Determination of the Branching Fraction for $B \rightarrow X_c \ell \nu$ Decays and of $|V_{cb}|$
from Hadronic-Mass and Lepton-Energy Moments

of the lower limit on the lepton energy, using data recorded with the BABAR hadronic-mass and lepton-energy distributions in semileptonic $B$-meson decays, measured as a function of the Cabibbo-Kobayashi-Maskawa matrix element $|V_{cb}|$, and other heavy-quark parameters from a simultaneous fit to moments of the hadronic-mass and lepton-energy distributions in semileptonic $B$-meson decays, measured as a function of $|V_{cb}|$, the branching fraction, $B_{\ell\nu}$, and other heavy-quark parameters from a simultaneous fit to moments of the hadronic-mass and lepton-energy distributions in semileptonic $B$-meson decays, measured as a function of the lower limit on the lepton energy, using data recorded with the BABAR detector. Using heavy-quark expansions (HQEs) to order $1/m_b^3$, we extract $B_{\ell\nu} = (10.61 \pm 0.16_{\text{exp}} \pm 0.06_{\text{HQE}})\%$ and
The Cabibbo-Kobayashi-Maskawa (CKM) matrix element $V_{cb}$ is one of the fundamental parameters of the standard model and thus its precise measurement with well understood uncertainties is important. At the parton level, the weak decay rate for $b \rightarrow c \ell \nu$ can be calculated accurately; it is proportional to $|V_{cb}|^2$ and depends on the quark masses $m_b$ and $m_c$. The semileptonic $B$-meson decay rate is determined from measurements of the average $B$ lifetime and the semileptonic branching fraction. To relate the semileptonic $B$-meson decay rate to $|V_{cb}|$, the parton-level calculations have to be corrected for effects of strong interactions. Heavy-quark expansions (HQEs) [1] have become a useful tool for calculating perturbative and nonperturbative QCD corrections [2] and for estimating their uncertainties. In the kinetic-mass scheme [3] for example, these expansions in $1/m_b$ and $\alpha_s(m_b)$ (the strong coupling constant) to order $\mathcal{O}(1/m_b^3)$ contain six parameters: the running kinetic masses of the $b$ and $c$ quarks, $m_b(\mu)$ and $m_c(\mu)$, and four nonperturbative parameters. We determine these parameters from a fit to the moments of the hadronic-mass and electron-energy distributions in semileptonic $B$ decays to charm particles, $B \rightarrow X_c \ell \nu$. This fit yields measurements of the inclusive branching fraction $\mathcal{B}_{\ell \nu}$ and of $|V_{cb}|$, significantly improved compared to earlier BABAR measurements [4]. It also allows us to test the consistency of the data with the HQEs employed and to check for the possible impact of higher-order contributions. Moment measurements and the extraction of $|V_{cb}|$ based on HQEs [5] were first performed by the CLEO Collaboration [6]. More recently, global fits to a variety of moments were presented [7–9], using HQEs in different mass schemes.

This analysis makes use of moments measured by the BABAR Collaboration [10,11]. The moments are derived from the inclusive hadronic-mass ($m_X$) and electron-energy ($E_\ell$) distributions in $B \rightarrow X_c \ell \nu$ decays, averaged over charged and neutral $B$ mesons produced at the $Y(4S)$ resonance. We have subtracted the charmless contributions based on the branching fraction $\mathcal{B}_{u\ell\nu} = (0.22 \pm 0.05)\%$ [12]. All moments are measured as functions of $E_{cut}$, a lower limit on the lepton energy (for energy moments we only use electrons, and for mass moments we also use muons). The moments are corrected for detector effects and QED radiation [13]. The hadronic-mass distribution is measured in events tagged by the fully reconstructed hadronic decay of the second $B$ meson. The hadronic-mass moments are defined as $M_n^1 (E_{cut}) = \langle m_X^{2n} \rangle_{E<E_{cut}}$ and the second and third moments as $M_n^2 (E_{cut}) = \langle E_\ell - M_1^1 (E_{cut}) \rangle_{E_{cut}>E_{cut}}$ with $n = 2, 3$. In addition, we use the partial branching fraction $\mathcal{B}_n^1 (E_{cut}) = \int_{E_{cut}}^{E_{max}} (d\mathcal{B}_{\ell \nu}/dE_\ell) dE_\ell$. All measured moments are shown in Fig. 1.

In the kinetic-mass scheme the HQE to $\mathcal{O}(1/m_b^3)$ for the rate of $B \rightarrow X_c \ell \nu$ decays can be expressed as [14]

\begin{equation}
\Gamma_{\ell \nu} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 (1 + A_{ew}) A_{\text{pert}}(r, \mu) \times \left[ z_0(r) \left( 1 - \frac{\mu^2}{2 m_b^2} + \frac{\rho_0 + \rho_{1S}}{2 m_b} \right) \right] - 2(1 - r)^4 \frac{\mu_G^2 + \rho_0 + \rho_{1S}}{m_b^3} + d(r) \frac{\rho_3^3}{m_b^3} + \mathcal{O}(1/m_b^3) \right].
\end{equation}

The leading nonperturbative effects arise at $\mathcal{O}(1/m_b^3)$ and are parametrized by $\mu_G^2(\mu)$ and $\mu_{1S}(\mu)$, the expectation values of the kinetic and chromomagnetic dimension-five operators. At $\mathcal{O}(1/m_b^3)$, two additional parameters enter, $\rho_0(\mu)$ and $\rho_{1S}(\mu)$, the expectation values of the Darwin ($D$) and spin-orbit ($LS$) dimension-six operators. These parameters depend on the scale $\mu$ that separates short-distance from long-distance QCD effects; the calculations are performed for $\mu = 1$ GeV [3]. Electroweak corrections are $1 + A_{ew} \equiv 1 + \alpha/\pi \ln(M_Z/m_b)^2 = 1.014$ and perturbative QCD corrections are estimated to be $A_{\text{pert}}(r, \mu) \equiv 0.91 \pm 0.01$ [14]. The ratio $r = m_c^2/m_b^2$ enters in the phase-space factor $z_0(r) = 1 - 8r + 8r^2 - r^4 - 12r^2 \ln r$ and the function $d(r) = 8r^2 + 34/3 - 32r/3 - 8r^2 + 32r^3/3 - 10r^4/3$.

This analysis uses linearized expressions for the HQEs [15]. Specifically, the dependence of $|V_{cb}|$ on the true values of heavy-quark parameters, expanded around a priori estimates of these parameters, is

\begin{equation}
|V_{cb}| \equiv \left\{ \frac{\mathcal{B}_{\text{ew}}}{(0.1032)} \times \left[ 1 + 0.30(\alpha_s(m_b) - 0.22) \right] \times \left[ 1 - 0.66(m_b - 4.60) + 0.39(m_c - 1.15) + 0.013(\mu_G^2 - 0.40) + 0.09(\rho_0^3 - 0.20) + 0.05(\mu_{1S}^3 - 0.35) - 0.01(\rho_{1S}^3 - 0.15) \right] \right\}^{0.0417} \times \mathcal{B}_{\text{ew}} \times \tau_B \times \tau_0 \times \tau_z
\end{equation}

Here $m_b$ and all other parameters of the expansion are in GeV/$c^2$; $\tau_B$ refers to the average lifetime of $B$ mesons produced at the $Y(4S)$; we use $\tau_B = f_0 \tau_0 + (1 - f_0) \tau_z = (1.608 \pm 0.012) \text{ps}$, taking into account the lifetimes [16] of neutral and charged $B$ mesons, $\tau_0$ and $\tau_z$, and their
relative production rates (defined in terms of $f_0 = 0.488 \pm 0.013$ \cite{17}, the fraction of $B^{0}\bar{B}^{0}$ pairs).

HQEs in terms of the same heavy-quark parameters are available for the hadronic-mass and electron-energy moments. The dependence on the heavy-quark parameters has again been linearized using the same $a$ priori estimates of the parameters \cite{14,15}. We have verified that the differences between the linearized expressions and the full theoretical calculation are very small in all cases. We use these linear equations to determine the unknown heavy-quark parameters, the total branching fraction $B_{ctr}$, and $|V_{cb}|$ from a simultaneous $\chi^2$ fit to the measured moments and the partial branching fraction, all as a function of the cutoff lepton-energy, $E_{\text{cut}}$.

In total, we have measured four hadronic-mass moments for each of seven different values of $E_{\text{cut}}$, ranging from 0.9 to 1.5 GeV, and three electron-energy moments plus the partial branching fraction at five values of $E_{\text{cut}}$, ranging from 0.6 to 1.5 GeV \cite{10,11}. Since many of these individual moments are highly correlated we select for the fitting procedure a set of moments for which the correlation coefficients do not exceed 95%. Thus we only use half of the 28 mass moments, and retain 13 of the 20 energy moments.

The global fit takes into account the statistical and systematic errors and correlations of the individual measurements, as well as the uncertainties of the expressions for the individual moments. We assess the uncertainty of the calculated moments by varying, as suggested in \cite{15}, in the linearized expressions [given for $|V_{cb}|$ in Eq. (2)] the $a$ priori estimates for $\mu_+^2$ and $\mu_+^2$ by $\pm 20\%$ and for $\rho_+^3$ and $\rho_+^3$ by $\pm 30\%$. We assume that for a given moment these variations are fully correlated for all values of $E_{\text{cut}}$, but uncorrelated for different moments. The resulting fit, shown in Fig. 1, describes the data well with $\chi^2 = 15.0$ for 20 degrees of freedom. Tables I and II list the fitted parameters, their errors and correlations.

Beyond the uncertainties that are included in the fit, the moment measurements ($\delta_{\text{exp}}$) and approximations of the HQEs ($\delta_{\text{HQE}}$), we have identified two additional sources of errors. The limited knowledge of the expression for the decay rate, including various perturbative corrections and higher-order nonperturbative corrections, introduces an error in $|V_{cb}|$, assessed to be 1.5% (refered to as $\delta_{\Gamma}$ in Table I) \cite{14}. By comparison, the impact of the uncertainty in $\alpha_s$ ($\delta_{\alpha_s}$) is estimated to be relatively small. For $M_{\alpha}^2(E_{\text{cut}})$ moments, perturbative corrections of order $\alpha_s^2$ are included with $\alpha_s(m_b) = 0.22 \pm 0.04$, whereas for $M_{\alpha}^3(E_{\text{cut}})$ moments, they are calculated only to $\mathcal{O}(\alpha_s)$ with $\alpha_s(m_b) = 0.3 \pm 0.1$. We estimate the error on the perturbative corrections by varying $\alpha_s$ within the stated errors. The choice of the scale $\mu$ is estimated

![Table I](https://example.com/table1.png)

**Table I.** Fit results and error contributions from the moment measurements, approximations to the HQEs, and additional theoretical uncertainties from $\alpha_s$ terms and other perturbative and nonperturbative terms contributing to $\Gamma_{ctr}$. Theoretical uncertainties are indicated as shaded bands. The solid data points mark the measurements included in the fit. The vertical bars indicate the experimental errors; in some cases they are comparable in size to the data points. Moment measurements for different $E_{\text{cut}}$ are highly correlated.

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![Figure 1](https://example.com/fig1.png)

**FIG. 1** (color online). The measured hadronic-mass (a)–(d) and electron-energy (e)–(h) moments as a function of the cutoff energy, $E_{\text{cut}}$, compared with the result of the simultaneous fit (line), with the theoretical uncertainties \cite{15} indicated as shaded bands. The solid data points mark the measurements included in the fit. The vertical bars indicate the experimental errors; in some cases they are comparable in size to the data points. Moment measurements for different $E_{\text{cut}}$ are highly correlated.
to have a very small impact on $|V_{cb}|$ and the branching fraction [14].

A series of tests has been performed to verify that the fit results are unbiased. Specifically, we enlarged and reduced the estimated theoretical uncertainties by a factor of 2 and verified that the changes in the fitted parameters were small compared to the errors of the standard fit. We have also checked that the choice of the set of moments that are used in the fit does not significantly affect the result. In particular, an energy cutoff above 1.2 GeV might have introduced larger theoretical uncertainties and a potential bias, and moments for lower values of $E_{\text{cut}}$ might have been affected by higher backgrounds. We found no evidence for any such effects.

The fit results are fully compatible with independent estimates [15] of $\mu_{c'}^2 = (0.35 \pm 0.07)$ GeV$^2$, based on the $B^- - B$ mass splitting, and of $\rho_{1S}^3 = (-0.15 \pm 0.10)$ GeV$^2$, from heavy-quark sum rules [18].

Figure 2 shows the $\Delta \chi^2 = 1$ ellipses for $|V_{cb}|$ versus $m_b$ and $\mu_{\tau}^2$, for a fit to all moments and separate fits to the electron-energy moments and the hadronic-mass moments, but including the partial branching fractions in both. The lepton-energy and hadronic-mass moments have slightly different sensitivity to the fit parameters, but the results for the separate fits, $|V_{cb}| = (41.4 \pm 0.7) \times 10^{-3}$ and $|V_{cb}| = (41.6 \pm 0.8) \times 10^{-3}$, are fully compatible with each other and with the global fit to all moments. Changes in the other fit parameters are also consistent within the stated errors. Since the expansions for the two sets of moments are sensitive to different theoretical uncertainties and assumptions, in particular, the differences in the treatment of the perturbative corrections, the observed consistency of the separate fits indicates that such differences are small compared with the experimental and assumed theoretical uncertainties.

In conclusion, we have extracted $|V_{cb}|$, the semileptonic branching fraction, and the heavy-quark masses,

$$|V_{cb}| = (41.4 \pm 0.4_{\text{exp}} \pm 0.4_{\text{HQE}} \pm 0.6_{\text{th}}) \times 10^{-3},$$

$$B_{\text{cev}} = (10.61 \pm 0.16_{\text{exp}} \pm 0.06_{\text{HQE}})\%,$$

$$m_b(1 \text{ GeV}) = (4.61 \pm 0.05_{\text{exp}} \pm 0.04_{\text{HQE}} \pm 0.02_{\text{th}}) \text{ GeV},$$

$$m_c(1 \text{ GeV}) = (1.18 \pm 0.07_{\text{exp}} \pm 0.06_{\text{HQE}} \pm 0.02_{\text{th}}) \text{ GeV},$$
as well as the nonperturbative parameters in the kinetic-mass scheme up to order $(1/m_b^2)$ (see Table I). The total semileptonic branching fraction is $B_{\text{cev}} + B_{\text{uev}} = (10.83 \pm 0.16_{\text{exp}} \pm 0.06_{\text{HQE}})\%$. The errors refer to contributions from the experimental errors on the moment measurements, the HQE uncertainties included in the fit, and additional theoretical uncertainties, $\delta_{\text{th}} = \sqrt{\delta_{\text{th}}^2 + \delta_{\text{2th}}^2}$, derived from Refs. [14,15].

Based on a large set of hadronic-mass and electron-energy moments and a consistent set of HQE calculations, we have also been able to assess the uncertainties in the $O(1/m_b^2)$ terms from the data without constraints to any a priori values. The fitted values of the parameters are consistent with theoretical estimates [3,14]. The uncertainties on the quark masses are much smaller than those of previous measurements [16]. Our measurements of $m_{b}$ and $m_{c}$ are highly correlated, the mass difference is $m_{b} - m_{c} = (3.436 \pm 0.025_{\text{exp}} \pm 0.018_{\text{HQE}} \pm 0.010_{\text{th}}) \text{ GeV}$. The result on $|V_{cb}|$ is in agreement with previous measurements using HQEs, either for a different mass scheme and with fixed terms of $O(1/m_b^2)$ [8], or for the kinetic-mass scheme, but with external constraints on almost all HQE parameters [9], as well as with an analysis combining both of these measurements [7]. It would be interesting to compare the results of this analysis with fits

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<p>|TABLE II. Correlation coefficients for the fit parameters.|
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FIG. 2 (color online). Fit results (crosses) with contours corresponding to $\Delta \chi^2 = 1$ for two pairs of the eight free parameters (a) $m_{b}$ and $\mu_{\tau}^2$ versus $V_{cb}$, separately for fits using the hadronic mass, the electron energy, and all moments.
based on recent calculations performed in the 1S mass scheme [19].

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\begin{itemize}
\item *Also with Università della Basilicata, Potenza, Italy.
\item †Also with IFIC, Instituto de Física Corpuscular, CSIC-Universidad de Valencia, Valencia, Spain.
\item ‡Deceased.
\end{itemize}

\begin{enumerate}
\item [10] \textit{BABAR} Collaboration, B. Aubert \textit{et al.}, hep-ex/0403031 [Phys. Rev. D (to be published)].
\end{enumerate}