( T \) and a vertex \( v \in T \) of valence 1, there is a tree \( T' \), obtained from \( T \) by deleting the vertex \( v \) and merging the two edges \( e_1 \) and \( e_2 \) on either side of \( v \) into one new edge \( e \). There is an operad map, i.e. an arrow \( \sigma_v : T \to T' \) in \( \Omega \),
which sends \( v \) to the unit \( 1_c \). For example:

![Diagram](image)

An arrow in \( \Omega \) of this kind will be called a degeneracy (map).

(ii) Given a tree \( T \), and a vertex \( v \) in \( T \) with exactly one inner edge attached to it, one can obtain a new tree \( T/v \) by deleting \( v \) and all the external edges attached to it. The operad \( \Omega(T/v) \) associated to \( T/v \) is simply a suboperad of the one associated to \( T \), and this inclusion of operads defines an arrow in \( \Omega \) denoted

\[ \partial_v : T/v \to T. \]

An arrow in \( \Omega \) of this kind is called an outer face (map). For example

![Diagram](image)

Moreover, for any tree \( T \) with exactly one vertex \( v \), each edge \( e \) of \( T \) (necessarily outer), there is an outer face map

\[ e : \eta \to T \]

sending the unique edge of \( \eta \) to \( e \).

(iii) Given a tree \( T \) and an inner edge \( e \) in \( T \), one can obtain a new tree \( T/e \) by contracting the edge \( e \). There is a canonical map of operads \( \partial_e : \Omega(T/e) \to \Omega(T) \) which sends the new vertex in \( T/e \) (obtained by merging the two vertices attached to \( e \)) into the appropriate composition of these two vertices in \( \Omega(T) \). An arrow \( \partial_e : T/e \to T \) in \( \Omega \) of this kind is called an inner face (map). For example

![Diagram](image)
(iv) Given two trees $T$ and $T'$, any isomorphism $T \to T'$ of trees, sending inputs to inputs and output to output, of course defines an isomorphism of operads $\Omega(T) \to \Omega(T')$, and hence is an isomorphism $T \xrightarrow{\cong} T'$ in $\Omega$. For example, if $C_n$ denotes the corolla with just one vertex, $n$ inputs, and one output, then we might name its input edges $e_1, \ldots, e_n$.

Any permutation $\varphi \in \Sigma_n$ defines an automorphism of $C_n$ in $\Omega$.

For a tree $T$, let its degree $|T|$ be the number of vertices in $T$. Then degeneracy maps decrease degree by 1, face maps (outer or inner) increase degree by 1, and isomorphisms preserve degree. Any map $T \xrightarrow{f} T'$ in $\Omega$ can be written as $f = \delta \varphi \sigma$, where $\delta$ is a composition of (inner or outer) faces, $\varphi$ is an isomorphism, and $\sigma$ is a composition of degeneracies. (This composition is unique up to isomorphism).

4. Skeletal filtration

As for any presheaf category, any dendroidal set $X$ is a colimit of representables, of the form

$$X = \varinjlim \Omega[T]$$

(see \cite{M}). We wish to refine this a little, in a way similar to the skeletal filtration for simplicial sets. To this end, call a dendrex $x \in X_T$ of shape $T$ degenerate if there is a surjective map $T \xrightarrow{\alpha} T'$ in $\Omega$ (a composition of degeneracies) such that $x = \alpha^*(x')$ for some $x' \in X_{T'}$. Here $\alpha$ should not be an isomorphism of course, so that $T'$ has strictly fewer vertices then $T$ and $\alpha$ is a non-empty composition of degeneracies.

Given a dendroidal set $X$ we denote by $Sk_n(X)$ the sub dendroidal set of $X$ generated by all non-degenerate dendrices $x \in X_T$ where $|T| \leq n$. An arbitrary dendroidal set $X$ is clearly the colimit (union) of the sequence

$$Sk_0(X) \subseteq Sk_1(X) \subseteq Sk_2(X) \subseteq \cdots \quad (1)$$

We call this the skeletal filtration of $X$. This filtration extends the skeletal filtration for simplicial sets in the precise sense that for any dendroidal set $X$ and any simplicial set $S$, there are canonical isomorphisms

$$i^* Sk_n(X) = Sk_n(i^* X)$$

and

$$i_! Sk_n(S) = Sk_n(i_! S).$$
Consider now the following diagram:
\[
\prod_{x,T} \partial \Omega[T] \xrightarrow{\partial} S_{n}(X) \\
\prod_{x,T} \Omega[T] \xrightarrow{\Omega} S_{n+1}(X)
\]
where the sum is taken over isomorphism classes of pairs \((x,T)\) in the category of elements of \(X\) where \(T\) is a tree with \(n\) vertices and \(x \in X_T\) is non-degenerate, and \(\partial \Omega[T]\) is the boundary of \(\Omega[T]\), i.e., the union of its faces. We call the skeletal filtration of \(X\) *normal* if this square is a pushout for each \(n > 0\).

Following Cisinski [C] we call a dendroidal set *normal* if for each non-degenerate dendrex \(x \in X_T\), the only isomorphism fixing \(x\) is the identity. Cisinski (loc. cit.) proves that the normal dendroidal sets are precisely those whose skeletal filtrations are normal.

**Example 4.1.** If \(X\) is a simplicial set then \(i_!(X)\) admits a normal skeletal filtration and in fact that skeletal filtration is isomorphic to the usual skeletal filtration of \(X\). If \(\mathcal{P}\) is a \(\Sigma\)-free operad then \(\mathcal{N}_d(\mathcal{P})\) is normal. In particular if \(\mathcal{P}\) is the symmetrization of a planar operad then \(\mathcal{N}_d(\mathcal{P})\) is normal.

## 5. Inner Kan Complexes

We begin by introducing inner horns. For a tree \(T\), each face map \(\partial : T' \to T\) defines a monomorphism \(\Omega[T'] \to \Omega[T]\) between (representable) dendroidal sets. The union (pushout) of these subobjects is the boundary of \(\Omega[T]\), denoted
\[
\partial \Omega[T] \xrightarrow{\partial} \Omega[T],
\]
as above. If \(e\) is an inner edge of \(T\), then the union of all the faces except
\[
\partial_e : T/e \xrightarrow{\partial} T
\]
defines a subobject of the boundary, denoted
\[
\Lambda^e[T] \xrightarrow{\Lambda^e} \Omega[T],
\]
and called the *inner horn* associated to \(e\) (and to \(T\)). This terminology and notation extends the one
\[
\Lambda^k[n] \xrightarrow{\partial k[n]} \partial \Delta[n] \xrightarrow{\partial} \Delta[n]
\]
for simplicial sets, in the sense that
\[
i_!(\Lambda^k[n]) = \Lambda^k[i[n]]
\]
\[
i_!(\partial \Delta[n]) = \partial \Omega[i[n]]
\]
as subobjects of \(i_!(\Delta[n]) = \Omega[i[n]]\).

A dendroidal set \(K\) is said to be a (dendroidal) *inner Kan complex* if, for any tree \(T\) and any inner edge \(e\) in \(T\), the map
\[
K_T = \text{Hom}(\Omega[T], K) \to \text{Hom}(\Lambda^e[T], K)
\]
is a surjection of sets. It is called a strict inner Kan complex if this map is a bijection (for any $T$ and $e$ as above). For example, we will see (Proposition 5.3 below) that the dendroidal nerve of an operad is always a strict inner Kan complex. This terminology is analogous to the one introduced by Boardman and Vogt, who say a simplicial set $X$ satisfies the restricted Kan condition if, for any $0 < k < n$, the map $\text{Hom}(\Delta[n], X) \to \text{Hom}(\Lambda^k[n], X)$ is a surjection ([BY, Definition 4.8, page 102]). In more recent work ([JL], [12]) Joyal develops the general theory of simplicial sets satisfying the restricted Kan condition. Joyal uses the terminology quasi-categories for such simplicial sets so as to stress the analogy with category theory. In fact a simplicial set $X$ is a quasi-category iff $\bar{n}(X)$ is a dendroidal inner Kan complex, and for any dendroidal inner Kan complex $K$, the restriction $i^*(K)$ is a quasi-category in the sense of Joyal.

Let us call a map $u : U \to V$ of dendroidal sets an anodyne extension if it can be obtained from the set of inner horn inclusions by coproducts, pushouts, compositions, and retracts ([cf., GZ], p. 60). Then obviously, the surjectivity property for inner Kan complexes extends to anodyne extensions, in the sense that the map of sets

$$u^* : \text{Hom}(V, K) \to \text{Hom}(U, K),$$

given by composition with $u$, is again surjective. Similarly, the map $u^*$ is a bijection for any strict inner Kan complex.

For a tree $T$ let $I(T)$ be the set of inner edges of $T$. For a non-empty subset $A \subseteq I(T)$ let $\Lambda^A[T]$ be the union of all faces of $\Omega[T]$ except those obtained by contracting an edge from $A$. Note that if $A = \{e\}$ then $\Lambda^A[T] = \Lambda^e[T]$.

**Lemma 5.1.** For any non-empty $A \subseteq I(T)$ the inclusion $\Lambda^A[T] \to \Omega[T]$ is anodyne.

**Proof.** By induction on $n = |A|$. If $n = 1$ then $\Lambda^A[T] \to \Omega[T]$ is an inner horn inclusion, thus anodyne. Assume the proposition holds for $n < k$ and suppose $|A| = k$. Choose an arbitrary $e \in A$ and put $B = A \setminus \{e\}$. The map $\Lambda^A[T] \to \Omega[T]$ factors as

$$\Lambda^A[T] \xrightarrow{\Lambda^B[T]} \Omega[T]$$

The vertical map is anodyne by the induction hypothesis and it therefore suffices to prove that $\Lambda^A[T] \to \Lambda^B[T]$ is anodyne. The following diagram expresses that map as a pushout

$$\Lambda^B[T/e] \xrightarrow{\Lambda^A[T]} \Omega[T/e] \xrightarrow{\Lambda^B[T]}$$

and since the map $\Lambda^B[T/e] \to \Omega[T/e]$ is anodyne (by the induction hypothesis), the proof is complete. □

We denote by $\Lambda^I[T]$ the dendroidal set $\Lambda^A[T]$ where $A = I(T)$, that is $\Lambda^I[T]$ is the union of all outer faces of $\Omega[T]$. By the above proposition the inclusion $\Lambda^I[T] \to \Omega[T]$ is anodyne.
We now consider grafting of trees. For two trees $T$ and $S$, and a leaf $l$ of $T$, let $T_{o_l}S$ be the tree obtained by grafting $S$ onto $T$ by identifying $l$ with the root (output edge) of $S$. Then there are obvious inclusions $\Omega[S] \to \Omega[T_{o_l}S]$ and $\Omega[T] \to \Omega[T_{o_l}S]$, the pushout (union) of which we denote by $\Omega[T] \cup_l \Omega[S] \to \Omega[T_{o_l}S]$.

**Lemma 5.2. (Grafting)** For any two trees $T$ and $S$ and any leaf $l$ of $T$, the inclusion $\Omega[T] \cup_l \Omega[S] \to \Omega[T_{o_l}S]$ is anodyne.

**Proof.** Let us write $R = T_{o_l}S$. The case where $T = \eta$ or $S = \eta$ is trivial, we therefore assume that this is not the case. We proceed by induction on $n = |T| + |S|$, the sum of the degrees of $T$ and $S$. The cases $n = 0$ or $n = 1$ are taken care of by our assumption that $T \neq \eta \neq S$. For the case $n = 2$ the same assumption implies that the inclusion $\Omega[T] \cup_l \Omega[S] \to \Omega[R]$ is an inner horn inclusion. In any case it is anodyne. Assume then that the result holds for $2 \leq n < k$ and suppose $|T| + |S| = k$.

Recall that $\Lambda^I[R]$ is the union of all the outer faces of $\Omega[R]$. First notice that $\Omega[T] \cup_l \Omega[S] \to \Omega[R]$ factors as

$$
\Omega[T] \cup_l \Omega[S] \to \Lambda^I[R] \\
\Omega[R]
$$

and the vertical arrow is anodyne by a previous result. We now show that $\Omega[T] \cup_l \Omega[S] \to \Lambda^I[R]$ is anodyne by exhibiting it as a pushout of an anodyne extension. Recall (MW) that an external cluster is a vertex $v$ with the property that one of the edges adjacent to it is inner while all the other edges adjacent to it are outer. Let $Cl(T)$ (resp. $Cl(S)$) be the set of all external clusters in $T$ (resp. $S$) which do not contain $l$ (resp. the root of $S$). For each $C \in Cl(T)$ the face of $\Omega[R]$ corresponding to $C$ is isomorphic to $\Omega[(T/C)_{o_l}S]$ and the map $\Omega[T/C] \cup_l \Omega[S] \to \Omega[(T/C)_{o_l}S]$ is anodyne by the induction hypothesis. Similarly for every $C \in Cl(S)$ the face of $\Omega[R]$ that corresponds to $C$ is isomorphic to $\Omega[T_{o_l}(S/C)]$ and the map $\Omega[T] \cup_l \Omega[S/C] \to \Omega[T_{o_l}(S/C)]$ is anodyne by the induction hypothesis. The following diagram is a pushout

$$
\coprod_{C \in Cl(T)}(\Omega[T/C] \cup_l \Omega[S]) \amalg \coprod_{C \in Cl(S)}(\Omega[T] \cup_l \Omega[S/C]) \to \Omega[T] \cup_l \Omega[S] \\
\coprod_{C \in Cl(T)}(\Omega[(T/C)_{o_l}S]) \amalg \coprod_{C \in Cl(S)}(\Omega[T_{o_l}(S/C)]) \to \Lambda^I[R]
$$

where the map on the left is the coproduct of all of the anodyne extensions just mentioned. Since anodyne extensions are closed under coproducts, it follows that the map on the left of the pushout is anodyne and thus also the one on the right, which is what we set out to prove. This concludes the proof.

We end this section with two remarks on strict inner Kan complexes.

**Proposition 5.3.** The dendroidal nerve of any operad is a strict inner Kan complex.
Proof. Let $\mathcal{P}$ be an operad. A dendrex $x \in N_d(\mathcal{P})$ is a map $x : \Omega[T] \to N_d(\mathcal{P})$ which is a map of operads $\Omega(T) \to \mathcal{P}$. If we choose a planar representative for $T$ then $\Omega(T)$ is specifically given in terms of generators and is a free operad. It follows that $x$ is equivalent to a labeling of the (planar representative) $T$ as follows. The edges are labeled by colors of $\mathcal{P}$ and the vertices are coloured by operations in $\mathcal{P}$ where the input of such an operation is the tuple of labels of the incoming edges to the vertex and the output is the label of the outgoing edge from the vertex. Any inner horn $\Lambda^k[T] \to N_d(\mathcal{P})$ is easily seen to be equivalent to such a labeling of the tree $T$ and thus determines a unique filler. 

Proposition 5.4. Any strict inner Kan complex is 2-coskeletal.

Proof. Let $X$ be a strict inner Kan complex. Let $Y$ be any dendroidal set and assume $f : Sk_2 Y \to Sk_2 X$ is given. We first show that $f$ can be extended to a dendroidal map $\hat{f} : Y \to X$. Suppose $f$ was extended to a map $f_k : Sk_k Y \to Sk_k X$ for $k \geq 2$. Let $y \in Sk_{k+1}(Y)$ be a non-degenerate dendrex and assume $y \notin Sk_k(Y)$. So $y \in Y_T$ and $T$ has exactly $k + 1$ vertices. Choose an inner horn $\Lambda^\alpha[T]$ (such an inner horn exist since $k \geq 2$). The set $\{\beta^y\}_{\beta \neq \alpha}$ where $\beta$ runs over all faces of $T$, defines a horn $\Lambda^\alpha[T] \to Y$. Since this horn factors through the $k$-skeleton of $Y$ we obtain, by applying $f_k$, a horn $\Lambda^\alpha[T] \to X$ in $X$ given by $\{f(\beta^y)\}_{\beta \neq \alpha}$. Let $f_{k+1}(y) \in X_T$ be the unique filler of that horn. By construction we have for each $\beta \neq \alpha$ that

$$\beta^y f_{k+1}(y) = f(\beta^y)$$

it thus remains to show the same for $\alpha$. The dendrices $f(\alpha^y)$ and $\alpha^y f_{k+1}(y)$ both have the same boundary and they are both of shape $S$ where $S$ has $k$ vertices. Since $k \geq 2$, $S$ has an inner face, but then it follows that both $f(\alpha^y)$ and $\alpha^y f_{k+1}(y)$ are fillers for the same inner horn in $X$ which proves that they are equal. By repeating the process for all dendrices in $Sk_{k+1}(Y)$ it follows that $f_k$ can be extended to $f_{k+1} : Sk_{k+1}(Y) \to Sk_{k+1}(X)$. This holds for all $k \geq 2$ which implies that $f$ can be extended to $\hat{f} : Y \to X$. To show uniqueness of $\hat{f}$ assume that $g$ is another extension of $f$. Suppose it has been shown that $\hat{f}$ and $g$ agree on all dendrices of shape $T$ where $T$ has at most $k$ vertices, and let $y \in X_S$ be a dendrex of shape $S$ where $S$ has $k + 1$ vertices. But then the dendrices $\hat{f}(y)$ and $g(y)$ are dendrices in $X$ that have the same boundary. Since $k \geq 2$ it follows that these dendrices are both fillers for the same inner horn and so are the same. This proves that $\hat{f} = g$. 

6. THE OPERAD GENERATED BY AN INNER KAN COMPLEX

We recall that $\tau_d : dSet \to Operad$ denotes the left adjoint to the dendroidal nerve functor $N_d$. In this section, we will give a more explicit description of the operad $\tau_d(X)$ in the case where $X$ is an inner Kan complex. This description extends the one in [BV] of the category generated by a simplicial set satisfying the restricted Kan condition. It will lead to a proof of the following converse of Proposition 5.3.

Theorem 6.1. For any strict inner Kan complex $X$, the canonical map $X \to N_d(\tau_d(X))$ is an isomorphism.

Proposition 5.3 and Theorem 6.1 together state that a dendroidal set is a strict inner Kan complex iff it is the nerve of an operad.
Consider an inner Kan complex $X$. For the description of $\tau_d(X)$, we first fix some notation. For each $n \geq 0$ let $C_n$ be the $n$-corolla:

![Diagram of a tree with nodes labeled 0, 1, and n]

and for each $0 \leq i \leq n$ recall that $i : \eta \to C_n$ denotes the obvious (outer face) map in $\Omega$ that sends the unique edge of $\eta$ to the edge $i$ in $C_n$. An element $f \in X_{C_n}$ will be denoted by

![Diagram of a tree with a node labeled f]

If $C_n'$ is another $n$-corolla together with an isomorphism $\alpha : C_n' \to C_n$ then we will usually write $f$ again instead of $\alpha^*(f)$. We will use this convention quite often in the coming definitions and constructions, and in each case there will be an obvious choice for the isomorphism $\alpha$ given by the planar representation of the trees at question, which will usually not be mentioned.

**Definition 6.2.** Let $X$ be an inner Kan complex and let $f, g \in X_{C_n}, n \geq 0$. For $1 \leq i \leq n$ we say that $f$ is homotopic to $g$ along the edge $i$, and write $f \sim_i g$, if there is a dendrex $H$ of shape

![Diagram of a triangle with nodes labeled 1, i, and n]

whose three faces are:

![Diagrams of three triangles with nodes labeled 1, i, n, 0, and an id]

$1 \sim_i 0$.
where the third one denotes a degeneracy. Similarly we will say that \( f \) is homotopic to \( g \) along the edge \( 0 \) and write \( f \sim_0 g \) if there is a dendrex of shape
\[
\begin{array}{c}
\begin{array}{cc}
1 & n \\
\downarrow & \downarrow \\
0 & 0' \\
\end{array}
\end{array}
\]
whose three faces are:
\[
\begin{array}{ccc}
\begin{array}{cc}
0 & 1 \\
\downarrow & \downarrow \\
0' & g \\
\end{array} & \begin{array}{cc}
1 & n \\
\downarrow & \downarrow \\
0 & 0' \\
\end{array} & \begin{array}{cc}
1 & f \\
\downarrow & \downarrow \\
0 & 0' \\
\end{array}
\end{array}
\]
When \( f \sim_i g \) for some \( 0 \leq i \leq n \) we will refer to the corresponding \( H \) as a homotopy from \( f \) to \( g \) along \( i \) and will sometimes write \( H : f \sim_i g \).

**Proposition 6.3.** Let \( X \) be an inner Kan complex. For each \( 0 \leq i \leq n \) the relation \( \sim_i \) on the set \( X_{C_n} \) is an equivalence relation.

**Proof.** First we prove reflexivity. For \( 1 \leq i \leq n \) let
\[
\begin{array}{ccc}
\begin{array}{cc}
1 & i \\
\downarrow & \downarrow \\
0 & n \\
\end{array} & \begin{array}{cc}
1 & i \\
\downarrow & \downarrow \\
0 & n \\
\end{array} & \begin{array}{cc}
1 & i \\
\downarrow & \downarrow \\
0 & n \\
\end{array}
\end{array}
\]
and for \( i = 0 \) let
\[
\begin{array}{ccc}
\begin{array}{cc}
1 & 0 \\
\downarrow & \downarrow \\
0 & n \\
\end{array} & \begin{array}{cc}
1 & 0 \\
\downarrow & \downarrow \\
0 & n \\
\end{array} & \begin{array}{cc}
1 & 0 \\
\downarrow & \downarrow \\
0 & n \\
\end{array}
\end{array}
\]
be the obvious degeneracies. It then follows that for any \( f \in X_{C_n} \) the dendrex \( \sigma_i^*(f) \) is a homotopy from \( f \) to \( f \), thus \( f \sim_i f \).
To prove symmetry assume $f \sim g$ for some $1 \leq i \leq n$ and let $H_{fg}$ be a homotopy from $f$ to $g$ along $i$. Consider the tree $T$:

![Tree Diagram]

For the inner horn $\Lambda^i[T]$, corresponding to the edge $i$ in the tree above, we now describe a map $\Lambda^i[T] \to X$. Such a map is given by specifying three dendrices in $X$ of certain shapes such that their faces match in a suitable way. We describe this map by explicitly writing the mentioned dendrices and their faces:

\[
\begin{align*}
H_i & \quad H_f & \quad H_{fg} \\
\begin{array}{c}
\text{id} \\
i \\
i \\
i \\
\end{array} & \quad \begin{array}{c}
\text{id} \\
i \\
i \\
i \\
\end{array} & \quad \begin{array}{c}
\text{id} \\
i \\
i \\
i \\
\end{array}
\end{align*}
\]

with inner faces of these dendrices:

\[
\begin{align*}
\begin{array}{c}
\text{id} \\
i \\
i \\
i \\
\end{array} & \quad \begin{array}{c}
\text{id} \\
i \\
i \\
i \\
\end{array} & \quad \begin{array}{c}
\text{id} \\
i \\
i \\
i \\
\end{array}
\end{align*}
\]

where $H_i$ is a double degeneracy of $i$, $H_f$ is a homotopy from $f$ to $f$ (along the branch $i$) and $H_{fg}$ is the given homotopy from $f$ to $g$. It is easily checked that the faces indeed match so that we have a horn in $X$. Let $x$ be a filler for that horn and
consider $H_{gf} = \partial^*_i(x)$. This dendrex can be pictured as

\[
\begin{array}{c}
\bullet & \bullet \\
1 & n \\
g \downarrow & \downarrow \\
0 & 0
\end{array}
\]

with inner face:

\[
\begin{array}{c}
\bullet & \bullet \\
1 & n \\
f \downarrow & \downarrow \\
0 & 0
\end{array}
\]

and is thus a homotopy from $g$ to $f$ along $i$, so that $g \sim_i f$. For $i = 0$ a similar proof works.

To prove transitivity let $f \sim_i g$ and $g \sim_i h$ for $1 \leq i \leq n$. Let $H_{fg}$ be a homotopy from $f$ to $g$ and let $H_{gh}$ be a homotopy from $g$ to $h$. We again consider the tree $T$:

\[
\begin{array}{c}
\bullet & \bullet & \bullet \\
1 & i & n \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

This time we look at $\Lambda^i_j[T]$. The following describes a map $\Lambda^i_j[T] \rightarrow X$ in $X$:

\[
\begin{array}{ccc}
H_i & H_{gh} & H_{fg} \\
\begin{array}{c}
\bullet & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow \\
1 & i & n \\
0 & 0 & 0
\end{array} & \begin{array}{c}
\bullet & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow \\
1 & i & n \\
0 & 0 & 0
\end{array} & \begin{array}{c}
\bullet & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow \\
1 & i & n \\
0 & 0 & 0
\end{array}
\end{array}
\]
with inner faces being:

Let \( x \) be a filler for that horn and let \( H_{fh} = \partial^n_i(x) \). This dendrex can be pictured as follows:

with inner face:

and is thus a homotopy from \( f \) to \( h \) so that \( f \sim_i h \). The proof for \( i = 0 \) is similar.

**Lemma 6.4.** Let \( X \) be an inner Kan complex. The relations \( \sim_0, \cdots, \sim_n \) on \( X_{C_n} \) are all equal.

**Remark 6.5.** On the basis of this lemma, we will later just write \( f \sim g \) instead of \( f \sim_i g \).

**Proof.** Suppose \( H : f \sim g \) for \( 1 \leq i \leq n \) and let \( 1 \leq i < j \leq n \). We consider the tree \( T \):

and the inner horn \( \Lambda^i[T] \). The following then describes a map \( \Lambda^i[T] \to X \):
where $H^i_f : f \sim_j f$ and $H^i_f : f \sim_i f$. The inner faces of the three dendrises are

Let $x$ be a filler for this horn, then $\partial^i_1(x)$ is the following dendrex

with inner face:

and is thus a homotopy from $g$ to $f$ along the $j$-th branch. Thus $g \sim_j f$ and so $f \sim_j g$ as well. The other cases to be considered follow in a similar way.

Given an inner Kan complex $X$ and vertices $x_1, \ldots, x_n, x \in X_\eta$, let us write

$X(x_1, \ldots, x_n; x) \subseteq X(C_n)$

for the set of dendrices $x$ of shape $C_n$ with $0^*(x) = x$ and $i^*(x) = x_i$ for $i = 0, \ldots, n$. Here $i : \eta \to C_n$ denotes the map in $\Omega$ sending the unique edge of $\eta$ to the one of $C_n$ with name $i$. The equivalence relation $\sim$ on $X(C_n)$ given by the preceding lemma defines a quotient of $X(C_n)$ which we will denote by

$Ho(X)(x_1, \ldots, x_n; x) = X(x_1, \ldots, x_n; x)/\sim$. 
This defines a coloured collection $\text{Ho}(X)$, and a canonical quotient map of collections $\text{Sk}_1(X) \to \text{Ho}(X)$. We will now proceed to prove the following.

**Proposition 6.6.** There is a unique structure of a (symmetric, coloured) operad on $\text{Ho}(X)$ for which the map of collections $\text{Sk}_1(X) \to \text{Ho}(X)$ extends to a map of dendroidal sets $X \to N_d(\text{Ho}(X))$. The latter map is an isomorphism whenever $X$ is a strict inner Kan complex.

To prepare for the proof of this proposition, we begin by defining the composition operations $\circ_i$ of the operad $\text{Ho}(X)$. Let $X$ be an inner Kan complex and let $f \in X_{C_n}$ and $g \in X_{C_m}$ be two dendrices in $X$. We will say that a dendrex $h \in X_{C_{n+m-1}}$ is a $\circ_i$-composition of $f$ and $g$ if there is a dendrex $\gamma$ in $X$ as follows:

![Diagram](attachment:image.png)

with inner face

![Diagram](attachment:image.png)

We will denote this situation by $h \sim f \circ_i g$ and call $\gamma$ a *witness* for the composition.

**Remark 6.7.** Notice that for $1 \leq i \leq n$ we have by definition that $H : f \sim_i g$ iff $H$ is a witness for the composition $g \sim f \circ_i id$. Similarly for $i = 0$ we have that $H : f \sim_0 g$ iff $H$ is a witness for the composition $g \sim id \circ f$.

**Lemma 6.8.** In an inner Kan complex $X$, if $h \sim f \circ_i g$ and $h' \sim f \circ_i g$ then $h \sim h'$.

**Proof.** Let $\gamma$ be a witness for the composition $h \sim f \circ_i g$ and $\gamma'$ one for the composition $h' \sim f \circ_i g$. We consider the tree $T$:

![Diagram](attachment:image.png)
and the inner horn $\Lambda^i[T]$. Let $H_g : g \sim_i g$ and consider the following map $\Lambda^i[T] \to X$

\[ \gamma \quad \gamma' \quad H_g \]

with inner faces

Let $x$ be a filler for this horn. The face $\partial^n_1 x$ is then the dendrex

\[ \begin{array}{c}
\end{array} \]

whose inner face is

\[ \begin{array}{c}
\end{array} \]

which proves that $h \sim h'$.

**Lemma 6.9.** In an inner Kan complex $X$, let $f \sim f'$ and $g \sim g'$. If $h \sim f \circ_i g$ and $h' \sim f' \circ_i g'$ then $h \sim h'$.

**Proof.** Let $H$ be a homotopy from $f$ to $f'$ along the edge $i$, $H'$ a homotopy from $g'$ to $g$ along the root, and $\gamma$ a witness for the composition $h \sim f \circ_i g$. We now
consider the tree $T$:

and the inner horn $\Lambda^i[T]$. The following is then a map $\Lambda^i[T] \to X$ in $X$:

with inner faces:

The missing face of a filler for this horn is then:
with inner face

\[
\begin{array}{c}
1'' \quad m'' \quad n \\
\downarrow \quad \downarrow \quad \downarrow \\
1 \quad 1' \quad n \\
\end{array}
\]

which proves that \( h \sim f' \circ_i g' \), and thus by the previous result also that \( h \sim h' \). \( \Box \)

We now proceed to prove Proposition 5.6:

**Proof.** (of Proposition 5.6) Lemma 5.8 implies that for

\[ [f] \in \text{Ho}(X)(x_1, \cdots, x_n; x) \]

and

\[ [g] \in \text{Ho}(X)(y_1, \cdots, y_m; x_i) \]

the assignment

\[ [f] \circ_i [g] = [f \circ_i g] \]

is well-defined. This provides the \( \circ_i \) operations in the operad \( \text{Ho}(X) \). The \( \Sigma_n \) actions are defined as follows. Given a permutation \( \sigma \in \Sigma_n \) let \( \sigma : C_n \to C_n \) be the obvious induced map in \( \Omega \). The map \( \sigma^* : X_{C_n} \to X_{C_n} \) restricts to a function

\[ \sigma^* : X(x_1, \cdots, x_n; x) \to X(x_{\sigma(1)}, \cdots, x_{\sigma(n)}; x) \]

and it is trivial to verify that this map respects the homotopy relation. We thus obtain a map

\[ \sigma^* \text{Ho}(X)(x_1, \cdots, x_n; x) \to \text{Ho}(X)(x_{\sigma(1)}, \cdots, x_{\sigma(n)}; x). \]

We now need to show that these structure maps make the coloured collection \( \text{Ho}(X) \) into an operad. The verification is simple and we exemplify it by proving associativity. Let \([f] \in \text{Ho}(X)(x_1, \cdots, x_n; x)\), \([g] \in \text{Ho}(X)(y_1, \cdots, y_m; x_i)\) and \([h] \in \text{Ho}(X)(z_1, \cdots, z_k; y_m)\). We need to prove that \([f] \circ ([g] \circ [h]) = ([f] \circ [g]) \circ [h]\) (with the obvious choice for subscripts on the \( \circ \)) which is the same as showing that \( f \circ (g \circ h) \sim (f \circ g) \circ h \) for any choice for compositions \( \psi \sim g \circ h \) and \( \varphi \sim f \circ g \). Consider the tree \( T \) given by

\[
\begin{array}{c}
1'' \quad k'' \\
1' \quad j \\
1 \quad i \quad m' \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \\
\end{array}
\]
and consider the anodyne extension \( \Lambda^i[T] \to \Omega[T] \), cf. Lemma 5.1. The two given compositions \( \psi \sim g \circ h \) and \( \varphi \sim f \circ g \) define a map \( \Lambda^i[T] \to X \) depicted by

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (0) at (0,0) {0};
  \node (1) at (-1,1) {1};
  \node (i) at (-1,2) {i};
  \node (m) at (-1,3) {m};
  \node (g) at (-2,3) {g};
  \node (j) at (-2,4) {j};
  \node (n) at (-2,5) {n};
  \node (f) at (-3,5) {f};
  \coordinate (1') at (-1.5,1); \coordinate (m') at (-1.5,3);

  \draw[->] (0) to (1);
  \draw[->] (0) to (i);
  \draw[->] (0) to (m);
  \draw[->] (1) to (i);
  \draw[->] (1) to (m);
  \draw[->] (1') to (i);
  \draw[->] (1') to (m');
  \draw[->] (m') to (g);
  \draw[->] (m') to (j);
  \draw[->] (m') to (n);
  \draw[->] (g) to (f);

\end{tikzpicture}
\end{array}
\]

whose inner faces are respectively \( \psi \) and \( \varphi \). Let \( x \in X_T \) be a dendrex extending this map. Let \( c : C_m \to T \) be the map obtained by contracting both \( i \) and \( j \) and \( \rho = c^*x \). It now follows that \( \partial^*_i x \) is a witness for the composition \( \rho \sim \psi \circ h \) and \( \partial^*_j x \) is a witness for the composition \( \rho \sim f \circ \varphi \). That proves associativity. The other axioms for an operad follow in a similar manner.

Next, let us show that the quotient map \( q : Sk_1(X) \to Ho(X) \) extends to a map \( q : X \to N_d(Ho(X)) \) of dendroidal sets. Since we already know that \( N_d(X) \) is 2-coskeletal, it suffices to give its values for dendrices \( x \in X_T \) where \( T \) is a tree with two vertices. Let \( e \) be the inner edge of this tree. Then \( \Lambda^e[T] \to \Omega[T] \to X \) factors through \( Sk_1(X) \), so its composition \( \Lambda^e[T] \to N_d(Ho(X)) \) with \( q \) has a unique extension (Proposition 5.3), which we take to be \( q(x) : \Omega[T] \to N_d(Ho(X)) \). This defines \( q : Sk_2(X) \to Sk_2(N_d(Ho(X))) \), and hence all of \( q : X \to N_d(Ho(X)) \) by 2-coskeletality, as said.

Finally, when \( X \) is itself a strict inner Kan complex, then the homotopy relation is the identity relation, so \( Sk_1(X) \to Ho(X) \) is the identity map. Since \( X \) and \( N_d(Ho(X)) \) are now both strict inner Kan complexes, the extension \( q : X \to N_d(Ho(X)) \) is an isomorphism. \( \square \)

The following Proposition, together with Proposition 6.6, now provide the proof of Theorem 6.1.

**Proposition 6.10.** For any inner Kan complex \( X \), the natural map \( \tau_d(X) \to Ho(X) \) is an isomorphism of operads.

**Proof.** It suffices to prove that the map \( q : X \to N_d(Ho(X)) \) of Proposition 6.6 has the universal property of the unit of the adjunction. This means that for any operad \( \mathcal{P} \) and any map \( \varphi : X \to N_d(\mathcal{P}) \), there is a unique map of operads \( \psi : Ho(X) \to \mathcal{P} \) for which \( N_d(\psi)q = \varphi \). But \( \varphi \) induces a map \( Ho(X) \to Ho(N_d(\mathcal{P})) \) for which

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (Sk1) at (0,0) {Sk_1(X)};
  \node (SkP) at (1,0) {Sk_1 N_d(\mathcal{P})};
  \node (HoX) at (0,-1) {Ho(X)};
  \node (HoP) at (1,-1) {Ho(N_d(\mathcal{P}))};
  \node (qX) at (0,-2) {q_X};
  \node (qP) at (1,-2) {q_{\mathcal{P}}};

  \draw[->] (Sk1) to (SkP);
  \draw[->] (Sk1) to (HoX);
  \draw[->] (HoX) to (HoP);
  \draw[->] (HoP) to (qP);
  \draw[->] (Sk1) to (qX);

\end{tikzpicture}
\end{array}
\]

commutes, and \( Ho(N_d(\mathcal{P})) = \mathcal{P} \) while \( q_{\mathcal{P}} \) is the identity as we have seen in (the proof of) Proposition 6.6. So \( Ho(\varphi) \) in fact defines a map \( \psi : Ho(X) \to \mathcal{P} \) of collections. It is easily seen that \( \psi \) is a map of operads. It is unique because \( q_X \) is surjective. \( \square \)
7. Homotopy coherent nerves of operads

In this section, we assume $\mathcal{E}$ is a monoidal model category with a cofibrant unit $I$. We also assume that $\mathcal{E}$ is equipped with an interval in the sense of [BM]. Such an interval is given by maps

$$I \xrightarrow{0} H \xrightarrow{\epsilon} I$$

and

$$H \otimes H \xrightarrow{\triangledown} H$$

satisfying certain conditions. In particular, $H$ is an interval in Quillen’s sense ([Q]), so 0 and 1 together define a cofibration $I \xrightarrow{\epsilon} H$, and $\epsilon$ is a weak equivalence. Such an interval $H$ allows one to construct for each (coloured) operad $\mathcal{P}$ in $\mathcal{E}$ a ”Boardman-Vogt” resolution $W_H(\mathcal{P}) \to \mathcal{P}$. Each operad in $\text{Set}$ can be viewed as an operad in $\mathcal{E}$ (via the functor $\text{Set} \to \mathcal{E}$ which preserves coproducts and sends the one-point set to $I$), and hence has such a Boardman-Vogt resolution. When we apply this to the operads $\Omega(T)$, we obtain the homotopy coherent dendroidal nerve $hcN_d(\mathcal{P})$ of any operad $\mathcal{P}$ in $\mathcal{E}$, as the dendroidal set given by

$$hcN_d(\mathcal{P})_T = \text{Hom}(W_H(\Omega(T)), \mathcal{P})$$

where the Hom is that of operads in $\mathcal{E}$. See [MW] for a more detailed description and examples. Our goal here is to prove the following result.

**Theorem 7.1.** Let $\mathcal{P}$ be an operad in $\mathcal{E}$, with the property that for each sequence $c_1, \cdots, c_n; c$ of colours of $\mathcal{P}$, the object $\mathcal{P}(c_1, \cdots, c_n; c)$ is fibrant. Then $hcN_d(\mathcal{P})$ is an inner Kan complex.

**Remark 7.2.** As explained in [MW], our construction of the dendroidal homotopy coherent nerve specializes to that of the homotopy coherent nerve of an $\mathcal{E}$-enriched category, and for the case where $\mathcal{E}$ is the category of topological spaces or simplicial sets, one recovers the classical definition ([CP]). In particular, as a special case of Theorem 7.1, one obtains that for an $\mathcal{E}$-enriched category with fibrant Hom objects (in other words, for a locally fibrant $\mathcal{E}$-enriched category), its homotopy coherent nerve is a quasi-category in the sense of Joyal. This result was proved, for the case where $\mathcal{E}$ is simplicial sets, by Cordier and Porter in [CP].

Before embarking on the proof of Theorem 7.1, we need to be a bit more explicit about the operads of the form $W_H \Omega(T)$ involved in the definition of the homotopy coherent nerve. Recall first of all the functor

$$\text{Symm} : \text{Operad}(\mathcal{E})_{\text{sym}} \to \text{Operad}(\mathcal{E})$$

which is left adjoint to the forgetful functor from symmetric operads to non-symmetric (i.e., planar) ones. If $T$ is an object in $\Omega$ and $\bar{T}$ is a chosen planar representative of $T$, then $\bar{T}$ naturally describes a planar operad $\Omega(\bar{T})$ for which $\Omega(T) = \text{Symm}(\Omega(\bar{T}))$. Since the $W$-construction commutes with symmetrization (as one readily verifies), it follows that

$$W_H(T) = \text{Symm}(W_H \Omega(\bar{T})).$$

This latter operad $W_H \Omega(\bar{T})$ is easily described explicitly. The colours of $W_H(\Omega(\bar{T}))$...
are the colours of \( \Omega(T) \), i.e., the edges of \( T \). By a **signature**, we mean a sequence \( e_1, \ldots, e_n; e_0 \) of edges. Given a signature \( \sigma = (e_1, \ldots, e_n; e_0) \), we have that \( W_H(\Omega(T))(\sigma) = 0 \) whenever \( \Omega(T)(\sigma) = \phi \). And if \( \Omega(T)(\sigma) \neq \phi \), there is a subtree \( T_\sigma \) of \( T \) (and a corresponding planar subtree \( \overline{T}_\sigma \) of \( \overline{T} \)) whose leaves are \( e_1, \ldots, e_n \), and whose root is \( e_0 \). Then

\[
W_H \Omega(\overline{T})(e_1, \ldots, e_n; e_0) = \bigotimes_{f \in i(\sigma)} H,
\]

where \( i(\sigma) \) is the set of inner edges of \( T_\sigma \) (or of \( \overline{T}_\sigma \)). (This last tensor product is to be thought of as the "space" of assignments of lengths to inner edges in \( \overline{T}_\sigma \); it is the unit if \( i(\sigma) \) is empty.)

**Remark 7.3.** The composition operations in the operad \( W_H \Omega(\overline{T}) \) are given in terms of the \( \circ_i \)-operations as follows. For signatures \( \sigma = (e_1, \ldots, e_n; e_0) \) and \( \rho = (f_1, \ldots, f_m; e_i) \), the composition map

\[
\Omega(T)(e_1, \ldots, e_n; e_0) \otimes \Omega(T)(f_1, \ldots, f_m; e_i)
\]

\[
\circ_i
\]

\[
\Omega(T)(e_1, \ldots, e_{i-1}, f_1, \ldots, f_m, e_{i+1}, \ldots, e_n; e_0)
\]

is the following one. The trees \( \overline{T}_\sigma \) and \( \overline{T}_\rho \) can be grafted along \( e_i \) to form \( \overline{T}_\sigma \circ_{e_i} \overline{T}_\rho \), again a planar subtree of \( T \). In fact

\[
\overline{T}_\sigma \circ_{e_i} \overline{T}_\rho = \overline{T}_{\sigma \circ_i \rho}
\]

where \( \sigma \circ_i \rho \) is the signature \( (e_1, \ldots, e_{i-1}, f_1, \ldots, f_m, e_{i+1}, \ldots, e_n; e_0) \), and for the sets of inner edges we have

\[
i(\sigma \circ_i \rho) = i(\sigma) \cup i(\rho) \cup \{e_i\}.
\]

The composition map in (1) now is the map

\[
H \otimes i(\sigma) \otimes H \otimes i(\rho) \quad \xrightarrow{\cong} \quad H \otimes i(\sigma \circ_i \rho)
\]

where \( 1 : I \to H \) is one of the "endpoints" of the interval \( H \), as above.

This description of the operad \( W_H \Omega(\overline{T}) \) is functorial in the planar tree \( T \). In particular, we note that for an inner edge \( e \) of \( T \), the tree \( \overline{T}/e \) inherits a planar structure \( \overline{T}/e \) from \( T \), and \( W_H \Omega(\overline{T}/e) \to W_H \Omega(\overline{T}) \) is the natural map assigning length 0 to the edge \( e \) whenever it occurs (in a subtree given by a signature).

**Proof.** (Of Theorem 7.1) Consider a tree \( T \) and an inner edge \( e \) in \( T \). We want to solve the extension problem

\[
\Lambda^e[T] \xrightarrow{\varphi} hcN_d(\mathcal{P})
\]

\[
\Omega[T]
\]
Fix a planar representative $\hat{T}$ of $T$. Then the desired map $\psi : \Omega[T] \to hcN_d(\mathcal{P})$ corresponds to a map of planar operads
\[ \hat{\psi} : W_H\Omega(\hat{T}) \to \mathcal{P}. \]
Each face $S$ of $T$ inherits a planar structure $\hat{S}$ from $\hat{T}$, and the given map $\varphi : \Lambda^e[T] \to hcN_d(\mathcal{P})$ corresponds to a map of operads in $\mathcal{E}$,
\[ \hat{\varphi} : W_H(\Lambda^e[T]) \to \mathcal{P}, \]
where $W_H(\Lambda^e[T])$ denotes the colimit of operads in $\mathcal{E}$,
\[ W_H(\Lambda^e[T]) = \text{colim} W(\Omega(\hat{S})) \] (2)
over all but one of the faces of $T$. In other words, $\varphi$ corresponds to a compatible family of maps
\[ \hat{\varphi}_S : W_H(\Omega(\hat{S})) \to \mathcal{P} \]
Let us now show the existence of an operad map $\hat{\psi}$ extending the $\hat{\varphi}_S$ for all faces $S \neq T/e$. First, the colours of $\Omega(\hat{T})$ are the same as those of the colimit in (2), so we already have a map $\psi_0 = \varphi_0$ on colours:
\[ \psi_0 : (\text{Edges of } T) \to (\text{Colours of } \mathcal{P}) \]
Next, if $\sigma = (e_1, \cdots, e_n, e_0)$ is a signature of $T$ for which $W_H(\Omega(\hat{T})) \neq \emptyset$, and if $T_{\sigma} \subseteq T$ is not all of $T$, then $T_{\sigma}$ is contained in an outer face $S$ of $T$. So $W_H(\Omega(\hat{T}))(\sigma) = W_H(\Omega(T_{\sigma}))(\sigma) = W_H(\Omega(\hat{S}))(\sigma)$, and we already have a map
\[ \hat{\varphi}_S(\sigma) : W_H(\Omega(\hat{T}))(\sigma) \to \mathcal{P}(\sigma), \]
given by $\hat{\varphi}_S : W_H(\Omega(\hat{S})) \to \mathcal{P}$. Thus, the only part of the operad map $\hat{\psi} : W_H(\Omega(\hat{T})) \to \mathcal{P}$ not determined by $\varphi$ is the one for the signature $\tau$ where $T_{\tau} = T$; i.e., $\tau = (e_1, \cdots, e_n, e_0)$ where $e_1, \cdots, e_n$ are all the input edges of $\hat{T}$ (in the planar order) and $e_0$ is the output edge. For this signature, $\hat{\psi}(\tau)$ is to be a map
\[ \hat{\psi} : W_H(\Omega(\hat{T}))(\tau) = H^{\otimes i(\tau)} \to \mathcal{P}(\tau) \]
which (i) is compatible with the $\hat{\psi}(\sigma) = \hat{\varphi}_S(\sigma)$ for other signatures $\sigma$; and (ii) together with these $\hat{\psi}(\sigma)$ respects operad composition. The first condition determines $\hat{\psi}(\tau)$ on the subobject of $H^{\otimes i(\tau)}$ which is given by a value $0$ on one of the tensor-factors marked by an edge $e_i$ other than the given $e$. The second condition determines $\hat{\psi}(\tau)$ on the subobject of $H^{\otimes i(\tau)}$ which is given by a value $1$ on one of the factors. Thus, if we write $1$ for the map $I \xrightarrow{1} H$ and $\partial H \xrightarrow{H^{\otimes k}} H$ for the map $I \coprod I \to H$, and define $\partial H^{\otimes k} \xrightarrow{H^{\otimes k}} H^{\otimes k}$ by the Leibniz rule (i.e., $\partial(A \otimes B) = \partial(A) \otimes B \cup A \otimes \partial(b)$), then the problem of finding $\hat{\psi}(\tau)$ comes down to an extension problem of the form
\[ \partial(H^{\otimes i(\sigma)} - \{e\} \otimes H) \cup H^{\otimes i(\sigma)} - \{e\} \otimes I \xrightarrow{\hat{\psi}(\sigma)} \mathcal{P}(\tau) \]
\[ \xrightarrow{\hat{\psi}(\sigma)} H^{\otimes i(\sigma)} - \{e\} \otimes H \approx H^{\otimes i(\sigma)} \]
This extension problem has a solution, because $\mathcal{P}(\tau)$ is fibrant by assumption, and because the left hand map is a trivial cofibration (by repeated use of the push-out
product axiom for monoidal model categories). This concludes the proof of the theorem.

8. Grothendieck construction for dendroidal sets

Let $\mathcal{S}$ be a Cartesian category. A functor $X : \mathcal{S}^{op} \to dSet$ is called a diagram of dendroidal sets. In [MW] a construction was given of the dendroidal set $\int_X S$. This construction was then applied to the specific diagram of dendroidal sets $X : Set^{op} \to dSet$, where for a set $A$, $X(A)$ was the dendroidal set of weak $n$-categories having $A$ as set of objects. The dendroidal set $\int_X S$ was defined to be the dendroidal set of weak $n$-categories. Our aim in this section is to prove that for a given diagram of dendroidal sets $X : \mathcal{S}^{op} \to dSet$, if each $X(S)$ is an inner Kan complex then $\int_X S$ is also an inner Kan complex. For the convenience of the reader we repeat here the definition of $\int_X S$.

It will be convenient to consider dendroidal collections. A dendroidal collection is a collection of sets $X = \{X_T\}_{T \in \Omega}$. Each dendroidal set has an obvious underlying dendroidal collection. A map of dendroidal collections $X \to Y$ is a collection of functions $\{X_T \to Y_T\}_{T \in \Omega}$. Given a Cartesian category $\mathcal{S}$, consider the dendroidal nerve $N_d(\mathcal{S})$ where $\mathcal{S}$ is regarded as an operad via the Cartesian structure. There is a natural way of associating an object of $\mathcal{S}$ with each dendrex of $N_d(\mathcal{S})$. For a tree $T$ in $\Omega$, let $\text{leaves}(T)$ be the set of leaves of $T$, and for a leaf $l$, write $l : \eta \to T$ also for the map sending the unique edge in $\eta$ to $l$ in $T$. Then, since $\mathcal{S}$ is assumed to have finite products, each dendrex $t \in N_d(\mathcal{S})_T$ defines an object

$$in(t) = \prod_{l \in \text{leaves}(T)} l^*(t)$$

in $\mathcal{S}$. Notice that if $\alpha : S \to T$ is a composition of face maps, then by using the canonical symmetries and the projections in $\mathcal{S}$ there is a canonical arrow $in(\alpha) : in(t) \to in(\alpha^*t)$ for any $t \in X_T$.

**Definition 8.1.** Let $X : \mathcal{S}^{op} \to dSet$ be a diagram of dendroidal sets. The dendroidal set $\int_X S$ is defined as follows. A dendrex $\Omega[T] \to \int_X S$ is a pair $(t, x)$ such that $t \in N_d(\mathcal{S})_T$ and $x$ is a map of dendroidal collections

$$x : \Omega[T] \to \prod_{S \in ob(\mathcal{S})} X(S)$$

satisfying the following conditions. For each $r \in \Omega[T]_R$ (that is an arrow $r : R \to T$), we demand that $x(r) \in X(in(r^*t))$. Furthermore we demand the following compatibility conditions to hold. For any $r \in \Omega[T]_R$ and any map $\alpha : U \to R$ in $\Omega$

$$\alpha^*(x(r)) = X(in(\alpha)) x(\alpha^*(r)).$$

**Theorem 8.2.** Let $X : \mathcal{S}^{op} \to dSet$ be a diagram of dendroidal sets. If for any $S \in ob(\mathcal{S})$ the dendroidal set $X(S)$ is a (strict) inner Kan complex then so is $\int_X S$.

**Proof.** Let $T$ be a tree and $e$ an inner edge. We consider the extension problem

$$\Lambda^e[T] \to \int_X S$$

$$\Omega[T]$$
The horn \( \Lambda^c[T] \to \int_S X \) is given by a compatible collection \( \{(r, x_R) : \Omega(R) \to \int_S X\}_{R \neq T/e} \). We wish to construct a dendrrix \( (t, x_T) : \Omega[T] \to \int_S X \) extending this family. First notice that the collection \( \{r\}_{R \neq T/e} \) is an inner horn \( \Lambda^c[T] \to \mathcal{N}_d(S) \) (actually this horn is obtained by composing with the obvious projection \( \int_S X \to \mathcal{N}_d(S) \) sending a dendrrix \((t, x)\) to \( t \)). We already know \( \mathcal{N}_d(S) \) to be an inner Kan complex (actually a strict inner Kan complex) and thus there is a (unique) filler \( t \in \mathcal{N}_d(S)_T \) for the horn \( \{r\}_{R \neq T/e} \). We now wish to define a map of dendroidal collections \( x_T : \Omega[T] \to \prod_{S \in \mathcal{N}_d(S)} X(S) \) that will extend the given maps \( x_R \) for \( R \neq T/e \). This condition already determines the value of \( x_T \) for any dendrrix \( r : U \to T \) other then \( id : T \to T \) and \( \alpha : T/e \to T \), since for each such \( r \), the tree \( U \) factors through one of the faces \( R \neq T/e \). To determine \( x_T(id_T) \) and \( x_T(\alpha) \) consider the family \( \{y_R = x_R(id : R \to R)\}_{R \neq T/e} \). By definition we have that \( y_R \in X(in(r))_.R \). For each such \( R \) let \( \alpha_R : R \to T \) be the corresponding face map in \( \Omega \). Since \( \alpha^*t = r \) we obtain the map \( in(\alpha_R) : in(r) \to in(t) \). We can now pull back the collection \( \{y_R\}_{R \neq T/e} \) using \( X(in(\alpha_R)) \) to obtain a collection \( \{y_{R'} = X(in(\alpha_R)(y_R))\}_{R' \neq T/e} \). This collection is now a horn \( \Lambda^c[T] \to X(in(T)) \) (this follows from the compatibility conditions in the definition of \( \int_S X \)). Since \( X(in(t)) \) is inner Kan there is a filler \( u \in X(in(t))_T \) for that horn. We now define \( x_T(id : T \to T) = u \) and \( x_T(\alpha : T/e \to T) = \alpha^*(u) \). Notice that since \( c \) is inner we have that \( in(t) = in(\alpha) \) and thus the image of these dendrrixes are in the correct dendroidal set, namely \( X(in(t)) \). It follows from our construction that this makes \((t, x_T)\) a dendrrix \( \Omega[T] \to \int_S X \) which extends the given horn. This concludes the proof. 

\[ \square \]

9. The exponential property

Our aim in this section is to prove the following theorem concerning the closed monoidal structure of dendroidal sets.

**Theorem 9.1.** Let \( K \) and \( X \) be dendroidal sets, and assume \( X \) is normal. If \( K \) is a (strict) inner Kan complex, then so is \( \text{Hom}_{\mathcal{H}Set}(X, K) \).

The internal \( \text{Hom} \) here is defined by the universal property, giving a bijective correspondence between maps \( Y \otimes X \to K \) and \( Y \to \text{Hom}(X, K) \) for any dendroidal set \( Y \), and natural in \( Y \). We recall from Section 2 that \( \otimes \) is defined in terms of the Boardman-Vogt tensor product of operads. We remind the reader that for two (coloured) operads \( \mathcal{P} \) and \( \mathcal{Q} \) with respective sets of colours \( C \) and \( D \), this tensor product operad \( \mathcal{P} \otimes_{BV} \mathcal{Q} \) has the product \( C \times D \) as its set of colours, and is described in terms of generators and relations as follows. The operations in \( \mathcal{P} \otimes_{BV} \mathcal{Q} \) are generated by the operations

\[
p \otimes d \in \mathcal{P} \otimes_{BV} \mathcal{Q}((c_1, d), \ldots, (c_n, d); (c, d))
\]

for any \( p \in \mathcal{P}(c_1, \ldots, c_n; c) \) and any \( d \in D \), and

\[
c \otimes q \in \mathcal{P} \otimes_{BV} \mathcal{Q}((c, d_1), \ldots, (c, d_m); (c, d))
\]

for any \( q \in \mathcal{Q}(d_1, \ldots, d_m; d) \) and any \( c \in C \). The relations between these state, first of all, that for fixed \( c \in C \) and \( d \in D \), the maps \( p \to p \otimes d \) and \( c \to c \otimes q \) are maps of operads. Secondly, there is an interchange law stating that, for \( p \) and \( q \) as above, the composition \( p \otimes d(c \otimes q, \ldots, c \otimes q) \) in

\[
\mathcal{P} \otimes_{BV} \mathcal{Q}((c_1, d_1), \ldots, (c_1, d_m), \ldots, (c_n, d_1), \ldots, (c_n, d_m); (c, d))
\]
and \( c \otimes q(p \otimes d, \cdots, p \otimes d) \) in

\[
\mathcal{P} \otimes_{BV} Q((c_1, d_1), \cdots, (c_n, d_1), \cdots, (c_1, d_m), \cdots, (c_n, d_m); (c, d))
\]

are mapped to each other by the obvious permutation \( \tau \in \Sigma_{n \times m} \) which puts the two sequences of input colours in the same order. The tensor product of dendroidal sets is then uniquely determined (up to isomorphism) by the fact that it preserves colimits in each variable separately, together with the identity

\[
\Omega[S] \otimes \Omega[T] = N_d(\Omega(S) \otimes_{BV} \Omega(T))
\]

stated in Section 2, which gives the tensor product of two representable dendroidal sets.

First of all, let us prove that Theorem 9.1 follows by a standard argument from the following proposition.

**Proposition 9.2.** For any two objects \( S \) and \( T \) of \( \Omega \), and any inner edge \( e \) in \( S \), the map

\[
\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \to \Omega[S] \otimes \Omega[T]
\]

is an anodyne extension.

In the proposition above, the union is that of subobjects of the codomain, which is the same as the pushout over the intersection \( \Lambda^e[S] \otimes \partial\Omega[T] \).

**Proof.** (of Theorem 9.1 from Proposition 9.2) The theorem states that for any tree \( S \) and any inner edge \( e \in S \), any map of dendroidal sets

\[
\varphi : \Lambda^e[S] \otimes X \to K
\]

extends to some map (uniquely in the strict case)

\[
\psi : \Omega[S] \otimes X \to K.
\]

By writing \( X \) as the union of its skeleta,

\[
X = \lim_{\longrightarrow} Sk_n(X)
\]

as in Section 4, and using the fact that the skeletal filtration is normal, we can build this extension \( \psi \) by induction on \( n \). For \( n = 0 \), \( Sk_0(X) \) is a sum of copies of \( \Omega[\eta] \), the unit for the tensor product, so obviously the restriction \( \varphi_0 : \Lambda^e[S] \otimes Sk_0(X) \to K \) extends to a map

\[
\psi_0 : \Omega[S] \otimes Sk_0(X) \to K.
\]
Suppose now that we have found an extension \( \psi_n : \Omega[S] \otimes Sk_n(X) \to K \) of the restriction \( \varphi_n : \Lambda^e[S] \otimes Sk_n(X) \to K \). Consider the following diagram:

\[
\begin{array}{ccc}
\prod \Lambda^e[S] \otimes \partial \Omega[T] & \longrightarrow & \prod \Lambda^e[S] \otimes \Omega[T] \\
\downarrow & & \downarrow \\
\Lambda^e[S] \otimes Sk_n(X) & \longrightarrow & \Lambda^e[S] \otimes Sk_{n+1}(X) \\
\downarrow & & \downarrow \\
\prod \Omega[S] \otimes \partial \Omega[T] & \longrightarrow & \prod \Omega[S] \otimes \Omega[T] \\
\downarrow & & \downarrow \\
\Omega[S] \otimes Sk_n(X) & \longrightarrow & \Omega[S] \otimes Sk_{n+1}(X)
\end{array}
\]

In this diagram, the top and bottom faces are pushouts given by the normal skeletal filtration of \( X \). Now inscribe the pushouts \( U \) and \( V \) in the back and front face, fitting into a square

\[
\begin{array}{ccc}
U & \longrightarrow & \prod \Omega[S] \otimes \Omega[T] \\
\downarrow & & \downarrow \\
V & \longrightarrow & \Omega[S] \otimes Sk_{n+1}(X)
\end{array}
\]

The maps \( \psi_n : \Omega[S] \otimes Sk_n(X) \to K \) and \( \varphi_{n+1} : \Lambda^e[S] \otimes Sk_{n+1}(X) \to K \) together define a map \( V \to K \). So, to find \( \psi_{n+1} \), it suffices to prove that

\[
V \longrightarrow \Omega[S] \otimes Sk_{n+1}(X)
\]

is anodyne. But, by a diagram chase argument, the square above is a pushout, so in fact, it suffices to prove that \( U \longrightarrow \prod \Omega[S] \otimes \Omega[T] \) is anodyne. The latter map is a sum of copies of anodyne extensions as in the statement of the proposition. \( \square \)

**Corollary 9.3.** The monoidal structure on the category of coloured operads given by the Boardman-Vogt tensor product is closed (see [MW]). It is related to the closed monoidal structure on dendroidal sets by two natural isomorphisms

\[
\tau_d(N_dP \otimes N_dQ) = P \otimes_{BV} Q
\]

and

\[
N_d(Hom(Q, R)) = Hom(N_dQ, N_dR)
\]

for any operads \( P, Q \) and \( R \).

**Proof.** The first isomorphism was proved in [MW]. The second isomorphism follows from the first one together with (the strict version of) Theorem 9.1, Theorem 6.1, and the fact that \( N_d \) is fully faithful. \( \square \)

In the rest of this section, we will be concerned with the proof of Proposition 9.2, and we fix \( S, T \) and \( e \) as in the statement of the proposition from now on. Our strategy will be as follows. First, let us write

\[
A_0 \subseteq \Omega[S] \otimes \Omega[T]
\]
for the dendroidal set given by the image of $\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T]$. We are going to construct a sequence of dendroidal subsets

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_N = \Omega[S] \otimes \Omega[T]$$

such that each inclusion is an anodyne extension. This will be done by writing $\Omega[S] \otimes \Omega[T]$ as a union of representables, as follows. We will explicitly describe a sequence of trees

$$T_1, T_2, \ldots, T_N$$

together with canonical monomorphisms (all called)

$$m : \Omega[T_i] \to \Omega[S] \otimes \Omega[T],$$

and we will write $m(T_i) \subseteq \Omega[S] \otimes \Omega[T]$ for the dendroidal subset given by the image of this monomorphism. We will then define

$$A_{i+1} = A_i \cup m(T_{i+1}) \quad (i = 0, \ldots, N-1)$$

and prove that each $A_i \to A_{i+1}$ thus constructed is anodyne. For the rest of this section, we will fix planar structures on the trees $S$ and $T$. These will then induce a natural planar structure on each of the trees $T_i$, and avoid unnecessary discussion involving automorphisms in the category $\Omega$.

To define the $T_i$, let us think of the vertices of $S$ as white (drawn $\circ$) and those of $T$ as black (drawn $\bullet$). The edges of $T_i$ are (labelled by) pairs $(a,x)$ where $a$ is an edge of $S$ and $x$ one of $T$. We refer to $a$ as the $S$-colour of this edge $(a,x)$, and to $x$ as its $T$-colour. There are two kinds of vertices in $T_i$ (corresponding to the generators for $\Omega[S] \otimes \Omega[T]$ coming from vertices of $S$ or of $T$). There are white vertices in $T_i$ labelled

$$\begin{array}{c}
(a_1, x) \\
\vdots \\
(a_n, x) \\
\downarrow \\
(v, x) \\
\downarrow \\
(b, x)
\end{array}$$

where $v$ is a vertex in $S$ with input edges $a_1, \ldots, a_n$ and output edge $b$, while $x$ is an edge of $T_i$; and there are black vertices in $T_i$ labelled

$$\begin{array}{c}
(a, x_1) \\
\vdots \\
(a, x_m) \\
\downarrow \\
(w) \\
\downarrow \\
(a, y)
\end{array}$$

where $w$ is a vertex in $T$ with input edges $x_1, \ldots, x_m$ and output edge $y$, while $a$ is an edge in $S$. Moreover, each such tree $T_i$ is maximal in the sense that its output (root) edge is labelled $(r_S, r_T)$ where $r_S$ and $r_T$ are the roots of $S$ and $T$, and its input edges are labelled by all pairs $(a, x)$ where $a$ is an input edge of $S$ and $x$ one of $T$.

All the possible such trees $T_i$ come in a natural (partial) order. The minimal tree $T_1$ in the poset is the one obtained by stacking a copy of the black tree $T$ on top of each of the input edges of the white tree $S$. Or, more precisely, on the bottom of
$T_1$ there is a copy $S \otimes r_T$ of the tree $S$ all whose edges are renamed $(a, r_T)$ where $r_T$ is the output edge at the root of $T$. For each input edge $b$ of $S$, a copy of $T$ is grafted on the edge $(b, r)$ of $S \otimes r$, with edges $x$ in $T$ renamed $(b, x)$. The maximal tree $T_N$ in the poset is the similar tree with copies of the white tree $S$ grafted on each of the input edges of the black tree. Pictorially $T_1$ looks like

\begin{center}
\includegraphics[width=0.3\textwidth]{t1.png}
\end{center}

and $T_N$ looks like

\begin{center}
\includegraphics[width=0.3\textwidth]{tn.png}
\end{center}

The intermediate trees $T_k$ ($1 < k < N$) are obtained by letting the black vertices in $T_1$ slowly percolate in all possible ways towards the root of the tree. Each $T_k$ is obtained from an earlier $T_l$ by replacing a configuration

\begin{center}
\includegraphics[width=0.3\textwidth]{tk.png}
\end{center}
in $T_l$ by

\[
\begin{array}{c}
\begin{array}{c}
\text{(B)} \\
\end{array}
\end{array}
\]

in $T_k$. More explicitly, if $v$ and $w$ are vertices in $S$ and $T$,

then the edges in (A) are named

\[
\begin{array}{c}
\begin{array}{c}
\text{(a, y)} \\
\text{(b, y)} \\
\text{(a, x_j)} \\
\text{(b, x_j)} \\
\end{array}
\end{array}
\]

and those in (B) are named

\[
\begin{array}{c}
\begin{array}{c}
\text{(a, x_j)} \\
\text{(b, y)} \\
\end{array}
\end{array}
\]

We will refer to these trees $T_i$ as the percolation schemes for $S$ and $T$, and if $T_k$ is obtained from $T_l$ by replacing (A) by (B), then we will say that $T_l$ is obtained by a single percolation step.

**Example 9.4.** Many of the typical phenomena that we will encounter already occur for the following two trees $S$ and $T$; here, we have singled out one particular edge $e$ in $S$, we’ve numbered the edges of $T$ as 1, $\cdots$, 5, and denoted the colour $(e, i)$ in $T_l$ by $e_i$.

\[
S = \begin{array}{c}
\begin{array}{c}
\text{e} \\
\end{array}
\end{array} \\
T = \begin{array}{c}
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4} \\
\text{5} \\
\end{array}
\end{array}
\]

There are 14 percolation schemes $T_1, \cdots, T_{14}$ in this case. Here is the complete list of them:
As claimed, there is a partial order on the percolation schemes $T_1, \ldots, T_N$ for $S \otimes T$, in which $T_1$ (copies of $T$ on top of $S$) is the minimal element and $T_N$ (copies of $S$ on top of $T$) the maximal one. The partial order is given by defining $T \leq T'$ whenever the percolation scheme $T'$ can be obtained from the percolation scheme $T$ by a sequence of percolations. For example, the poset structure on the percolation
trees above is:

\[
\begin{array}{c}
T_1 \\
| \\
T_2 \\
| \\
T_3 & T_4 \\
| & | \\
T_7 & T_5 & T_9 \\
| & | & |
T_8 & T_{10} & T_{12} \\
| & | & |
T_{11} & T_{13} \\
| & |
T_{14}
\end{array}
\]

The planar structures of \(S\) and \(T\) provide a way to refine this partial order by a linear order. It is not important exactly how this is done, but we shall from now on assume that the percolation schemes for \(S\) and \(T\) are ordered \(T_1, \ldots, T_N\) where \(T_i\) comes before \(T_j\) only if \(T_i \leq T_j\) in the partial order.

**Lemma 9.5.** (and notation) Each percolation scheme \(T_i\) is equipped with a canonical monomorphism

\[
m : \Omega[T_i] \rightarrow \Omega[S] \otimes \Omega[T].
\]

The dendroidal subset given by the image of this monomorphism will be denoted

\[
m(T_i) \subseteq \Omega[S] \otimes \Omega[T].
\]

**Proof.** The vertices of the dendroidal set \(\Omega[T_i]\) are the edges of the tree \(T_i\). The map \(m\) is completely determined by asking it to map an edge named \((a, x)\) in \(T_i\) to the vertex with the same name in \(\Omega[S] \otimes \Omega[T]\). This map is a monomorphism. In fact, any map

\[
\Omega[R] \rightarrow X,
\]

from a representable dendroidal set to an arbitrary one, is a monomorphism as soon as the map \(\Omega[R]_\eta \rightarrow X_\eta\) on vertices is.

Before we continue, we need to introduce a bit of terminology for trees, i.e., for objects of \(\Omega\). Let \(R\) be such a tree. A map \(R' \rightarrow R\) which is a composition of basic face maps (maps of type (ii) or (iii) in Section 3) will also be referred to as a face of \(R\), just like for simplicial sets. If it is a composition of inner faces (resp. outer faces), the map \(R' \rightarrow R\) will be called an inner face (resp. outer face) of \(R\). A top face of \(R\) is an outer face map \(\partial_k : R' \rightarrow R\) where \(R'\) is obtained by deleting a
top vertex from \( R \). An initial segment \( R' \rightarrow R \) is a composition of top faces (it is a special kind of outer face of \( R \)). If \( v \) is the vertex above the root of \( R \) and \( e \) is an input edge of \( v \), then \( R \) contains a subtree \( R' \) whose root is \( e \). We'll refer to an inclusion of this kind as a bottom face of \( R \) (it is again a special kind of outer face). In all these cases, we'll often leave the monomorphism \( R' \rightarrow R \) implicit, and apply the same terminology not only to the map \( R' \rightarrow R \) but also to the tree \( R' \).

For example, for the tree \( T \) constructed above

\[
T = \begin{array}{c}
3 \\
2 \\
1 \\
4 \\
5
\end{array}
\]

The following sub-trees are examples of, respectively, a top face, an initial segment, a bottom face, and an inner face:

\[
\begin{array}{c}
2 \\
1 \\
4 \\
5
\end{array}
\quad
\begin{array}{c}
2 \\
1 \\
4 \\
3 \\
5
\end{array}
\quad
\begin{array}{c}
2 \\
1 \\
4 \\
3 \\
5
\end{array}
\quad
\begin{array}{c}
2 \\
1 \\
4 \\
3 \\
5
\end{array}
\]

**Remark 9.6.** We observe the following simple properties, which we will repeatedly use in the proofs of the lemmas below. In stating these properties and below, we denote by \( m(R) \) the image of the composition of the inclusion \( \Omega[R] \rightarrow \Omega[T_i] \) given by a subtree (a face) \( R \) of \( T_i \) and the canonical monomorphism

\[
m : \Omega[T_i] \rightarrow \Omega[S] \otimes \Omega[T].
\]

(i) Let \( R \) be a subtree of \( T_i \). If \( m(R) \subseteq A_0 \) then \( R \) misses a \( T \)-colour, or an \( S \)-colour other than \( e \), or a stump of either \( S \) or \( T \). Here, a stump is a top vertex of valence zero (i.e., without input edges). We say that \( R \) “misses” such a stump \( v \in S \), for example, if \( m(R) \subseteq \partial_v[S] \otimes \Omega[T] \). The tree \( R \) is a sub-tree of \( T_i \), where edges are coloured by pairs \((a, x)\), where \( a \) is an \( S \)-colour and \( x \) a \( T \)-colour. By saying that \( R \) ”misses” a \( T \)-colour \( y \), we mean that none of the colours \((a, x)\) occurring in \( R \) has \( x = y \) as second coordinate. "Missing an \( S \)-colour" is interpreted similarly.

(ii) This implies in particular that for any bottom face \( R \rightarrow T_i \) of any percolation scheme \( T_i \) the dendroidal set \( m(R) \) is contained in \( A_0 \), because it must miss either the root colour \( r_s \) (in case the root of \( T_i \) is white), or the root colour \( r_T \) (in case the root of \( T_i \) is black), and \( r_S \neq e \) because \( e \) is assumed inner.

(iii) If \( F, G \) are faces of \( T_i \), then \( F \) is a face of \( G \) iff \( m(F) \subseteq m(G) \). (This is clear from the fact that the map from \( \Omega[R] \) onto its image \( m(R) \) is an isomorphism of dendroidal sets.)

(iv) Let \( Q \) and \( R \) be initial segments of \( T_i \), and let \( F \) be an inner face of \( Q \). If \( m(F) \subseteq m(R) \) then also \( m(Q) \subseteq m(R) \) (and hence \( Q \) is a face of \( R \), by (iii)). In fact, let \( \text{Inn}(Q) \) denote the set of all inner edges of \( Q \) and \( Q/\text{Inn}(Q) \rightarrow Q \) the inner
face of $Q$ given by contracting all these. Then if $m(Q/\text{Inn}(Q)) \subseteq m(R/\text{Inn}(R))$, it follows by comparing labels of input edges of $Q$ and $R$ that $Q$ is a face of $R$.

These remarks prepare the ground for the following lemma. Recall that $A_k = A_0 \cup m(T_1) \cup \cdots \cup m(T_k)$, where $m(T_i)$ is the image in $\Omega[S] \otimes \Omega[T]$ of the dendroidal set $\Omega[T_i]$.

**Lemma 9.7.** Let $R, Q_1, \cdots, Q_p$ be a family of initial segments in $T_{k+1}$ and write $B = m(Q_1) \cup \cdots \cup m(Q_p) \subseteq \Omega[S] \otimes \Omega[T]$. Suppose 
(i) For every top face $F$ of $R$, $m(F) \subseteq A_k \cup B$.
(ii) There exists an edge $\xi$ in $R$ such that for every inner face $F \rightarrow R$, if $m(F)$ is not contained in $A_k \cup B$ then neither is $m(F/\xi)$.

Then the inclusion $A_k \cup B \rightarrow A_k \cup B \cup m(R)$ is anodyne.

We call $\xi$ a characteristic edge of $R$ with respect to $Q_1, \cdots, Q_p$.

**Proof.** If $m(R) \subseteq A_k \cup B$ there is nothing to prove. If not, then by (ii), $m(R/\xi)$ is not contained in $A_k \cup B$. Let

$$\xi = \xi_0, \xi_1, \cdots, \xi_n$$

be all the inner edges in $R$ such that the dendroidal set $m(R/\xi_i)$ is not contained in $A_k \cup B$. For a sub-sequence $\xi_{i_1}, \cdots, \xi_{i_p}$ of these $\xi_0, \cdots, \xi_n$, we have the dendroidal subset of $\Omega[S] \otimes \Omega[T]$,

$$m(R/\xi_{i_1}, \cdots, \xi_{i_p}),$$

obtained by contracting each of $\xi_{i_1}, \cdots, \xi_{i_p}$ and composing with $m : \Omega[T_{k+1}] \rightarrow \Omega[S] \otimes \Omega[T]$. We are going to consider a sequence of anodyne extensions

$$A_k \cup B = B_0 \rightarrow B_1 \rightarrow \cdots \rightarrow B_{2^n} = A_k \cup B \cup m(R)$$

by considering images of faces of $\Omega[R]$ of this type (1).

Consider first

$$R_{(0)} = m(R/\xi_{i_1}, \cdots, \xi_{i_p}).$$

If $m(R_{(0)})$ is contained in $A_k \cup B$, let $B_1 = B_0 = A_k \cup B$. Otherwise, let $B_1$ be the pushout

$$m(\Lambda^0 R_{(0)}) \rightarrow B_0 \rightarrow m(R_{(0)}) \rightarrow B_1$$

Notice that $m(\Lambda^0 R_{(0)})$ is indeed contained in $B_0 = A_k \cup B$. For, any outer face $F$ of $R_{(0)}$ is a face of an outer face $G$ of $R$

$$F \rightarrow R/(\xi_1, \cdots, \xi_n) = R_{(0)}$$

if $G$ is a top face, then $m(G) \subseteq A_k \cup B$ by assumption (i); and if $G$ is a bottom face, it already factors through $A_0 \subseteq A_k$ (cf Remark 9.6 (ii) before the lemma). On the other hand, if $F \subseteq R_{(0)}$ is an inner face of $R_{(0)}$ given by contracting an edge $\zeta$ in $R/(\xi_1, \cdots, \xi_n)$, then $F$ is a face of $R/(\zeta)$. So if $m(F) \subseteq B_0$ then $m(R/(\zeta))$ wouldn’t
be contained in \( B_0 \) either, and hence \( \zeta \) must be one of \( \xi_0, \ldots, \xi_n \). But \( \xi_1, \ldots, \xi_n \) are no longer edges in \( R/(\xi_1, \ldots, \xi_n) \), so \( \zeta \) must be \( \xi_0 \). This shows that for any inner face \( F \) of \( R(0) \) other then \( R(0)/(\xi_0) \), the dendroidal set \( m(F) \) is contained in \( B_0 \), as claimed.

Next, consider all sub-sequences \( (\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_n) \), and the faces

\[
R(i) = R/(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_n) \quad i = 1, \ldots, n
\]

We will define

\[
B_2, \ldots, B_{n+1}
\]

by considering these \( R(1), \ldots, R(n) \). Suppose \( B_3, \ldots, B_i \) have been defined. Consider \( R(i) \) to form \( B_{i+1} \). If its image \( m(R(i)) \) is contained in \( B_i \), let \( B_{i+1} = B_i \). Otherwise, \( m(R(i)) \to \Omega[S] \otimes \Omega[T] \) does not factor through \( B_i \), and a fortiori doesn’t factor through \( A_k \cup B = B_0 \) either. So by assumption (ii), we have that \( m(R(i)/(\xi_0)) \not\subseteq A_k \cup B \). But then \( m(R_i/(\xi_0)) \) is not contained in \( B_i \) either, because by Remark 9.6(iv), if \( m(R_i/(\xi_0)) \) would be contained in one of \( m(R(0)), \ldots, m(R(1-i)) \), then \( R_i/(\xi_0) \) would be a face of one of \( R(0), \ldots, R(1-i) \), which is obviously not the case. On the other hand, \( \xi_0 \) is the only edge of \( R(i) \) for which \( m(R(i)/(\xi_0)) \) is not contained in \( B_i \) (indeed, the only other candidate would be \( \xi_i \), but \( R(i)/\xi_i = R(0) \) and \( m(R(0)) \subseteq B_1 \)). So, we can form the pushout

\[
\begin{array}{ccc}
m(\Lambda^0 R(i)) & \longrightarrow & B_i \\
\downarrow & & \downarrow \\
m(R(i)) & \longrightarrow & B_{i+1}
\end{array}
\]

Next, consider for each \( i < j \) the tree

\[
R_{ij} = R/(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_j, \ldots, \xi_n),
\]

and order these lexicographically, say as

\[
R^2_1, \ldots, R^2_u, \quad (u = \binom{n}{2}).
\]

We are going to form anodyne extensions of \( B_{n+1} \) by using these trees.

\[
B_{n+1} \longrightarrow B_{n+2} \longrightarrow \cdots \longrightarrow B_{n+1+u},
\]

treating \( R^2_p \) in the step to form \( B_{n+p} \to B_{n+p+1} \) (for each \( p = 1, \ldots, u \)). Suppose \( B_{n+p} \) has been formed, and consider \( R^2_p = R_{ij} \) say. If \( m(R^2_p) \subseteq B_{n+p} \) then let \( B_{n+p+1} = B_{n+p} \). If not, then surely \( m(R^2_p) \not\subseteq A_k \cup B \), so by assumption (ii) \( m(R^2_p/(\xi_0)) = m(R/(\xi_0, \xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_j, \ldots, \xi_n)) \) is not contained in \( A_k \cup B \). On the other hand, Remark 9.6(iv) implies that \( m(R^2_p/(\xi_0)) \) cannot be contained in any of \( m(R_1), \ldots, m(R_n), m(R^2_i), \ldots, m(R^2_{p-1}) \) either. So \( m(R^2_p/(\xi_0)) \) is not contained in \( B_{n+p} \). As before, \( \xi_0 \) is the only inner edge \( \zeta \) for which \( m(R^2_p/(\zeta)) \) is not contained in \( B_{n+p} \). So we can form the pushout

\[
\begin{array}{ccc}
m(\Lambda^0(R^2_p)) & \longrightarrow & B_{n+p} \\
\downarrow & & \downarrow \\
m(R^2_p) & \longrightarrow & B_{n+p+1}
\end{array}
\]
Next consider for each $i_1 < i_2 < i_3$ the tree
\[ R_{(i_1 i_2 i_3)} = R/\langle \xi_1, \cdots, \hat{\xi}_{i_1}, \cdots, \hat{\xi}_{i_2}, \cdots, \xi_{i_3} \rangle \]
and adjoin the pushout along
\[ m(\Lambda^k R_{(i_1 i_2 i_3)}) \rightarrow m(R_{(i_1 i_2 i_3)}) \]
if necessary. Continuing in this way for all $l = 0, 1, \cdots, n-1$ and all sub-sequences $i_1 < \cdots < i_l$ and corresponding trees
\[ R/\langle \xi_1, \cdots, \hat{\xi}_{i_1}, \cdots, \hat{\xi}_{i_l}, \cdots, \xi_{n} \rangle, \]
we end up with a sequence of anodyne extensions
\[ B_1 \rightarrow \cdots \rightarrow B_q \]
where $q = \Sigma^n_{i=0} \binom{n}{i} = 2^n - 1$, and where $m(R/\langle \xi_i \rangle)$ is contained in $B_q$ for each $i = 1, \cdots, n$. In the final step, and exactly as before, we let $B_{2^n} = B_{2^n-1}$ if $m(R) \subseteq B_{2^n-1}$; and if not, we form the pushout
\[
\begin{array}{c}
m(\Lambda^3(R)) \quad \xrightarrow{B_{2^n-1} \quad m(R)} \quad B_{2^n}
\end{array}
\]
Then $B_{2^n}$ is the pushout of $A_0 \cup B$ and $m(R)$ over $(A_0 \cup B) \cap m(R)$ (because every face $F$ of $R$ for which $m(F)$ is contained in $A_0 \cup B$ occurs in some corner of the pushouts taken in the construction of the $B_i$). This proves the lemma. \qed

Consider the tree $T_{k+1}$, and look at all lowest occurrences of the $S$-colour $e$ (Recall $e$ is the fixed edge in $S$, occurring in the statement of Proposition 9.2). More precisely, let $e_i = (e, x_i)$ for $i = 1, \cdots, t$ be all the edges in $T_{k+1}$ whose $S$-colour is $e$, while the $S$-colour of the edge immediately below it isn’t. This means that $(e, x_i)$ is an edge having a white vertex at its bottom. Let $\beta_i$ be the branch in $T_{k+1}$ from the root to and including this edge $e_i$. Each such $\beta_i$ is an initial segment in $T_{k+1}$, to which we will refer as the spine through $e_i$. For example, this is a picture of a spine in $T_{k+1}$,

\[
\begin{array}{c}
w_i \quad e_i \quad v_i
\end{array}
\]
corresponding to the edge $e$ in $S$
Lemma 9.8. Let \( R, Q_1, \ldots, Q_p \) be initial segments in \( T_{k+1} \), as in the preceding lemma, and suppose condition (i) of that lemma is satisfied. Then for any spine \( \beta_i \) contained in \( R \), the edge \( e_i \) is characteristic for \( R \) with respect to \( Q_1, \ldots, Q_p \).

Proof. We have to check condition (ii) of Lemma 9.7. So, suppose \( F \) is an inner face of \( R \), and suppose \( m(F(e_i)) \) is contained in \( A_k \cup B = A_0 \cup m(T_1) \cup \cdots \cup m(T_k) \cup m(Q_1) \cup \cdots \cup m(Q_p) \subseteq m(T_{k+1}) \). Since \( m(F(e_i)) \) is isomorphic to the representable dendroidal set \( \Omega[F(e_i)] \), it must be contained in one of the dendroidal sets constituting this union. But, if \( m(F(e_i)) \) is contained in \( A_0 \), then by Remark 9.6 (ii) \( m(F) \) is also contained in \( A_0 \). (the only colour occurring in \( F \) but possibly not in \( F(e_i) \) is the \( S \)-colour \( e \)). And, if \( m(F(e_i)) \) is contained in \( m(T_j) \) for some \( j \leq k \), then there must be a tensor product relation applying to the image of \( F(e_i) \), which allows a black vertex to move up so as to get into an earlier \( T_j \), as in:

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{array}
\end{array}
\]

where the left tree is in \( T_{k+1} \), the right tree is in \( T_j \), and the middle one in \( F(e_i) \).

But then the same relation must apply to the image of \( F \), because the edge \( e_i \), having a white vertex at its root, cannot contribute to this relation. Finally, if \( m(F(e_i)) \) is contained in \( m(Q_l) \) for some \( l \leq p \), then by Remark 9.6 (iv), we have \( m(R) \subseteq m(Q_1) \). So a fortiori, \( m(F) \) is contained in \( m(Q_i) \). This proves the lemma. \( \square \)

Recall that our aim is to prove for \( A_k = A_0 \cup m(T_1) \cup \cdots \cup m(T_k) \) that each inclusion

\[ A_k \hookrightarrow A_{k+1} \]

is anodyne. Consider the tree \( T_{k+1} \), and let \( \beta_1, \ldots, \beta_l \) be all the spines contained in it. We shall prove by induction that \( A_k \hookrightarrow A_k \cup m(R_1) \cup \cdots \cup m(R_q) \) is anodyne, for any family \( R_1, \ldots, R_q \) of initial segments each of which contains at least one such spine. The induction will be on the number of such initial segments as well as on their size. When applied to the maximal initial segment \( T_{k+1} \) itself, this will show that \( A_k \hookrightarrow A_k \cup m(T_{k+1}) = A_{k+1} \) is anodyne, as claimed. The precise form of induction is given by the following lemma.

Lemma 9.9. Fix \( l \) with \( 0 \leq l \leq t \). Let \( Q_1, \ldots, Q_p \) be a family of initial segments in \( T_{k+1} \), each containing at least one and at most \( l \) spines. Let \( R_1, \ldots, R_q \) be initial segments, each of which contains \( l+1 \) spines. Then the inclusion

\[ A_k \hookrightarrow A_k \cup B \cup C \]

for \( B = m(Q_1) \cup \cdots \cup m(Q_p) \) and \( C = m(R_1) \cup \cdots \cup m(R_q) \), is anodyne.

Proof. We can measure the size of each of the initial segments \( R_j \) by counting the number of vertices in \( R_j \), which are not on one of the \( l+1 \) spines. If this number is not bigger than \( u \), we say that \( R_j \) has size at most \( u \), and write \( size(R_j) \leq u \). Let \( \Lambda(l, u) \) be the assertion that the lemma holds for \( l \), for any families \( \{ Q_i \} \) and \( \{ R_j \} \), where the \( R_j \) all have \( size(R_j) \leq u \). We will prove \( \Lambda(l, u) \) by induction, first on \( l \) and then on \( u \).
Case $l = 0$: This is the case where there are no $Q$’s, i.e., $p = 0$. For $l = 0$, first consider the case where $u = 0$ also. Then each $R_i$ is itself a spine, say $\beta_i$, with top inner edge $e_i$ running from a copy of $v$ to a copy $w_i$ of $w$. We will prove that each of the inclusions

$$A_k \cup m(R_1) \cup \cdots \cup m(R_{i-1}) \rightarrow A_k \cup m(R_1) \cup \cdots \cup m(R_i)$$

for $i = 0, \ldots, q$, is anodyne. If $R_i = \beta_i$ coincides with one of the earlier spines $R_j$, $j < i$, then there is nothing to prove. If $R_i$ is a different spine, then its outer top face is contained in $A_0$ because it misses the vertex $v'$ which is above $e$ in $S$. So condition (i) of Lemma 9.7 is satisfied, where $R_i, R_1, \ldots, R_{i-1}$ take the role of $R, Q_1, \ldots, Q_p$ in that lemma. By Lemma 9.8, the edge $e_i \in R_i$ is characteristic so Lemma 9.7 gives that $A_k \cup m(R_1) \cup \cdots \cup m(R_{i-1}) \rightarrow A_k \cup m(R_1) \cup \cdots \cup m(R_i)$ is anodyne, as claimed. The composition of these inclusions will then be anodyne also, which proves the statement $\Lambda(0, 0)$.

Suppose now that $\Lambda(0, u)$ has been proved, and consider families $R_1, \ldots, R_q$ of initial segments which are each of size not bigger than $u + 1$. Suppose that among these, $R_1, \ldots, R_{q'}$ actually have size not bigger than $u$, while $R_{q'+1}, \ldots, R_q$ have size $u + 1$. We shall prove that

$$A_k \rightarrow A_k \cup m(R_1) \cup \cdots \cup m(R_q)$$

is anodyne, by induction on the number $r = q - q'$ of initial segments that have size $u + 1$. If $r = 0$, this holds by $\Lambda(0, u)$. Suppose we have proved this for any family with not more than $r$ initial segments of size $u + 1$, and consider such a family $R_1, \ldots, R_q$ where $q - q' = r + 1$. Write $\beta_q$ for the spine contained in $R_q$ (there is only one such because we are still in the case $l = 0$). For a top outer face $\partial_x(R_q)$ of $R_q$, either $x \neq w_q$ so that $\partial_x(R_q)$ still contains $\beta_q$, but has size at most $u$, or $x = w_q$ so that $m(\partial_x(R_q))$ is contained in $A_0$ because it misses the vertex $v'$ immediately above $e$ in $S$. Thus, if we let

$$P = m(R_1) \cup \cdots \cup m(R_{q-1}) \cup \bigcup_x m(\partial_x(R_q))$$

where $x$ ranges over all the top vertices in $R_q$, then by the fact that $\Lambda(0, u + 1)$ is assumed to hold for $r = (q - 1) - q'$,

$$A_k \rightarrow A_k \cup P$$

is anodyne. To prove that $A_k \cup P \rightarrow A_k \cup P \cup m(R_q)$ is anodyne as well, we can now apply Lemma 9.7. Indeed, the family of initial segments containing $P$ is made to contain the images of all the top faces of $R_q$, and $e_q \in R_q$ is characteristic by Lemma 9.8. This proves that $A_k \cup P \rightarrow A_k \cup P \cup m(R_q)$ is anodyne, as claimed. When composed with (1), we find that $A_k \rightarrow A_k \cup m(R_1) \cup \cdots \cup m(R_q)$ is anodyne. This proves $\Lambda(0, u + 1)$ and completes the inductive proof of $\Lambda(l, u)$ for $l = 0$ and all $u$.

Suppose now that we have proved $\Lambda(l', u)$ for all $l' \leq l$ and all $u$. We will now prove $\Lambda(l + 1, u)$ by induction on $u$.

Case $u = 0$: This is the assertion that for any given initial segments

$$Q_1, \ldots, Q_p, R_1, \ldots, R_q$$
of $T_{k+1}$, where the $Q_j$ contain at most $l$ spines while each $R_i$ is made up out of exactly $l + 1$ spines (and no other vertices), the inclusion

$$A_k \mapsto A_k \cup m(Q_1) \cup \cdots \cup m(Q_p) \cup m(R_1) \cup \cdots \cup m(R_q)$$

(2)
is anodyne. We shall prove by induction on $q$ that this holds for all $p$. For $q = 0$, the conclusion follows by the inductive assumption that $\Lambda(l, u)$ holds. Suppose the assertion holds for $q - 1$, and consider $R_q$. Each top vertex of $R_q$ lies at the end of a spine, so $\partial_x(R_q)$ contains at most $l$ spines. Let

$$D = m(Q_1) \cup \cdots \cup m(Q_p) \cup \bigcup_x \partial_x(R_q)$$

where $x$ ranges over the top vertices of $R_q$. Then, by the assumption for $q - 1$,

$$A_k \mapsto A_k \cup D \cup m(R_1) \cup \cdots \cup m(R_{q-1})$$

(3)
is anodyne. To prove that (2) is anodyne, it then suffices to apply Lemma 9.7, and show that $R_q$ has a characteristic edge with respect to the family of initial segments containing the union $D \cup m(R_1) \cup \cdots \cup m(R_{q-1})$ in (3). But by Lemma 9.8, any top edge $e_q$ of $R_q$ is characteristic. This proves $\Lambda(l + 1, u)$, for $u = 0$.

Case $u + 1$: Suppose now $\Lambda(l + 1, u)$ holds. To prove $\Lambda(l + 1, u + 1)$, consider families

$$Q_1, \ldots, Q_p, R_1, \ldots, R_{q'}, R_{q' + 1}, \ldots, R_q$$

(4)
of initial segments in $T_{k+1}$, where the $Q_i$ contain at most $l$ spines, the $R_i$ contain exactly $l + 1$ spines, the $R_1, \ldots, R_{q'}$ are of size not more than $u$, and $R_{q' + 1}, \ldots, R_q$ are of size exactly $u + 1$. We will show by induction on the last number $r = q - q'$ that for any such family, the inclusion

$$A_k \mapsto A_k \cup m(Q_1) \cup \cdots \cup m(Q_p) \cup m(R_1) \cup \cdots \cup m(R_q)$$

(5)
is anodyne. For $r = q - q' = 0$ there is nothing to prove, because this is the case covered by $\Lambda(l + 1, u)$. Suppose we have proved that (5) is anodyne for any family (4) with $q - q' \leq r$, and consider such a family with $q - q' = r + 1$. The initial segment $R_q$ has two kinds of top outer faces, namely the $\partial_x(R_q)$ which remove the top of a spine, and the $\partial_x(R_q)$ where $x$ does not lie on a spine. Outer faces of the first kind contain $l$ spines only, and outer faces of the second kind are of size not more than $u$. Let

$$D = m(Q_1) \cup \cdots \cup m(Q_p) \cup \bigcup_x m(\partial_x R_q)$$

where $x$ ranges over the top vertices of $R_q$ which are on a spine. Let

$$E = m(R_1) \cup \cdots \cup m(R_{q'}) \cup \bigcup_x m(\partial_x (R_q))$$

where $x$ ranges over the top vertices of $R_q$ which are not on a spine. Then, by the assumption that $\Lambda(l + 1, u + 1)$ has been established for families (4) where $q - q' \leq r$, we see that

$$A_k \mapsto A_k \cup D \cup E \cup R_{q' + 1} \cup \cdots \cup R_{q-1}$$

(6)
is anodyne. The union $D \cup E$ is made to contain all the images $\partial_x (R_q)$ of top faces $\partial_x (R_q)$ of $R_q$, and by Lemma 9.8, any edge $e_q$ on the top of a spine $\beta_q$ in $R_q$ is
characteristic with respect to the family of initial segments making up the union on the right-hand-side of (6). So by Lemma 9.7, the map

\[ A_k \cup D \cup E \cup R_{q+1} \cup \cdots \cup R_{q-1} \rightarrow A_k \cup D \cup E \cup R_{q+1} \cup \cdots \cup R_q \]

is anodyne. When composed with (6), this gives (5), and proves the case \( u + 1 \).

This established \( \Lambda(l+1, u+1) \) and completes, for \( l+1 \), the induction on \( u \), thus completing the proof. \( \square \)

References


[F] Z. Fiedorowicz, Construction of \( E_n \)-operads, preprint, \texttt{math.AT/9808089}


