ON THE UNIVERSAL ENVELOPING ALGEBRA OF A LIE-RINEHART ALGEBRA

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Abstract. We review the extent to which the universal enveloping algebra of a Lie-Rinehart algebra resembles a Hopf algebra, and refer to this structure as a Rinehart bialgebra. We then prove a Cartier-Milnor-Moore type theorem for such Rinehart bialgebras.

1. Lie-Rinehart algebras

Throughout this paper k will denote a field of characteristic 0. In this section we will briefly recall the definition of Lie-Rinehart algebra and mention some basic examples. Lie-Rinehart algebras were introduced and studied by Herz [2], Palais [12] and Rinehart [14]. They form the algebraic counterpart of the more geometric notion of Lie algebroid, which has become better known. It was Huebschmann [3] who gave Lie-Rinehart algebras their name, and also emphasized their advantages over Lie algebroids in contexts where singularities arise. See also [4] for an excellent survey.

Let R be a unital *commutative* algebra over k. A *Lie-Rinehart algebra over* R is a Lie algebra L over k, equipped with a structure of a unital left R-module and a homomorphism of Lie algebras $\rho\colon L \to \mathrm{Der}_k(R)$ into the Lie algebra of derivations on R, which is a map of left R-modules satisfying the following Leibniz rule

$$[X, rY] = r[X,Y] + \rho(X)(r)Y$$

for any $X, Y \in L$ and $r \in R$. We shall write $\rho(X)(r) = X(r)$.

Note that in [3] and [14], a Lie-Rinehart algebra over a fixed R is referred to as (k,R)-Lie algebra.

Example 1.1. (1) If $R = k$, then $\mathrm{Der}_k(R) = \{0\}$, and a Lie-Rinehart algebra over R is simply a Lie algebra over k.

(2) For arbitrary R, the Lie algebra $\mathrm{Der}_k(R)$ is itself a Lie-Rinehart algebra over R if one takes ρ to be the identity map.

(3) Let $k = \mathbb{R}$. If $\pi\colon E \to M$ is a vector bundle over a smooth manifold M, then the structure of a Lie-Rinehart algebra on the $C^\infty(M)$-module $\Gamma(E)$ of sections of E is the same as the structure of a Lie algebroid on E.

(4) Let L be a Lie-Rinehart algebra over R and $\tau : R \to S$ a homomorphism of unital commutative k-algebras. Then we can form a Lie-Rinehart algebra $\tau_!(L)$ over S as the kernel of the map φ,

$$0 \longrightarrow \tau_!(L) \longrightarrow (S \otimes_R L) \oplus \mathrm{Der}_k(S) \xrightarrow{\varphi} \mathrm{Hom}_k(R,S)\ ,$$

defined by

$$\varphi(s \otimes X, D)(r) = s\tau(X(r)) - D(\tau(r))\ .$$

The bracket on $\tau_!(L)$ is given by

$$[(s \otimes X, D), (t \otimes Y, E)] = (st \otimes [X,Y] + D(t) \otimes Y - E(s) \otimes X, [D,E])\ ,$$

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while the representation of $\tau_!(L)$ on S is given by the projection. In the special case where τ is a localization $R \to R_{\mathfrak{p}}$ of R at a prime ideal $\mathfrak{p}$ of R, one can check that $\tau_!(L)$ is isomorphic to the localization $R_{\mathfrak{p}} \otimes_R L = L_{\mathfrak{p}}$, with the bracket

$$[s^{-1}X, t^{-1}Y] = (st)^{-1}[X,Y] - s^{-1}t^{-2}X(t)Y + t^{-1}s^{-2}Y(s)X$$

and representation $\rho_{\mathfrak{p}} : L_{\mathfrak{p}} \to \mathrm{Der}_k(R_{\mathfrak{p}})$ induced by $\rho : L \to \mathrm{Der}_k(R)$ and the canonical map $R_{\mathfrak{p}} \otimes_R \mathrm{Der}_k(R) \to \mathrm{Der}_k(R_{\mathfrak{p}})$. This agrees with the definition given in [14].

The Lie-Rinehart algebras over R form a category

$$\mathsf{LieRinAlg}_R\ ,$$

where a morphism $\phi\colon L \to L'$ is a homomorphism of Lie algebras over k as well as a map of R-modules which intertwines the representations, $\rho' \circ \phi = \rho$. The operation $\tau_!$, induced by a homomorphism $\tau : R \to S$ of unital commutative k-algebras, is a functor $\mathsf{LieRinAlg}_R \to \mathsf{LieRinAlg}_S$. Moreover, for another homomorphism $\sigma\colon S \to T$ of unital commutative k-algebras there is a canonical map $\sigma_!(\tau_!(L)) \to (\sigma \circ \tau)_!(L)$. Using these canonical maps, the categories $\mathsf{LieRinAlg}_R$, for varying k-algebras R, can be assembled into one big (fibered) category

$$\mathsf{LieRinAlg}\ ,$$

in which a map $(R,L) \to (S,K)$ is a pair (τ,ϕ), consisting of a homomorphism of unital k-algebras $\tau : R \to S$ and a homomorphism $\phi\colon \tau_!(L) \to K$ of Lie-Rinehart algebras over S.

2. The universal enveloping algebra

Let L be a Lie-Rinehart algebra over R. The left R-module $R \oplus L$ has a natural Lie algebra structure, given by

$$[(r,X),(s,Y)] = (X(s) - Y(r), [X,Y])$$

for any $r,s \in R$ and $X,Y \in L$. Let $U(R \oplus L)$ be its universal enveloping algebra over k, obtained as the quotient of the tensor algebra of $R \oplus L$ over k with respect to the usual ideal. Write $i : R \oplus L \to U(R \oplus L)$ for the canonical inclusion and $\bar{U}(R \oplus L)$ for the subalgebra of $U(R \oplus L)$ generated by $i(R \oplus L)$. The *universal enveloping algebra* of the Lie-Rinehart algebra L over R (see [14]) is the quotient algebra

$$\mathscr{U}(R,L) = \bar{U}(R \oplus L)/I$$

over k, where I is the two-sided ideal in $\bar{U}(R \oplus L)$ generated by the elements $i(s,0) \cdot i(r,X) - i(sr,sX)$, for all $r,s \in R$ and $X \in L$. The natural map $\iota_R\colon R \to \mathscr{U}(R,L)$, $r \mapsto i(r,0) + I$, is a homomorphism of unital k-algebras, while $\iota_L : L \to \mathscr{U}(R,L)$, $X \mapsto i(0,X) + I$, is a homomorphism of Lie algebras. Furthermore, we have $\iota_R(r)\iota_L(X) = \iota_L(rX)$ and $[\iota_L(X), \iota_R(r)] = \iota_R(X(r))$.

The universal enveloping algebra $\mathscr{U}(R,L)$ is characterized by the following universal property: if A is any unital k-algebra, $\kappa_R : R \to A$ a homomorphism of unital k-algebras and $\kappa_L : L \to A$ a homomorphism of Lie algebras such that $\kappa_R(r)\kappa_L(X) = \kappa_L(rX)$ and $[\kappa_L(X), \kappa_R(r)] = \kappa_R(X(r))$ for any $r \in R$ and $X \in L$, then there exists a unique homomorphism of unital algebras $f\colon \mathscr{U}(R,L) \to A$ such that $f \circ \iota_R = \kappa_R$ and $f \circ \iota_L = \kappa_L$.

In particular, the universal property of $\mathscr{U}(R,L)$ implies that there exists a unique representation

$$\varrho\colon \mathscr{U}(R,L) \to \mathrm{End}_k(R)$$

such that $\varrho \circ \iota_L = \rho$ and $\varrho \circ \iota_R$ is the canonical representation given by the multiplication in R. Since the canonical representation of R is faithful, we see that the map ι_R is injective. We shall therefore identify $\iota_R(R)$ with R, $\iota_R(r) = r$. We shall often denote $\iota_L(X)$ by $\iota(X)$ or simply by X. In this notation, the algebra $\mathscr{U}(R,L)$ is generated by elements $X \in L$ and $r \in R$, while $r \cdot X = rX$ and $[X,r] = X \cdot r - r \cdot X = X(r)$ in $\mathscr{U}(R,L)$. As a k-linear space, $\mathscr{U}(R,L)$ is generated by R and the powers $\iota(L)^n$, $n = 1,2,\ldots$. The algebra $\mathscr{U}(R,L)$ also has a natural filtration

$$R = \mathscr{U}_{(0)}(R,L) \subset \mathscr{U}_{(1)}(R,L) \subset \mathscr{U}_{(2)}(R,L) \subset \cdots ,$$

where $\mathscr{U}_{(n)}(R,L)$ is spanned by R and the powers $\iota(L)^m$, for $m = 1,2,\ldots,n$. We define the associated graded algebra as

$$\mathrm{gr}(\mathscr{U}(R,L)) = \bigoplus_{n=0}^{\infty} \mathscr{U}_{(n)}(R,L)/\mathscr{U}_{(n-1)}(R,L) ,$$

where we take $\mathscr{U}_{(-1)}(R,L) = \{0\}$. It is a commutative unital algebra over R.

Example 2.1. (1) Let V be a left R-module. With zero bracket and representation, V is a Lie-Rinehart algebra. The corresponding universal enveloping algebra $\mathscr{U}(R,V)$ is in this case the symmetric algebra $S_R(V)$ (see the appendix below).

(2) For any Lie algebra over k, the universal enveloping algebra $\mathscr{U}(k,L)$ is the classical universal enveloping algebra $U(L)$ of L.

(3) If G is a Lie groupoid over a smooth manifold M and $\mathfrak{g}$ its Lie algebroid, then the algebra of right invariant tangential differential operators on G is a concrete model for the universal enveloping algebra $\mathscr{U}(C^\infty(M),\Gamma(\mathfrak{g}))$ (see [11]).

As for the classical universal enveloping algebra of a Lie algebra, there is a Poincaré-Birkhoff-Witt theorem for the universal enveloping algebra of a Lie-Rinehart algebra [14]: if the Lie-Rinehart algebra L is projective as a left R-module, then the natural map $\theta\colon S_R(L) \to \mathrm{gr}(\mathscr{U}(R,L))$ is an isomorphism of algebras. In particular, this implies that $\iota_L\colon L \to \mathscr{U}(R,L)$ is in this case injective.

3. Rinehart bialgebras

The universal enveloping algebra of a Lie algebra is a Hopf algebra, as is the group ring of a discrete group. In this section we will identify the algebraic structure common to the universal enveloping algebra of a Lie-Rinehart algebra and the convolution algebra of an étale groupoid. This structure has occurred in the literature under various names, see [5, 6, 7, 9, 10, 16, 17]. We suggest the name *Rinehart bialgebra*.

Let R be a unital commutative algebra over k as before. All modules considered will be unital left R-modules, and $\otimes_R$ will always denote the tensor product of left R-modules. Suppose that A is a unital k-algebra which *extends* R, i.e. such that R is a unital subalgebra of A. In particular, A is an R-R-bimodule. The left R-module $A \otimes_R A$ (tensor product of A, viewed as a left R-module, with itself) is also a right R-module in two ways. Observe, that $A \otimes_R A$ is not necessarily an algebra in a natural way unless R lies in the centre of A. Following [5], we denote by $A \bar{\otimes}_R A$ the submodule of $A \otimes_R A$ given by the kernel of the map ϑ,

$$0 \longrightarrow A\bar{\otimes}_R A \longrightarrow A \otimes_R A \xrightarrow{\vartheta} \mathrm{Hom}_k(R, A \otimes_R A) ,$$

defined by $\vartheta(a \otimes b)(r) = ar \otimes b - a \otimes br$. The R-module $A\bar{\otimes}_R A$ has a natural structure of a k-algebra. If R is in the centre of A, then $A\bar{\otimes}_R A = A \otimes_R A$.

Definition 3.1. A *Rinehart bialgebra over* R is a unital k-algebra A which extends R, with a *compatible* structure of a cocommutative coalgebra in the category of left R-modules.

If we denote the comultiplication by $\Delta\colon A \to A\otimes_R A$ and the counit by $\epsilon\colon A \to R$, the compatibility conditions are

(i) $\Delta(A) \subset A\bar{\otimes}_R A$,
(ii) $\epsilon(1) = 1$,
(iii) $\Delta(1) = 1 \otimes 1$,
(iv) $\epsilon(ab) = \epsilon(a\epsilon(b))$, and
(v) $\Delta(ab) = \Delta(a)\Delta(b)$

for any $a, b \in A$.

Observe that the condition (v) makes sense because of (i) and the fact that $A\bar{\otimes}_R A$ is a k-algebra. Note, however, that (iv) does not express that ϵ is an algebra map.

The Rinehart bialgebras over R form a category

$$\mathsf{RinBiAlg}_R \ ,$$

in which a morphism $f\colon A \to B$ is a map which is at the same time a homomorphism of k-algebras with unit and a homomorphism of coalgebras in the category of left R-modules.

Example 3.2. (1) If a unital k-algebra A extends R such that R lies in the centre of A, a Rinehart bialgebra structure on A is the same as an ordinary R-bialgebra structure on A.

(2) Let $\mathscr{U}(R,L)$ be the universal enveloping algebra of a Lie-Rinehart algebra L over R (Section 2). The universal property implies that there exists a unique homomorphism of algebras

$$\Delta\colon \mathscr{U}(R,L) \to \mathscr{U}(R,L)\bar{\otimes}_R\mathscr{U}(R,L) \subset \mathscr{U}(R,L) \otimes_R \mathscr{U}(R,L)$$

such that $\Delta(r) = 1 \otimes r = r \otimes 1$ and $\Delta(X) = 1 \otimes X + X \otimes 1$ for any $r \in R$ and $X \in L$. With the counit $\epsilon\colon \mathscr{U}(R,L) \to R$ given by

$$\epsilon(u) = \varrho(u)(1) \ ,$$

one can check that $\mathscr{U}(R,L)$ is a Rinehart bialgebra over R. Furthermore, a morphism of Lie-Rinehart algebras $\phi : L \to L'$ induces, by the universal property, a morphism of Rinehart bialgebras $\mathscr{U}(R,\phi)\colon \mathscr{U}(R,L) \to \mathscr{U}(R,L')$, and this gives a functor

$$\mathscr{U} : \mathsf{LieRinAlg}_R \to \mathsf{RinBiAlg}_R \ .$$

(3) The convolution algebra of smooth functions with compact support on an étale Lie groupoid G over a compact manifold M is a Rinehart bialgebra over $C^\infty(M)$ (see [9, 10]).

Let A be a Rinehart bialgebra over R. Then A splits as an R-module as

$$A = R \oplus \bar{A} \ ,$$

where $\bar{A} = \ker \epsilon$. The R-submodule $\bar{A}$ is a subalgebra of A and carries a cocommutative coassociative coproduct $\bar{\Delta}\colon \bar{A} \to \bar{A} \otimes_R \bar{A}$, defined by

$$\bar{\Delta}(a) = \Delta(a) - a \otimes 1 - 1 \otimes a \ .$$

The bialgebra A can be reconstructed from $\bar{A}$, $\bar{\Delta}$ and the multiplication on $\bar{A}$.

The R-module $\bar{A}$ has a filtration

$$\{0\} = \bar{A}_0 \subset \bar{A}_1 \subset \bar{A}_2 \subset \cdots$$

with $\bar{A}_n = \ker \bar{\Delta}^{(n)}$, where $\bar{\Delta}^{(n)}$ denotes the iterated coproduct $\bar{A} \to \bar{A} \otimes_R \cdots \otimes_R \bar{A}$ ($n+1$ copies). We refer to this filtration as the *primitive filtration* of $\bar{A}$, and also write $A_n = R \oplus \bar{A}_n$ ($n \geq 0$). The submodule $\bar{A}_1$ is called the submodule of *primitive* elements, and also denoted by $\mathscr{P}(A)$. We observe that $\mathscr{P}(A)$ is a Lie-Rinehart algebra over R. Its Lie bracket is given by the commutator in A, and its representation $\rho\colon \mathscr{P}(A) \to \mathrm{Der}_k(R)$ is given by $\rho(a)(r) = \epsilon(ar)$. Indeed, note that

$$\begin{aligned} ar &= (\epsilon \otimes 1)(\Delta(ar)) \\ &= (\epsilon \otimes 1)(\Delta(a)\Delta(r)) \\ &= (\epsilon \otimes 1)((a \otimes 1 + 1 \otimes a)(r \otimes 1)) \\ &= (\epsilon \otimes 1)(ar \otimes 1 + r \otimes a) \\ &= \epsilon(ar) + ra\ , \end{aligned}$$

and from this it follows easily that ρ is an R-linear homomorphism of Lie algebras.

We call A (or $\bar{A}$) *cocomplete* if $A = \bigcup_{n=0}^{\infty} A_n$ (or $\bar{A} = \bigcup_{n=0}^{\infty} \bar{A}_n$). The Rinehart algebra A is *graded projective* if each of the subquotients $A_{n+1}/A_n = \bar{A}_{n+1}/\bar{A}_n$ is a projective R-module. We will use these notions in several theorems stated below.

Example 3.3. It follows from the Poincaré-Birkhoff-Witt theorem that the primitive filtration of the universal enveloping algebra $\mathscr{U}(R, L)$, associated to a Lie-Rinehart algebra L over R, coincide with its natural filtration if L is projective as a left R-module. Furthermore, the universal enveloping algebra $\mathscr{U}(R, L)$ is in this case cocomplete and graded projective.

Remark 3.4. Let A be a cocomplete graded projective Rinehart bialgebra over R. In particular, this implies that A and all A_n are projective R-modules. We write $\mathrm{gr}(\bar{A}) = \bigoplus_{n=1}^{\infty} \mathrm{gr}_n(\bar{A})$, where $\mathrm{gr}_n(\bar{A}) = \bar{A}_n/\bar{A}_{n-1}$. There is a cocommutative coassociative comultiplication $\bar{\Delta}^{\mathrm{gr}}$ on $\mathrm{gr}(\bar{A})$, induced by $\bar{\Delta}$, such that

$$\bar{\Delta}^{\mathrm{gr}}(\mathrm{gr}_n(\bar{A})) \subset \bigoplus_{p+q=n} \mathrm{gr}_p(\bar{A}) \otimes \mathrm{gr}_q(\bar{A})$$

and $\ker(\bar{\Delta}^{\mathrm{gr}}) = \mathrm{gr}_1(\bar{A})$. Furthermore, the non-counital coalgebra $\mathrm{gr}(\bar{A})$ is cocomplete, i.e. $\bigcup_{n=1}^{\infty} \ker((\bar{\Delta}^{\mathrm{gr}})^{(n)}) = \mathrm{gr}(\bar{A})$ (see the appendix). Note that any morphism $f : A \to B$ of cocomplete graded projective Rinehart bialgebras over R induces a morphism of non-counital coalgebras $\mathrm{gr}(\bar{f})\colon \mathrm{gr}(\bar{A}) \to \mathrm{gr}(\bar{B})$.

4. A Cartier-Milnor-Moore theorem

We have already seen that the universal enveloping algebra construction defines a functor

$$\mathscr{U} : \mathsf{LieRinAlg}_R \to \mathsf{RinBiAlg}_R\ .$$

In the other direction, there is a functor

$$\mathscr{P} : \mathsf{RinBiAlg}_R \to \mathsf{LieRinAlg}_R\ ,$$

which assigns to a Rinehart bialgebra A its Lie-Rinehart algebra $\mathscr{P}(A)$ of primitive elements.

Theorem 4.1. *The functor $\mathscr{U}$ is left adjoint to $\mathscr{P}$. Furthermore, the functors $\mathscr{U}$ and $\mathscr{P}$ restrict to an equivalence between the full subcategory of Lie-Rinehart algebras over R which are projective as left R-modules and that of cocomplete graded projective Rinehart bialgebras over R.*

The second part of the theorem in particular implies the following property of the counit of the adjunction, which is an analogue of the Cartier-Milnor-Moore theorem for Hopf algebras:

Corollary 4.2. *Let A be a Rinehart bialgebra over R. If A is cocomplete and graded projective, then there is a canonical isomorphism of Rinehart bialgebras $\mathscr{U}(R, \mathscr{P}(A)) \to A$.*

Proof of Theorem 4.1. For a Lie-Rinehart algebra L over R, the canonical map $L \to \mathscr{U}(R,L)$ clearly lands in the submodule of primitive elements, and this defines the unit of the adjunction,

$$\alpha_L \colon L \to \mathscr{P}(\mathscr{U}(R,L)) \ .$$

For a Rinehart bialgebra A over R, the inclusion $\mathscr{P}(A) \to A$ induces, by the universal property of the universal enveloping algebra, a canonical algebra map

$$\beta_A \colon \mathscr{U}(R,\mathscr{P}(A)) \to A \ ,$$

which in fact is clearly a map of Rinehart bialgebras. This defines the counit of the adjunction.

The first part of the theorem states that these two maps satisfy the triangular identities

$$\mathscr{P}(\beta_A) \circ \alpha_{\mathscr{P}(A)} = \mathrm{id}_{\mathscr{P}(A)}$$

and

$$\beta_{\mathscr{U}(R,L)} \circ \mathscr{U}(R,\alpha_L) = \mathrm{id}_{\mathscr{U}(R,L)} \ .$$

These both hold by the (uniqueness part of the) universal property of the universal enveloping algebra.

If L is projective as a left R-module, then the Poincaré-Birkhoff-Witt theorem implies that $\mathscr{U}(R,L)$ is graded projective and cocomplete (because the natural and primitive filtrations coincide), and that α_L is an isomorphism. For a graded projective cocomplete Rinehart bialgebra A over R, the R-module $\mathscr{P}(A)$ is obviously projective, and it remains to show that in this case the map $\beta = \beta_A$ is an isomorphism.

It suffices to prove that the map of reduced non-counital coalgebras

$$\bar{\beta} \colon \bar{\mathscr{U}}(R,\mathscr{P}(A)) \to \bar{A}$$

is an isomorphism. For this, in turn, it is enough to show that the induced map of non-counital coalgebras

$$\mathrm{gr}(\bar{\beta}) \colon \mathrm{gr}(\bar{\mathscr{U}}(R,\mathscr{P}(A))) \to \mathrm{gr}(\bar{A})$$

is an isomorphism. The projection $\mathrm{gr}(\bar{A}) \to \mathrm{gr}_1(\bar{A}) = \mathscr{P}(A)$ induces a map of non-counital coalgebras $\gamma \colon \mathrm{gr}(\bar{A}) \to \bar{S}_R(\mathscr{P}(A))$, by the universal property of $\bar{S}_R(\mathscr{P}(A))$ (Proposition A.2). Now consider the diagram

$$\begin{tikzcd}
\mathrm{gr}(\bar{\mathscr{U}}(R,\mathscr{P}(A))) \arrow[r, "\mathrm{gr}(\bar{\beta})"] \arrow[dr, "\bar{\theta}^{-1}"'] & \mathrm{gr}(\bar{A}) \arrow[d, "\gamma"] \\
 & \bar{S}_R(\mathscr{P}(A))
\end{tikzcd}$$

where $\bar{\theta}^{-1}$ is the inverse of the Poincaré-Birkhoff-Witt isomorphism θ (Section 2) restricted to $\bar{S}_R(\mathscr{P}(A))$. All maps in the diagram are maps of non-counital coalgebras. Thus, to see that the diagram commutes, it suffices (again by the universal property of $\bar{S}_R(\mathscr{P}(A))$ stated in Proposition A.2) that

$$\mathrm{pr}_1 \circ \gamma \circ \mathrm{gr}(\bar{\beta}) = \mathrm{pr}_1 \circ \bar{\theta}^{-1}$$

for the projection $\mathrm{pr}_1 : \bar{S}_R(\mathscr{P}(A)) \to \mathscr{P}(A)$, which is clear from the explicit definitions. To finish the proof, recall from Remark 3.4 that $\mathrm{gr}(\bar{A})$ is cocomplete and $\ker(\bar{\Delta}^{\mathrm{gr}}) = \mathscr{P}(A)$, so that by Lemma A.1 the map γ is injective, while by the commutativity of the diagram it is also surjective. Thus γ is an isomorphism, and hence so is $\mathrm{gr}(\bar{\beta})$. $\square$

Appendix A. Cocommutative non-counital coalgebras

The main goal of this appendix is to prove some elementary properties of cocommutative non-counital coalgebras over a ring, in particular the universal property of the symmetric coalgebra. Although these results are well-known, they are usually stated in the literature in the context of coalgebras over a field (see e.g. [1, 8, 13, 15]).

As before, k denotes a field of characteristic 0 and R a unital commutative k-algebra. A cocommutative non-counital coalgebra over R is an R-module C, together with a cocommutative coassociative comultiplication. $\delta : C \to C \otimes_R C$. (Note that we do not assume that C has a counit.) With the obvious maps, these cocommutative non-counital coalgebras over R form a category. For example, for a Rinehart bialgebra A over R, the pair $(\bar{A}, \bar{\Delta})$ is an object of this category.

As discussed in this special case already, any cocommutative non-counital coalgebra (C, δ) carries a *primitive* filtration

$$\{0\} = C_0 \subset C_1 \subset C_2 \subset \cdots ,$$

where C_n is the kernel of the iterated coproduct $\delta^{(n)} \colon C \to C \otimes_R \cdots \otimes_R C$ ($n+1$ copies). We say that C is *cocomplete* if $C = \bigcup_{n=0}^{\infty} C_n$. If C is cocomplete and C_n are all flat R-modules, then C is a flat R-module as well. Notice that in this case $\delta \colon C \to C \otimes_R C$ maps each C_{n+1} into $C_n \otimes_R C_n$.

Lemma A.1. *Let $f : C \to D$ be a morphism of cocomplete cocommutative non-counital coalgebras over R, and assume that all the submodules C_n, D_n are flat. If $f_1 = f|_{C_1} \colon C_1 \to D_1$ is injective, then so is $f \colon C \to D$.*

Proof. We prove by induction that $f_n = f|_{C_n} : C_n \to D_n$ is injective. Assuming this has been proved for f_n, injectivity of f_{n+1} follows from the diagram

$$\begin{tikzcd} C_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\delta_C"'] & D_{n+1} \arrow[d, "\delta_D"] \\ C_n \otimes_R C_n \arrow[r, "f_n \otimes f_n"] & D_n \otimes_R D_n \end{tikzcd}$$

since the map at the bottom is injective by the flatness assumption. Indeed, we have $\ker(f_{n+1}) \subset \ker(\delta_C) = C_1$ and $C_1 \cap \ker(f) = \{0\}$ by assumption. □

Let V be an R-module. We denote by $S_R(V)$ the symmetric algebra on V. It is a graded algebra, where $S^n_R(V)$ is the space $(V \otimes_R \cdots \otimes_R V)_{\Sigma_n}$ of coinvariants of the n-fold tensor product. The algebra $S_R(V)$ is the free commutative unital R-algebra on V. Therefore, the map $\Delta \colon V \to S_R(V) \otimes_R S_R(V)$, given by $\Delta(v) = 1 \otimes v + v \otimes 1$, extends uniquely to a unital algebra homomorphism $\Delta \colon S_R(V) \to S_R(V) \otimes_R S_R(V)$, giving $S_R(V)$ the structure of a commutative and cocommutative bialgebra over R. (In fact, this is the Rinehart bialgebra $\mathscr{U}(R, L)$, where L is V viewed as a Lie-Rinehart algebra with zero bracket and representation.)

The bialgebra $S_R(V)$ also has a universal property as a coalgebra, most easily stated in terms of the R-module $\bar{S}_R(V) = \bigoplus_{n=1}^{\infty} S^n_R(V)$, which is a cocomplete cocommutative non-counital R-algebra with coproduct $\bar{\Delta}(c) = \Delta(c) - 1 \otimes c - c \otimes 1$.

Proposition A.2. *Let (C, δ) be a cocomplete cocommutative non-counital coalgebra over R and V an R-module. Then any morphism of R-modules $C \to V$ is the first component of a unique homomorphism $C \to \bar{S}_R(V) = \bigoplus_{n=1}^{\infty} S^n_R(V)$ of non-counital coalgebras over R.*

Proof. Let $g_1 : C \to V = S^1_R(V)$ be an R-linear map. We will define a map $g_n \colon C \to S^n_R(V)$, for each $n \geq 2$, in such a way that these together form a coalgebra map $g = (g_n)\colon C \to \bar{S}_R(V) = \bigoplus_{n=1}^{\infty} S_R^n(V)$. Note that for g to be a coalgebra map, there can be at most one such g_n, since it will have to make the diagram

$$\begin{tikzcd} C \arrow[r, "g_n"] \arrow[d, "\delta^{(n+1)}"'] & S_R^n(V) \arrow[d, "\bar{\Delta}^{(n+1)}"] \\ C^{\otimes_R n} \arrow[r, "g_1^{\otimes n}"] & S_R^1(V)^{\otimes_R n} \end{tikzcd}$$

commute. In fact, we can use this diagram to define g_n, because C is cocommutative and $\bar{\Delta}^{(n+1)}$ gives an isomorphism between $S_R^n(V)$ and the subspace of Σ_n-invariants of $S_R^1(V)^{\otimes_R n}$. (Here we used the assumption that the characteristic of k is 0.)

The cocompleteness of C implies that g is well defined. To see that g is indeed a map of coalgebras, we show by induction on n that the diagram

$$\begin{tikzcd} C \arrow[r, "g_n"] \arrow[d, "\delta"'] & S_R^n(V) \arrow[d, "\bar{\Delta}"] \\ C \otimes_R C \arrow[r, "(g_p \otimes g_{n-p})"] & \bigoplus_{0<p<n} (S_R^p(V) \otimes S_R^{n-p}(V)) \end{tikzcd} \tag{1}$$

commutes. For $n = 1$ there is nothing to prove, and for $n = 2$ this holds by definition of g_2. Take $n > 2$, and suppose that the commutativity of (1) has been proved for all $m < n$. Now consider the following diagram

$$\begin{tikzcd} C \arrow[r, "g_n"] \arrow[d, "\delta"'] & S_R^n(V) \arrow[d, "\bar{\Delta}"] \\ C^{\otimes_R 2} \arrow[r, "(g_{p_1} \otimes g_{p_2})"] \arrow[d, shift right, "\delta\otimes 1"'] \arrow[d, shift left, "1\otimes\delta"] & \bigoplus_{p_1+p_2=n} (S^{p_1} \otimes_R S^{p_2}) \arrow[d, shift right, "\bar{\Delta}\otimes 1"'] \arrow[d, shift left, "1\otimes\bar{\Delta}"] \\ C^{\otimes_R 3} \arrow[r, "(g_{p_1} \otimes g_{p_2} \otimes g_{p_3})"] \arrow[d, shift right=2, "\delta\otimes 1\otimes 1"'] \arrow[d] \arrow[d, shift left=2, "1\otimes 1\otimes\delta"] & \bigoplus_{p_1+p_2+p_3=n} (S^{p_1} \otimes_R S^{p_2} \otimes_R S^{p_3}) \arrow[d, shift right=2, "\bar{\Delta}\otimes 1\otimes 1"'] \arrow[d] \arrow[d, shift left=2, "1\otimes 1\otimes\bar{\Delta}"] \\ \vdots \arrow[d, shift right=2] \arrow[d, shift right] \arrow[d, shift left=2, "\cdots"'] & \vdots \arrow[d, shift right=2] \arrow[d, shift right] \arrow[d, shift left=2, "\cdots"'] \\ C^{\otimes_R n} \arrow[r, "g_1^{\otimes n}"] & (S_R^1(V))^{\otimes_R n} \end{tikzcd} \tag{2}$$

where all $p_i \geq 1$ and $S^{p_i} = S_R^{p_i}(V)$. Our aim is to prove that the top square in (2) commutes. By the induction hypothesis, the two parallel squares at the second level commute, as do the parallel squares at each lower level. Furthermore, the outer square commutes by the definition of g_n, as the vertical compositions $C \to C^{\otimes_R n}$ all agree and define $\delta^{(n+1)}$, and similarly for all the vertical compositions $S_R^n(V) \to (S_R^1(V))^{\otimes_R n}$. Thus, to show that the top square commutes, it suffices to prove that the joint kernel of all possible vertical compositions

$$\bigoplus_{p_1+p_2=n} (S^{p_1} \otimes_R S^{p_2}) \to (S_R^1(V))^{\otimes_R n}$$

is zero, which can easily be checked (note that $\bar{\Delta}\colon S_R^n(V) \to \bigoplus_{p_1+p_2=n} (S^{p_1} \otimes_R S^{p_2})$ is split injective map of left R-modules if $n \geq 2$). $\square$

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