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On the Freyd Cover of a Topos

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A theory is said to have the disjunction-property (*DP*) if whenever a disjunction $\phi \vee \psi$ is provable in the theory, either ϕ or ψ must be provable. As is well-known, many theories for intuitionistic arithmetic and analysis have the *DP*. The *DP* for intuitionistic type theory was first established by Friedman. More recently, a purely topos theoretic proof has been given by Freyd. An extensive discussion of both methods can be found in [4]. Although Freyd's construction is much more elegant, A. Ščedrov and P. Scott have shown that the two methods are essentially the same in [7].

A question that arises immediately is the following: If one adds new symbols and a particular set of axioms T to the logical axioms and rules, does the resulting higher-order theory still have the *DP*? Some instances of this question in which T consists of a single axiom have been considered in [5]. In this note, we will obtain a syntactic description of a class of theories that have the *DP* by investigating some of the logical properties of the Freyd cover, thus extending the results of [5].

The results will *not* cover many of the higher-order analogues of theories of intuitionistic arithmetic and analysis which are known to have the *DP*. One reason for this is that, from a more logical point of view, the Freyd cover lacks many nice properties. For an alternative type of cover that fills this gap, the reader is referred to [6].

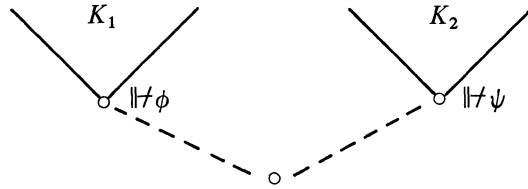
In the first section of this paper, we will motivate the Freyd cover from a more logical perspective. There is probably nothing new in this, but it still is important to realize that what is really going on is a straightforward generalization of more traditional methods used in the model theory of first-order

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intuitionistic logic. Thus, the above-mentioned result of Ščedrov and Scott should not come as a surprise. This perspective also opens the way to connections with, for example, (higher-order analogues of) the Aczel-slash, and the Kleene-slash (see [8]).

In the second section, we examine preservation-properties of the Freyd cover, and prove the main result.

1 Motivating the Freyd cover Everybody knows how to prove the disjunction property for intuitionistic propositional logic (or Heyting’s Arithmetic, etc.): If ϕ and ψ are two nonprovable formulas, just take two Kripke models $K_1 \Vdash \not\phi$ and $K_2 \Vdash \not\psi$, and add a new bottom node (this operator on Kripke models is called the Smorynski operator).



Then the bottom node cannot force $\phi \vee \psi$, so $\phi \vee \psi$ is not provable either (for details, see [8]).

Looking at this topologically, what we did was take two sheaf-models over spaces X_1 and X_2 , take their topological sum $X_1 + X_2$, and define a new space $X = (X_1 + X_2) \cup \{*\}$, where $*$ $\notin X_1 + X_2$ is a closed point of X whose only neighbourhood is the whole space X .

But this is precisely the situation for applying the theorem of Artin glueing [2], which says that you can get $Sh(X)$, the category of sheaves over X , by glueing along the global sections functor Γ ,

$$Sh(X_1 + X_2) \cong Sh(X_1) \times Sh(X_2) \xrightarrow{\Gamma} Sets \cong Sh(*)$$

This is easily generalized for topoi, using the elementary form of Artin glueing ([3], Section 4.2): Given two topoi \mathcal{E}_1 and \mathcal{E}_2 , let $\mathcal{E}_1 \times \mathcal{E}_2 \xrightarrow{\Gamma} Sets$ be the global sections-functor $(1, -)$, and glue along Γ , i.e., make the comma category $(Sets \downarrow \Gamma)$. This topos $(Sets \downarrow \Gamma)$ is the Freyd cover of $\mathcal{E}_1 \times \mathcal{E}_2$, and will be denoted by $\mathcal{E}_1 * \mathcal{E}_2$. Objects of this topos are triples (X, E, ϕ) , where X is a set, $E = (E_1, E_2)$ is an object of $\mathcal{E}_1 \times \mathcal{E}_2$, and ϕ is a function $X \rightarrow \Gamma E$. Recall (see [9]) that we have a geometric morphism

$$\mathcal{E}_1 \times \mathcal{E}_2 \rightarrow \mathcal{E}_1 * \mathcal{E}_2$$

with inverse image the forgetful functor $\mathcal{E}_1 * \mathcal{E}_2 \xrightarrow{U} \mathcal{E}_1 \times \mathcal{E}_2$, $U(X, E, \phi) = E$, and with direct image the cofree coalgebra functor $\mathcal{E}_1 \times \mathcal{E}_2 \xrightarrow{G} \mathcal{E}_1 * \mathcal{E}_2$, $GE = (\Gamma E, E, id_{\Gamma E})$. This geometric morphism is an open inclusion, so U is logical, and G preserves exponents.

We now want to reason as in the case of the Smorynski operator, roughly as follows: given two nonprovable formulas ϕ and ψ of intuitionistic higher-order logic, find topoi \mathcal{E}_1 and \mathcal{E}_2 with interpretations \mathcal{I}_1 in \mathcal{E}_1 and \mathcal{I}_2 in \mathcal{E}_2

such that $\mathcal{C}_1 \not\models_{\mathcal{I}_1} \phi$ and $\mathcal{C}_2 \not\models_{\mathcal{I}_2} \psi$. Then the product $\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2$ is an interpretation in $\mathcal{C}_1 \times \mathcal{C}_2$ such that $\mathcal{C}_1 \times \mathcal{C}_2 \not\models_{\mathcal{I}} \phi$ and $\mathcal{C}_1 \times \mathcal{C}_2 \not\models_{\mathcal{I}} \psi$. We now want to transport this interpretation \mathcal{I} along G and obtain an interpretation $\bar{\mathcal{I}}$ in $\mathcal{C}_1 * \mathcal{C}_2$ with the property that $U \circ \bar{\mathcal{I}} = \mathcal{I}$. Since U is logical (and therefore preserves validity), $\mathcal{C}_1 * \mathcal{C}_2 \not\models_{\bar{\mathcal{I}}} \phi$ and $\mathcal{C}_1 * \mathcal{C}_2 \not\models_{\bar{\mathcal{I}}} \psi$. From a simple inspection of the subobject-classifier in the comma-topos $\mathcal{C}_1 * \mathcal{C}_2$ (the terminal object in $\mathcal{C}_1 * \mathcal{C}_2$ is indecomposable, see [5]) it then follows that $\mathcal{C}_1 * \mathcal{C}_2 \not\models_{\bar{\mathcal{I}}} \phi \vee \psi$. Below, we will discuss the problem of

(1) how to make $\bar{\mathcal{I}}$ out of \mathcal{I} ?

Often, one starts with a theory T and two nonprovable formulas $T \not\vdash \phi$ and $T \not\vdash \psi$, and finds $\mathcal{C}_1, \mathcal{I}_1$ and $\mathcal{C}_2, \mathcal{I}_2$ such that $\mathcal{C}_1 \models_{\mathcal{I}_1} T$ and $\mathcal{C}_2 \models_{\mathcal{I}_2} T$, $\mathcal{C}_2 \not\models_{\mathcal{I}_1} \phi$, $\mathcal{C}_1 \not\models_{\mathcal{I}_2} \psi$. To show that T has the DP, one then wants $\mathcal{C}_1 * \mathcal{C}_2$ to be a model of T under the interpretation $\bar{\mathcal{I}}$, too. So we want to know

(2) for which theories T does it hold that whenever $(\mathcal{C}_1, \mathcal{I}_1)$ and $(\mathcal{C}_2, \mathcal{I}_2)$ are models of T , so is $(\mathcal{C}_1 * \mathcal{C}_2, \bar{\mathcal{I}})$?

(1) and (2) will be dealt with in the next section.

But before we turn to this, let us be more explicit about *interpretations*. We take a version of higher-order logic of the kind described in [1], which is sound and complete for interpretations in topoi. The language has two ingredients: sorts and constants. We have a set of ground sorts $\{s_i \mid i \in I\}$, from which we can build up the set of sorts inductively: every groundsort is a sort, and if s_1, \dots, s_n, t are sorts, $[s_1, \dots, s_n]$ is a sort (the sort of n -place relations taking arguments of sorts s_1, \dots, s_n , respectively), and $[s_1, \dots, s_n \rightarrow t]$ is a sort (the sort of functions taking n arguments of sorts s_1, \dots, s_n , respectively, to a value of sort t). We also have a set of constants $\{c_j \mid j \in J\}$, together with an assignment $c \mapsto \#(c)$ of a sort to each constant. An interpretation \mathcal{I} of the language in a topos \mathcal{C} assigns to each groundsort an object $\mathcal{I}(s)$ of \mathcal{C} ; \mathcal{I} is then extended to all sorts by setting

$$\begin{aligned} \mathcal{I}([s_1, \dots, s_n]) &= \Omega^{\mathcal{I}(s_1) \times \dots \times \mathcal{I}(s_n)}, \\ \mathcal{I}(s_1, \dots, s_n \rightarrow t) &= \mathcal{I}(t)^{\mathcal{I}(s_1) \times \dots \times \mathcal{I}(s_n)}. \end{aligned}$$

Further, \mathcal{I} assigns an arrow $\mathcal{I}(c): 1 \rightarrow \mathcal{I}(\#c)$ to each constant c . The interpretation of terms and formulas is then defined in the standard way (see, e.g., [1]).

Note that abstraction terms (terms of the form $\{ \langle x_1, \dots, x_n \mid \phi \rangle \}$) are eliminable in formulas. Therefore we will in the sequel assume that *formulas do not contain abstraction terms*.

Below, we will use the word *term* only in the following sense: variables and constants are terms, and if $\sigma_1, \dots, \sigma_n$ are terms and f is a functional term of the appropriate sort, $f(\sigma_1, \dots, \sigma_n)$ is a term. Thus, no quantifiers, connectives, or abstraction $\{ \cdot \mid \cdot \}$ can occur in terms. Note that every formula of the higher-order language is equivalent to one which is built up from atomic formulas of the form $R(\sigma_1, \dots, \sigma_n)$ or $\sigma_1 = \sigma_2$, where $\sigma_1, \dots, \sigma_n$ are terms in this sense and R is a relational term in this sense, by the usual clauses for the

quantifiers and connectives. It is important to be explicit about this, as will appear in the sequel.

2 Preservation properties of the Freyd cover We consider a slightly more general situation: let \mathcal{C} and \mathcal{F} be topoi, and let $\mathcal{C} \xrightarrow{d} \mathcal{F}$ be a left-exact functor. We then have a geometric morphism $\mathcal{C} \rightarrow (\mathcal{F} \downarrow d)$ given by the forgetful functor $U: (\mathcal{F} \downarrow d) \rightarrow \mathcal{C}$ and the cofree coalgebra functor $G: \mathcal{C} \rightarrow (\mathcal{F} \downarrow d)$; U is logical, G preserves exponents, and $U \circ G = id_{\mathcal{C}}$. Suppose that we have an interpretation \mathcal{I} of the logical language in \mathcal{C} . We want to construct an interpretation $\bar{\mathcal{I}}$ in $(\mathcal{F} \downarrow d)$ (cf. (1) above).

First note that $G\Omega_{\mathcal{C}}$ is a retract of $\Omega_{(\mathcal{F} \downarrow d)}$: the classifying morphism $G\Omega_{\mathcal{C}} \xrightarrow{e} \Omega_{(\mathcal{F} \downarrow d)}$ of $Gtrue: 1 \simeq G1 \rightarrow G\Omega_{\mathcal{C}}$ is splitmono, with splitting $\Omega_{(\mathcal{F} \downarrow d)} \xrightarrow{\lambda} G\Omega_{\mathcal{C}}$ (the transpose of $U\Omega_{\mathcal{F} \downarrow d} \xrightarrow{\cong} \Omega_{\mathcal{C}}$).

For a groundsort s we define an object $\bar{\mathcal{I}}(s)$ of $(\mathcal{F} \downarrow d)$ by

$$\bar{\mathcal{I}}(s) = G\mathcal{I}(s)$$

$\bar{\mathcal{I}}$ is then uniquely (up to isomorphism) extended to all sorts. We then construct by induction on the sort s morphisms k_s and e_s

$$G\mathcal{I}(s) \xrightarrow{k_s} \bar{\mathcal{I}}(s) \xrightarrow{e_s} G\mathcal{I}(s)$$

with $e_s \circ k_s = 1_{G\mathcal{I}(s)}$, and $U(k_s) = U(e_s) = 1_{\mathcal{I}(s)}$. If s is a groundsort, then $k_s = e_s = 1_{G\mathcal{I}(s)}$. If $s = [t_1, \dots, t_n]$, and we have defined k_{t_i} and e_{t_i} ($i = 1, \dots, n$), then k_s and e_s are defined as the compositions

$$\rho^{\bar{\mathcal{I}}(t_1) \times \dots \times \bar{\mathcal{I}}(t_n)} \circ G\Omega_{\mathcal{C}}^{e_{t_1} \times \dots \times e_{t_n}}$$

and

$$\lambda^{G\mathcal{I}(t_1) \times \dots \times G\mathcal{I}(t_n)} \circ \Omega_{(\mathcal{F} \downarrow d)}^{k_{t_1} \times \dots \times k_{t_n}}.$$

If $s = [t_1, \dots, t_n \rightarrow r]$, and we have defined k_{t_i} , e_{t_i} ($i = 1, \dots, n$), k_r , e_r , then k_s and e_s are the following two compositions

$$\mathcal{I}(r)^{e_{t_1} \times \dots \times e_{t_n}} \circ k_r^{G\mathcal{I}(t_1) \times \dots \times G\mathcal{I}(t_n)}$$

and

$$G\mathcal{I}(t)^{k_{t_1} \times \dots \times k_{t_n}} \circ e_r^{\bar{\mathcal{I}}(t_1) \times \dots \times \bar{\mathcal{I}}(t_n)}.$$

$\bar{\mathcal{I}}$ is then defined for constants as follows: if $\#c = s$, then

$$\bar{\mathcal{I}}(c) = 1 \simeq G1 \xrightarrow{G\mathcal{I}(c)} G\mathcal{I}(s) \xrightarrow{k_s} \bar{\mathcal{I}}(s).$$

This completes the definition of $\bar{\mathcal{I}}$. Note that $U \circ \bar{\mathcal{I}} = \mathcal{I}$. Since U is logical, we immediately have

2.1 Lemma *Let ϕ be an arbitrary formula, with free variables among x_1, \dots, x_n . Then*

$$U\left(\llbracket \phi \rrbracket_{\bar{\mathcal{I}}} \xrightarrow{\quad} \prod_{i=1}^n \mathcal{I}(\#x_i)\right) = \left(\llbracket \phi \rrbracket_{\mathcal{I}} \xrightarrow{\quad} \prod_{i=1}^n \mathcal{I}(\#x_i)\right),$$

and similarly for terms.

For an atomic formula $R(\tau_1, \dots, \tau_n)$, where R is a relational constant, and τ_1, \dots, τ_n are terms (recall the convention at the end of Section 1) with free variables among x_1, \dots, x_k , and $\mathcal{L}(\#x_i) = A_i$, $\llbracket R(\tau_1, \dots, \tau_n) \rrbracket_{\mathcal{L}}$ defines a subobject of $A_1 \times \dots \times A_k$ in \mathcal{E} , or a morphism $A_1 \times \dots \times A_k \rightarrow \Omega$, or $1 \rightarrow \Omega^{A_1 \times \dots \times A_k}$. Now what is $\llbracket R(\tau_1, \dots, \tau_n) \rrbracket_{\bar{\mathcal{L}}}$ in $(\mathcal{F} \downarrow d)$? We will show that the association

$$(1) \quad \llbracket R(\tau_1, \dots, \tau_n) \rrbracket_{\mathcal{L}} \mapsto \llbracket R(\tau_1, \dots, \tau_n) \rrbracket_{\bar{\mathcal{L}}}$$

corresponds to the following operation on subobjects

$$(2) \quad \Phi: \mathcal{E}(A, \Omega) \rightarrow (\mathcal{F} \downarrow d)(\bar{A}, \Omega)$$

(here $A = \mathcal{L}(s_1) \times \dots \times \mathcal{L}(s_k)$, $\bar{A} = \bar{\mathcal{L}}(s_1) \times \dots \times \bar{\mathcal{L}}(s_k)$, for suitable s_1, \dots, s_k): Φ associates with $1 \xrightarrow{f} \Omega^A$ the composition

$$1 \simeq G1 \xrightarrow{Gf} (\Omega^A) \xrightarrow{k} \Omega^{\bar{A}}$$

where k is the splitmono for $[s_1, \dots, s_n]$. (In the sequel, we will usually omit the indices on the morphisms k_s and e_s .)

For the proof that (1) is the same as (2), first observe that for any term σ with free variables among x_1, \dots, x_n , ($\mathcal{L}(\#x_i) = A$, $\bar{\mathcal{L}}(\#x_i) = \bar{A}$, $k_{A_i} = k_{\#x_i}$) the following diagram commutes (the proof is an easy induction on σ):

$$\begin{array}{ccc} \bar{A}_1 \times \dots \times \bar{A} & \xrightarrow{\llbracket \sigma \rrbracket_{\bar{\mathcal{L}}}} & \bar{B} \\ \uparrow k_{A_1} \times \dots \times k_{A_n} & & \uparrow k_B \\ GA_1 \times \dots \times GA_n & \xrightarrow{\llbracket \sigma \rrbracket_{\mathcal{L}}} & GB \end{array}$$

Now suppose for ease of notation that R is a one-place relational constant, say with $\mathcal{L}(R): 1 \rightarrow \Omega_{\mathcal{E}}^B$, and write $\mathcal{L}(\sigma): 1 \rightarrow B^A$ for the transpose of $\llbracket \sigma \rrbracket_{\mathcal{L}}: A \rightarrow B$. Then the claim that

$$\Phi(\llbracket R(\sigma) \rrbracket_{\mathcal{L}}) = \llbracket R(\sigma) \rrbracket_{\bar{\mathcal{L}}}$$

follows easily, if we can show that the following compositions (i) and (ii) are identical:

$$\begin{aligned} (i) \quad & \bar{A} \xrightarrow{1 \times G\mathcal{L}(\sigma)} \bar{A} \times G(B^A) \xrightarrow{1 \times k} \bar{A} \times \bar{B}^{\bar{A}} \xrightarrow{ev} \bar{B} \xrightarrow{1 \times G\mathcal{L}(R)} \bar{B} \times G(\Omega_{\mathcal{E}}^B) \xrightarrow{1 \times k} B \times \Omega_{\mathcal{F} \downarrow d}^{\bar{B}} \xrightarrow{ev} \Omega_{\mathcal{F} \downarrow d} \\ (ii) \quad & \bar{A} \xrightarrow{1 \times G\mathcal{L}(R)} \bar{A} \times G(\Omega^B) \xrightarrow{1 \times G(\Omega^{\llbracket \sigma \rrbracket_{\mathcal{L}}})} \bar{A} \times G(\Omega^A) \xrightarrow{1 \times k} \bar{A} \times \Omega^{\bar{A}} \xrightarrow{ev} \Omega. \end{aligned}$$

But from the definition of k it follows that (1) is identical to

$$\begin{aligned} & \bar{A} \xrightarrow{1 \times G\mathcal{L}(\sigma)} \bar{A} \times G(B^A) \xrightarrow{e \times 1} GA \times G(B^A) \xrightarrow{Gev} GB \xrightarrow{k} \bar{B} \xrightarrow{1 \times G\mathcal{L}(R)} \bar{B} \times G(\Omega^B) \\ & \xrightarrow{e \times 1} GB \times G(\Omega^B) \xrightarrow{Gev} G\Omega \xrightarrow{e} \Omega \end{aligned}$$

and since $e \circ k = id$, this is identical to

$$\bar{A} \xrightarrow{e} GA \xrightarrow{1 \times G\mathcal{J}(\sigma)} GA \times G(B^A) \xrightarrow{Gev} GB \xrightarrow{1 \times G\mathcal{J}(R)} GB \times G(\Omega^B) \xrightarrow{Gev} G\Omega \xrightarrow{\rho} \Omega.$$

Similarly, one shows that (2) is identical to

$$\bar{A} \xrightarrow{1 \times G\mathcal{J}(R)} \bar{A} \times G(\Omega^B) \xrightarrow{e \times 1} GA \times G(\Omega^B) \xrightarrow{G\llbracket \sigma \rrbracket_{\mathcal{J}} \times 1} GB \times G(\Omega^B) \xrightarrow{Gev} G\Omega \xrightarrow{\rho} \Omega.$$

And clearly, the latter two compositions are identical, since $\mathcal{J}(\sigma)$ is the transpose of $\llbracket \sigma \rrbracket_{\mathcal{J}}$. As is easily seen, this proves the correspondence of (1) and (2) not only for R a single constant, but also more generally for R a term without variables (i.e., R built up from constants by functional application only).

Let us now turn to the properties of the operation Φ . First a notational convention: a subobject of A is either represented by a mono $B \twoheadrightarrow A$, or its classifying morphism $A \xrightarrow{f} \Omega$, or its transpose $1 \xrightarrow{\hat{f}} \Omega^A$. In all these cases we will write $\Phi(B)$, $\Phi(f)$, $\Phi(\hat{f})$ for the corresponding representation of the subobject given by the original definition of Φ .

2.2 Lemma Φ preserves conjunction (and hence Φ is orderpreserving).

By “ Φ preserves conjunction” we mean that if $f, g: A \rightarrow \Omega$ in \mathcal{E} , then $\Phi(\wedge_{\mathcal{E}} \circ (f, g)) = \wedge_{(\mathcal{F} \downarrow d)} \circ (\Phi(f), \Phi(g))$; similarly for the other cases to be considered below.

Proof: We have to show that

$$G(\Omega^A \times \Omega^A) \xrightarrow{G(\wedge^A)} G(\Omega^A) \xrightarrow{k} \Omega^{\bar{A}} = G(\Omega^A \times \Omega^A) \xrightarrow{k \times k} \Omega^{\bar{A}} \times \Omega^{\bar{A}} \xrightarrow{\wedge^{\bar{A}}} \Omega^{\bar{A}}.$$

Passing to the transposed maps, the left-hand side becomes

$$\begin{aligned} \bar{A} \times G(\Omega^A \times \Omega^A) &\xrightarrow{1 \times G(\wedge^A)} \bar{A} \times G(\Omega^A) \xrightarrow{e \times 1} GA \times G(\Omega^A) \xrightarrow{Gev} G\Omega \xrightarrow{\rho} \Omega \\ &= \bar{A} \times G(\Omega^A \times \Omega^A) \xrightarrow{(e, e) \times 1} GA \times GA \times G\Omega^A \\ &\quad \times G\Omega^A \xrightarrow{(Gev \circ (\pi_1, \pi_3), Gev \circ (\pi_2, \pi_4))} G\Omega \times G\Omega \xrightarrow{G\wedge} G\Omega \xrightarrow{\rho} \Omega. \end{aligned}$$

Similarly, the right-hand side becomes

$$\begin{aligned} \bar{A} \times G(\Omega^A \times \Omega^A) &\xrightarrow{(e, e) \times 1} GA \times GA \times G\Omega^A \times G\Omega^A \xrightarrow{Gev \times Gev} G\Omega \\ &\quad \times G\Omega \xrightarrow{\rho \times \rho} \Omega \times \Omega \xrightarrow{\wedge} \Omega. \end{aligned}$$

Therefore, it suffices to show that

$$\begin{array}{ccc} G\Omega \times \Omega & \xrightarrow{\rho \times \rho} & \Omega \times \Omega \\ \downarrow G\wedge_{\mathcal{E}} & & \downarrow \wedge_{\mathcal{F} \downarrow d} \\ G\Omega & \xrightarrow{\rho} & \Omega \end{array}$$

commutes. But this follows easily from the fact that ρ classifies $G1 \xrightarrow{Gtrue} G\Omega$.

Note that from the fact that $\Phi: Sub_{\mathcal{E}}(A) \rightarrow Sub_{\mathcal{F} \downarrow d}(\bar{A})$ is orderpreserving, it immediately follows that for U and $V \in Sub_{\mathcal{E}}(A)$,

$$\begin{aligned} \Phi(U) \vee \Phi(V) &\leq \Phi(U \vee V) \\ \Phi(U \Rightarrow V) &\leq \Phi(U) \Rightarrow \Phi(V). \end{aligned}$$

2.3 Lemma Φ preserves \top_A , the largest subobject of A . Also, Φ preserves \perp_A , the smallest subobject, provided d preserves the initial object 0.

Proof: Following the same method as in the proof of Lemma 2.2, we see that it suffices to show that $G1 \xrightarrow{G \text{ true}} G\Omega \xrightarrow{\rho} \Omega = 1 \xrightarrow{\text{true}} \Omega$ (which is clear from the definition of ρ) and that $G1 \xrightarrow{G \text{ false}} G\Omega \xrightarrow{\rho} \Omega = 1 \xrightarrow{\text{false}} \Omega$. This latter identity only holds if G preserves the initial, or, equivalently, if d does. For in that case $\rho \circ G \text{ false}$ classifies the subobject $G0 \cong 0 \twoheadrightarrow 1 \cong G1$, since both squares of the diagram below are pullback

$$\begin{array}{ccccc}
 G1 & \xrightarrow{G \text{ false}} & G\Omega & \xrightarrow{\rho} & \Omega \\
 \uparrow & & \uparrow G \text{ true} & & \uparrow \text{true} \\
 G0 & \longrightarrow & G1 & \longrightarrow & 1
 \end{array}$$

2.4 Remark: The properties of Φ that have been stated above also follow easily from the following alternative description of Φ : If $U \twoheadrightarrow A$ is a subobject of A , then $\Phi(U) \cong e^{-1}(GU)$; that is, the following diagram is pullback

$$\begin{array}{ccc}
 \bar{A} & \xrightarrow{e} & GA \\
 \uparrow & & \uparrow \\
 \Phi(U) & \longrightarrow & GU
 \end{array}$$

2.5 Lemma Φ preserves negation (provided d preserves 0).

Proof: From the fact that $\Phi(U \Rightarrow V) \leq \Phi(U) \Rightarrow \Phi(V)$, and $\Phi(\perp_A) = \perp_{\bar{A}}$, it follows that $\Phi(\neg U) \leq \neg \Phi(U)$.

As for the converse, it again suffices (as in the proof of Lemma 2.2) to show that the subobject classified by $G\Omega \xrightarrow{\rho} \Omega \xrightarrow{\neg} \Omega$ is contained in the subobject classified by $G\Omega \xrightarrow{G\neg} G\Omega \xrightarrow{\rho} \Omega$. So make two pullbacks:

$$\begin{array}{ccc}
 G\Omega & \xrightarrow{\rho} & \Omega & \xrightarrow{\neg} & \Omega \\
 \uparrow g & & \uparrow \text{false} & & \uparrow \text{true} \\
 P & \xrightarrow{!} & 1 & \longrightarrow & 1
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccccc}
 G\Omega & \xrightarrow{G\neg} & G\Omega & \xrightarrow{\rho} & \Omega \\
 \uparrow G \text{ false} & & \uparrow G \text{ true} & & \uparrow \text{true} \\
 G1 & \longrightarrow & 1 & \longrightarrow & 1
 \end{array}$$

Now $P \leq G1$ in $Sub(G\Omega)$, for $\rho \circ G \text{ false} \circ ! = \text{false} \circ ! = \rho \circ g$, so $G\perp \circ ! = g$, since ρ is mono.

We now turn to the quantificational structure. Let's first consider universal quantification. Recall that $\Omega^B \xrightarrow{\forall_B} \Omega$ is the classifier of the exponentially transposed of $B \rightarrow 1 \xrightarrow{\text{true}} \Omega$. Universal quantification $Sub_{\mathcal{C}}(A \times B) \rightarrow Sub_{\mathcal{C}}(A)$

is then defined by composing an arrow $1 \rightarrow \Omega^{A \times B}$ with $(\forall_B)^A: \Omega^{A \times B} \cong (\Omega^B)^A \rightarrow \Omega^A$.

2.6 Lemma Φ preserves universal quantification; that is, for a subobject $U \rightrightarrows A \times B$ in E , $\Phi(\forall_B(U)) \cong \forall_{\bar{B}}(\Phi(U))$.

Proof: It again suffices to show that

$$(i) \quad G(\Omega^{A \times B}) \xrightarrow{G(\forall_B)^A} G(\Omega^A) \xrightarrow{k} \Omega^{\bar{A}} = G(\Omega^{A \times B}) \xrightarrow{k} \Omega^{\bar{A} \times \bar{B}} \xrightarrow{(\forall_{\bar{B}})^{\bar{A}}} \Omega^{\bar{A}}.$$

It is easy to see that this would follow from

$$(ii) \quad G(\Omega^B) \xrightarrow{G(\forall_B)} G\Omega \xrightarrow{\rho} \Omega = G(\Omega^B) \xrightarrow{k} \Omega^{\bar{B}} \xrightarrow{\forall_{\bar{B}}} \Omega.$$

Since the left-hand side in (ii) classifies $G(\ulcorner true_B \urcorner)$,

$$\begin{array}{ccccc} G(\Omega^B) & \xrightarrow{G(\forall_B)} & G\Omega & \xrightarrow{\rho} & \Omega \\ \uparrow G(\ulcorner true_B \urcorner) & & \uparrow G \text{ true} & & \uparrow \text{true} \\ G1 & \longrightarrow & G1 & \longrightarrow & 1 \end{array}$$

it suffices to show that the left-hand square of the diagram below is pullback

$$\begin{array}{ccccc} G(\Omega^B) & \xrightarrow{k} & \Omega^{\bar{B}} & \xrightarrow{\forall_{\bar{B}}} & \Omega \\ \uparrow G(\ulcorner true_B \urcorner) & & \uparrow \ulcorner true_{\bar{B}} \urcorner & & \uparrow \\ G1 & \longrightarrow & 1 & \longrightarrow & 1 \end{array}$$

But since k is mono, we only have to show that it commutes which is easy.

As for the existential quantifier, recall that $\Omega^B \xrightarrow{\exists_B} \Omega$ is the classifier of the image of $\in_B \xrightarrow{e_B} \Omega^B \times B \xrightarrow{\pi_1} \Omega^B$. (We will write $\bigcirc \exists_B$ for this image.)

2.7 Lemma For a subobject $U \in \text{Sub}_{\mathcal{E}}(A \times B)$, $\exists_{\bar{B}} \Phi(U) \leq \Phi(\exists_B(U))$.

Proof: As before, we have to show that the subobject of $G(\Omega^B)$ classified by $G(\Omega^B) \xrightarrow{k} \Omega^{\bar{B}} \xrightarrow{\exists_{\bar{B}}} \Omega$ is contained in that classified by $G(\Omega^B) \xrightarrow{G(\exists_B)} G\Omega \xrightarrow{\rho} \Omega$.

Now $\rho \circ G\exists_B$ classifies the image of $G\in_B \rightrightarrows GB \times G\Omega^B \xrightarrow{\pi} G\Omega^B$. Let P be the subobject of $G(\Omega^B)$ classified by $\exists_{\bar{B}}$. Pullbacks preserve epi-mono-factorizations, so P is the image of the pullback of $\in_{\bar{B}} \rightarrow \Omega^{\bar{B}} \times \bar{B} \xrightarrow{\pi} \Omega^{\bar{B}}$ along k , or, the image of $\pi \circ q$ in the diagram below

$$\begin{array}{ccccc} Q & \xrightarrow{q} & \bar{B} \times G\Omega^B & \xrightarrow{\pi} & G\Omega^B \\ \downarrow & \text{pb.} & \downarrow 1 \times k & & \downarrow k \\ \in_{\bar{B}} & \longrightarrow & \bar{B} \times \Omega^{\bar{B}} & \xrightarrow{\pi} & \Omega^{\bar{B}} \end{array}$$

q is the pullback of $1 \xrightarrow{true} \Omega$ along $ev \circ (1 \times k) = \rho \circ Gev \circ (e \times 1)$

$$\begin{array}{ccccccc}
 B \times G\Omega^B & \xrightarrow{e \times 1} & GB \times G\Omega^B & \xrightarrow{Gev} & G\Omega & \xrightarrow{\rho} & \Omega \\
 \uparrow q & & \uparrow Ge_B & & \uparrow Gtrue & & \uparrow true \\
 Q & \longrightarrow & G\in_B & \longrightarrow & G1 & \longrightarrow & 1
 \end{array}$$

We have to show that $P \leq G(\exists_B)$, or, that $\pi \circ q$ factors through $G\exists_B$, or, that $G\exists_B \circ \pi \circ q = Gtrue$. But $\pi \circ q = \pi \circ (e \times 1) \circ q = G\pi \circ Ge_B \circ s$, and, by definition, $\exists_B \circ \pi \circ e_B = true$, so $G\exists_B \circ G\pi \circ Ge_B \circ s = Gtrue$.

2.8 Lemma *Let $\Delta_A \twoheadrightarrow A \times A$ be the diagonal. Then*

- (i) $\Phi(\Delta_A) \geq \Delta_{\bar{A}}$
- (ii) if e_A is iso, $\Phi(\Delta_A) = \Delta_{\bar{A}}$.

Proof: Immediate from Remark 2.4.

We now return to question (2) of Section 1. Let us call an atomic formula *simple* if it is \top or \perp , or it is either of the form $\sigma_1 = \sigma_2$, where σ_1 and σ_2 are terms (in the sense explained at the end of Section 1!), or of the form $R(\sigma_1, \dots, \sigma_n)$, where $\sigma_1, \dots, \sigma_n$ are terms, and R is a relational term without (free) variables occurring in it. Furthermore, we call an occurrence of $=$ in a formula *basic* if it occurs in a subformula $\sigma_1 = \sigma_2$, where σ_1 and σ_2 are terms whose sorts are nonrelational, that is, have been built up from groundsorts without using the rule to make $[s_1, \dots, s_n]$ from s_1, \dots, s_n .

2.9 Theorem *Let T be a theory which has a set of axioms of the form $\forall \bar{x}(\phi(\bar{x}) \rightarrow \psi(\bar{x}))$, where the atomic parts of ϕ and ψ are simple, and*

- \exists, \forall , and nonbasic $=$ occur only positively in ϕ , and only negatively in ψ
- \rightarrow occurs only negatively in ϕ , and only positively in ψ .

Then

- (i) if $(\mathcal{E}, \mathcal{L})$ is a model of T and $\mathcal{E} \xrightarrow{d} \mathcal{F}$ is a left-exact functor which preserves the initial object, then $((\mathcal{F} \downarrow d), \bar{\mathcal{L}})$ is a model of T ,
- (ii) T has the disjunction-property.

Proof: (ii) follows from (i), and (i) follows easily from the properties of Φ that have been collected in the preceding lemmas.

We conclude with some remarks. First of all, it should be pointed out that the same techniques can be used to prove a result similar to Theorem 2.9 for theories having the existence property. Secondly, observe that the axioms of Higher-order Heyting's Arithmetic (*HHA*) are not preserved. In other words, if the language has a basic sort N for the natural numbers, and the theory T includes *HHA* $\mathcal{L}(N)$ must be the natural number object of \mathcal{E} for $(\mathcal{E}, \mathcal{L})$ to be a model of T , but $\bar{\mathcal{L}}(N) = G\mathcal{L}(N)$ is, in general, not the natural number object of

($\mathcal{F} \downarrow d$). There are several ways to improve on this, one of them being contained in [6], so we will not go into this here.

Finally, a word about occurrences of the identity, which also illustrates the conditions on atomic formulas. Suppose, for example, that we have a constant f of a functional sort $[[s] \rightarrow [s]]$ that is interpreted in $(\mathcal{C}, \mathcal{A})$ by $\mathcal{A}(f): \Omega^A \rightarrow \Omega^A$, and that $\mathcal{A}(f)$ equals the identity. Then $\mathcal{C} \models \forall U: \Omega^A \cdot f(U) = U$, and the identity-symbol occurring in $\forall U: \Omega^A \cdot f(U) = U$ is nonbasic, so its preservation is not covered by the theorem. This is how it should be, since $\mathcal{A}(f)$ is $\Omega^A \xrightarrow{e} G(\Omega^A) \xrightarrow{k} \Omega^A$ in this case, which is not the identity-arrow. Rewriting $\forall U: \Omega^A \cdot f(U) = U$ as $\forall U: \Omega^A \forall x: A(f(U)(x) \leftrightarrow U(x))$ does not help, since now the atomic part $f(U)(x)$ is not simple.

REFERENCES

- [1] Boileau, A. and A. Joyal, "La logique des topos," *The Journal of Symbolic Logic*, vol. 46 (1981), pp. 6-16.
- [2] Grothendieck, A., *Théorie des Topos et Cohomologie Étale des Schémas*, Springer, New York, 1972. (See esp. IV, Section 9.5.)
- [3] Johnstone, P., *Topos Theory*, Academic Press, New York, 1977.
- [4] Lambek, J. and P. J. Scott, "Intuitionistic type theory and the free topos," *Journal of Pure and Applied Algebra*, vol. 19 (1980), pp. 215-257.
- [5] Lambek, J. and P. J. Scott, "Independence of premisses and the free topos," pp. 109-227 in *Constructive Mathematics*, ed., F. Richman, Springer-Verlag, New York, 1980.
- [6] Moerdijk, I., "Glueing topoi and higher order disjunction and existence," to appear in *Proceedings of the Brouwer Centenary Conference*, eds., D. van Dalen and A. Troelstra, North-Holland, Amsterdam, 1982.
- [7] Ščedrov, A. and P. J. Scott, "Comparing the proofs of the disjunction and the existence property," to appear in *Proceedings of the Brouwer Centenary Conference*, eds., D. van Dalen and A. Troelstra, North-Holland, Amsterdam, 1982.
- [8] Smorynski, C., "Applications of Kripke models," pp. 324-391 in *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, ed., A. Troelstra, Springer, New York, 1973.
- [9] Wraith, G., "Artin glueing," *Journal of Pure and Applied Algebra*, vol. 4 (1974), pp. 345-348.

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