Morita equivalence for continuous groups

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This paper is essentially concerned with the following problem. Let $G$ be a topological group. A $G$-set is a set $S$ equipped with a continuous action $S \times G \to S$ (where $S$ is given the discrete topology). These $G$-sets form a category $BG$. It is well known that if $G$ and $H$ are discrete groups, $BG$ is equivalent to $BH$ iff $G$ is isomorphic to $H$. However, this result cannot be extended much beyond the discrete case. The problem is: when is $BG$ equivalent to $BH$, for topological groups $G$ and $H$?

It is of some interest to compare this with the results on Morita equivalence for commutative rings [14]. The analogous problem is: given rings (commutative, with 1) $R$ and $S$, and an equivalence between their categories of modules: $\text{Mod}(R) \simeq \text{Mod}(S)$, what is the relation between $R$ and $S$? An essential ingredient here is that any functor $\text{Mod}(R) \to \text{Mod}(S)$ which preserves co-limits is of the form $- \otimes_R P$, for a $R$-$S$-bimodule $P$. (For an exposition, see, e.g. chapter II of [2].)

One of the problems in the context of $BG$ is that, unlike $\text{Mod}(R)$, $BG$ does not have a single generator. However, for every open subgroup $U \leq G$, the set of right cosets $G/U$ is an object of $BG$, and the objects of this form collectively generate $BG$. So if $F : BG \to BH$ is a functor which preserves co-limits, one may construct the space

$$P = \lim_{U} F(G/U),$$

where the inverse limit is taken over all open subgroups $U$, ordered by inclusion. Since $H$ acts on each set $F(G/U)$ on the right, there is a continuous action of $H$ on $P$.

To proceed, it turns out that $G$ is somewhat too small: let

$$M(G) = \lim_{U} G/U.$$ 

$M(G)$ is a topological monoid, with multiplication defined by

$$(x \cdot y)_{U} = x_{U} \cdot y_{x_{U}^{-1}ux_{U}}$$

where $x = (Ux_{U})_{U}$, $y = (Uy_{U})_{U}$ are points of $M(G)$. $M(G)$ acts on $P$ from the left, by

$$(x \cdot p)_{U} = F(G/x_{U}^{-1}UXU^{-1}Ux_{U}G/U)(px_{U}^{-1}ux_{U}),$$

where $x = (Ux_{U})_{U} \in M(G)$, $p = (p_{U})_{U} \in P$ and $x_{U} : G/x_{U}^{-1}UXU^{-1}Ux_{U}G/U$ stands for the map $x_{U}^{-1}UXU^{-1}Ux_{U}g \mapsto Ux_{U}g$.

If $S$ is a $G$-set, the action $S \times G \to S$ can uniquely be extended to a continuous action $S \times M(G) \to S$, since for $s \in S$ and $x = (Ux_{U})_{U} \in M(G)$, the sequence $(s \cdot x_{U})_{U}$ eventually becomes constant. Analogously to the module case, one can then show that if $F$ preserves co-limits and finite limits,

$$F \simeq - \otimes_{M(G)} P;$$

(3)
i.e. that for each $G$-set $S$, $F(S)$ is isomorphic to the quotient of $S \times P$ obtained by identifying $(s \cdot x, p)$ and $(s, x \cdot p)$ for $s \in S$, $p \in P$, $x \in M(G)$ (see 3.4 of this paper).

There is one necessary condition, though: the inverse limit (1) may be very badly behaved if $G$ is too big. However, if $G$ has a countable neighbourhood basis at the unit-element, for instance, then (3) will follow.

This size condition on $G$ can be eliminated if one takes the inverse limits (1) and (2) in a slightly different category of generalized spaces, or locales (as discussed in [5, 8, 9], and others). Having taken this step, one may as well consider the problem of when $BG$ and $BH$ are equivalent categories for group objects in this larger category of generalized spaces, so-called continuous groups. Another reason for working with continuous groups, rather than just topological groups, is a theorem due to A. Joyal and M. Tierney, which states that any connected atomic topos with a point is equivalent to a category of the form $BG$, which is Galois theory, generalized from Galois categories in the sense of Grothendieck [4] to arbitrary connected atomic toposes with a point.

As a consequence, our results will imply a description of the category of pointed connected atomic toposes and geometric morphisms, purely in terms of continuous groups and certain 'bimodules'; see Theorem 4.4 below. These results are valid over an arbitrary base topos.

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1. The classifying topos of a continuous group

Throughout this paper, $\mathcal{S}$ denotes an arbitrary but fixed base topos; we will abuse the language in the usual way and act as if $\mathcal{S}$ is 'the' category of sets.

1.1. Spaces and locales. We will follow the terminology of [9]. So a locale is a complete lattice $A$ which satisfies the distributivity law

$$a \land \bigvee_{i \in I} b_i = \bigvee_{i \in I} a \land b_i.$$  

A morphism of locales $A \to B$ is a function which preserves finite meets and arbitrary sups. This defines the category of locales. Its dual is the category of spaces. If $X$ is a space, we write $\mathcal{O}(X)$ for the corresponding locale, the elements of which are called opens of $X$. So a map of spaces $X \rightarrow Y$ is by definition a locale morphism $f^{-1}: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$. Cts$(X, Y)$ denotes the set of maps from $X$ to $Y$.

We will assume that the reader is familiar with the basic properties of locales and spaces; see [8, 9]. Change-of-base techniques play an important role in the sequel, and some familiarity with the methods (rather than the results) as exemplified in [13] will be very helpful; cf. also [12], in particular 5.3. In applying change-of-base techniques in the context of quotients of locales, it is necessary that these quotients be stable. To this end, we make the following important observation.

1.2. Lemma. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a coequalizer of spaces. If $f$ and $g$ are open, then so is $g$. Moreover, in this case

$$T \times X \xrightarrow{1 \times f} T \times Y \xrightarrow{1 \times g} T \times Z$$

is a coequalizer, for any space $T$. 

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Proof. The coequalizer $Z$ can be constructed as

$$\mathcal{O}(Z) = \{ U \in \mathcal{O}(Y) \mid f^{-1}(U) = g^{-1}(U) \},$$

and $q^{-1} : \mathcal{O}(Z) \rightarrow \mathcal{O}(Y)$ is the inclusion. Clearly if $f^{-1}$ and $g^{-1}$ preserve arbitrary infima and implications $\wedge$, so does $q^{-1}$. So $q$ is open if $f$ and $g$ are.

Now take a space $T$. If $f$ is open, so is $1 \times f : T \times X \rightarrow T \times Y$, and if $f : \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ is the left adjoint of $f^{-1}$, then $1 \otimes f$, is left adjoint to $1 \otimes f^{-1}$,

$$\mathcal{O}(T) \otimes \mathcal{O}(X) \xrightarrow{1 \otimes f} \mathcal{O}(T) \otimes \mathcal{O}(Y)$$

(recall that $\mathcal{O}(T \times X) = \mathcal{O}(T) \otimes \mathcal{O}(X)$, etc.; cf. [9]), and similarly for $g$.

Let $\mathcal{E} = \{ U \in \mathcal{O}(T) \otimes \mathcal{O}(Y) \mid (1 \otimes f)^{-1}(U) = (1 \otimes g)^{-1}(U) \}$. Clearly $1 \otimes q^{-1} : \mathcal{O}(T) \otimes \mathcal{O}(Z) \rightarrow \mathcal{O}(T) \otimes \mathcal{O}(Y)$ factors through $\mathcal{E}$. We claim that $\mathcal{E}$ is precisely the image of $1 \otimes q^{-1}$. To see this, take $U \in \mathcal{E}$. For $V \in \mathcal{O}(T)$, let

$$W_V = \bigvee \{ O \in \mathcal{O}(Y) \mid V \otimes O \leq U \}.$$

Then

$$U = \bigvee \{ V \otimes W_V \mid V \in \mathcal{O}(Y) \}.$$

Now take $V \in \mathcal{O}(Y)$. Then

$$V \otimes g, f^{-1}(W_V) = (1 \otimes g,)(1 \otimes f^{-1})(V \otimes W_V) \leq (1 \otimes g,)(1 \otimes f^{-1})(U) = (1 \otimes g,)(1 \otimes g^{-1})(U) \leq U.$$

So $g, f^{-1}(W_V) \leq W_V$, and hence by adjointness, $f^{-1}(W_V) \leq g^{-1}(W_V)$. By symmetry, we conclude that $f^{-1}(W_V) = g^{-1}(W_V)$, i.e. $W_V \in \mathcal{O}(Z)$. Consequently, $U$ is in the image of $1 \otimes q^{-1}$. This shows that $\mathcal{E} \cong \mathcal{O}(T) \otimes \mathcal{O}(Z)$, and hence that $T \times X \xrightarrow{\sim} T \times Y \rightarrow T \times Z$ is a coequalizer.

1.3. Continuous groups. A continuous group is by definition a group object in the category of spaces. In this paper, we will always assume that $G$ is open, i.e. that $G \rightarrow 1$ is an open map of spaces. If $G$ is a continuous group and $U$ is an open subgroup, then the quotient $G/U$ defined as the coequalizer

$$G \times U \xrightarrow{\pi_U} G \xrightarrow{p_U} G/U$$

is discrete. Moreover, (1) is a stable coequalizer and $p_U$ is an open surjection, by 1.2.

A continuous monoid is a space $M$ equipped with an associative multiplication $M \times M \rightarrow M$ which has a two-sided unit $1 \xrightarrow{1} M$.

The following construction plays a fundamental role in this paper. Let $G$ be a continuous group, and let

$$M(G) = \lim_{\longrightarrow U} G/U.$$  

(2)

be the inverse limit in the category of spaces, taken over all open subgroups $U \subseteq G$, partially ordered by inclusion. If $U \subseteq V$, the projection $G/U \rightarrow G/V$ is a surjection of sets, so it follows that the projection

$$M(G) \xrightarrow{\pi_U} G/U$$

is a coequalizer.
is an open surjection (surjectivity follows from [6]; that \( \pi_U \) is also open is a special case of theorem 5.1(ii) from [10]).

1.4. Lemma. The construction of \( M(G) \) is stable; i.e. if \( \mathcal{F} \rightarrow \mathcal{E} \) is a geometric morphism and \( G \) is a continuous group in \( \mathcal{E} \), then

\[
M(f^*G) \cong f^*(M(G)),
\]

where \( f^* \) is the pullback functor from spaces in \( \mathcal{E} \) to spaces in \( \mathcal{F} \).

Proof. \( f^* \) commutes with inverse limits, and with coequalizers of the form (1) since these are stable. Moreover, the open subgroups of \( f^*(G) \) in \( \mathcal{F} \) which are of the form \( f^*(U) \) for an open subgroup \( U \subset G \) in \( \mathcal{E} \), are co-final in the (internal) system of all open subgroups of \( f^*(G) \).

1.5. Lemma. Let \( G \) be a continuous group. Then \( M(G) \) has the structure of a continuous monoid, and the canonical map

\[
G \rightarrow M(G)
\]

defined by \( \pi_U \circ \theta = p_U \) preserves the multiplication.

Proof. (a) Let us first consider the classical case of a topological group \( G \) in sets, and define \( M(G) = \lim G/U \) where the inverse limit is taken in topological spaces (rather than ‘generalized’ spaces). Writing the elements of \( M(G) \) as sequence \( x = \{Ux\} \) of cosets \( (x \in G) \), we can define an associative multiplication by

\[
(x \cdot y)_U = x_U \cdot y_U \cdot U _{x_U} y_U,
\]

and it is easy to check that this makes \( M(G) \) into a topological monoid.

(b) The same formula (1) works in the context of a continuous group in the base topos \( \mathcal{S} \), by the method of test spaces and change-of-base. Indeed, to define a multiplication \( M(G) \times M(G) \rightarrow M(G) \), we may equivalently define an operation

\[
\text{Cts}(T, M(G)) \times \text{Cts}(T, M(G)) \xrightarrow{\mu_T} \text{Cts}(T, M(G)),
\]

natural in the test space \( T \). So suppose we are given maps \( T \xrightarrow{\nu} M(G) \), and consider the pullback

\[
\begin{array}{ccc}
T' & \xrightarrow{(x', y')} & \Pi_U(G \times G) \\
\downarrow t & & \downarrow p \\
T & \xrightarrow{(x, y)} & M(G) \times M(G) \xrightarrow{\pi} \Pi_U(G/U \times G/U)
\end{array}
\]

where the product \( \Pi_U(\ldots) \) is taken over all open subgroups \( U \subset G \), while \( p \) and \( \pi \) are the obvious maps induced by the projections \( G \xrightarrow{p_U} G/U, M(G) \xrightarrow{\pi_U} G/U \). Since each \( p_U \) is an open surjection, so is \( p \), and hence \( t \) is an open surjection. Let \( \text{Sh}(T') \rightarrow \mathcal{S} \) be the canonical geometric morphism. Then \( x' \) and \( y' \) define points of \( \gamma^*(\Pi_U(G) \cong \Pi_U(\gamma^*G)) \) in \( \text{Sh}(T') \), and formula (1) defines a point \( x' \cdot y' \) of \( M(\gamma^*G) = \gamma^*(M(G)) \), because open subgroups of the form \( \gamma^*(U) \) are co-final among open subgroups of \( \gamma^*(G) \), cf. 1.4. Since \( t \) is an open surjection, it is a coequalizer of its kernel-pair ([10], p. 66), and it is not hard to see that this implies that \( x' \cdot y' \) factors through \( t \), so as to produce the required map \( T \xrightarrow{\mu_T} M(G) \). Further details are routine.
1.6. Remark. This type of change-of-base argument can essentially be reduced to the slogan that we may use point-set definitions (like (1)) and arguments in the context of stable constructions of generalized spaces. In the sequel, we will often (implicitly) refer to this slogan, and suppress further details.

1.7. $G$-sets. Let $G$ be a continuous group. A $G$-set is a set $S$ (an object of $\mathcal{S}$), equipped with a map of spaces $S \times G \to S$ satisfying the usual identities for an action; here $S$ is regarded as a discrete space. With the obvious notion of a map of $G$-sets, we obtain the category $BG$, called the classifying topos of $G$. $BG$ is an atomic topos (see [1] for information about atomic toposes). The full subcategory of $BG$ consisting of the $G$-sets $G/U$ (with action induced by the multiplication of $G$) where $U$ ranges over the open subgroups of $G$, forms an atomic site for $BG$ which we denote by $\mathcal{S}(G)$. If $U$ and $V$ are open subgroups of $G$ and $x: 1 \to G$ is a point of $G$ such that $U \subseteq x^{-1}Vx$, then $x$ defines a map $G/U \to G/V$ in $\mathcal{S}(G)$ by $Ug \to Vxg$ (point-set notation). We simply write $G/U \to G/V$ for this map.

The construction works over any base topos: if $G$ is a continuous group in a topos $\mathcal{E}$, the topos of objects of $\mathcal{E}$ equipped with an action by $G$ is denoted by $B(\mathcal{E}, G)$. The canonical geometric morphism

$$B(\mathcal{E}, G) \xrightarrow{\gamma_G} \mathcal{E}$$

is given by $\gamma_G^*(E) = E$ with the trivial $G$-action. $\gamma_G$ has a canonical section

$$\mathcal{E} \xrightarrow{\nu_G} B(\mathcal{E}, G),$$

whose inverse image is the forgetful functor.

The construction is stable under change of base (recall that we assume $G \to 1$ to be open).

1.8. Lemma. If $\mathcal{F} \to \mathcal{E}$ is a geometric morphism and $G$ is a continuous group in $\mathcal{E}$, then there is a natural equivalence

$$B(\mathcal{F}, f^*G) \xrightarrow{\sim} \mathcal{F} \times_{\mathcal{E}} B(\mathcal{E}, G).$$


The following theorem is due to Joyal and Tierney[9].

1.9. Theorem. Let $\mathcal{F} \xrightarrow{f} \mathcal{E}$ be a connected atomic geometric morphism. If $f$ has a section $p$, then there is a continuous group $G$ in $\mathcal{E}$ and an equivalence

$$\mathcal{F} \xrightarrow{\sim} B(\mathcal{E}, G)$$

(\text{the section $p$ corresponds to the canonical section $p_G$ under the equivalence}).

This is a representation theorem for pointed atomic connected toposes (over any base, $\mathcal{E}$ in this case). The aim of this paper is to investigate how geometric morphisms between pointed connected atomic toposes can be described in terms of the corresponding groups.
1.10. **Remark.** Let \( G \) be a continuous group, and \( \mathcal{S} \xrightarrow{p_G} BG \) be the canonical point of the topos \( BG \). Then \( M(G) \) is the space of endomorphisms of \( p_G \), in the sense that

\[
\begin{array}{ccc}
Sh(M(G)) & \xrightarrow{i} & \mathcal{S} \\
\downarrow & & \downarrow \quad \quad \downarrow \\
\mathcal{S} & \xrightarrow{p_G} & BG
\end{array}
\]

is a lax pullback of toposes.

Obviously, the definition of the category \( BG \) also makes sense if \( G \) is just a continuous monoid, and we extend the notation to this context. (If \( M \) is a continuous monoid, the corresponding topos \( BM \) is in general no longer atomic.)

1.11. **Lemma.** Let \( G \) be a continuous group. The canonical map \( G \xrightarrow{\theta} M(G) \) induces an equivalence of categories

\[
BM(G) \xrightarrow{\sim} BG.
\]

**Proof.** Let \( S \) be a \( G \)-set. Density of \( \theta \) easily implies that the action \( S \times G \rightarrow S \) has a unique factorization through \( S \times \theta \):

\[
\begin{array}{ccc}
S \times G & \xrightarrow{\theta} & S \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
S \times M(G)
\end{array}
\]

1.12. **Example.** It follows that if \( G \) and \( H \) are continuous groups such that \( M(G) \) and \( M(H) \) are isomorphic, then \( BG \) is equivalent to \( BH \). The converse is false. For instance, if \( S \) is an infinite set, \( M(\text{Aut}(S)) \) is the monoid of monomorphisms \( S \rightarrow \rightarrow S \) (constructed as a locale), and \( B \text{Aut}(S) \) classifies infinite decidable sets (and monomorphisms between them). So \( B \text{Aut}(S) \) is equivalent to \( B \text{Aut}(S') \) for any two infinite sets \( S \) and \( S' \).

2. **Bispaces**

Let \( G \) be a continuous group (in a topos \( \mathcal{S} \)), and let \( M(G) \) be the associated monoid. An \( M(G) \)-space is a space \( P \) with an action

\[
M(G) \times P \rightarrow P.
\]

Such an \( M(G) \)-space \( P \) is called *open* if \( P \) is an open space (i.e. \( \rightarrow \rightarrow 1 \) is open) and the action \( M(G) \times P \rightarrow P \) is an open map.

If \( X \) is a space with an action \( X \times M(G) \rightarrow X \) of \( M(G) \) on the right, one can form the tensor-product \( X \otimes_{M(G)} P \), as the coequalizer of spaces

\[
X \times M(G) \times P \xrightarrow{\alpha} X \times P \rightarrow X \otimes_{M(G)} P
\]

where \( \alpha = X \times \cdot, \beta = \cdot \times P \).

2.1. **Lemma.** Let \( P \) be an open \( M(G) \)-space, and suppose that the diagonal action map

\[
M(G) \times M(G) \times P \xrightarrow{\mu_{ij}} M(G) \times P \rightarrow P
\]

is open (here \( \mu_{ij} \) is the composite \( M(G) \times M(G) \times P \xrightarrow{\nu_{ij}} M(G) \times P \rightarrow P \)). Let \( S \) be a \( G \)-
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set, regarded as a discrete $M(G)$-space (cf. 1.11). Then $S \otimes_{M(G)} P$ is discrete, and the coequalizer

\[ S \times M(G) \times P \xrightarrow{\alpha} S \times P \xrightarrow{\beta} S \otimes_{M(G)} P \]

(2)

is stable.

Proof. $- \otimes_{M(G)} P$ commutes with sums, so by writing $S$ as the sum of its orbits, we may restrict ourselves to the case where $S = G/U$ for some open subgroup $U \subseteq G$. Let $N_U$ be the corresponding basic open of $M(G)$, i.e.

\[
\begin{array}{ccc}
1 & \xrightarrow{1} & G \\
\downarrow & & \downarrow \pi_U \\
N_U & \xrightarrow{N_U} & M(G)
\end{array}
\]

is a pullback. Then $G/U \otimes_{M(G)} P$ is the same as the coequalizer

\[ N_U \times N_U \times P \xrightarrow{\mu_{13}, \mu_{23}} P \xrightarrow{q} G/U \otimes_{M(G)} P, \]

(3)

where $\mu_{ij}$ is multiplication of the $i$th and $j$th factor. Note that $q$ is open since $\mu_{13}$ and $\mu_{12}$ are (cf. 1.2).

Since $P \rightarrow 1$ is open, so is $G/U \otimes_{M(G)} P \rightarrow 1$. Moreover, $G/U \otimes_{M(G)} P$ has an open diagonal, as follows by considering the diagram (with $r = \mu_{ij} \circ q$)

\[
\begin{array}{ccc}
N_U \times N_U \times P & \xrightarrow{(\mu_{13}, \mu_{23})} & P \times P \\
\downarrow r & & \downarrow q \times q \\
G/U \otimes_{M(G)} P & \xrightarrow{\Delta} & (G/U \otimes_{M(G)} P) \times (G/U \otimes_{M(G)} P).
\end{array}
\]

Indeed, $q \times q$ is open since $q$ is, and $(\mu_{13}, \mu_{23})$ is open by assumption. Since $r$ is surjective, it follows that $\Delta$ must be open. Consequently, $G/U \otimes_{M(G)} P$ is discrete (cf. [9], §iv.2).

Finally, it follows from 1.2 that the coequalizer (2) is stable.

2.2. Notation. Note that $G/U \otimes_{M(G)} P$ can also be constructed as the coequalizer

\[ N_U \times P \xrightarrow{\pi_U} P \rightarrow G/U \otimes_{M(G)} P. \]

(5)

Therefore, we will also write $P/N_U$ for $G/U \otimes_{M(G)} P$.

An $M(G)$-space is called flat if it is an open $M(G)$-space such that the diagonal action map $M(G) \times M(G) \times P \rightarrow P \times P$ from 2.1 is open, and the induced functor (by 2.1)

\[ - \otimes_{M(G)} P : BG \rightarrow \mathcal{S} \]

is left-exact. Notice that since $- \otimes_{M(G)} P$ obviously commutes with co-limits, a flat $M(G)$-space gives rise to a point of the topos $BG$. 
2-3. Bispaces. Let $G$ and $H$ be continuous groups. An $M(G)-M(H)$-bispace is an open $M(G)$-space $P$ equipped with another open action $P \times M(H) \to P$ of $M(H)$ on the right, such that the two actions are associative, i.e.

$$
\begin{array}{ccc}
M(G) \times P \times M(H) & \xrightarrow{M(G) \times \ast} & M(G) \times P \\
\downarrow \times M(H) & & \downarrow \\
P \times M(H) & \xrightarrow{\ast} & P
\end{array}
$$

commutes. An $M(G)-M(H)$-bispace is called flat if it is flat as an $M(G)$-space.

A map of $M(G)-M(H)$-bispaces $P \xrightarrow{z} P'$ is a map of spaces which commutes with both actions. This defines a category

$$
\text{Flat}(M(G),M(H))
$$

of flat $M(G)-M(H)$-bispaces.

By 2-1 and 1-11, each flat $M(G)-M(H)$-space $P$ induces a functor $- \otimes_{M(G)} P : BH \to BH$, which is the inverse image of a geometric morphism $t(P) : BH \to BG$. This defines a functor

$$
t : \text{Flat}(M(G),M(H)) \to \text{Hom}_\mathcal{S}(BH,BG)
$$

where for toposes $\mathcal{E}$ and $\mathcal{F}$ over $\mathcal{S}$, $\text{Hom}_\mathcal{S}(\mathcal{F},\mathcal{E})$ denotes the category whose objects are geometric morphisms $\mathcal{F} \xrightarrow{f} \mathcal{E}$ over $\mathcal{S}$, and whose maps $f \Rightarrow f'$ are natural transformations $f^* \Rightarrow f'^*$ (over $\mathcal{S}$).

If $K$ is another continuous group, $P$ is an $M(G)-M(H)$-bispace and $Q$ is an $M(H)-M(K)$-bispace, one can construct the coequalizer

$$
P \times M(H) \times Q \xrightarrow{P \times \ast \times Q} P \times Q \xrightarrow{\ast} P \otimes_{M(H)} Q.
$$

(1)

This coequalizer is stable by 1-2, so $P \otimes_{M(H)} Q$ inherits an action of $M(G)$ on the left and one from $M(K)$ on the right, in the obvious way.

2-4. Lemma. Let $P$ be a flat $M(G)-M(H)$-bispace, and $Q$ a flat $M(H)-M(K)$-bispace. Then $P \otimes_{M(H)} Q$ is a flat $M(G)-M(K)$-bispace, and there is a natural isomorphism of functors $BK \to BG$:

$$
- \otimes_{M(G)} (P \otimes_{M(H)} Q) \xrightarrow{\sim} (- \otimes_{M(G)} P) \otimes_{M(H)} Q.
$$

2-5. Remark. The preceding lemma says that the functors

$$
t : \text{Flat}(M(G),M(H)) \to \text{Hom}_\mathcal{S}(BH,BG)
$$

send tensor products to compositions of geometric morphisms, i.e. that the square

$$
\begin{array}{ccc}
\text{Flat}(M(G),M(H)) \times \text{Flat}(M(H),M(K)) & \xrightarrow{t \times t} & \text{Hom}_\mathcal{S}(BH,BG) \times \text{Hom}_\mathcal{S}(BK,BH) \\
\otimes_{M(H)} & & \circ \\
\text{Flat}(M(G),M(K)) & \xrightarrow{t} & \text{Hom}_\mathcal{S}(BG,BK)
\end{array}
$$

of categories and functors commutes up to natural isomorphism.
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Proof of 2.4. It is easy to see that $P \otimes_{M(H)} Q$ is an open bispace, i.e. that the three maps $P \otimes_{M(H)} Q \rightarrow 1$, $M(G) \times (P \otimes_{M(H)} Q) \rightarrow P \otimes_{M(H)} Q$ and $(P \otimes_{M(H)} Q) \times M(K) \rightarrow P \otimes_{M(H)} Q$ are all open.

To see that the diagonal action map (cf. 2.1)

$$M(G) \times M(G) \times (P \otimes_{M(H)} Q) \xrightarrow{\mu} (P \otimes_{M(H)} Q) \times (P \otimes_{M(H)} Q)$$

(1)

is open, consider the pullback

$$\begin{array}{ccc}
R & \xrightarrow{\phi} & M(G) \times M(G) \times P \\
\downarrow & & \downarrow \alpha \\
M(G) \times M(G) & \xrightarrow{\beta} & P \times P,
\end{array}$$

where $\pi_1 \circ \alpha$ is the composite $M(G) \times M(G) \times P \xrightarrow{\pi_1} M(G) \times P \xrightarrow{\pi} P$, and similarly for $\beta$. Now $\alpha$ is open by hypothesis (this is part of the definition of ‘flat’), so $\psi$ is open. Consider the diagram

$$\begin{array}{ccc}
M(G) \times M(G) \times R \times Q & \xrightarrow{1 \times \psi \times 1} & (M(G) \times M(G) \times P) \times (M(H) \times M(H) \times Q) \\
\downarrow 1 \times \phi \times 1 & & \downarrow \nu \\
M(G) \times M(G) \times M(G) \times M(G) \times P \times Q & \xrightarrow{m \times \pi} & (P \times P) \times (Q \times Q) \\
\downarrow m \times \pi & & \downarrow (\pi \times \pi) \circ \tau \\
M(G) \times M(G) \times (P \otimes_{M(H)} Q) & \xrightarrow{\mu} & (P \otimes_{M(H)} Q) \times (P \otimes_{M(H)} Q)
\end{array}$$

where $\pi$ is the quotient map $P \times Q \rightarrow P \otimes_{M(H)} Q$, $\tau$ interchanges the second and third coordinates, $\nu$ is the product of the two diagonal actions (cf. 2.1) and $\pi_1 \circ m$ is the product of the first and third coordinates, $\pi_2 \circ m$ that of the second and last. $\pi$, $\nu$ and $\psi$ are open, hence so is

$$(\pi \times \pi) \circ \tau \circ \nu \circ (1 \times \psi \times 1) = \mu \circ (m \times \pi) \circ (1 \times \phi \times 1).$$

So $\mu$ is open, if we can show that $(m \times \pi) \circ (1 \times \phi \times 1)$ is a surjection. But there is a factorization

$$\begin{array}{ccc}
M(G) \times M(G) \times R \times Q & \xrightarrow{1 \times \phi \times 1} & (M(G) \times M(G) \times P) \times (M(H) \times M(H) \times Q) \\
\downarrow (m \times \pi) \circ (1 \times \phi \times 1) & & \\
M(G) \times M(G) \times P \times Q & \xrightarrow{1 \times \pi} & M(G) \times M(G) \times (P \otimes_{M(H)} Q)
\end{array}$$

where, writing $1 \xrightarrow{\epsilon} M(G)$ and $1 \xrightarrow{\epsilon} M(H)$ for the units, $P \xrightarrow{\epsilon} R$ is given by $\phi \circ \delta = (e, e, 1)$, $\psi \circ \delta = (e', e', 1)$.

This shows that (1) is an open map.

We now complete the proof by showing that there is a natural isomorphism

$$S \otimes_{M(G)} (P \otimes_{M(H)} Q) \cong S \otimes_{M(G)} (P \otimes_{M(H)} Q)$$
of $H$-sets, for each $G$-set $S$. Writing $S$ as the sum of its orbits, it suffices to show that for each open subgroup $U \subseteq G$,

$$(P \otimes_{M(H)} Q)/N_U \cong (P/N_U) \otimes_{M(H)} Q$$

(cf. 2.2). Consider the diagram

$$
\begin{array}{c}
N_U \times N_U \times P \times M(H) \times Q \twoheadrightarrow P \times M(H) \times Q \twoheadrightarrow P/N_U \times M(H) \times Q \\
\downarrow \quad \downarrow \quad \downarrow \\
N_U \times N_U \times P \times Q \twoheadrightarrow P \times Q \twoheadrightarrow P/N_U \times Q \\
\downarrow \quad \downarrow \\
(N_U \times N_U \times P) \otimes_{M(H)} Q \twoheadrightarrow P \otimes_{M(H)} Q \twoheadrightarrow (P/N_U) \otimes_{M(H)} Q
\end{array}
$$

Here the rows come from applying $- \times (M(H) \times Q)$, $- \times Q$ and $- \otimes_{M(H)} Q$ respectively to the stable coequalizer (3) in 2.1; the middle vertical one is (1) above 2.4, and the left-hand vertical coequalizer is isomorphic to the result of applying $N_U \times N_U \times -$ to the coequalizer (1) above 2.4, since this coequalizer is stable:

$$(N_U \times N_U \times P) \otimes_{M(H)} Q \cong (N_U \times N_U) \times (P \otimes_{M(H)} Q).$$

Taking first the coequalizers of the two upper rows, then of the last column, gives the same result as first taking the two left-hand coequalizers, and then taking the coequalizer along the bottom. In other words, the composition of the bottom and the isomorphism (6) is a coequalizer, i.e.

$$(P \otimes_{M(H)} Q)/N_U \cong (P/N_U) \otimes_{M(H)} Q$$

as was to be shown.

3. Geometric morphisms

Let $G$ and $H$ be continuous groups. We will now show that each geometric morphism $BH \xrightarrow{\phi} BG$ is induced by a flat $M(G)$-$M(H)$-bispace, as in 2.3. Given such a $\phi$, let

$$P = \lim \phi^*(G/U),$$

where the inverse limit is taken over all open subgroups of $G$, ordered by inclusion. Write $P \xrightarrow{\pi_U} \phi^*(G/U)$ for the projection; each $\pi_U$ is an open surjection ([10], 5.1 (ii)). Each $\phi^*(G/U)$ is an $H$-set. This action can be uniquely extended to an action $\phi^*(G/U) \times M(H) \to \phi^*(G/U)$ (cf. 1.11), so by passing to the limit, we get an action $P \times M(H) \to P$. It is easy to see that

$$P \times M(H) \twoheadrightarrow P$$

is an open map.

$M(G)$ acts on $P$ from the left, $M(G) \times P \twoheadrightarrow P$. This action is described in point-set notation by

$$\pi_U(x \cdot p) = \phi^*(G/\varphi^{-1}_U) \cdot x \cdot U \cdot \varphi^{-1}_U \cdot G/U \cdot (p_{/U})_{/ \varphi^{-1}_U},$$

where $x = \{x_U\} \in \lim G/U = M(G)$, and $p = \{p_U\}$ is a point of $P$. (2) can be taken as a definition of the map $M(G) \times P \to P$, by using test-spaces as in 1.5. It follows
easily from functoriality of $\phi^*: \mathbb{S}(G) \to BH$ that these two actions on $P$ satisfy the associativity condition of 2.3. So we have

3.1. Lemma. Let $BH \xrightarrow{\phi} BG$ be a geometric morphism, and $P = \lim_{U} \phi^*(G/U)$ as above. Then $P$ has the structure of an $M(G) \cdot M(H)$-space.

We list some of the properties of $P$. Note that (ii) below together with (1) above imply that $P$ is an open bispace.

3.2. Lemma. Let $BH \xrightarrow{\phi} BG$ and $P$ be as in 3.1.
(i) The action of $M(G)$ on $P$ is free, i.e.

$$M(G) \times P \xrightarrow{\Delta \times P} M(G) \times M(G) \times P \xrightarrow{\mu_{12}} P$$

is an equalizer, where $\mu_U$ is $M(G) \times M(G) \times P \xrightarrow{\pi_0} M(G) \times P \to P$.

(ii) The action is transitive, in the sense that

$$M(G) \times M(G) \times P \xrightarrow{\mu_{12}, \mu_{23}} P \times P$$

is an open surjection. Moreover:

(iii) For any open subgroup $U \subseteq G$, the horizontal map in

$$N_U \times N_U \times P \xrightarrow{\mu_{12}, \mu_{23}} P \times P \xrightarrow{\phi^*(G/U)}$$

is an open surjection over $\phi^*(G/U)$.

As a preparation for the proof of 3.2, let us observe the following. The ‘point-set interpretation’ of (iii) is roughly that for all $p = (p_U)_U$, $q = (q_U)_U$ in $P$ with $p_U = q_U$, there are $x, y \in N_U \subseteq M(G)$ and an $r \in P$ with $r_U = p_U = q_U$ such that $x \cdot r = p$, and $y \cdot r = q$. Now consider the following special case of (iii), which contains the key construction.

3.3. Lemma. Let $BH \xrightarrow{\phi} BG$ and $P = \lim_{U} \phi^*(G/U)$ be as above, and suppose that $G$ has a countable co-final system of open subgroups $U_0 \supseteq U_1 \supseteq U_2 \supseteq \ldots$. Let $U \subseteq G$ be an open subgroup, and let $p, q$ be points of $P$ with $\pi_U(p) = \pi_U(q)$. Then there is an open surjection $\mathcal{S} \xrightarrow{\alpha} \mathcal{S}$ such that in $\mathcal{S}'$, there are points $x, y$ of $N_U \subseteq M(G)$ and a point $r$ of $P$ with $\pi_U(r) = \pi_U(p) = \pi_U(q)$, and

$$x \cdot r = p, \quad y \cdot r = q.$$  

(Here we write $M(G)$ for $\alpha^*(M(G)) = M(\alpha^*(G))$, etc.)

Proof. Assume for the moment that $\mathcal{S} = \text{Sets}$, and that $G$ has enough points. We may suppose that $U = U_0$. By induction, we will define a strictly increasing
sequence $0 = n_0 < n_1 < \ldots$ of natural numbers, sequences $(x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}}$ of points of $G$, and a sequence $(r_{n_k})_k, r_{n_k} \in \phi^*(G/U_{n_k})$, such that

$$x_k, \quad y_k \in U,$$

$$x_{k+1} \in U_k x_k, \quad y_{k+1} \in U_k y_k,$$

$$U_{n_k} \subseteq x_k^{-1}U_k x_k \cap y_k^{-1}U_k y_k,$$

$$\phi^*(G/U_{n_{k+1}} \xrightarrow{1} G/U_{n_k})(r_{n_{k+1}}) = r_{n_k},$$

$$\phi^*(G/U_{n_k} \xrightarrow{x_k} G/U_k)(r_{n_k}) = p_k,$$

$$\phi^*(G/U_{n_k} \xrightarrow{y_k} G/U_k)(r_{n_k}) = q_k.$$

Then if we define $r_m = \phi^*(G/U_{n_k} \xrightarrow{1} G/U_m)(r_{n_k})$ when $n_{k-1} < m < n_k$, we obtain points $x, y \in M(G)$ and $r \in P$ such that

$$x \cdot r = p, \quad y \cdot r = q.$$  

Put $n_0 = 0, x_0 = y_0 = 1, r_{n_0} = p_0 = q_0$.  

Suppose $n_k, x_k, y_k, r_{n_k}$ have been defined. By successive applications of flatness of $\phi^*: \mathcal{F}(G) \to \text{Sets}$, we construct a commutative diagram from right to left, as follows:

Since $\phi^*(G/U_{n_{k+1}} \xrightarrow{1} G/U_k)(p_{k+1}) = \phi^*(G/U_{n_k} \xrightarrow{x_k} G/U_k)(r_{n_k}) = p_k$, there are an open subgroup $W \subseteq G$ and points $a, b \in G$, and a $\xi \in \phi^*(G/W)$, such that

$$W \subseteq a^{-1}U_k a \cap b^{-1}U_{n_k} b,$$

and

$$\phi^*(G/W \xrightarrow{a} G/U_k)(\xi) = p_{k+1}, \quad \phi^*(G/W \xrightarrow{b} G/U_{n_k})(\xi) = r_{n_k}.$$

Similarly, there are $c, d \in G$ and $W' \subseteq d^{-1}U_k d \cap c^{-1}U_{n_k} c$ and a $\xi \in \phi^*(G/W')$, such that

$$\phi^*(G/W' \xrightarrow{c} G/U_{n_k})(\xi) = r_{n_k}, \quad \phi^*(G/W' \xrightarrow{d} G/U_k)(\xi) = q_k.$$

By the same argument, we find $e, f \in G$ and $V \subseteq e^{-1}W e \cap f^{-1}W f$, and an $\eta \in \phi^*(G/V)$ with

$$\phi^*(G/V \xrightarrow{e} G/W)(\eta) = \xi, \quad \phi^*(G/V \xrightarrow{f} G/W')(\eta) = \xi.$$

Write $g = (be)^{-1}$, let $n_{k+1}$ be so large that

$$U_{n_{k+1}} \subseteq g^{-1}V g,$$
and let \( r_{n_k+1} \in \phi^*(G/U_{n_{k+1}}) \) be an element with \( \phi^*(G/U_{n_{k+1}}) \rightarrow G/V \) \((r_{n_{k+1}}) = \eta\). Furthermore, let \( x_{k+1} = d \cdot f \cdot g \) and \( y_{k+1} = a \cdot e \cdot g \). It now easily follows that (0)--(4) still hold.

This proves the lemma for the case where \( \mathcal{S} = \text{Sets} \) and \( G \) has enough points, even without extending the base to some \( \mathcal{S}' \). Note, however, that we used the axiom of dependent choices to define the sequences \( x, y \) and \( r \).

In the general case, one has to apply successive base extensions to find points of \( G \) to represent maps \( G/O \rightarrow G/O' \) in \( \mathcal{S}(G) \), for open subgroups \( O \) and \( O' \), and instead of dependent choices one has to pass to another base extension, given by the tree of finite data \( \{(x_k, y_k, r_k, n_0, \ldots, n_k)_{k \leq m} \mid m \in \mathbb{N}\} \) satisfying (0)--(4). This can be done using lemma C of [13].

**Proof of 3.2.** (i) is easy and left to the reader. (ii) follows from (iii). To prove (iii), it is not hard to see that it suffices to show that for any base extension \( \mathcal{S} \rightarrow \mathcal{T} \) and any points \( p, q \) of \( e^*(P) \) with \( \pi_v(p) = \pi_v(q) \), there is an open surjection \( \mathcal{T} \rightarrow \mathcal{S} \) such that in \( \mathcal{T} \) there are points \( x, y \) of \( \beta^*e^*(M(G)) \) and \( r \) of \( \beta^*e^*(P) \) such that in \( \mathcal{T} \) one has \( x \cdot r = p \) and \( y \cdot r = q \).

But this follows from 3.3, because given \( \mathcal{S} \rightarrow \mathcal{T} \) and \( p, q \), we can start with an open surjective base extension \( \mathcal{S}' \rightarrow \mathcal{S} \) such that in \( \mathcal{S}' \), \( G \) (or rather \( \gamma^*e^*(G) \)) has a countable co-final system of open subgroups, by lemma 5.3 of [10].

3.4. **PROPOSITION.** Let \( G \) and \( H \) be continuous groups, and let \( BH \rightarrow BG \) be a geometric morphism. Construct the \( M(G) \)-\( M(H) \)-bispace \( P = \lim \phi^*(G/U) \) as above. Then \( P \) is a flat \( M(G) \)-\( M(H) \)-bispace, and there is a natural isomorphism of functors \( BM(G) = BG \rightarrow BH \),

\[- \otimes_{M(G)} P \cong \phi^*.\]

**Proof.** It follows from 3.1 and 3.2 that both actions \( M(G) \times P \rightarrow P \) and \( P \times M(H) \rightarrow P \) are open, and that the diagonal action \( M(G) \times M(G) \times P \rightarrow P \times P \) is open. Since both \( \phi^* \) and \( - \otimes_{M(G)} P \) preserve co-limits, it is enough to show that there is an isomorphism of \( H \)-sets, \( \phi^*(G/U) \rightarrow P/N_U \), natural in \( G/U \). (Recall from 2.1, 2.2 that \( P/N_U \) is discrete, and that this is a stable quotient.) Write \( \pi_U \) for the projection, and \( q_U \) for the quotient map as in the diagram below.

\[
P \rightarrow \phi^*(G/U) \quad \quad \quad \pi_U \quad \quad \quad q_U \quad \quad \quad \tau_U
\]

\[
P/N_U \rightarrow P
\]

Clearly, \( \pi_U \) factors through \( q_U \), say by \( \tau_U \), and \( \tau_U \) is an open surjection since \( \pi_U \) and \( q_U \) are. It also follows from 3.2(iii) that \( \tau_U \) is mono. Thus \( \tau_U \) is an isomorphism.

If \( \phi \) and \( \psi \) are geometric morphisms \( BH \rightarrow BG \) and \( \phi \rightarrow \psi \) is a 2-cell, i.e. a natural transformation \( \alpha : \phi^* \rightarrow \psi^* \), then clearly there is an induced map of bispaces \( P \rightarrow Q \), where \( P = \lim \phi^*(G/U) \), \( Q = \lim \psi^*(G/U) \) as before. Thus we obtain a functor \( l : \text{Hom}_\mathcal{S}(BH, BG) \rightarrow \text{Flat}(M(G), M(H)) \).
3.5. Lemma. The functor $l: \text{Hom}_\varphi(BG, BH) \rightarrow \text{Flat}(M(G), M(H))$ is fully faithful.

Proof. If $\alpha: \phi^* \rightarrow \psi^*$ is a natural transformation, the map $l(\alpha): l(\phi) \rightarrow l(\psi)$ of bispaces makes

\[
\begin{array}{ccc}
l(\phi) & \xrightarrow{l(\alpha)} & l(\psi) \\
\downarrow \pi_U & & \downarrow \pi_U \\
\phi^*(G/U) & \xrightarrow{\alpha_U} & \psi^*(G/U)
\end{array}
\]

commute, so clearly $l$ is faithful.

On the other hand, if $l(\phi) \xrightarrow{\zeta} l(\psi)$ is a map of bispaces, $\zeta$ preserves the action by $M(G)$, so induces a map $l(\phi)/N_U \xrightarrow{\zeta_U} l(\psi)/N_U$ of $H$-sets, for each open subgroup $U \subseteq G$. The map $\zeta_U$ are the components of a natural transformation $\phi^* \xrightarrow{\alpha} \psi^*$, modulo the isomorphism of 2.4:

\[
l(\phi)/N_U \xrightarrow{\zeta_U} \phi^*(G/U) \xrightarrow{\alpha_U} \psi^*(G/U)
\]

so $l$ is full.

4. Torsors

Let $G$ and $H$ be continuous groups, and let $P$ be a flat $M(G)$-$M(H)$-bispace, as in Section 2. Let

\[\hat{P} = \varprojlim P/N_U,\]

where $P/N_U$ is the quotient of $P$ by the action of $N_U \subseteq M(G)$, cf. 2.2, and the inverse limit is taken over all open subgroups $U \subseteq G$, ordered by inclusion. Write

\[P \xrightarrow{\theta} \hat{P}\]

for the canonical map. $\hat{P}$ is an $M(G)$-$M(H)$-space, and $\theta$ is a map of bispaces.

4.1. Lemma. For each open subgroup $U \subseteq G$, $\theta$ induces an isomorphism

\[P/N_U \xrightarrow{\simeq} \hat{P}/N_U.\]  (1)

Proof. We proved this already: let $\phi^* = - \otimes_{M(G)} P: BG \rightarrow BH$. It follows from 3.4 that $\phi^* \cong - \otimes_{M(G)} \hat{P}$. However, it is also easy to see directly that (1) is an isomorphism: consider the diagram

\[
\begin{array}{ccc}
N_U \times P & \xrightarrow{\pi_2} & P \\
\downarrow 1 \times \theta & & \downarrow \theta \\
N_U \times \hat{P} & \xrightarrow{\pi_2} & \hat{P}
\end{array}
\]

The rows are coequalizers, and $\hat{q}_U \circ \theta = \hat{q}_U \circ \theta \circ \pi_2$, so there is a unique $\alpha$ with $\alpha q_U = \hat{q}_U \theta$. And $\pi_U \circ = \pi_U \circ \pi_2$, so there is a unique $\beta$ with $\beta q_U = \pi_U$. Then $\beta\alpha q_U = \hat{q}_U \theta$.\]
Morita equivalence for continuous groups

\[ \beta q_U \theta = \pi_U \theta = q_U, \]  
so \( \beta \alpha = \text{id} \) since \( q_U \) is epi. On the other hand, \( \alpha \beta q_U \theta = \alpha \pi_U \theta = \alpha q_U = q_U \theta. \) Since \( \bar{P}/N_U \) is discrete and \( \theta \) is dense, it follows that \( \alpha \beta q_U = q_U. \) So \( \alpha \beta = \text{id} \) since \( q_U \) is epi.

4-2. Definition. A flat \( M(G)\)-\( M(H) \)-bispase is called an \( M(G)\)-\( M(H) \)-torsor if \( P \xrightarrow{u} \bar{P} \) is an isomorphism. (Notice that by definition, \( P \) is a torsor iff \( P \cong \mathbb{U}(P). \) ) We write

\[ \text{Tor} (M(G), M(H)), \]

for the full subcategory of \( \text{Flat} (M(G), M(H)) \) consisting of torsors.

Clearly, \( \text{Tor} (M(G), M(H)) \) is a reflective subcategory of \( \text{Flat} (M(G), M(H)) \): if \( P \) is a flat \( M(G)\)-\( M(H) \)-bispase, and \( T \) is an \( M(G)\)-\( M(H) \)-torsor, then for any map \( P \xrightarrow{u} T \) of bispases there is a unique \( \hat{\alpha} : \bar{P} \rightarrow T \) with \( \hat{\alpha} \circ \theta_T = \alpha. \) If \( P \in \text{Tor} (M(G), M(H)) \) and \( Q \in \text{Tor} (M(H), M(K)) \), we write

\[ P \hat{\otimes}_{M(H)} Q, \]

for \( (P \otimes_{M(H)} Q) \).

An \( M(G)\)-\( M(H) \)-torsor is called invertible if there exists an \( M(H)\)-\( M(G) \)-torsor \( Q \) such that there are isomorphisms of bispases

\[ P \hat{\otimes}_{M(H)} Q \cong M(G), \]

\[ Q \hat{\otimes}_{M(G)} P \cong M(H). \]

The isomorphism classes of invertible \( M(G)\)-\( M(G) \)-torsors form a group under the operation \( - \hat{\otimes} M(G) -, \) analogous to the Picard group of a commutative ring. This group is denoted by \( \text{Pic} (G) \).

4-3. Theorem. Let \( G \) and \( H \) be continuous groups. There is an equivalence of categories (natural in \( G \) and \( H \)) between geometric morphisms \( BH \rightarrow BG \) and \( M(G)\)-\( M(H) \)-torsors:

\[ \text{Hom}_{\mathcal{G}} (BH, BG) \xrightarrow{\sim} \text{Tor} (M(G), M(H)). \]

Proof. The equivalence is given by the functors

\[ \text{Hom}_{\mathcal{G}} (BH, BG) \xrightarrow{l} \xrightarrow{t} \text{Tor} (M(G), M(H)). \]

Indeed, \( l \) obviously maps into torsors (cf. 3-4) and is fully faithful (3-5). That \( l \circ t \) is isomorphic to the identity follows immediately from the construction, and Definition 4-2.

One may collect the torsors for different groups \( G \) and \( H \) in one large (bi-)category \( \text{Tor} \): the objects are the continuous groups, the morphisms (1-cells) from \( G \) to \( H \) are \( M(G)\)-\( M(H) \)-torsors, and the 2-cells are maps of bispases, i.e.

\[ \text{Hom} (G, H) := \text{Tor} (M(G), M(H)) \]

as categories. Composition is given by tensor product (cf. 2-3), followed by reflection:

\[ \text{Tor} (M(G), M(H)) \times \text{Tor} (M(H), M(K)) \xrightarrow{\hat{\otimes} M(H)} \text{Tor} (M(G), M(K)). \]

(Recall that composition in a bicategory is only associative up to a coherent isomorphism, cf. [3].)
4.4. Theorem. The category of atomic connected toposes with a point (over an arbitrary base topos $\mathcal{S}$) is dual to the category $\text{Tor}_{\mathcal{S}}^*$ of continuous groups and torsors in $\mathcal{S}$.

Proof. This follows from 1.9, 2.4 (and 2.5) and 4.3.

If $\mathcal{E}$ is an $\mathcal{S}$-topos, $\text{Aut}_{\mathcal{S}}(\mathcal{E})$ denotes the group of isomorphism classes of self-equivalences $\mathcal{E} \xrightarrow{\sim} \mathcal{E}$.

4.5. Corollary. There is a natural isomorphism of groups

$$\text{Aut}_{\mathcal{S}}(BG) \xrightarrow{\sim} \text{Pic}(G)$$

for any continuous group $G$ in $\mathcal{S}$.

4.6. Corollary ('Morita equivalence'). Let $G$ and $H$ be continuous groups. The following statements are equivalent:

1. $BG$ and $BH$ are equivalent categories.
2. There are an invertible $M(G)$-$M(H)$-torsor $P$ and an invertible $M(H)$-$M(G)$-torsor $Q$ such that (as bispaces)
   $$P \otimes_{M(H)} Q \cong M(G),$$
   $$Q \otimes_{M(G)} P \cong M(H).$$
3. There are a flat $M(G)$-$M(H)$-bispace $P$ and a flat $M(H)$-$M(G)$-bispace $Q$ such that
   $$P \otimes_{M(H)} Q \cong M(G),$$
   $$Q \otimes_{M(G)} P \cong M(H).$$

Let $\mathcal{S} \xrightarrow{f} \mathcal{E}$ be a geometric morphism. Then $f$ induces a functor

$$f^*: (\mathcal{E}\text{-spaces}) \to (\mathcal{S}\text{-spaces})$$

from internal spaces in $\mathcal{E}$ to internal spaces in $\mathcal{S}$; if $X$ is a space in $\mathcal{E}$, $f^*(X)$ is the space in $\mathcal{S}$ making

$$\text{Sh}_{\mathcal{E}}(f^*X) \xrightarrow{f} \text{Sh}_{\mathcal{E}}(X)$$

into a pullback. $f^*$ has a left-adjoint, so it preserves all projective limits ([7, 9]); moreover, it sends open maps to open maps. Since the quotients involved in the definition of flat bispace (2.3) and torsor (4.2) are stable (1.2), as is the construction of $M(G)$ (1.4), it follows that $f^*$ induces two functors

$$f^*: \text{Flat}_{\mathcal{S}}(M(G), M(H)) \to \text{Flat}_{\mathcal{S}}(M(f^*(G)), M(f^*(H))),$$
$$f^*: \text{Tor}_{\mathcal{S}}(M(G), M(H)) \to \text{Tor}_{\mathcal{S}}(M(f^*(G)), M(f^*(H))),$$

where $\text{Flat}_{\mathcal{S}}(M(G), M(H))$ is the category of internal flat bispaces in $\mathcal{E}$, for continuous groups $G$ and $H$ in $\mathcal{S}$, etc.

On the other hand, $\mathcal{S} \xrightarrow{f} \mathcal{E}$ induces a functor

$$\text{Hom}_{\mathcal{E}}(\mathcal{X}, \mathcal{Y}) \xrightarrow{f^*} \text{Hom}_{\mathcal{S}}(\mathcal{X} \times \mathcal{S}, \mathcal{Y} \times \mathcal{S})$$

by pullback, for $\mathcal{E}$-toposes $\mathcal{X}$ and $\mathcal{Y}$. Since everything involved in Theorem 4.3 is stable, we conclude:
4.7. Proposition. Let $\mathcal{F} \xrightarrow{f} \mathcal{E}$ be a geometric morphism, and let $G$ and $H$ be continuous groups in $\mathcal{E}$. Then

$$
\begin{array}{c}
\text{Hom}_\mathcal{E}(B(\mathcal{E}, H), B(\mathcal{E}, G)) \xrightarrow{f^*} \text{Hom}_\mathcal{E}(B(\mathcal{F}, f^*H), B(\mathcal{F}, f^*G)) \\
\text{Tor}_\mathcal{E}(M(G), M(H)) \xrightarrow{f^*} \text{Tor}_\mathcal{F}(M(f^*G), M(f^*H))
\end{array}
$$

commutes up to isomorphism (modulo the equivalence $B(\mathcal{F}, f^*H) \simeq \mathcal{F} \times B(\mathcal{E}, H)$, cf. 1.8).

If $G$ is a continuous group in a topos $\mathcal{E}$, we write $\text{Tor}_\mathcal{E}(M(G))$ for $\text{Tor}_\mathcal{E}(M(G), M(1))$, where $1$ is the trivial group.

4.8. Corollary ('$BG$ classifies $M(G)$-torsors'). Let $G$ be a continuous group in a topos $\mathcal{E}$, and let $\mathcal{F} \xrightarrow{f} \mathcal{E}$ be a geometric morphism. There is a natural equivalence between geometric morphisms $\mathcal{F} \to B(\mathcal{E}, G)$ over $\mathcal{E}$ and internal $M(f^*G)$-torsors in $\mathcal{F}$:

$$
\text{Hom}_\mathcal{E}(\mathcal{F}, B(\mathcal{E}, G)) \xrightarrow{\sim} \text{Tor}_\mathcal{F}(M(f^*G)).
$$

5. Pointed maps

Let $G$ and $H$ be continuous groups. $BG$ and $BH$ have canonical points $p_G : \mathcal{F} \to BG$ and $p_H : \mathcal{F} \to BH$ (1.7). A pointed map $BH \to BG$ is a pair $(f, \alpha)$, where $BH \xrightarrow{f} BG$ is a geometric morphism over $\mathcal{F}$ and $\alpha : p_G \Rightarrow fp_H$ is a 2-isomorphism. If $(f, \alpha)$ and $(g, \beta)$ are such pointed maps, a 2-cell $(f, \alpha) \Rightarrow (g, \beta)$ is a 2-cell $u : f \Rightarrow g$ such that $(u \cdot p_H) \circ \alpha = \beta$. This defines a category

$$
\text{Hom}_\mathcal{F}(BH, BG).
$$

Notice that since $p_H^*$ is faithful, there can be at most one 2-cell $u$ between two objects $(f, \alpha)$ and $(g, \beta)$, and that $u$ must be an isomorphism. So the 2-cells define an equivalence relation on $\text{Hom}_\mathcal{F}(BH, BG)$. Let $[BH, BG]$ be the set of equivalence classes.

5.1. Theorem. Let $G$ and $H$ be continuous groups. There is a natural isomorphism between equivalence classes of pointed geometric morphisms $BH \to BG$ and continuous homomorphisms $M(G) \to M(H)$:

$$
\text{Hom}(M(G), M(H)) \xrightarrow{\sim} [BH, BG].
$$

Proof. Let $(f, \alpha)$ be a pointed geometric morphism, where $BH \xrightarrow{f} BG$ and $\alpha : p_G \Rightarrow fp_H$. $f$ is induced by an $M(G)$-$M(H)$-torsor $P$, and $\alpha$ gives a natural transformation $p_G^* \Rightarrow - \otimes_{M(G)} P$, with components

$$
\alpha_S : S \xrightarrow{\sim} S \otimes_{M(G)} P.
$$

So for each open subgroup $U \subseteq G$, $\alpha_U : G/U \xrightarrow{\sim} P/N_U$, and this gives an isomorphism $\dot{\alpha}$ of $M(G)$-spaces $\lim G/U \to \lim P/N_U$, i.e.

$$
\dot{\alpha} : M(G) \to P.
Since $M(H)$ acts on $P$ from the right, there is a unique $\delta_a: M(H) \rightarrow M(G)$ such that

$$M(H) \xrightarrow{(\alpha(1), id)} P \times M(H)$$

$$\downarrow \delta_a \quad \downarrow \alpha$$

$$M(G) \xrightarrow{\delta} \quad P$$

(commutes (where $\alpha(1)$ is the map $1 \rightarrow M(G) \xrightarrow{\alpha} P$); so in point-set notation, $\delta_a$ is defined by $\delta_a(y) \cdot \alpha(1) = \alpha(1) \cdot y$, or equivalently $\alpha(\delta_a(y)) = \alpha(1) \cdot y$).

It is easy to see that $\delta_a$ is a homomorphism of continuous monoids.

Now suppose $u: (f, \alpha) \simto (g, \beta)$ is a 2-cell. If $f$ is induced by the torsor $P$ and $g$ by the torsor $Q$, and $M(G) \xrightarrow{\delta} P, M(G) \xrightarrow{\hat{\beta}} Q$ are the corresponding isomorphisms of $M(G)$-spaces, then $u$ induces a map $\tilde{u}$ of bispaces such that

$$M(G) \xrightarrow{\delta} \quad P$$

$$\downarrow \hat{\beta} \quad \downarrow \tilde{u}$$

$$M(G) \xrightarrow{\delta} \quad Q$$

(commutes. Since $\tilde{u}$ is in particular a map of right $M(H)$-spaces, it easily follows from commutativity of (1) and (2) that $\delta_a = \delta_b$.

Conversely, a continuous homomorphism $\delta: M(H) \rightarrow M(G)$ obviously makes $M(G)$ into a right $M(H)$-space, and hence into an $M(G)$-$M(H)$-torsor.

As a corollary, we obtain a result from [11].

5.2. Corollary. Let $G$ be a prodiscrete group, and $H$ an arbitrary continuous group. Then there is a natural isomorphism

$$\text{Hom}(H, G) \simto [BH, BG].$$

Proof. If $G$ is prodiscrete, then $\theta_G: G \rightarrow M(G)$ is an isomorphism (cf. [11]). Since $\theta_H: H \rightarrow M(H)$ is dense, it induces an isomorphism $\text{Hom}(M(H), G) \simto \text{Hom}(H, G)$. The result now follows from 5.1.

One may also consider the full subcategory of $\text{Hom}_\varphi(BH, BG)$ whose objects come from $\text{Hom}_\varphi(BH, BG)_+$, i.e. the full subcategory of $\text{Hom}_\varphi(BH, BG)$, consisting of those $BH \xrightarrow{f} BG$ for which there exists some unspecified isomorphism $f p_H \cong p_G$. Denote this category by $\text{Hom}_\varphi(BH, BG)_+$. The set of continuous homomorphisms $M(H) \xrightarrow{\phi} M(G)$ can also be made into a category: the maps $\phi \Rightarrow \psi$ are points $g: 1 \rightarrow M(G)$ such that the identity $\phi(x) \cdot g = g \cdot \psi(x)$ holds.

5.3. Corollary. Let $G$ and $H$ be continuous groups. There is a natural equivalence of categories

$$\text{Hom}(M(H), M(G)) \simto \text{Hom}_\varphi(BH, BG)_+.$$
Morita equivalence for continuous groups

\( g = \beta^{-1} \delta \alpha(1) : 1 \to M(G) \). It is easy to check that (in point-set notation)\( \delta_a(x) \cdot g = g \cdot \delta_b(x) \). Further details are straightforward.

If \( G \) is discrete, \( M(G) = G \) and for any \( M(G) \cdot M(H) \)-torsor \( P \) there is an isomorphism \( M(G) = G \to P \). So we obtain the following well known corollary as a special case.

5.4. COROLLARY. Let \( G \) be a discrete group, \( H \) any continuous group. Then

\[
\text{Hom}(H, G) \overset{\sim}{\longrightarrow} \text{Hom}_{\mathscr{P}}(BH, BG)
\]

is an equivalence of categories.

Needless to say, it is easier to prove 5.4 directly.

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REFERENCES