Abstract. Basic Harish-Chandra series are asymptotically free meromorphic solutions of the system of basic hypergeometric difference equations associated to root systems. The associated connection coefficients are explicitly computed in terms of Jacobi theta functions. We interpret the connection coefficients as the transition functions for asymptotically free meromorphic solutions of Cherednik’s root system analogs of the quantum Knizhnik-Zamolodchikov equations. They thus give rise to explicit elliptic solutions of root system analogs of dynamical Yang-Baxter and reflection equations. Applications to quantum $c$-functions, basic hypergeometric functions, reflectionless difference operators and multivariable Baker-Akhiezer functions are discussed.

1. Introduction

The monodromy of the Heckman-Opdam system of hypergeometric differential equations associated to root systems is explicitly computed in [28]. The key step is the derivation of explicit expressions of the connection coefficients for the Harish-Chandra series solution of the system in terms of Gamma functions. We prove the basic hypergeometric analog of this result by determining explicit expressions of the connection coefficients for basic Harish-Chandra series in terms of Jacobi theta functions.

The basic Harish-Chandra series is a self-dual, meromorphic solution of the system of basic hypergeometric difference equations associated to root systems, characterized by its plane wave asymptotics deep in a fixed Weyl chamber. The system of basic hypergeometric difference equations is the spectral problem of the commuting Ruijsenaars-Macdonald-Koornwinder-Cherednik difference operators, whose Laurent polynomial solutions are the celebrated symmetric Macdonald-Koornwinder polynomials.

The difference Cherednik-Matsuo correspondence relates the spectral problem to Cherednik’s quantum affine Knizhnik-Zamolodchikov (KZ) equations associated to minimal principal series representations of the affine Hecke algebra, which are root system analogs of the Frenkel-Reshetikhin-Smirnov quantum KZ equations. Thus the connection coefficients are transition functions for asymptotically free meromorphic solutions of quantum KZ equations. This point of view leads to the interpretation of the connection matrices as elliptic solutions to root system analogs of dynamical Yang-Baxter equations.

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In the remainder of the introduction we give a detailed description of the main results, including precise references to the literature. We start by fixing some basic notations in Subsection 1.1. We discuss the two relevant compatible systems of difference equations in Subsection 1.2. The basic Harish-Chandra series are discussed in Subsection 1.3. In Subsection 1.4 we formulate the associated connection coefficients problem and give the explicit expressions of the connection coefficients. In Subsection 1.5 we also discuss the relation to modified dynamical Yang-Baxter equations. Applications to basic hypergeometric functions and $c$-functions, reflectionless basic Harish-Chandra series and multivariable Baker-Akhiezer functions are discussed in Subsections 1.6, 1.7 and 1.8 respectively.

1.1. Initial data. We start with the introduction of the initial data of the Cherednik-Macdonald theory \cite{macdonald, cherednik} on Macdonald-Koornwinder polynomials. The setup follows closely the conventions of the recent exposition \cite{macdonald}, which provides a uniform framework for all the known cases of the theory.

The initial datum is given by a triple $(D, \kappa, q)$ with $D$ the root system datum, $\kappa$ the associated free parameters, and $0 < q = e^\tau < 1$ the deformation parameter. The root system datum $D = (R_0, \Delta_0, \bullet, \Lambda, \tilde{\Lambda})$ consists of

1. a finite, reduced crystallographic root system $R_0$ in the Euclidean space $(E, (\cdot, \cdot))$, irreducible within the Euclidean subspace $V$ spanned by $R_0$,
2. a basis $\Delta_0 = (\alpha_1, \ldots, \alpha_n)$ of the root system $R_0$,
3. $\bullet \in \{u,t\}$ ("u" standing for untwisted and "t" for twisted),
4. full lattices $\Lambda, \tilde{\Lambda} \subseteq E$ such that
   $$Q \subseteq \Lambda, \quad (\Lambda, Q^\vee) \subseteq \mathbb{Z},$$
   $$\tilde{Q} \subseteq \tilde{\Lambda}, \quad (\tilde{\Lambda}, \tilde{Q}^\vee) \subseteq \mathbb{Z},$$

with $Q$ and $\tilde{Q}$ (respectively $Q^\vee$ and $\tilde{Q}^\vee$) the root lattice (respectively co-root lattice) of $R_0$ and of the dual root system

$$\bar{R}_0 := \begin{cases} R_0^u = \{ \alpha^u := \frac{2a}{|\alpha|^2} \alpha \in R_0 \} & \text{if } \bullet = u, \\
R_0 & \text{if } \bullet = t \end{cases}$$

respectively.

We write

$$\mu_\alpha := \begin{cases} 1 & \text{if } \bullet = u, \\
\frac{|\alpha|^2}{2} & \text{if } \bullet = t \end{cases}$$

and $\tilde{\alpha} := \mu_\alpha \alpha^\vee$ for $\alpha \in R_0$. Then $\bar{R}_0 = \{ \tilde{\alpha} \}_{\alpha \in R_0}$. We write $R_0^+$ and $R_0^-$ for the positive and negative roots in $R_0$ with respect to the basis $\Delta_0$.

We attach to the root system datum $D$ an irreducible affine root system $R(D)$. It is built from the reduced affine root system $R^\bullet := \{ \alpha^{(r)} := \mu_\alpha rc + \alpha \}_{r \in \mathbb{Z}, \alpha \in R_0}$, where $\mu_\alpha rc + \alpha$ stands for the affine linear function $z \mapsto \mu_\alpha r + (\alpha, z)$ on $E$, by adding the multiple $2\alpha^{(r)}$ if $(\Lambda, \alpha^\vee) = 2\mathbb{Z}$. A basis of $R(D)$ is obtained by adding to $\Delta_0$ the simple affine root $\alpha_0 := \mu_\psi c - \psi$ with $\psi$ the highest (respectively highest short) root of $R_0$ relative to the basis $\Delta_0$ if $\bullet = u$ (respectively $\bullet = t$).

The free parameters $\kappa$ are represented by the function values $\kappa_\alpha (a \in R(D))$ of a $W$-invariant function $\kappa : R(D) \to \mathbb{R}$, where $W := W_0 \ltimes \tilde{\Lambda}$ is the extended affine Weyl group. We call $\kappa$ a multiplicity function and set $\kappa_{2\alpha^{(r)}} := \kappa_{\alpha^{(r)}}$ if $2\alpha^{(r)} \notin R(D)$.
With this convention the values $\kappa_\alpha, \kappa_2\alpha, \kappa_{\alpha(1)}, \kappa_{2\alpha(1)} \ (\alpha \in R_0)$ uniquely determine the multiplicity function $\kappa$.

Here are three important examples of root system data. The $\text{GL}_{n+1}$ root system datum is $D = (R_0, \Delta_0, \bullet, \mathbb{Z}^{n+1}, \mathbb{Z}^{n+1})$, where $R_0$ is the root system of type $A_n$ with its standard realization in $E = \mathbb{R}^{n+1}$. Cherednik [15] developed his theory on Macdonald polynomials mainly for reduced semisimple root system data, in which case $D = (R_0, \Delta, \bullet, P, \overline{P})$ with $V = E$ and with the lattices taken to be the weight lattices $P$ and $\overline{P}$ of $R_0$ and $\overline{R_0}$ respectively. Here semisimple refers to the fact that $V = E$, reduced to the fact that $R(D) = R^*$ and similarly for the associated dual affine root system, see Subsection 2.2. The Koornwinder case of the Macdonald-Koornwinder theory [37, 46, 53, 48] corresponds to the root system datum of type $B_n$ (see Section 2.2). The four (dual) AW parameters associated to a root $\alpha \in R_0$ comprise either one, two or four of the free parameters, reflecting the fact that the associated local rank one reduction of the Macdonald-Koornwinder theory relates to the theory of continuous $q$-ultraspherical polynomials, continuous $q$-Jacobi polynomials and Askey-Wilson polynomials respectively. The case at hand can be read off from the root system datum as follows.

### Continuous $q$-ultraspherical case:

Let $\Lambda, \alpha^\vee = (\Lambda, \alpha^\vee)$ be the root system datum, then $\kappa_{2\alpha(1)} = \kappa_{2\alpha} = \kappa_\alpha = \kappa_{\alpha(1)}$.

### Continuous $q$-Jacobi case:

Let $\Lambda, \alpha^\vee = (\Lambda, \alpha^\vee)$ be the root system datum, then $\kappa_{2\alpha} = \kappa_\alpha$ and $\kappa_{2\alpha(1)} = \kappa_\alpha(1)$.

### Askey-Wilson case:

Let $\Lambda, \alpha^\vee = (\Lambda, \alpha^\vee)$ be the root system datum, then $\kappa_{\alpha(1)} = \kappa_\alpha$ and $\kappa_{2\alpha(1)} = \kappa_{2\alpha}$.

The Askey-Wilson case only occurs when $D$ is the Koornwinder root system datum $D = (R_0, \Delta_0, t, Q, Q)$ with $R_0$ of type $A_1$ or of type $B_n$ ($n \geq 2$) and $\alpha \in R_0$ a short root. For reduced semisimple root system datum, one is dealing with the continuous $q$-ultraspherical case for all roots $\alpha \in R_0$. Continuous $q$-Jacobi cases only occur in the untwisted theory $\bullet = u$, see [60].
1.2. Integrable difference equations. Consider the trigonometric function
\[ A(z) := \frac{(1 - a_\psi q^{\psi(z)})(1 - b_\psi q^{\psi(z)})(1 - c_\psi q^{\psi(z)})(1 - d_\psi q^{\psi(z)})}{(1 - q^{2\psi(z)})(1 - q^{2\psi(z)})} \times \prod_{\alpha \in R_0^+ : \langle \alpha, \bar{\alpha} \rangle = 1} (1 - a_\alpha q^{\alpha(z)})(1 - b_\alpha q^{\alpha(z)}) \]
(1.3)
in \( z \in E_\mathbb{C} := \mathbb{C} \otimes_{\mathbb{R}} E \), where we canonically extend the (affine) roots to complex affine linear functions on \( E_\mathbb{C} \). The symmetric Macdonald-Koornwinder polynomials associated to the initial datum \((D, \kappa, q)\) are trigonometric Laurent polynomial eigenfunctions of the difference operator
\[ (Lf)(z) := q^{-\langle \rho, \bar{\psi} \rangle} \sum_{w \in W_0/W_{0,\psi}} A(w^{-1} z) f(z + w \bar{\psi} - f(z)) + \sum_{w \in W_0/W_{0,\psi}} q^{-\langle \rho, w \bar{\psi} \rangle} f(z) \]
acting on meromorphic functions \( f(z) \) in \( z \in E_\mathbb{C} \), where
\[ \rho := \frac{1}{2} \sum_{\alpha \in R_0^+} (\kappa_\alpha + \kappa_{\alpha(1)}) \bar{\alpha}^\vee, \]
\( W_0 \subseteq GL_\mathbb{C}(E_\mathbb{C}) \) is the Weyl group of \( R_0 \) and \( W_{0,\psi} \) the stabilizer subgroup of \( \psi \).

In fact, for the \( GL_{n+1} \) root system datum, the difference operator \( L \) is a quantum conserved integral of Ruijsenaars’ \textsuperscript{50} quantum relativistic integrable many body system. For reduced semisimple root system datum, \( L \) is the Macdonald \textsuperscript{40} difference operator associated to a quasi-miniscule weight. For the Koornwinder root system datum, \( L \) is the Koornwinder’s \textsuperscript{37} multivariable analog of the Askey-Wilson \textsuperscript{11} second order difference operator. Higher order difference operators, mutually commuting and commuting with \( L \), have been constructed using the theory of double affine Hecke algebras, see, e.g., \textsuperscript{13, 41, 60}. We recall their construction in Subsection \textsuperscript{3.3} We call \( L \) and the associated higher order difference operators Ruijsenaars-Macdonald-Koornwinder-Cherednik (RMKC) difference operators.

The associated spectral problem is a compatible system of basic hypergeometric difference equations. It is the natural generalization of the Heckman-Opdam \textsuperscript{28} system of hypergeometric differential equations associated to root systems to the basic hypergeometric level. It has a natural upgrade to a bispectral problem, see Subsection \textsuperscript{3.4}.

The bispectral quantum KZ equations
\[ C_{(\tau(\nu), \tau(\lambda))}(z, \xi) f(z - \nu, \xi - \lambda) = f(z, \xi), \quad \nu \in \tilde{\Lambda}, \lambda \in \Lambda \]
(1.4)
associated to the initial datum \((D, \kappa, q)\) form an explicit compatible system of linear difference equations for meromorphic functions \( f(z, \xi) \) in \((z, \xi) \in E_\mathbb{C} \times E_\mathbb{C} \) taking values in a complex \#\( W_0 \)-dimensional vector space \( \mathcal{V} \). Here \( \tau(\nu) \) (respectively \( \tau(\lambda) \)) stands for the element \( \nu \in \tilde{\Lambda} \) (respectively \( \lambda \in \Lambda \)) viewed as element of \( W = W_0 \times \tilde{\Lambda} \) (respectively \( \tilde{W} := W_0 \times \Lambda \)). The explicit expressions for \( C_{(\tau(\nu), \tau(\lambda))}(z, \xi) \) are given in Theorem \textsuperscript{3.1}. For the twisted theory \( \bullet = t \) with \( \Lambda = \tilde{\Lambda} \), the bispectral quantum KZ equations (1.4) have been defined and studied before in \textsuperscript{43, 44, 59}.

For fixed \( \xi \in E_\mathbb{C} \), the restricted compatible system of difference equations
\[ C_{(\tau(\nu), \tau(0))}(z, \xi) f(z - \nu) = f(z), \quad \nu \in \tilde{\Lambda} \]
(1.5)
for $V$-valued meromorphic functions $f(z)$ in $z \in E_C$ are Cherednik’s [9, 10] quantum affine KZ equations associated to the minimal principal series representation of the affine Hecke algebra with central character $q^z$, see [44, 58] for details (here $q^z$ is interpreted as element of the complex algebraic torus $\text{Hom}(\hat{\Lambda}, \mathbb{C}^*)$ by $\nu \mapsto q^{(\nu, \xi)}$). For the $GL_{n+1}$ root system datum, the quantum affine KZ equations (1.5) become Frenkel-Reshetikhin-Smirnov [23, 54] type quantum KZ equations, which were derived in [23] as the consistency conditions satisfied by matrix coefficients of products of quantum affine algebra intertwiners. From the physics point of view, they form the consistency conditions of correlation functions for integrable two dimensional lattice models from statistical physics. See [18, 31] for detailed expositions.

The Cherednik-Matsuo correspondence [42, 11] relates solutions of the affine KZ equations to solutions of the Heckman-Opdam system of hypergeometric differential equations. Its difference analog [10, 12, 33, 58] embeds the solution space of the quantum affine KZ equations (1.5) into the solution space of a spectral problem of the RMKC operators. See Subsection 3.5 for the definition of the associated embedding $\chi$. The difference Cherednik-Matsuo correspondence was obtained in [10, Thm. 3.4(a)] for reduced semisimple root datum (untwisted case), see also [12]. Subsequently Kato [33, Thm. 4.6] showed that $\chi$ maps solutions of the quantum affine KZ equations to eigenfunctions of $L$ by different methods. The surjectivity of $\chi$ was claimed in [10, Thm. 3.4(b)] and [12, Thm. 4.3(b)]. It was proved for generic spectral parameter $\xi \in E_C$ in [58, Thm. 5.16(b)] using an extension of the methods from [11, 49] for the classical Cherednik-Matsuo correspondence.

In Subsection 3.5 we show that the difference Cherednik-Matsuo correspondence gives rise to an embedding of the solution space of the bispectral quantum KZ equations to the solution space of a bispectral problem of the RMKC operators. This extends the results from [43, 44, 59], which dealt with the twisted case.

1.3. Basic Harish-Chandra series. Extending the results from [44, 43, 59], we prove in Subsection 3.2 the existence of an asymptotically free, meromorphic solution $\Phi_{\text{KZ}}$ of the bispectral quantum KZ equations (1.4) and establish its basic properties (selfduality, description of singularities). Via the difference Cherednik-Matsuo correspondence it leads to the existence of asymptotically free meromorphic eigenfunctions of the RMKC operator $L$. More precisely, we will establish the following result.

**Theorem 1.1.** There exists a unique meromorphic function $\Phi(\cdot, \cdot) = \Phi(\cdot, \cdot; D, \kappa; q)$ on $E_C \times E_C$ satisfying

1. the eigenvalue equations

\[
L \Phi(\cdot, \xi) = \left( \sum_{w \in W_0/W_{0, \kappa}} q^{\delta(w^{-1} \xi)} \right) \Phi(\cdot, \xi),
\]

viewed as identity of meromorphic functions in $(\cdot, \xi) \in E_C \times E_C$ (the unspecified first entry is to emphasize that this is the space on which the RMKC operator $L$ is acting),

2. the asymptotic expansion

\[
\Phi(z, \xi) = \frac{W(z, \xi)}{S(z)S(\xi)} \sum_{\alpha \in Q^+} \Gamma_\alpha(\xi) q^{-\alpha(z)}, \quad Q^+ := \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i
\]
with
(a) the plane wave \( W(z, \xi) = q^{(\alpha - \xi, \rho + w_0 z)} \), where \( w_0 \in W_0 \) is the longest Weyl group element and
\[
\rho := \frac{1}{2} \sum_{\alpha \in R_0^+} (\kappa_\alpha + \kappa_2) \alpha \cdot
\]
is a dual version of \( \rho \).
(b) the series \( \Psi(z, \xi) := \sum_{\alpha \in Q^+} \Gamma_\alpha(\xi) q^{-\alpha(z)} \) converging normally for \( (z, \xi) \) in compacta of \( E_C \times E_C \) (hence it defines a holomorphic function in \( (z, \xi) \in E_C \times E_C \)),
(c) the holomorphic functions \( S(\cdot) \) and \( \tilde{S}(\cdot) \) on \( E_C \), capturing the singularities of \( \Phi(\cdot, \cdot) \), explicitly given by
\[
S(z) := \prod_{\alpha \in R_0^+} (q^2 a_{\alpha}^{-1} q^{-\alpha(z)}; q^2 \alpha^{-1} q^{-\alpha(z)}; q^2)_\infty,
\]
\[
\tilde{S}(\xi) := \prod_{\alpha \in R_0^+} (q^2 a_{\alpha}^{-1} q^{-\xi(\alpha)}; q^2 \alpha^{-1} q^{-\xi(\alpha)}; q^2)_\infty,
\]
where
\[
(x_1, \ldots, x_m; q)_\infty := \prod_{j=1}^m (1 - q^i x_j),
\]
(d) the normalization
\[
\Gamma_0(\xi) = \prod_{\alpha \in R_0^+} (q^2 a_{\alpha}^{-2\xi(\alpha)}; q^2)_\infty.
\]

The function \( \Phi(\cdot, \cdot) \) is the natural generalization to the present basic hypergeometric context of Harish-Chandra series, see \cite{28} and \cite{27} Part 1. We therefore call \( \Phi(\cdot, \cdot) \) the basic Harish-Chandra series associated to the initial datum \((D, \kappa, q)\). It is automatically an eigenfunction of the higher order RMKC operators, see Subsection 5.0. Specializing \( \xi \) to a polynomial spectral point turns \( \Phi(z, \xi) \) into the self-dual symmetric Macdonald-Koornwinder polynomial associated to \((D, \kappa, q)\) (the proof from \cite{27}, §4] and \cite{59} §3.3] generalizes easily to the present setup, with the Macdonald-Koornwinder polynomials associated to \((D, \kappa, q)\) as defined in \cite{60}.

**Remark 1.2.** If \( R_0 \) is of rank one then explicit expressions of the basic Harish-Chandra series in terms of basic hypergeometric series are known, see \cite{59} §5. For \( R_0 \) of rank two explicit basic hypergeometric expressions are known only for the \( GL_3 \) root system datum, see \cite{47}. Explicit expressions for the coefficients \( \Gamma_\alpha(\xi) \) \((\alpha \in Q^+)\) of the power series expansion of \( \Phi(z, \xi) \) are only known in higher rank cases if \( D \) is the \( GL_{n+1} \) system datum, see \cite{47}.

The above characterization of the basic Harish-Chandra series is easy to establish. Its existence was proved for \( \bullet = t \) and \( \Lambda = \Lambda \) in \cite{44} \cite{43} \cite{59}. These methods are extended to the present context in Section 3.

The particular choice of normalization of the basic Harish-Chandra series (see Theorem 1.12d)) is to ensure the self-duality of the basic Harish-Chandra series (which does not have a classical analog).
Theorem 1.3. Let $\tilde{\Phi}(\cdot, \cdot)$ be the basic Harish-Chandra series associated to the dual initial datum $(\tilde{D}, \tilde{\kappa}, q)$, where $\tilde{D} := (\tilde{R}_0, \tilde{\Delta}_0, \tilde{\Lambda}, \Lambda)$ with $\tilde{\Delta}_0 := (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ and with $\tilde{\kappa}$ the dual set of free parameters as defined in Subsection 2.3 (its associated AW parameters are the dual AW parameters). Then

$$\Phi(z, \xi) = \tilde{\Phi}(\xi, z).$$

The selfduality of $\Phi$ implies that $\Phi$ solves a bispectral problem, see Subsection 3.6. We prove Theorem 1.1 and Theorem 1.3 in Subsection 3.6.

1.4. Connection coefficients and root system analogs of elliptic $R$-matrices. The RMKC operators are $W_0$-equivariant, resulting in the $W_0 \times W_0$-invariance of the solution space of the associated bispectral problem. It leads to the following definition of connection matrices.

Definition/Theorem 1.4. Let $\mathcal{F}$ be the space of meromorphic $\tilde{\Lambda} \times \Lambda$-translation invariant meromorphic functions $f(z, \xi)$ in $(z, \xi) \in E_C \times E_C$. For $\sigma \in W_0$ there exists a unique matrix

$$M^\sigma(z, \xi) = (m^\sigma_{\tau_1, \tau_2}(z, \xi))_{\tau_1, \tau_2 \in W_0}$$

with coefficients $m^\sigma_{\tau_1, \tau_2}$ in $\mathcal{F}$ such that

$$\Phi(\sigma^{-1}z, \tau^{-1}_2 \xi) = \sum_{\tau_1 \in W_0} m^\sigma_{\tau_1, \tau_2}(z, \xi) \Phi(z, \tau_1^{-1}\xi)$$

as meromorphic functions in $(z, \xi) \in E_C \times E_C$. We call $M^\sigma$ the connection matrix associated to $\sigma \in W_0$ and $M := \{M^\sigma\}_{\sigma \in W_0}$ the connection cocycle.

For fixed $\xi \in E_C$ the connection cocycle is Cherednik’s monodromy cocycle [9, Cor. 5.3] (see also [12, §4]) for the quantum affine KZ equations (1.5), represented as matrix with respect to a suitable basis of asymptotically free solutions. We will first establish the theorem in the context of the bispectral quantum KZ equations. Applying the difference Cherednik-Matsuo correspondence then provides the current formulation in terms of basic Harish-Chandra series. See Subsections 3.3 and 3.6 for the details.

The cocycle property of the connection cocycle is

$$M^{\sigma_1 \sigma_2}(z, \xi) = M^{\sigma_1}(z, \xi) M^{\sigma_2}(\sigma^{-1}_1 z, \xi), \quad \sigma_1, \sigma_2 \in W_0$$

with $M^e$ the identity matrix, where $e \in W_0$ is the neutral element. Note furthermore that $m^e_{\tau_1, \tau_2}(z, \xi) = m^{\tau^{-1}_2 \tau_1, e}(z, \tau^{-1}_2 \xi)$.

The following theorem provides explicit expressions of the entries of the connection matrices in terms of theta functions. Write

$$\theta(x_1, \ldots, x_m; q) := \prod_{j=1}^m (x_j, q/x_j; q)_\infty$$

for products of the normalized Jacobi theta function $\theta(x; q)$ in base $q$ and define for $\alpha \in R_0$ the following two meromorphic functions in $(x, y) \in \mathbb{C} \times \mathbb{C}$,

$$e_\alpha(x, y) := q^{-\frac{1}{2 \rho_\alpha}(\kappa_\alpha + \kappa_2 \alpha - x)(\kappa_\alpha + \kappa_2 \alpha - y)} \frac{\theta(\alpha, q^{-y}; q^2) \theta(q^2 y, d_\alpha y^{-x}; q^2_\alpha)}{\theta(q^2 y, d_\alpha q^{-x}; q^2_\alpha)}$$

for products of the normalized Jacobi theta function $\theta(x; q)$ in base $q$. Define for $\alpha \in R_0$ the following two meromorphic functions in $(x, y) \in \mathbb{C} \times \mathbb{C}$,

$$e_\alpha(x, y) := q^{-\frac{1}{2 \rho_\alpha}(\kappa_\alpha + \kappa_2 \alpha - x)(\kappa_\alpha + \kappa_2 \alpha - y)} \frac{\theta(\alpha, q^{-y}; q^2) \theta(q^2 y, d_\alpha y^{-x}; q^2_\alpha)}{\theta(q^2 y, d_\alpha q^{-x}; q^2_\alpha)}$$

for products of the normalized Jacobi theta function $\theta(x; q)$ in base $q$. Define for $\alpha \in R_0$ the following two meromorphic functions in $(x, y) \in \mathbb{C} \times \mathbb{C}$,
and its dual version
\[ \tilde{\epsilon}_\alpha(x, y) := q^{-\frac{1}{2} \kappa_\alpha - x} \theta\left(a_\alpha q^y, b_\alpha q^y, c_\alpha q^y, d_\alpha q^{-x}/a_\alpha; q^2\right) / \theta(q^{2y}, d_\alpha q^{-x}; q^2) \]

The meromorphic functions \( \epsilon_\alpha(x, y) \) and \( \tilde{\epsilon}_\alpha(x, y) \) only depend on \( W_0 \alpha \). Furthermore, \( \epsilon_\alpha(x, y) \) and \( \tilde{\epsilon}_\alpha(x, y) \) are \( 2\mu_\alpha \)-translation invariant in both \( x \) and \( y \), which follows from repeated application of the functional equation

(1.8) \[ \theta(q^r x; q) = (-q^{-1} x^{-r} q^{-1}) \theta(x; q), \quad r \in \mathbb{Z}. \]

For \( i \in \{1, \ldots, n\} \) let \( s_i \in W_0 \) be the orthogonal reflection associated to the simple root \( \alpha_i \).

**Theorem 1.5.** Fix \( i \in \{1, \ldots, n\} \). Let \( i^* \in \{1, \ldots, n\} \) such that \( \alpha_i^* = -w_0 \alpha_i \). Then \( m_{s_i}^{s_i} = 0 \) if \( \tau_1 \notin \{\tau_2, \tau_2 s_i^*\} \) and

\[
\begin{align*}
 m_{s_i, e}^{s_i}(z, \xi) &= \frac{\epsilon_{\alpha_i}(\alpha_i(z), \xi) - \tilde{\epsilon}_{\alpha_i}(\xi, \alpha_i(z))}{\epsilon_{\alpha_i}(\xi, -\alpha_i(z))}, \\
 m_{s_i^*, e}^{s_i}(z, \xi) &= \frac{\epsilon_{\alpha_i^*}(\alpha_i(z), -\xi) \tilde{\epsilon}_{\alpha_i^*}(\xi, -\alpha_i(z))}{\epsilon_{\alpha_i^*}(\xi, -\alpha_i(z)).}
\end{align*}
\]

Note that it is not immediately clear that the right hand sides of (1.9) are \( \hat{\Lambda} \times \Lambda \)-translation invariant since the \( \epsilon_\alpha(x, y) \) are only \( 2\mu_\alpha \)-translation invariant in \( x \) and \( y \). The \( \hat{\Lambda} \times \Lambda \)-translation invariance can be verified directly using quadratic transformation formulas, see [61, 87].

The proof of Theorem 1.5 which follows from rank reduction [39] and the explicit formulas [61] for the connection coefficients in rank one, is discussed in Section 3.

The expression for the nontrivial entries \( m_{s_i, e}^{s_i} \) and \( m_{s_i^*, e}^{s_i} \) of the connection matrix \( M^{s_i} \) simplifies for the continuous \( q \)-ultraspherical case \( \hat{\Lambda} = \mathbb{Z} \) by a direct application of the so called addition formula [63] for theta functions

(1.10) \[ \theta(x, x/\lambda, \mu \nu, \mu/\nu; q) = \theta(x, x/\lambda, \lambda \mu, \lambda/\mu; q) \]

which plays a fundamental role in the theory [24, Chpt. 11] of elliptic hypergeometric functions (see [52, App. A] for a detailed discussion of (1.10)).

**Proposition 1.6.** If \( \hat{\Lambda} = \mathbb{Z} \) then

\[
\begin{align*}
 m_{s_i, e}^{s_i}(z, \xi) &= q^{\frac{1}{2} \kappa_{\alpha_i} - \tilde{\alpha}_i(\xi)} \frac{\theta(a_{\alpha_i} q^\xi, q^{-\alpha_i}(\xi), a_{\alpha_i} q^{-\xi}; q_{\alpha_i})}{\theta(q^{\alpha_i}(\xi), q^{-\alpha_i}(\xi), q_{\alpha_i}); q_{\alpha_i})}, \\
 m_{s_i^*, e}^{s_i}(z, \xi) &= q^{\frac{1}{2} \kappa_{\alpha_i} - \tilde{\alpha}_i(\xi)} \frac{\theta(a_{\alpha_i} q^{-\xi}, q^{\alpha_i}(\xi), a_{\alpha_i} q^{\xi}; q_{\alpha_i})}{\theta(q^{\alpha_i}(\xi), q^{-\alpha_i}(\xi), q_{\alpha_i}); q_{\alpha_i})},
\end{align*}
\]

where \( \mu_i := \mu_{\alpha_i}, q_i := q_{\alpha_i}, \kappa_i := \kappa_{\alpha_i} \), and \( a_i := a_{\alpha_i} \).

Such a simplification of \( m_{s_i, e}^{s_i} \) is apparently not possible for the continuous \( q \)-Jacobi cases and the Askey-Wilson case.

Recall that the connection coefficients are also the transition functions for the asymptotically free meromorphic solutions of the quantum KZ equations (1.5). Hence the explicit computation of the connection coefficients may be seen as a quantum analog of the explicit computation of the monodromy of the trigonometric KZ equations from [8, 11, 49].
The cocycle property (1.7) of the connection cocycle $M(z, \xi)$ comprises root system analogs of dynamical Yang-Baxter type equations. We clarify this point of view for the important special example $D = (R_0, \Delta_0, t, Q, Q)$ with $R_0$ of type $B_n$ ($n \geq 3$). Choose the ordering of the basis $\Delta_0$ in such a way that the braid relations of the associated simple reflections $s_i$ are given by

\[
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad 1 \leq i \leq n - 2,
\]

\[
s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1},
\]

\[
s_i s_j = s_j s_i, \quad |i - j| > 1.
\]

The cocycle condition (1.7) of $M(z, \xi)$ then yields

\[
M^{s_i}(z, \xi) M^{s_{i+1}}(s_i z, \xi) M^{s_i}(s_{i+1} z, \xi) =
\]

\[
= M^{s_{i+1}}(z, \xi) M^{s_i}(s_i z, \xi) M^{s_{i+1}}(s_{i+1} z, \xi)
\]

for $1 \leq i \leq n - 2$ and

\[
M^{s_{n-1}}(z, \xi) M^{s_n}(s_{n-1} z, \xi) M^{s_{n-1}}(s_n s_{n-1} z, \xi) M^{s_n}(s_n s_{n-1} z, \xi) =
\]

\[
= M^{s_n}(z, \xi) M^{s_{n-1}}(s_n z, \xi) M^{s_n}(s_{n-1} s_n z, \xi) M^{s_{n-1}}(s_{n-1} s_n z, \xi).
\]

It is natural to view the equations (1.11) and (1.12) as modifications of the dynamical Yang-Baxter equation [25] [21] [20] and the dynamical reflection equation [4] [2] [3] [35] respectively, with $z$ playing the role of spectral parameter and $\xi$ the role of dynamical parameter. This viewpoint can be understood from the interpretation of the connection coefficients as the transition functions for asymptotically free meromorphic solutions of the quantum KZ equations (1.5). For example, for the $GL_{n+1}$ root system datum, the quantum KZ equations (1.5) coincide with Frenkel-Reshetikhin-Smirnov type quantum KZ equations. The corresponding transition functions are governed by the elliptic solution [30] [32] of the star triangle equation associated to the integrable $A_{n-1}(1)$ face model (see [23 §6 & §7] and [62, 153, 36]) which, in turn, is known [21] to be equivalent to Felder’s [21] Prop. 1 elliptic solution of the dynamical Yang-Baxter equation. Note also the resemblance of the explicit expression [21] Prop. 1 of Felder’s elliptic solution of the dynamical Yang-Baxter equation with the explicit expression of the connection matrix $M^{s_i}(z, \xi)$ from Proposition (1.6).

Theorem (1.5) now also provides an explicit expression of the solution $M^{s_n}(z, \xi)$ of the associated modified dynamical reflection equation (1.12). Note that it depends on four free parameters (the Askey-Wilson parameters associated to the short simple root $\alpha$). It is expected to provide elliptic dynamical $K$-matrices for the $A_n^{(1)}$ face models, hence giving rise to new families of integrable $A_n^{(1)}$ face type models with reflecting boundary conditions. Thus far, $A_n^{(1)}$ face models with reflecting boundary conditions have only been constructed for the eight vertex solid-on-solid model [22] with respect to a diagonal solution of the dynamical reflection equation.

1.5. Quantum $c$-functions and basic hypergeometric functions. The basic hypergeometric function $E_{sph}(z, \xi)$ is a distinguished $W_0 \times W_0$-invariant, meromorphic solution of the bispectral problem of the RMKC operators in case that the root system datum is of the form $D = (R_0, \Delta_0, t, \Lambda, \Lambda)$ and $\kappa_0 > 0$. It was constructed in [14] [10] [57, 59] as reproducing kernel of a difference Fourier transform. It admits an explicit series expansion in symmetric Macdonald-Koornwinder polynomials. Just as for the basic Harish-Chandra series, specializing $\xi$ to a polynomial spectral point
turns $E_{sph}(z, \xi)$ into the pertinent selfdual Macdonald-Koornwinder polynomial, see, e.g., [59 Thm. 2.20]. If $R_0$ has rank one then $E_{sph}$ arises as quantum spherical function on noncompact quantum groups, see [35].

The basic hypergeometric function is the natural basic hypergeometric analog of the Heckman-Opdam [25] hypergeometric function associated to root systems, cf. [16 Thm. 4.4] and [59]. The Heckman-Opdam hypergeometric function is defined in a completely different fashion, see [28] and [27 Part I]; the explicit computation of the monodromy representation of the system of hypergeometric differential equations is used to define the Heckman-Opdam hypergeometric function as the explicit expansion in Harish-Chandra series which is fixed under the monodromy representation. The explicit expansion is the $c$-function expansion of the Heckman-Opdam hypergeometric function; the coefficients are expressed in terms of the Harish-Chandra $c$-function [25 Def. 6.4].

In the current basic hypergeometric context, the analog of the $c$-function expansion of the basic hypergeometric function $E_{sph}(z, \xi)$ has been derived in [59] using the asymptotic analysis of the basic hypergeometric function from [16]. We now recall the explicit $c$-function expansion of $E_{sph}(z, \xi)$ from [59] and relate it to the explicit computation of the connection cocycle.

From [59 Thm. 4.6] the $c$-function expansion of $E_{sph}(z, \xi)$ is

\[(1.13) \quad E_{sph}(z, \xi) = \sum_{w \in W_0} c_{sph}(z, w\xi) \Phi(z, w\xi)\]

with the quantum analog $c_{sph} \in \mathcal{F}$ of the Harish-Chandra $c$-function defined as follows.

The higher rank theta function is the holomorphic function

$$\theta_\Lambda(z) := \sum_{\lambda \in \Lambda} q^{\frac{1}{2} |\lambda|^2} q^{(\lambda, z)}$$

in $z \in E_C$. Write $\delta^\vee_\alpha := \frac{1}{2} \sum_{\alpha \in R^+_0, s} \alpha^\vee$, where $R^+_0, s \subset R^+_0$ is the subset of positive short roots in $R_0$. If $R_0$ is simply laced then we set $R^+_0, s = R^+_0$.

**Definition 1.7.** The quantum $c$-function $c_{sph} \in \mathcal{F}$ associated to the root system datum $D = (R_0, \Delta_0, t, \Lambda, \Lambda)$ is

$$c_{sph}(z, \xi) := W(z, \xi)^{-1} \frac{\partial_\Lambda (\rho + (\kappa_2 n_0 - \kappa_0) \delta^\vee_\xi + z + w_0\xi)}{\partial_\Lambda ((\kappa_2 n_0 - \kappa_0) \delta^\vee_\xi + z)} \prod_{\alpha \in R^+_0} \frac{\theta(a_\alpha q^{\alpha(\xi)}, b_\alpha q^{\alpha(\xi)}, \bar{c}_\alpha q^{\alpha(\xi)}, \bar{d}_\alpha q^{\alpha(\xi)}; q_0^2)}{\theta(q^{2\alpha(\xi)}; q_0^2)}.$$

If furthermore $(\Lambda, \alpha^\vee) = \mathbb{Z}$ for all $\alpha \in R_0$, then

$$c_{sph}(z, \xi) = W(z, \xi)^{-1} \frac{\partial_\Lambda (\rho + z + w_0\xi)}{\partial_\Lambda (z)} \prod_{\alpha \in R^+_0} \frac{\theta(q^{2\kappa_\alpha + \alpha(\xi)}; q_0^2)}{\theta(q^{\alpha(\xi)}; q_0^2)}.$$

It follows from [59 Thm. 2.20 (iv) & Thm. 4.6] that for generic multiplicity functions $\kappa$ satisfying $\kappa_\alpha > 0$ for all $\alpha \in R(D)$, the basic hypergeometric function
\( E_{\text{sph}} \) satisfies
\[
E_{\text{sph}}(z, \rho) = \frac{1}{\vartheta_{\Lambda}((\kappa 2a_0 - \kappa 0)\delta_s^\nu - \rho)} \prod_{\alpha \in R_0^+} \frac{(a_\alpha q^{\alpha(\rho)}, \bar{a}_\alpha q^{\alpha(\rho)}, \bar{c}_\alpha q^{\alpha(\rho)}, \bar{d}_\alpha q^{\alpha(\rho)}; q^2_\alpha)_{\infty}}{(q^{2\alpha(\rho)}; q^2_\alpha)_{\infty}}.
\]

If one renormalizes \( E_{\text{sph}} \) such that \( E_{\text{sph}}(z, \rho) = 1 \), which is the convention used in [59], then it becomes selfdual, see [59] Thm. 2.20 (iii). In Remark 5.1 we will precisely match the present notations to the ones used in [59].

For arbitrary initial data \((D, \kappa, q)\), if \( \epsilon \in \mathcal{F} \) then the meromorphic function
\[
(1.14) \quad E(z, \xi) := \sum_{w \in W_0} \epsilon(z, w\xi)\Phi(z, w\xi)
\]
in \((z, \xi) \in E_C \times E_C \) is \( W_0 \)-invariant in \( z \) if and only \( \epsilon \in \mathcal{F} \) satisfies
\[
(1.15) \quad \epsilon(z, \xi) = m^0_{\nu, s}(z, \xi)\epsilon(s_iz, \xi) + m^0_{\nu, s}(z, s_i\xi)\epsilon(s_iz, s_i\xi)
\]
for all \( i \in \{1, \ldots, n\} \) in view of Theorem 1.5. In Section 5 we give a direct proof that \( E_{\text{sph}} \) indeed satisfies (1.15) if \( D = (R_0, \Delta_0, t, \Lambda, \Lambda) \). It leads to the following higher rank analog of the addition formula (1.10) for Jacobi theta functions.

**Proposition 1.8.** Suppose \( D = (R_0, \Delta_0, t, \Lambda, \Lambda) \) and let \( i \in \{1, \ldots, n\} \) such that \((\Lambda, \alpha_i^+) = Z \). Then
\[
\theta(q^{\tilde{\alpha}_i, (\xi)}; q^{2\kappa_i - \alpha_i(z)}, q_i)\vartheta_{\Lambda}(\rho + z + w_0\xi) =
\]
\[
= \theta(q^{2\kappa_i}, q^{\tilde{\alpha}_i, (\xi)} - \alpha_i(z), q_i)\vartheta_{\Lambda}(\rho + s_i z + w_0\xi)
\]
\[
- q^{\tilde{\alpha}_i, (\xi)}\theta(q^{2\kappa_i - \tilde{\alpha}_i, (\xi)}, q^{-\alpha_i(z)}, q_i)\vartheta_{\Lambda}(s_\rho + z + w_0\xi).
\]

The definition of the basic hypergeometric function as reproducing kernel of a difference Fourier transform is restricted to the twisted equal lattice case, \( D = (R_0, \Delta_0, t, \Lambda, \Lambda) \). With the explicit expressions of the connection coefficients now available, it is natural to try to extend the method employed in the construction of the Heckman-Opdam hypergeometric function and define the appropriate analog of the basic hypergeometric function for all root system data \( D \) as the expansion (1.14) for a distinguished solution \( \epsilon \in \mathcal{F} \) of the equations (1.15). This is a subtle matter, since the equations (1.14) do not determine \( \epsilon \in \mathcal{F} \) uniquely, cf. Subsection 1.6. We do not pursue this issue in the present paper, although we will make some initial steps in the analysis of the equations (1.15) in Section 5.

1.6. **Reflectionless basic Harish-Chandra series.** In, e.g., [51] [52] [17], reflectionless analytic difference operators are studied. These are difference analogs of one-dimensional Schrödinger operators admitting a meromorphic eigenfunction \( \phi(\cdot, p) \) in \((\cdot, p) \in \mathbb{C}^2 \) with eigenvalue \( e^p + e^{-p} \) and having plane wave asymptotics
\[
\phi(x, p) \sim e^{\sqrt{-1}xp}, \quad \Re(x) \to \infty,
\]
\[
\phi(x, p) \sim \alpha(p)e^{\sqrt{-1}xp} + \beta(p)e^{-\sqrt{-1}xp}, \quad \Re(p) \to -\infty
\]
with \( \beta \equiv 0 \). This has the following analog for RMKC operators.

Write \( \ell(\nu) := \min\{(\nu, \tilde{\alpha}_i^+)\}_{i=1}^n \) for \( \nu \in \Lambda \). Then for generic \( \xi \in E_C \),
\[
\Phi(z - \nu, \xi) = \frac{\ell(\rho, w_0\xi, z - \nu)}{\mathcal{S}(\xi)}e^{-\rho + \nu, \xi, z - \nu}(1 + \mathcal{O}(q^l(\nu)))
\]
as $\ell(\nu) \to \infty$, uniformly for $z$ in compacta of $E_{\mathbb{C}}$, in view of Theorem 1.10. It describes the plane wave asymptotics of $\Phi(z, \xi)$ as common eigenfunction of the RMKC operators for the real part $\Re(z)$ of $z$ deep in the negative fundamental Weyl chamber $E_- := \{ \nu \in E \mid \alpha(\nu) < 0 \ \forall \alpha \in R^+_0 \}$. By Theorem 1.4 $\Phi(z, \xi)$ has plane wave asymptotics for $\Re(z)$ deep in an arbitrary Weyl chamber $\tau(E_-) (\tau \in W_0)$. In particular, for the Weyl chambers neighboring $E_-,$

$$
\Phi(z - s_i \nu, \xi) \sim m_{s_i, e}^\nu(s_i z, \xi)\frac{q(\rho, \rho - \xi)}{\tilde{S}(\xi)}e^{-(s_i \rho + w_0 s_i \xi, z - s_i \nu)}
+ m_{s_i, e}^\nu(s_i z, \xi)\frac{q(\rho, \rho - s_i \xi)}{\tilde{S}(s_i \xi)}e^{-(s_i \rho + w_0 \xi, z - s_i \nu)}
$$

as $\ell(\nu) \to \infty$ for generic $z, \xi \in E_{\mathbb{C}}$ ($i = 1, \ldots, n$) by Theorem 1.10. Thus we come to the following analog of reflectionless in the context of RMKC operators.

**Definition 1.9.** We say that the RMKC operators are reflectionless if $m_{s_i, e}^\nu \equiv 0$ for $i = 1, \ldots, n$. In this case the associated basic Harish-Chandra series $\Phi$ is said to be reflectionless.

By Theorem 1.5 the RMKC operators are reflectionless if and only if

$$
(1.17) \quad e_\alpha(x, y) = \tilde{e}_\alpha(y, x)
$$

as meromorphic functions in $(x, y) \in \mathbb{C} \times \mathbb{C}$ for all $\alpha \in R_0$. The following result now follows by straightforward computations using the functional equation $(1.8)$ for the normalized Jacobi theta function $\theta(x; q)$ (cf. [61, Prop. 3.1] for a weaker statement).

**Proposition 1.10.** Suppose that the multiplicity function $\kappa$ satisfies

$$
(1.18) \quad \kappa_\alpha \pm \kappa_{\alpha(1)}, \kappa_{2\alpha} \pm \kappa_{2\alpha(1)}, \kappa_\alpha \pm \kappa_{2\alpha}, \kappa_{\alpha(1)} \pm \kappa_{2\alpha(1)} \in \mu_\alpha \mathbb{Z},
\kappa_\alpha + \kappa_{2\alpha} + \kappa_{\alpha(1)} + \kappa_{2\alpha(1)} \in 2\mu_\alpha \mathbb{Z}
$$

for all $\alpha \in R_0$. Then the RMKC operators are reflectionless, and $m_{s_i, e}^\nu \equiv 1$ for $i = 1, \ldots, n$.

If the multiplicity function $\kappa$ satisfies $(1.18)$, then so does the dual multiplicity function $\tilde{\kappa}$ (see Lemma 2.1). Note the following special cases:

**Continuous $q$-ultraspherical case:** $(\Lambda, \alpha^\vee) = \mathbb{Z} = (\tilde{\Lambda}, \tilde{\alpha}^\vee)$. Then $(1.18)$ reduces to the simple condition

$$
\kappa_\alpha \in \frac{\mu_\alpha}{2} \mathbb{Z}.
$$

**Continuous $q$-Jacobi case:** $(\Lambda, \alpha^\vee) = \mathbb{Z}$ and $(\tilde{\Lambda}, \tilde{\alpha}^\vee) = 2\mathbb{Z}$, then $(1.18)$ reduces to

$$
\kappa_\alpha, \kappa_{\alpha(1)} \in \frac{\mu_\alpha}{2} \mathbb{Z}, \quad \kappa_\alpha + \kappa_{\alpha(1)} \in \mu_\alpha \mathbb{Z},
$$
or $(\Lambda, \alpha^\vee) = 2\mathbb{Z}$ and $(\tilde{\Lambda}, \tilde{\alpha}^\vee) = \mathbb{Z}$, then $(1.18)$ reduces to

$$
\kappa_\alpha, 2\kappa_\alpha \in \frac{\mu_\alpha}{2} \mathbb{Z}, \quad \kappa_\alpha + 2\kappa_\alpha \in \mu_\alpha \mathbb{Z}.
$$

Theorem 1.4 and Proposition 1.10 give the following result.
Corollary 1.11. If the multiplicity function $\kappa$ satisfies \[1.18\], then the reflectionless basic Harish-Chandra series $\Phi$ satisfies

\[1.19\] $\Phi(wz, w_0w_0\xi) = \Phi(z, \xi) \quad \forall w \in W_0$.

The analysis of quantum $c$-functions simplifies in the present context of reflectionless RMKC operators, since the conditions \[1.15\] for $i = 1, \ldots, n$ for $c \in F$ are equivalent to the invariance property

$\Psi(wz, w_0w_0\xi) = \Psi(z, \xi) \quad \forall w \in W_0$

if $\kappa$ satisfies \[1.18\]. Consequently, under the assumption \[1.18\] on the multiplicity function $\kappa$,

\[1.20\] $\Phi_+(z, \xi) := \sum_{w \in W_0} \Phi(z, w\xi)$

is a $W_0 \times W_0$-invariant meromorphic function of the bispectral problem of the reflectionless RMKC operators. Note that in the twisted equal lattice case $D = (R_0, \Delta_0, t, \Lambda, \Lambda)$ with $\kappa$ satisfying the reflectionless conditions \[1.18\], $\Phi_+$ does not coincide with the basic hypergeometric function $E_{sph}$.

1.7. Multivariable Baker-Akhiezer functions. In \[6,7\], multivariable Baker-Akhiezer functions associated to RMKC operators are defined under suitable restrictions on the multiplicity function $\kappa$ for reduced semisimple root data $D = (R_0, \Delta, \bullet, P, P)$ and for the Koornwinder root system datum. The conditions \[7, \S2.1.3\] on the multiplicity function $\kappa$ for the multivariable Baker-Akhiezer function to be defined then read

(BA1) if $\alpha \in R_0$ with $(\Lambda, \alpha^\vee) = Z = (\bar{\Lambda}, \bar{\alpha}^\vee)$ (continuous $q$-ultraspherical case) then $\kappa_\alpha \in \frac{m_2}{2}Z \leq 0$,

(BA2) if $\alpha \in R_0$ with $(\Lambda, \alpha^\vee) = 2Z = (\bar{\Lambda}, \bar{\alpha}^\vee)$ (Askey-Wilson case) then

$\kappa_\alpha = \kappa_{2\alpha}, \kappa_\alpha \pm \kappa_{\alpha(1)} \in \mu_\alpha Z \leq 0$,

$\kappa_{\alpha(1)} \pm \kappa_{2\alpha(1)}, \kappa_{\alpha} \pm \kappa_{2\alpha} \in 2\mu_\alpha Z \\
\kappa_{\alpha} + \kappa_{2\alpha} + \kappa_{\alpha(1)} + \kappa_{2\alpha(1)} \in 2\mu_\alpha Z$

(the continuous $q$-Jacobi cases $(\Lambda, \alpha^\vee) = Z$ and $(\bar{\Lambda}, \bar{\alpha}^\vee) = 2Z$, respectively $(\Lambda, \alpha^\vee) = 2Z$ and $(\bar{\Lambda}, \bar{\alpha}^\vee) = Z$, do not occur for reduced semisimple and Koornwinder root data).

Remark 1.12. (i) For the reduced semisimple root data $D$ the free parameters $m_\alpha$ in \[7, \S2.1.1\] corresponds to $-2\kappa_\alpha/\mu_\alpha$. For the Koornwinder root system datum $D$ the free parameters $m_i$ ($1 \leq i \leq 5$) in \[7, \S2.1.2\] are related to $\kappa$ by

$m_1 = -\kappa_\alpha - \kappa_{2\alpha}, \quad m_2 = -\frac{1}{2} - \kappa_{\alpha(1)} - \kappa_{2\alpha(1)}$, 

$m_3 = -\kappa_\alpha + \kappa_{2\alpha}, \quad m_4 = -\frac{1}{2} - \kappa_{\alpha(1)} + \kappa_{2\alpha(1)}$, 

$m_5 = -2\kappa_\beta$

where $\alpha \in R_0$ (resp. $\beta \in R_0$) is a short (resp. long) root and the root system $R_0$ is normalized such that long roots have squared length two. Here the fifth free parameter $m_5$ (resp. $\kappa_\beta$) should only be taken into account if the rank $n$ of $R_0$ is $\geq 2$.

(ii) The results of the previous subsection apply if $D$ is a reduced semisimple or a
Koornwinder root system datum and the multiplicity function \( \kappa \) satisfies (BA1) and (BA2), since conditions (BA1) and (BA2) imply the reflectionless conditions \( 1.18 \).

In particular, the RMKC operators are reflectionless and the basic Harish-Chandra series \( \Phi \) satisfies the invariance property \( 1.19 \).

The following result traces back to [39, §4.4]. We discuss its proof at the end of Subsection 3.6.

**Proposition 1.13.** Let \( (D, \kappa, q) \) be an initial datum with a reduced semisimple or a Koornwinder root system datum \( D \) and with multiplicity function \( \kappa \) satisfying (BA1) and (BA2). Let \( \psi(\lambda, x) \) be the multivariable Baker-Akhiezer function associated to \( (D, \kappa, q) \) (see [7, §3.1], in particular the definition below [7, (3.8)]), where we use the parameter correspondence as indicated in Remark 1.12(i). Then

\[
\Phi(z, \xi) = \text{cst} \psi(-w_0 \xi, z)
\]

as meromorphic functions in \((z, \xi) \in \mathbb{E}_C \times \mathbb{E}_C\) for some constant \( \text{cst} \in \mathbb{C}^* \).

**Remark 1.14.**

(i) The constant \( \text{cst} \) can easily be explicitly computed by comparing the normalizations of \( \Phi \) and \( \psi \), cf. Subsection 3.6.

(ii) Proposition 1.13 allows to rederive various fundamental properties of the multivariable Baker-Akhiezer function as direct consequences of the analogous properties of the basic Harish-Chandra series. For instance, the selfduality [7, Thm. 3.3(iii)] of the multivariable Baker-Akhiezer function \( \psi \) becomes a special case of the selfduality of the basic Harish-Chandra series \( \Phi \) (Theorem 1.3), while the \( W_0 \)-invariance [7, Lem 3.4(i)] of \( \psi \) is a special case of the \( W_0 \)-invariance of the reflectionless basic Harish-Chandra series (Corollary 1.11).

Proposition 1.13 opens the way to study the results [6, 7] on multivariable Baker-Akhiezer functions on the level of (reflectionless) basic Harish-Chandra series. In particular, one can now study the extra symmetries [7] (3.4)-(3.6) and the terminating series expansion property [7] (3.3) of the multivariable Baker-Akhiezer function \( \psi \) on the level of (reflectionless) basic Harish-Chandra series. For instance, the terminating series expansion of \( \psi \) becomes the following surprising property of the expansion coefficients \( \hat{\Gamma}_\alpha(\xi) \) \( 3.10 \) of the basic Harish-Chandra series.

**Corollary 1.15.** Let \( (D, \kappa, q) \) be an initial datum with \( D \) a reduced semisimple root system datum and \( \kappa \) a multiplicity function satisfying (BA1). Then

\[
\hat{\Gamma}_\alpha(\xi) = 0
\]

as meromorphic function in \( \xi \in \mathbb{E}_C \) unless \( \alpha \in \mathbb{Q}^+ \) is of the form \( \alpha = \frac{1}{2} \sum_{\beta \in R_0^+} l_\beta \beta \) with \( 0 \leq l_\beta \leq -4\kappa_\beta/\mu_\beta \) for all \( \beta \in R_0^+ \).

2. Notations

We continue the introduction of basic notations as started in Subsection 1.1. We refer to [60] for further details.

2.1. The affine root system. For fixed root system datum \( D = (R_0, \Delta_0, \bullet, \Lambda, \tilde{\Lambda}) \) with ambient Euclidean space \( E \) let \( \tilde{E} \) be the linear space of real affine linear functions on \( E \). Then \( \tilde{E} \simeq \mathbb{R}c \oplus E \), where \( a = \eta c + v \) with \( \eta \in \mathbb{R} \) and \( v \in E \) is interpreted as the affine linear function \( v' \mapsto \eta + (v, v') \).
Let $V$ be the real span of the roots. We view the linear space $\hat{V}$ of real affine linear functions on $V$ as the subspace of $\hat{E}$ which are constant on the orthocomplement $V^\perp$ of $V$ in $E$.

The extended affine Weyl group $W = W_0 \ltimes \tilde{\Lambda}$ acts on $E$ and on its complexification $E_\mathbb{C}$ with the canonical action of $W_0$ and with $\tilde{\Lambda}$ acting by translations $\tau(\nu)z := z + \nu$ ($\nu \in \tilde{\Lambda}$). It induces a linear $W$-action on $\hat{E}$. Note that

$$\tau(\nu)\alpha^{(r)} = \alpha^{(r-\langle \nu, \tilde{\alpha} \rangle)}$$

for $\alpha \in R_0$, $\nu \in \tilde{\Lambda}$ and $r \in \mathbb{Z}$, hence $R^\bullet$ and $R$ are $W$-invariant.

For an affine root $\alpha^{(r)} \in R^\bullet \subset \hat{V}$ let $s_{\alpha^{(r)}} \in W$ be the orthogonal reflection in the affine hyperplane $\ker(\alpha^{(r)})$. Then $s_{\alpha^{(r)}} = \tau(-r\tilde{\alpha})s_{\alpha}$, with $s_{\alpha} \in W_0$ the orthogonal reflection in the hyperplane $\alpha^\perp$. We write $s_i := s_{\alpha_i} \in W$ ($0 \leq i \leq n$) for the simple reflections. They generate the affine Weyl group $W$ and $\tilde{\nu}$-invariant.

We have $\hat{V} = V_0 \ltimes \tilde{\Lambda}$ and $\tilde{\Lambda}$ acts on $\hat{V}$ by translations $\tau(\nu)\tilde{\lambda} := \tilde{\lambda} + \nu$ ($\nu \in \tilde{\Lambda}$). We write $\tilde{\lambda} := \tau(\psi)s_\psi$. Then $s_0 = \tau(\psi)s_\psi$.

Let $R^\bullet_+$ and $R^\bullet_-$ be the positive and negative affine roots of $R^\bullet$ with respect to the basis $\Delta := (\alpha_0, \alpha_1, \ldots, \alpha_n)$ of $R^\bullet$. The length of $w \in W$ is defined by

$$l(w) := \#(R^\bullet_+ \cap w^{-1}R^\bullet_-), \quad w \in W.$$

We have $W \simeq \Omega \ltimes W^\bullet$ with $\Omega = \Omega(D)$ the subgroup

$$\Omega := \{w \in W \mid l(w) = 0\}.$$

For $\nu \in \tilde{\Lambda}$ let $u(\nu) \in W$ be the element of minimal length in $\tau(\nu)W_0$ and write $v(\nu) := u(\nu)^{-1}\tau(\nu) \in W_0$. Then

$$\Omega = \{u(\nu)\}_{\nu \in \tilde{\Lambda}^+},$$

with $\tilde{\Lambda}^+$ the set of dominant minimal weights in $\tilde{\Lambda}$,

$$\tilde{\Lambda}^+ := \{\nu \in \tilde{\Lambda} \mid \langle \nu, \tilde{\alpha} \rangle \in \{0, 1\} \quad \forall \alpha \in R_0^+\}.$$

The set of dominant weights in $\tilde{\Lambda}$ is

$$\tilde{\Lambda}^+ := \{\nu \in \tilde{\Lambda} \mid \langle \nu, \tilde{\alpha} \rangle \geq 0 \quad \forall \alpha \in R_0^+\}.$$

### 2.2. The dual affine root system.

The root system datum $\tilde{D} = (\tilde{R}_0, \tilde{\Delta}_0, \cdot, \tilde{\Lambda}, \Lambda)$ dual to $D$ gives rise to a dual reduced affine root system

$$\tilde{R}^\bullet = \{\tilde{\alpha}^{(r)} = \mu_\alpha r_\alpha + \tilde{\alpha}\}_{\alpha \in R_0, r \in \mathbb{Z}}$$

and its extension $\tilde{R} := R(\tilde{D})$. The associated extended affine Weyl group is $\tilde{W} := W_0 \ltimes \tilde{\Lambda}$. The additional simple affine root of $\tilde{R}$ is denoted by $\tilde{\alpha}_0$. Write $\tilde{s}_i := s_{\tilde{\alpha}_i}$ for $i \in \{0, \ldots, n\}$. Note that $\tilde{s}_i = s_i$ for $1 \leq i \leq n$ while $\tilde{s}_0 = \tau(\theta)s_\theta$ with $\theta \in R^n_0$ the highest short root, since $\tilde{\alpha}_0 = \mu_\theta e - \theta$. We write $\Omega := \Omega(D)$ so that $\tilde{W} = \Omega \ltimes \tilde{W}^\bullet$ with $\tilde{W}^\bullet = W_0 \ltimes Q$ the affine Weyl group associated to $\tilde{R}$.

For $\lambda \in \Lambda$ we write $\tilde{u}(\lambda) \in \tilde{W}$ for the shortest element in $\tau(\lambda)W_0$ and $\tilde{v}(\lambda) := \tilde{u}(\lambda)^{-1}\tau(\lambda) \in W_0$. The set of dominant weights in $\Lambda$ is

$$\Lambda^+ := \{\lambda \in \Lambda \mid \langle \lambda, \alpha \rangle \geq 0 \quad \forall \alpha \in R_0^+\}$$

and the set of dominant mimiscule weights in $\Lambda$ is

$$\Lambda^+_{\text{min}} := \{\lambda \in \Lambda \mid \langle \lambda, \alpha \rangle \in \{0, 1\} \quad \forall \alpha \in R_0^+\}.$$

As in the previous subsection, we have $\tilde{\Omega} = \{\tilde{u}(\lambda)\}_{\lambda \in \Lambda^+}$. 
2.3. Multiplicity functions. Let $\mathcal{M}(D)$ be the space of $W$-invariant functions $\kappa : R(D) \to \mathbb{R}$. Its value at $a \in R$ is denoted by $\kappa_a$. Recall the convention $\kappa_{2\alpha}(r) := \kappa_{\alpha}(r)$ if $2\alpha(r) \notin R$ (i.e. if $(\Lambda, \alpha^\vee) = \mathbb{Z}$). The involution $D \mapsto \bar{D}$ on root system data extends to multiplicity functions as follows (see [26, 60]).

**Lemma 2.1.** There exists a unique linear isomorphism $\mathcal{M}(D) \overset{\sim}{\to} \mathcal{M}(\bar{D})$, $\kappa \mapsto \bar{\kappa}$, satisfying $\bar{\kappa} = \kappa$ and $\bar{\kappa}_{\alpha}(1) = \kappa_{2\alpha}(1)$ for all $\alpha \in R_0$.

Recall from Subsection 1.1 that we associated to the initial datum $(D, \kappa, q)$ Askey-Wilson (AW) parameters $a_\alpha = a_\alpha(D, \kappa, q), \ldots, d_\alpha = d_\alpha(D, \kappa, q)$ for all $\alpha \in R_0$, as well as dual AW parameters $(\tilde{\kappa}, \tilde{a}_\alpha)$ for all $\alpha \in R_0$,

$$\tilde{a}_\alpha = a_\alpha(\bar{D}, \bar{\kappa}, q), \ldots, \tilde{d}_\alpha = d_\alpha(\bar{D}, \bar{\kappa}, q)$$

since $\mu_{\tilde{\alpha}} = \mu_{\alpha}$.

3. The difference integrable equations and asymptotic analysis

Basic Harish-Chandra series are meromorphic common eigenfunctions of the RMKC operators, characterized by suitably asymptotically free behaviour deep in an appropriate Weyl chamber. Basic Harish-Chandra series have been considered in various different contexts [10, 19, 39, 44, 43, 59, 47]. In [10] their existence was predicted based on the correspondence with solutions of quantum KZ equations. In [19, 33, 49, 54] the basic Harish-Chandra series were considered for $R_0$ of type $A_{n-1}$ using vertex operators. In [39] basic Harish-Chandra series were constructed as formal power series using classical methods from harmonic analysis. In the series of papers [44, 48, 59] Cherednik’s prediction was worked out in detail for the twisted case $\bullet = t$ and with $\tilde{\Lambda} = \Lambda$ by relating the basic Harish-Chandra series to asymptotically free solutions of (bispectral extensions) of quantum KZ equations through the difference Cherednik-Matsuo correspondence [10, 12, 33, 58]. In [47] a direct approach is undertaken to derive the fundamental properties of the basic Harish-Chandra series when $R_0$ is of type $A_n$. In this section we shortly discuss the extension of the methods from [44, 48, 59] to the present context, which includes the untwisted theory and has extra freedom in the choice of lattices. We only give the proof if it needs new arguments compared to the twisted case $\bullet = t$ with $\tilde{\Lambda} = \Lambda$.

So throughout this section $(D, \kappa, q)$ stands for an arbitrary choice of initial datum unless explicitly specified otherwise.

3.1. Bispectral quantum KZ equations. Define for $a \in R^*$ the meromorphic function $c_a(\cdot) = c_a(\cdot; D, \kappa, q)$ on $E_C$ by

$$c_a(z) := \frac{(1 - q^{\kappa_a + \kappa_2a + a(z)})(1 + q^{\kappa_a - \kappa_2a + a(z)})}{1 - q^{2a(z)}}.$$ 

We write $c_a(\cdot; \kappa, q) = c_a(\cdot; D, \kappa, q)$ and $\tilde{c}_i(\cdot; \tilde{\kappa}, q) = c_{\tilde{a}_i}(\cdot; \bar{D}, \bar{\kappa}, q)$ for $i \in \{0, \ldots, n\}$.

Let $\mathcal{M}$ be the field of meromorphic function on $E_C \times E_C$. The contragredient actions of $W$ and $\bar{W}$ on $E_C$ give rise to an action of $W \times \bar{W}$ on $\mathcal{M}$ by field automorphisms. Note that $\mathcal{F} = \mathcal{M}^{\tau(\tilde{\Lambda}) \times \tau(\Lambda)}$. 
Let $\mathcal{M} \otimes \mathcal{C} \text{End}_\mathcal{C}(\mathcal{V}) \simeq \text{End}_\mathcal{M}(\mathcal{M} \otimes \mathcal{C} \mathcal{V})$ be the space of $\text{End}_\mathcal{C}(\mathcal{V})$-valued meromorphic functions on $E_\mathcal{C} \times E_\mathcal{C}$, where $\mathcal{V} := \bigoplus_{\sigma \in W_0} \mathcal{C} \sigma$. Let $\chi : R_0 \to \{0, 1\}$ be the characteristic function of $R_0^+ - R_0$.

**Theorem 3.1.** There exists unique $C_{(w, w')} \in \text{End}_\mathcal{M}(\mathcal{M} \otimes \mathcal{C} \mathcal{V})$ ($(w, \bar{w}) \in W \times \bar{W}$) satisfying the cocycle conditions

\begin{align}
C_{(v, \nu)}(z, \xi) & = C_{(v, \bar{v})}(z, \xi)C_{(w, \bar{w})}(v^{-1}z, \bar{v}^{-1}\xi), \quad \forall (v, \bar{v}), (w, \bar{w}) \in W \times \bar{W} \\
\text{and} \quad C_{(e, e)}(z, \xi) & = \text{Id}_\mathcal{V}, \quad \text{satisfying for all} \ \sigma \in W_0,
\end{align}

(3.1)

\begin{align}
C_{(u, w_\nu, \sigma)}(z, \xi) & = q^{-w_\nu, \sigma}(z, \xi)C_{(e, w, \sigma)}(z, \xi), \\
\text{for} \quad 1 \leq i \leq n \quad \text{and} \quad \nu \in \bar{\Lambda}^+ \text{min and }
\end{align}

(3.2)

\begin{align}
C_{(e, \bar{w}_\lambda)}(z, \xi) & = q^{\bar{w}_\lambda, \lambda}(z, \xi)C_{(e, \bar{w}, \lambda)}(z, \xi), \\
\text{for} \quad 1 \leq i \leq n \quad \text{and} \quad \lambda \in \bar{\Lambda}^+ \text{min},
\end{align}

(3.3)

\begin{align}
(\nabla(w, \bar{w})f)(z, \xi) := C_{(w, \bar{w})}(z, \xi)f(w^{-1}z, \bar{w}^{-1}\xi)
\end{align}

(3.4)

defines a complex linear left action $\nabla = \nabla(D, \kappa, q)$ of $W \times \bar{W}$ on $\mathcal{M} \otimes \mathcal{C} \mathcal{V}$. Following \cite{[44, 43, 59]}, we arrive now at the definition of the bispectral quantum Khnizhnik-Zamolodchikov (KZ) equations.

**Definition 3.2.** We say that $f \in \mathcal{M} \otimes \mathcal{C} \mathcal{V}$ is a meromorphic solution of the bispectral quantum KZ equations if

\begin{align}
(\nabla(\tau(\nu), \tau(\lambda))f)(z, \xi) & = f \quad \forall \nu \in \bar{\Lambda}, \ \forall \lambda \in \Lambda.
\end{align}

(3.5)

We write $\text{Sol}_{KZ} = \text{Sol}_{KZ}(D, \kappa, q)$ for the vector space over $\mathcal{F}$ of meromorphic $\mathcal{V}$-valued functions $f \in \mathcal{M} \otimes \mathcal{C} \mathcal{V}$ satisfying the bispectral quantum KZ equations \cite{[35, 36]}.

Note that

\begin{align}
(\nabla(\tau(\nu), \tau(\lambda))f)(z, \xi) & = C_{(\tau(\nu), \tau(\lambda))}(z, \xi)f(z - \nu, \xi - \lambda), \quad \nu \in \bar{\Lambda}, \ \lambda \in \Lambda,
\end{align}

hence the bispectral quantum KZ equations form a compatible system of linear difference equations (an integrable difference connection). The solution space $\text{Sol}_{KZ}$
of the bispectral quantum KZ equations is a \( W_0 \times W_0 \)-invariant complex linear subspace of \( M \otimes \mathcal{V} \) with respect to the action \( \nabla|_{W_0 \times W_0} \) of \( W_0 \times W_0 \) on \( M \otimes \mathcal{V} \).

Note that the coefficients \( C_{\tau(\nu),\tau(\lambda)}(z,\xi) \) are in fact rational functions in \((q^z, q^\xi) \in \mathbb{T} := \text{Hom}(\Lambda \times \Lambda^*, \mathbb{C}^*)\),

where we interpret \( q^z \in \text{Hom}(\Lambda, \mathbb{C}^*) \) and \( q^\xi \in \text{Hom}(\Lambda^*, \mathbb{C}^*) \) as \( \lambda \mapsto q^{(\lambda, z)} \) (\( \lambda \in \Lambda \)) and \( \nu \mapsto q^{(\nu, \xi)} \) (\( \nu \in \Lambda^* \)) respectively. Hence the bispectral quantum KZ equations, restricted to meromorphic \( \mathcal{V} \)-valued functions on \( \mathbb{T} \), form a compatible system of \( q \)-difference equations (an integrable \( q \)-connection). It is in this form that quantum KZ type equations usually appear, see, e.g., [23] and references therein.

Remark 3.3 (Duality symmetry). Let \( j : M \otimes \mathcal{V} \to M \otimes \mathcal{V} \) be the complex linear map defined by

\[
j(f, \nu) := f(\nu, \cdot)
\]

for \( f \in M \) and \( \nu \in W_0 \), where \( f(z, \xi) := f(\xi, z) \). Set \( \nabla = \nabla(\hat{D}, \hat{\kappa}, q) \). Then

\[
j(\nabla(w, w)) = \nabla(\hat{w}, \hat{w}) \circ j \quad \forall (w, \hat{w}) \in W \times \hat{W}.
\]

In particular, \( j \) restricts to a complex linear isomorphism \( \text{Sol}_{KZ}(D, \kappa, q) \cong \text{Sol}_{KZ}(\hat{D}, \hat{\kappa}, q) \).

3.2. Asymptotically free solutions. For the \( \text{GL}_{n+1} \) root system datum, asymptotically free solutions of quantum KZ equations have been constructed using correlation functions for quantum affine algebras in [23], see also [18, §10]. In the present context we establish the existence of asymptotically free solutions using classical asymptotic methods going back to Birkhoff [5] (see [44] Appendix for a detailed discussion of this approach that fits the present context).

Repeating the arguments of [44, §10] one obtains the following asymptotically free solution of the bispectral quantum KZ equations. For \( \epsilon > 0 \) set

\[
B_\epsilon := \{ (z, \xi) \in E_\mathcal{V} \times E_\mathcal{V} \mid |q^{-\alpha_i(z)}|, |q^{-\alpha_i(\xi)}| < \epsilon \quad \forall i \in \{1, \ldots, n\} \}.
\]

Theorem 3.4. There exists a unique \( \Phi_{KZ}(\cdot, \cdot) = \Phi_{KZ}(\cdot, \cdot, D, \kappa, q) \in M \otimes \mathcal{V} \) such that

1. \( \Phi_{KZ} \in \text{Sol}_{KZ} \),
2. for some \( \epsilon > 0 \),

\[
\Phi_{KZ}(z, \xi) = W(z, \xi) \sum_{(\alpha, \beta) \in Q^+ \times \hat{Q}^+} \gamma_{(\alpha, \beta)} q^{-\alpha(z) - \beta(\xi)}
\]

for \( (z, \xi) \in B_\epsilon \), with the \( \mathcal{V} \)-valued sum \( \sum_{(\alpha, \beta) \in Q^+ \times \hat{Q}^+} \gamma_{(\alpha, \beta)} q^{-\alpha(z) - \beta(\xi)} \) converging normally on compacta of \( (z, \xi) \in B_\epsilon \),
3. \( \gamma_{(0, 0)} = v_{w_0} \).

Proof. Compared to the proofs in [44 Thm. 5.3] and [43 Thm. 5.4] an extra argument is needed to take care of the extra flexibility in the choice of lattices \( \Lambda \) and \( \Lambda^* \).

Since \( Q \subseteq \Lambda \) and \( \hat{Q} \subseteq \Lambda^* \) there exist sublattices

\[
M := \bigoplus_{i=1}^{n} \mathbb{Z}w_i \subseteq \Lambda, \quad \tilde{M} := \bigoplus_{i=1}^{n} \mathbb{Z}\tilde{w}_i \subseteq \tilde{\Lambda}
\]
with \( \varpi_i \) and \( \tilde{\varpi}_i \) satisfying \( (\varpi_i, \alpha_i^\gamma) \in \delta_{ij} \mathbb{Z}_{>0} \) and \( (\tilde{\varpi}_i, \tilde{\alpha}_i^\gamma) \in \delta_{ij} \mathbb{Z}_{>0} \) for \( i, j \in \{1, \ldots, n\} \). The arguments in [44 Thm. 5.3] and [43 Thm. 5.4] now lead to the proof of the existence and uniqueness of a meromorphic \( \mathcal{V} \)-valued series expansion \( \Phi_{KZ}(\cdot, \cdot) \) satisfying (2), (3) and satisfying the compatible system

\[
(3.6) \quad \nabla(\tau(\lambda), \tau(\nu))\Phi_{KZ} = \Phi_{KZ} \quad \forall (\lambda, \nu) \in M \times \tilde{M}
\]
of difference equations. Fix \( (\lambda', \nu') \in \Lambda \times \tilde{\Lambda} \) and set \( \Phi'_{KZ} := \nabla(\tau(\lambda'), \tau(\nu'))\Phi_{KZ} \). By the integrability of the bispectral quantum KZ equations it follows that \( \Phi'_{KZ} \) satisfies (3.6). Since \( \Phi'_{KZ} \) also satisfies properties (2) and (3) we conclude that \( \Phi'_{KZ} = \Phi_{KZ} \). Hence \( \Phi_{KZ} \in \text{Sol}_{KZ} \).

It is now possible to establish various properties of \( \Phi_{KZ} \) (duality, singularities) by a detailed analysis of the bispectral quantum KZ equations. It leads to the following result.

**Proposition 3.5.** (i) \( \Phi_{KZ} \) is selfdual,

\[
\Phi_{KZ}(z, \xi; D, \kappa, q) = \Phi_{KZ}(\xi, z; \tilde{D}, \tilde{\kappa}, q).
\]

(ii) The \( \mathcal{V} \)-valued meromorphic function

\[
\Psi_{KZ}(z, \xi) := \frac{S(z)\tilde{S}(\xi)}{W(z, \xi)} \Phi_{KZ}(z, \xi)
\]

has a \( \mathcal{V} \)-valued series expansion

\[
\Psi_{KZ}(z, \xi) = \sum_{(\alpha, \beta) \in Q^+ \times \tilde{Q}^+} \Gamma_{(\alpha, \beta)}^{KZ} q^{-\alpha(z)-\beta(\xi)},
\]

normally convergent for \( (z, \xi) \) in compacta of \( E_{\mathbb{C}} \times E_{\mathbb{C}} \). In particular, \( \Psi_{KZ}(z, \xi) \) is holomorphic in \( (z, \xi) \in \overline{E_{\mathbb{C}}} \times E_{\mathbb{C}} \) and \( \Gamma_{(0,0)}^{KZ} = Y_{(0,0)} = v_{w_0} \).

(iii) Define for \( \alpha \in Q^+ \) the \( \mathcal{V} \)-valued holomorphic function \( \Gamma_{\alpha}^{KZ}(\xi) \) in \( \xi \in E_{\mathbb{C}} \) by

\[
\Gamma_{\alpha}^{KZ}(\xi) := \sum_{\beta \in \tilde{Q}^+} \Gamma_{(\alpha, \beta)}^{KZ} q^{-\beta(\xi)},
\]

so that \( \Psi_{KZ}(z, \xi) = \sum_{\alpha \in Q^+} \Gamma_{\alpha}^{KZ}(\xi) q^{-\alpha(z)} \). Then

\[
\Gamma_0^{KZ}(\xi) = \prod_{\alpha \in \tilde{R}_0^+} \left(q_0^2 q^{-2\tilde{\alpha}(\xi)}; q_0^2\right)_\infty v_{w_0}.
\]

(iv) The bispectral quantum KZ equations are consistent,

\[
\text{dim}_\mathbb{R}(\text{Sol}_{KZ}) = \text{dim}_\mathbb{C}(\mathcal{V}) = \#W_0.
\]

Furthermore, \( \{\nabla(e, \sigma)\Phi_{KZ}\}_{e \in W_0} \) is a \( \mathcal{F} \)-basis of \( \text{Sol}_{KZ} \).

**Proof.** The proofs for (i), (ii) and (iv) as given in [43 43 53] for the twisted case \( \bullet = t \) with \( \tilde{\Lambda} = \Lambda \) generalize easily to the present context (for (i) use Remark 3.3). (iii) Similarly as in [43 43] for the twisted equal lattice case, the asymptotics as \( q^{-\alpha_i(z)} \to 0 \) for \( i = 1, \ldots, n \) shows that \( \Gamma_0^{KZ}(\xi) = \tilde{S}(\xi)K(\xi)v_{w_0} \) for a unique scalar valued meromorphic function \( K(\xi) \) in \( \xi \in E_{\mathbb{C}} \) having a convergent power series expansion

\[
K(\xi) = \sum_{\beta \in \tilde{Q}^+} k_\beta q^{-\beta(\xi)}, \quad k_0 = 1
\]

if \( |q^{-\tilde{\alpha}(\xi)}| \) is sufficiently small for all \( i \in \{1, \ldots, n\} \).
In the twisted equal lattice case \[14\] 143 59, \( K(\xi) \) was explicitly determined using the difference Cherednik-Matsuo correspondence, which puts the problem in the context of the bispectral problem of the (higher order) RMKC operators. We give here a new proof, which stays completely in the realm of the bispectral problem of the (higher order) RMKC operators.

We characterize \( K(\xi) \) as a formal power series in \( q^{-\tilde{\alpha}_1(\xi)}, \ldots, q^{-\tilde{\alpha}_n(\xi)} \) with constant coefficient 1 and solving an explicit system of difference equations in \( \xi \). To derive the difference equations, consider for \( \lambda \in \Lambda^+ \) the meromorphic function

\[
R_\lambda(z, \xi) := q^{\tilde{\rho} + \omega_0 z + \lambda} \left( C_{(\epsilon, \tau(\lambda))}(z, \xi) v_{\omega_0} \right)|_{v_{\omega_0}},
\]

where \( v|_{\omega_0} \) for \( v \in \mathcal{V} \) means picking the \( \omega_0 \)-component in the expansion of \( v \) as linear combination of the basis elements \( v_\sigma \) (\( \sigma \in W_0 \)) of \( \mathcal{V} \). Then

\[
R_\lambda(z, \xi) = \sum_{\alpha \in Q^+} q^{-\alpha(z)} r_{\lambda}(\alpha)(\xi)
\]

(finite sum) with \( r_{\lambda}(\alpha) \in \mathbb{C}[q^{-\tilde{\alpha}_1}, \ldots, q^{-\tilde{\alpha}_n}] \), which follows from the extension of [143 Lem. 5.3] to the present setup. As limit of the bispectral quantum KZ equations for \( \Phi_{\text{KZ}}(z, \xi) \) it follows that \( K(\xi) \) satisfies the difference equations

\[
 r_{\lambda}(0) K(\xi - \lambda) = K(\xi) \quad \forall \lambda \in \Lambda^+,
\]

which characterize \( K(\xi) \) as formal power series in the \( q^{-\tilde{\alpha}_i(\xi)} \) with constant coefficient 1.

Choosing a reduced expression of \( \tau(\lambda) \in \tilde{W} \) and using the cocycle condition (3.1) allows one to give an explicit expression of \( C_{(\epsilon, \tau(\lambda))}(z, \xi) \), from which it follows that

\[
r_{\lambda}(0)(\xi) = \prod_{a \in \tilde{R}^* \cap \tau(\lambda) \tilde{R}^*} c_a(\xi; -\tilde{\kappa}, q)^{-1}
\]

for \( \lambda \in \Lambda^+ \). Consequently

\[
K(\xi) = \prod_{a \in \tilde{R}^0, \tau \in \mathbb{Z}_{>0}} c_{\tilde{\alpha}(\tau)}(\xi; -\tilde{\kappa}, q)^{-1}
\]

\[
= S(\xi)^{-1} \prod_{a \in \tilde{R}^0} \left( q_{a_0}^2 q^{-2\tilde{\alpha}(\xi)}; q_{a_0}^2 \right)_\infty.
\]

3.3. Ruijsenaars-Macdonald-Koornwinder-Cherednik operators. We follow Cherednik’s [143] construction of higher order Ruijsenaars-Macdonald-Koornwinder-Cherednik (RMKC) operators, see also [111, 60]. For the precise definition of the affine Hecke algebra in the present context, we refer to [60] §2.4.

Let \( \tilde{w} \) be the contragredient action of \( w \in W \) on meromorphic functions on \( E_\mathbb{C} \),

\[
(\tilde{w} f)(z) := f(w^{-1} z).
\]

For \( i \in \{0, \ldots, n\} \) the Demazure-Lusztig type difference reflection operators

\[
(3.7) \quad \tilde{T}_i := q^{a_i} + q^{-a_i} c_i(\cdot; \kappa, q)(\tilde{\kappa}_i - \text{id})
\]

define a representation of the affine Hecke algebra \( H(W^\bullet; q^e) \) on the space of meromorphic functions on \( E_\mathbb{C} \), where \( q^e \) stands for the Hecke parameters \( q^{a_i} \).
(i = 0, . . . , n). The corresponding Hecke relation is
\[(\hat{T}_i - q^\kappa)(\hat{T}_i + q^{-\kappa}) = 0.\]
Recall that \(W \simeq \Omega \ltimes W^\bullet\) with \(\Omega \subset W\) the subgroup consisting of extended affine Weyl group elements of length zero. Then \([67, 20\text{.}4]\) and the operators \(\hat{u}(u \in \Omega)\) provide a representation of the extended affine Hecke algebra \(H(W; q^\alpha) \simeq \Omega \ltimes H(W^\bullet; q^\alpha)\), cf. \([60, \S 2.4]\).

Fix \(\nu \in \Lambda^+\) and suppose that \(\tau(\nu) = s_{i_1} \cdots s_{i_r} u \in W\) is a reduced expression \((0 \leq i_j \leq n, u \in \Omega)\). The associated operator
\[\hat{Y}^\nu := \hat{T}_{i_1} \cdots \hat{T}_{i_r} \hat{u}\]
is well defined and invertible. For arbitrary weight \(\nu \in \tilde{\Lambda}\) the Bernstein-Zelevinsky operator is defined as
\[\hat{Y}^\nu := \hat{Y}^\nu_1 (\hat{Y}^\nu_2)^{-1},\]
where the \(\nu_i \in \tilde{\Lambda}^+\) are such that \(\nu = \nu_1 - \nu_2\). The operators \(\hat{Y}^\nu (\nu \in \tilde{\Lambda})\) are well defined and mutually commute.

Fix \(\nu \in \tilde{\Lambda}^+\). There exists unique difference operators \(L_{\nu, \sigma}(\sigma \in W_0)\) such that
\[\sum_{\nu' \in W_0 \nu} \hat{Y}^\nu' = \sum_{\sigma \in W_0} L_{\nu, \sigma} \hat{\sigma}.\]

**Definition/Theorem 3.6 ([13])**. The difference operators
\[L_\nu := \sum_{\nu \in W_0} L_{\nu, \sigma}, \quad \nu \in \tilde{\Lambda}^+\]
are the higher order RMKC operators associated to the initial datum \((D, \kappa, q)\). They are \(W_0\)-equivariant and mutually commute.

The difference operators \(L_\nu\) can be made entirely explicit for miniscule dominant weights \(\nu \in \tilde{\Lambda}^+_{\text{min}}\) and for the quasi-miniscule dominant weight \(\nu = \tilde{\psi}\), see, e.g., \([13, 11, 60]\). We give here only the explicit formula for \(\nu = \tilde{\psi}\),
\[L_{\tilde{\psi}} f(z) = q^{-(\rho, \tilde{\psi})} \sum_{w \in W_0/W_0, \psi} c_{\tau(\tilde{\psi})}(w^{-1} z; \kappa, q)(f(z + w\tilde{\psi}) - f(z)) + \left(\sum_{w \in W_0/W_0, \psi} q^{-(\rho, w\tilde{\psi})}\right)f(z),\]
where for \(w \in W\),
\[c_w(z; \kappa, q) := \prod_{a \in R^\bullet_+ \cap w^{-1} R^\bullet_-} c_a(z; \kappa, q).\]
Recall the explicit difference operator \(L\) from Subsection \([12]\).

**Lemma 3.7.** \(L_{\tilde{\psi}} = L\).

**Proof.** It suffices to show that \(c_{\tau(\tilde{\psi})}(z; \kappa, q) = A(z)\), with \(A(z)\) the trigonometric function \([13]\). This follows from the fact that
\[R^\bullet_+ \cap \tau(-\tilde{\psi}) R^\bullet_- = \{\psi, \psi^{(1)}\} \cup \{\alpha \in R^\bullet_0 \mid (\tilde{\psi}, \alpha^\vee) = 1\}.\]

\[\square\]
3.4. The bispectral problem for RMKC operators. We denote the higher order RMKC operators with respect to the dual initial datum \((\bar{D}, \bar{\kappa}, q)\) by \(\bar{\Lambda}_\lambda (\lambda \in \Lambda^+).\) Note that the RMKC operator \(\bar{L} := \bar{L}_\theta\) is explicitly given by

\[
\bar{L} = q^{-\bar{e}(\rho, \theta)} \sum_{w \in W_0/W_0, \nu} \bar{A}(w^{-1} z)(f(z + w\nu) - f(z)) + \left( \sum_{w \in W_0/W_0, \nu} q^{-\bar{e}(\rho, w\nu)} \right) f(z),
\]

with

\[
\bar{A}(z) = \frac{(1 - a_\theta q^{\tilde{\nu}(z)})(1 - b_\theta q^{\tilde{\nu}(z)})(1 - c_\theta q^{\tilde{\nu}(z)})(1 - d_\theta q^{\tilde{\nu}(z)})}{(1 - q^{2\theta}(z))(1 - q^{2\theta}(z))} \prod_{\alpha \in R_0^+ : (\theta, \alpha^\vee) = 1} \frac{(1 - \bar{a}_\alpha q^{\tilde{\nu}(z)})(1 - \bar{b}_\alpha q^{\tilde{\nu}(z)})}{(1 - q^{2\theta}(z))}
\]

(here we use that \(\bar{a}_\alpha = a_{\bar{\alpha}}(\bar{D}, \bar{\kappa}, q), \ldots, \bar{d}_\alpha = d_{\bar{\alpha}}(\bar{D}, \bar{\kappa}, q)\).

**Definition 3.8.** The system

\[
L_\nu f(:, \xi) = \left( \sum_{w \in W_0/W_0, \nu} q^{(w, \nu, \xi)} \right) f(:, \xi), \quad \nu \in \bar{\Lambda}^+,
\]

\[
\bar{L}_\lambda f(z, :) = \left( \sum_{w \in W_0/W_0, \lambda} q^{(w, \lambda, z)} \right) f(z, :), \quad \lambda \in \Lambda^+
\]

of difference equations for \(f \in \mathcal{M}\) is called the bispectral problem of the RMKC operators. We write \(\text{Sol}_{RMKC}\) for the vector space over \(\mathcal{F}\) consisting of \(f \in \mathcal{M}\) satisfying (3.8).

Since the (higher order) RMKC operators are \(W_0\)-equivariant, \(\text{Sol}_{RMKC} \subset \mathcal{M}\) is \(W_0 \times W_0\)-invariant with respect to the contragredient action of \(W_0 \times W_0\) on \(\mathcal{M}\).

3.5. The difference Cherednik-Matsuo correspondence. Recall that \(\mathcal{M} \otimes \mathbb{C} \mathcal{V}\) is a left \(W_0 \times W_0\)-module with respect to the \(\nabla\)-action [44]. Let \(\chi : \mathcal{M} \otimes \mathbb{C} \mathcal{V} \rightarrow \mathcal{M}\) be the \(\mathcal{M}\)-linear \(W_0 \times W_0\)-equivariant map

\[
\chi \left( \sum_{w \in W_0} f_w \otimes v_w \right) := q^{\kappa_w} \sum_{w \in W_0} q^{-\kappa_w} f_w,
\]

where \(\kappa_w := \sum_{\alpha \in R^+_0 \cap w^{-1} R^-_0} \kappa_{\alpha}\) for \(w \in W_0\).

**Theorem 3.9.** The map \(\chi\) restricts to an injective, \(\mathcal{F}\)-linear, \(W_0 \times W_0\)-equivariant map

\[
\chi : \text{Sol}_{KZ} \rightarrow \text{Sol}_{RMKC}.
\]

**Proof.** The proofs in [44] [45], extending the results of Cherednik [10] to the bispectral setting, generalize easily to the present setting. In particular, the injectivity is proved by showing that the meromorphic functions \(\chi(\nabla(e, w)\Phi_{KZ})\) \((w \in W_0)\) are \(\mathcal{F}\)-linear independent.


**Definition 3.10.** The basic Harish-Chandra series \(\Phi(:, :) = \Phi(:, , :; D, \kappa, q)\) is defined by

\[
\Phi := \chi(\Phi_{KZ}) \in \text{Sol}_{RMKC}.
\]
We are now in the position to prove all the fundamental properties of the basic Harish-Chandra series as stated in Section 1.

**Proof of Theorem 1.1.** The results on the asymptotically free solution \( \Phi_{KZ} \) of the bispectral quantum KZ equations from Subsection 3.2 show that the basic Harish-Chandra series \( \Phi \) satisfies all the properties as stated in Theorem 1.1 with \( \Gamma_\alpha(\xi) = \chi(\Gamma_{KZ}^\alpha(\xi)) (\alpha \in Q^+) \). In particular, the eigenvalue equation (1.6) for \( \Phi \) follows from the fact that \( \Phi \) solves the bispectral problem (3.8) of the RMKC operators since \( L = L^\wedge \). It thus suffices to prove the uniqueness claim.

This follows from the results in [12, §3] (untwisted case) and [39, §2] on the analog of the Harish-Chandra homomorphism and from the subsequent formal analysis of the basic Harish-Chandra series in [39, §4]. These results show that the eigenvalue equations (1.6) determine the expansion coefficients \( \Gamma_\alpha(\xi) (\alpha \in Q^+ \setminus \{0\}) \) uniquely in terms of \( \Gamma_0 \).

**Proof of Theorem 1.3.** The selfduality of the basic Harish-Chandra series \( \Phi \) follows immediately from the selfduality of \( \Phi_{KZ} \) (see Proposition 3.5(iv)).

**Proof of Theorem 1.4.** By Proposition 3.5(iv) there exists unique \( \Phi(\sigma_1, \tau_2 \in W_0) \) such that

\[
\nabla(\sigma, \tau_2)\Phi_{KZ} = \sum_{\tau_1 \in W_0} m_{\tau_1, \tau_2}(\nabla(e, \tau_1)\Phi_{KZ})
\]

for all \( \sigma, \tau_2 \in W_0 \). Applying the injective, \( W_0 \times W_0 \)-equivariant Cherednik map \( \chi \) shows that

\[
(3.9) \quad \Phi(\sigma^{-1}z, \tau_2^{-1}z) = \sum_{\tau_1 \in W_0} m_{\sigma, \tau_1, \tau_2}(z, \xi)\Phi(z, \tau_1^{-1}z)
\]
as meromorphic functions in \((z, \xi) \in E_C \times E_C\). By the injectivity of \( \chi|_{\text{Sol}_{KZ}} \) it follows that the equations (3.9) determine the \( m_{\sigma, \tau_1, \tau_2} \in F \) uniquely.

**Proof of Proposition 1.13.** Theorem 1.1 implies that

\[
(3.10) \quad \Phi(z, \xi) = q^{-q_{w_0}(z, \xi)} \sum_{\alpha \in Q^+} \tilde{\Gamma}_\alpha(\xi)q^{-q_{-\alpha}(z, \xi)}
\]
for generic \( \xi \in E_C \) if \( \mathbb{R}(z) \) is sufficiently deep in the negative fundamental Weyl chamber \( E_- \), with leading coefficient

\[
\tilde{\Gamma}_0(\xi) = q^{2\beta(\rho, 0, \xi)} \frac{\Gamma_0(\xi)}{S(\xi)}
\]

Such (formal) power series solutions of the spectral problem of the RMKC operators are unique up to normalization (see [39, Thm. 4.4]). These two observations are valid without any restrictions on the initial datum \((D, \kappa, q)\). Under the assumptions on \((D, \kappa, q)\) as stated in the proposition, comparison with the series expansion of the multivariable Baker-Akhiezer function \( \psi(-w_0z, \cdot) \) from [7, Rem. 3.6] shows that \( \Phi(z, \xi) = \text{cst}(\xi)\psi(-w_0z, \xi) \). A straightforward computation proves that the leading
coefficient \( \hat{\Delta}_0(\xi) \) of the power series expansion (3.11) of \( \Phi(\cdot, \xi) \) coincides with the leading coefficient \( \Delta'(\xi) = \Delta'(-w_0\xi, \cdot) \) of \( \psi(-w_0\xi, \cdot) \) up to a nonzero multiplicative constant (see [7, §2.1.4] for the definition of \( \Delta'(\xi) \)). This shows that \( \text{cst}(\xi) \) is independent of \( \xi \).

Remark 3.11. In the proof of Proposition [1.13] we could have used the selfduality of \( \Phi \) (Theorem [1.3]) and \( \psi \) ([7, Thm. 3.3(iii)]) to immediately conclude that \( \text{cst}(\xi) \) is independent of \( \xi \). The current proof has the advantage that the selfduality of the normalized multivariable Baker-Akhiezer function \( \psi \) becomes a consequence of the selfduality of \( \Phi \).

4. The Connection Cocycle

In this section we prove the explicit expressions for the connection coefficients as stated in Theorem [1.5] using rank reduction. The strategy is as follows.

Fix \( i \in \{1, \ldots, n\} \). Let \( i^* \in \{1, \ldots, n\} \) be the corresponding index such that \( -w_0\alpha_{i^*} = \alpha_i \). Let \( \delta_i \in \Lambda \) be a weight such that \( (\delta_i, \alpha_i^\vee) = 0 \) and \( (\delta_i, \alpha_j^\vee) > 0 \) if \( j \neq i \) (one can take for instance \( \delta_i = \sum_{j \neq i} \rho_j \) with \( \rho_j \in \Lambda \) as in the proof of Theorem [3.4]).

Recall the holomorphic function

\[
\Psi(z, \xi) = \sum_{\alpha \in Q^+} \Gamma_{\alpha}(\xi) q^{-\alpha(z)}
\]

from Theorem [1.1] such that

\[
\Phi(z, \xi) = \frac{\mathcal{W}(z, \xi)}{S(z)S(\xi)} \Psi(z, \xi).
\]

Define the holomorphic function \( S_i(z) \) in \( z \in \mathbb{C} \) by

\[
S_i(z) := (q^2a_i^{-1}q^{-x}, q^2b_i^{-1}q^{-y}, q^2c_i^{-1}q^{-x}, q^2d_i^{-1}q^{-y}; q^2)_\infty
\]

and the holomorphic function \( \Psi_i(x, \xi) \) in \( (x, \xi) \in \mathbb{C} \times E_\mathbb{C} \) by

\[
\Psi_i(x, \xi) := \sum_{r=0}^{\infty} \Gamma_{r\alpha_i}(\xi) q^{-rx}.
\]

Then

\[
\lim_{m \to \infty} S(z - m\delta_i) = S_i(\alpha_i(z)),
\]

\[
\lim_{m \to \infty} \Psi(z - m\delta_i, \xi) = \Psi_i(\alpha_i(z), \xi),
\]

uniformly on compacta. We define now the meromorphic function \( \Phi_i(x, \xi) \) in \( (x, \xi) \in \mathbb{C} \times E_\mathbb{C} \) by

\[
\Phi_i(x, \xi) := \frac{\mathcal{W}_i(x, \alpha_i, (\xi))}{S_i(x)S(\xi)} \Psi_i(x, \xi)
\]

with the one variable plane wave

\[
\mathcal{W}_i(x, y) := q^{\frac{1}{2}(\kappa_i+\kappa_{2\alpha_i}-(x+\kappa_{i+1})-y)}
\]

cf. [61] (2.1). We will prove that \( \Phi_1(\cdot, \xi) \) is the asymptotically free solution of the Askey-Wilson [1] second order difference operator, with associated AW parameters given by \( (a_i, b_i, c_i, d_i) \). This allows us to compute the connection coefficients using results from the classical theory [24] on basic hypergeometric series.
4.1. **Vanishing connection coefficients.** We first show that most of the connection coefficients are zero.

**Proposition 4.1.** Let $\tau_1, \tau_2 \in W_0$. Then $m_{\tau_1, \tau_2}^\sigma \equiv 0$ if $\tau_1 \not\in \{\tau_2, \tau_2s_1\}$.

**Proof.** Fix $\sigma \not\in \{e, s_1\}$. Since $m_{\tau_1, \tau_2}^\sigma(z, \xi) = m_{\tau_2^{-1}, \tau_1, e}^\sigma(z, \tau_2^{-1}\xi)$ it suffices to show that $m_{\tau_1, e}^\sigma \equiv 0$.

Rewriting the identity

$$\Phi(s_iz, \xi) = \sum_{\tau \in W_0} m_{\tau, e}^\sigma(z, \xi)\Phi(z, \tau^{-1}\xi)$$

using (4.1) gives

$$\Psi(s_iz, \xi) = \frac{S(siz)\tilde{S}(\xi)}{S(z)} \sum_{\tau \in W_0} m_{\tau, e}^\sigma(z, \xi)W(z, \tau^{-1}\xi)\Psi(z, \tau^{-1}\xi).$$

In (4.2) we replace $z$ by $z - m\tilde{\delta}_i$ and multiply the resulting identity by $q^{m(w_0\tilde{\delta}_i, -\sigma^{-1}\xi)} \frac{S(z - m\tilde{\delta}_i)}{S(s_iz - m\tilde{\delta}_i)}$.

It gives

$$\frac{S(z - m\tilde{\delta}_i)}{S(s_iz - m\tilde{\delta}_i)} q^{m(w_0\tilde{\delta}_i, -\sigma^{-1}\xi)} \Psi(s_iz - m\tilde{\delta}_i, \xi) =$$

$$= \frac{\tilde{S}(\xi)}{W(s_iz, \xi)} \sum_{\tau \in W_0} m_{\tau, e}^\sigma(z, \xi)W(z, \tau^{-1}\xi)q^{m(w_0\tilde{\delta}_i, \tau^{-1}\xi -\sigma^{-1}\xi)}\Psi(z - m\tilde{\delta}_i, \tau^{-1}\xi)$$

since $m_{\tau, e}^\sigma$ is $\bar{\Lambda} \times \Lambda$-translation invariant and

$$\frac{W(z - m\tilde{\delta}_i, \tau^{-1}\xi)}{W(s_iz - m\tilde{\delta}_i, \xi)} = q^{m(w_0\tilde{\delta}_i, \tau^{-1}\xi -\sigma^{-1}\xi)} \frac{W(z, \tau^{-1}\xi)}{W(s_iz, \xi)}.$$

Set

$$E_c^+ := \{\xi \in E_c \mid \Re(\alpha_i(\xi)) > 0 \quad \forall i\}$$

and fix generic $(z, \xi) \in E_c \times \sigma E_c^+$. Taking the limit $m \to \infty$ in (4.3) then gives

$$\sum_{\tau \in \{\sigma, \sigma s_1, \sigma s_2\}} m_{\tau, e}^\sigma(z, \xi)W(z, \tau^{-1}\xi)\frac{\Psi_i(\alpha_i(z), \tau^{-1}\xi)}{S(\tau^{-1}\xi)} = 0.$$ 

Let $\tilde{\omega}_i \in \bar{\Lambda}$ such that $(\tilde{\omega}_i, \tilde{\omega}_i^\ast) \in \delta_i, j\mathbb{Z}_{>0}$. Replace $z$ by $z - m\tilde{\omega}_i$ in (4.4) and multiply both sides of the identity by $q^{-m(\rho+w_0\sigma^{-1}\xi, \tilde{\omega}_i)}$. Then

$$\sum_{\tau \in \{\sigma, \sigma s_1, \sigma s_2\}} m_{\tau, e}^\sigma(z, \xi)W(z, \tau^{-1}\xi)q^{m(w_0(\tau^{-1}\xi -\sigma^{-1}\xi), \tilde{\omega}_i)}\Psi_i(\alpha_i(z) - m(\tilde{\omega}_i, \alpha_i), \tau^{-1}\xi) = 0.$$

Taking the limit $m \to \infty$ we get

$$m_{\tau, e}^\sigma(z, \xi)W(z, \sigma^{-1}\xi)\frac{\Gamma_0(\sigma^{-1}\xi)}{S(\sigma^{-1}\xi)} = 0.$$

Hence $m_{\tau, e}^\sigma(z, \xi) = 0$ for generic $(z, \xi) \in E_c \times \sigma E_c^+$. Since $m_{\tau, e}^\sigma$ is $\bar{\Lambda} \times \Lambda$-translation invariant, we get $m_{\tau, e}^\sigma \equiv 0$. □
4.2. Rank reduction. The aim is to compute \( \Phi_i(x, \xi) \) explicitly in terms of basic hypergeometric series. The following proposition is fundamental.

**Proposition 4.2.** (i) We have

\[
\Phi_i(-\alpha_i(z), \xi) = m_{c,z}^{s_i}(z, \xi)\Phi_i(\alpha_i(z), \xi) + m_{s_i,c,e}^{s_i}(z, \xi)\Phi_i(\alpha_i(z), s_i, \xi)
\]

as meromorphic functions in \((z, \xi) \in E_C \times E_C\).

(ii) We have the eigenvalue equations

\[
(M_i \Phi_i)(\cdot, \xi) = (q^{\tilde{\alpha}^\vee(s_i)}(\xi) + q^{-\tilde{\alpha}^\vee(s_i)}(\xi) - \tilde{a}_i - \tilde{a}_i^{-1})\Phi_i(\cdot, \xi)
\]

as meromorphic functions in \((\cdot, \xi) \in \mathbb{C} \times E_C\), with \( M_i \) the Askey-Wilson \([4] \) second order difference operator

\[
(M_i g)(x) := A_i(x)(g(x - 2\mu_i) - g(x)) + A_i(-x)(g(x + 2\mu_i) - g(x)),
\]

\[
A_i(x) := \frac{(1 - a_i q^{-x})(1 - b_i q^{-x})(1 - c_i q^{-x})(1 - d_i q^{-x})}{\tilde{a}_i(1 - q^{-2x})(1 - q^2 q^{-2x})}.
\]

**Proof.** (i) By Proposition 4.1,

\[
\Phi(s_i z, \xi) = m_{c,z}^{s_i}(z, \xi)\Phi(z, \xi) + m_{s_i,c,e}^{s_i}(z, \xi)\Phi(z, s_i, \xi).
\]

The result now follows by multiplying both sides of this identity by

\[
W_i(-\alpha_i(z), \tilde{\alpha}^\vee(\xi))
\]

replacing \( z \) by \( z - m_i \) and taking the limit \( m \to \infty \), using that the connection coefficients are \( \Lambda \times \Lambda \)-translation invariant and that

\[
\frac{W_i(s_i z, \xi)}{W_i(z, \xi)} = \frac{W_i(-\alpha_i(z), \tilde{\alpha}^\vee(\xi))}{W_i(\alpha_i(z), \tilde{\alpha}^\vee(\xi))}, \quad \frac{W_i(s_i z, \xi)}{W_i(z, s_i, \xi)} = \frac{W_i(-\alpha_i(z), \tilde{\alpha}^\vee(\xi))}{W_i(\alpha_i(z), -\tilde{\alpha}^\vee(\xi))}.
\]

(ii) Define a second difference operator \( N_i \) by

\[
(N_i g)(x) := B_i(x)g(x - \mu_i) + B_i(-x)g(y + \mu_i),
\]

\[
B_i(x) := \frac{(1 - a_i q^{-x})(1 - b_i q^{-x})}{q^{\alpha_i}(1 - q^{-2x})}.
\]

From (the proof of) \([39] \) Prop. 3.13 we obtain the following result.

**Case a.** If \( \alpha_i \notin W_0 \) then \((\tilde{Q}, \tilde{\alpha}^\vee_0) = \mathbb{Z} \) and

\[
N_i \Phi_i(\cdot, \xi) = (q^{\tilde{\alpha}^\vee(s_i)}(\xi) + q^{-\tilde{\alpha}^\vee(s_i)}(\xi)/2)\Phi_i(\cdot, \xi)
\]

as meromorphic functions in \((\cdot, \xi) \in \mathbb{C} \times E_C\).

**Case b.** If \( \alpha_i \in W_0 \) and \((\tilde{Q}, \tilde{\alpha}^\vee_0) = \mathbb{Z} \) then

\[
(N_i + q^{\tilde{\alpha}^\vee(s_i)}(\xi)/2 + q^{-\tilde{\alpha}^\vee(s_i)}(\xi)/2 + \text{cst}_i)\left(N_i - q^{\tilde{\alpha}^\vee(s_i)}(\xi)/2 - q^{-\tilde{\alpha}^\vee(s_i)}(\xi)\right)\Phi_i(\cdot, \xi) = 0
\]

as meromorphic functions in \( \mathbb{C} \times E_C \), where

\[
\text{cst}_i := q^{\tilde{\alpha}^\vee(s_i)}(\xi)/2 \sum_{\sigma} q^{-\hat{\sigma}(\sigma^{-1} w_0 \xi)}
\]

with the sum running over the \( \sigma \in W_0/W_0, \psi \) satisfying \((\sigma \psi, \tilde{\alpha}^\vee_0) = -1\).

**Case c.** If \( \alpha_i \in W_0 \) and \((\tilde{Q}, \tilde{\alpha}^\vee_0) = 2\mathbb{Z} \) then

\[
M_i \Phi_i(\cdot, \xi) = (q^{\tilde{\alpha}^\vee(s_i)}(\xi) + q^{-\tilde{\alpha}^\vee(s_i)}(\xi) - \tilde{a}_i - \tilde{a}_i^{-1})\Phi_i(\cdot, \xi)
\]

as meromorphic functions in \((\cdot, \xi) \in \mathbb{C} \times E_C\).
It thus suffices to show that in case a (resp. case b), (4.5) (resp. (4.7)) implies (4.3).

In both cases a and b we have $\kappa_{\alpha_i} = \kappa_{\alpha_i}$, $\kappa_{2\alpha_i} = \kappa_{2\alpha_i}$, since $(\tilde{A}, \tilde{a}) = Z$, hence $c_i = q_i \alpha_i$, $d_i = q_i \beta_i$ for the corresponding AW parameters. Consequently

$$M_i - q\tilde{a}_i^{\epsilon}(\xi) - q^{-\tilde{a}_i^{\epsilon}(\xi)} + \tilde{a}_i + \tilde{a}_i^{-1} = (N_i + q\tilde{a}_i^{\epsilon}(\xi)/2 + q^{-\tilde{a}_i^{\epsilon}(\xi)}(\xi/2)) (N_i - q\tilde{a}_i^{\epsilon}(\xi)/2 - q^{-\tilde{a}_i^{\epsilon}(\xi)}(\xi/2)), $$

see [61, §4]. This shows that (4.5) is correct for case a.

**Case b.** Fix generic $\xi \in \mathbb{C}$ and write

$$L_i := (N_i + q\tilde{a}_i^{\epsilon}(\xi)/2 + q^{-\tilde{a}_i^{\epsilon}(\xi)}(\xi/2) + c_{a_i}) (N_i - q\tilde{a}_i^{\epsilon}(\xi)/2 - q^{-\tilde{a}_i^{\epsilon}(\xi)}(\xi)), $$

so that $L_i \Phi_1(\cdot, \xi) = 0$. By [61, §5] there exists a unique meromorphic function $g(x)$ in $x \in \mathbb{C}$ satisfying

$$g(x) = \frac{W_i(x, \tilde{a}_i^{\epsilon}(\xi))}{S_i(x)S(x)} \sum_{r = 0}^{\infty} g_r q^{-rx} \quad (g_0 = \Gamma_0(\xi))$$

with the series converging normally on compacta of $x \in \mathbb{C}$, such that

$$N_i g = (q\tilde{a}_i^{\epsilon}(\xi)/2 + q^{-\tilde{a}_i^{\epsilon}(\xi)}(\xi)/2) g.$$ 

Then $g$ also satisfies $L_i g = 0$, and, together with the asymptotic properties (4.8) of $g$, this eigenvalue equation characterizes $g$. Hence $g = \Phi_1(\cdot, \xi)$. We conclude that $\Phi_1(\cdot, \xi)$ satisfies (4.5). As for case a, this implies that $\Phi_1(\cdot, \xi)$ also satisfies the desired eigenvalue equation (4.3). \[ \square \]

**Remark 4.3.** The factorization of the Askey-Wilson difference operator appearing in the above proof relates to quadratic transformation formulas for the associated eigenfunctions, see [61, §5]. In fact, the quadratic transformation formula [61, 5.1] guarantees the existence of the function $g$.

4.3. **Explicit expressions.** For generic $b_j$ the $r+1 \varphi_r$ basic hypergeometric series is defined by

$$r+1 \varphi_r \left( a_1, a_2, \ldots, a_{r+1}; q, z \right) := \sum_{j=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_j}{(q, b_1, \ldots, b_r; q)_j} z^j, \quad |z| < 1,$$

where $(a_1, \ldots, a_r; q)_j = \prod_{i=1}^{j} (1 - a_i + q)$ for $j \in \mathbb{Z}_{\geq 0} \cup \{ \infty \}$ (empty products are equal to one by convention). The very-well-poised $\,\phi_7$ series is defined by

$$8 W_7 \left( a_0; a_1, a_2, a_3, a_4, a_5; q, z \right) = 8 \varphi_7 \left( \frac{a_0, q a_0^2}{q - a_0}, -q a_0^2, a_1, \ldots, a_5; q, z \right)$$

$$\quad = \sum_{r=0}^{\infty} \frac{1 - a_0 q^{2r}}{1 - a_0} z^r \prod_{j=0}^{5} \frac{(a_j; q)_r}{(q a_0 a_j; q)_r}, \quad |z| < 1.$$ 

If $z = a_0^2 q^2/a_0 a_3 a_4 a_5$ (which is the case below) then it has a meromorphic continuation to $(a_0, \ldots, a_5) \in (\mathbb{C}^*)^5$ by [24, III.36], for which we will use the same notation.

The results from the previous subsection characterize $\Phi_1(\cdot, \xi)$ as asymptotically free eigenfunction of the AW second order difference operator $M_i$. In [61, Prop.
2.2] an explicit expression of this eigenfunction has been obtained (it essentially traces back to [29]). In our present notations, the result is as follows.

**Proposition 4.4.** We have

\[
\Phi_i(x, \xi) = \mathcal{W}_i(x, \tilde{\alpha}_i, (\xi)) \frac{\Gamma_0(\xi)}{S_i(x)S(\xi)} \\
\times \left( \frac{q^2a_i}{q^{-x-\tilde{\alpha}_i, (\xi)}} \cdot \frac{q^2b_i}{q^{-x-\tilde{\alpha}_i, (\xi)}} \cdot \frac{q^2c_i}{q^{-x-\tilde{\alpha}_i, (\xi)}} \cdot \left( \frac{\partial}{\partial q} q^{-x+2\tilde{\alpha}_i, (\xi)} \right) \right)_{q^2} \\
\times \mathcal{W}_7 \left( \frac{q^2}{d_i} q^{-x-2\tilde{\alpha}_i, (\xi)} \right) \mathcal{W}_7 \left( \frac{q^2}{d_i} q^{-x+\tilde{\alpha}_i, (\xi)} \right)
\]

as meromorphic functions in \((x, \xi) \in \mathbb{C} \times E_C\).

We are now in a position to determine the connection coefficients explicitly.

**Proof of Theorem 1.5.** Define meromorphic functions \(n^s_\pm (x, \xi)\) in \((x, \xi) \in \mathbb{C} \times E_C\) by

\[
n^s_+(x, \xi) := \frac{\epsilon_{\alpha_i}(x, \tilde{\alpha}_i, (\xi)) - \epsilon_{\alpha_i}(\alpha_i, (\xi), x)}{\epsilon_{\alpha_i}(\alpha_i, (\xi), -x)}
\]

\[
n^s_-(x, \xi) := \frac{\epsilon_{\alpha_i}(x, -\tilde{\alpha}_i, (\xi))}{\epsilon_{\alpha_i}(\alpha_i, (\xi), -x)}
\]

It follows from [21] Cor. 2.6 that

\[
\Phi_i(-x, \xi) = n^s_+(x, \xi) \Phi_i(x, \xi) + n^s_-(x, \xi) \Phi_i(x, s_i, \xi)
\]

as meromorphic functions in \((x, \xi) \in \mathbb{C} \times E_C\). Note furthermore that the \(n^s_\pm (\cdot, \xi)\) are 2\(\mu_i\)-translation invariant.

For meromorphic functions \(f\) and \(g\) on \(\mathbb{C}\) define the Wronskian \([f, g]\) to be the meromorphic function

\[
[f, g](x) := w_i(x) A_i(x) (f(x - 2\mu_i)g(x) - f(x)g(x - 2\mu_i)),
\]

with \(w_i(x)\) the weight function [1] of the Askey-Wilson polynomials,

\[
w_i(x) := \frac{(q^{2s}; q^{-x}; q^2)_{\infty}}{(a_i q^x, a_i q^{-x}, b_i q^x, b_i q^{-x}, c_i q^x, c_i q^{-x}, d_i q^x, d_i q^{-x}; q^2)_{\infty}}.
\]

Since \(w_i(x + 2\mu_i) A_i(x + 2\mu_i) = w_i(x) A_i(-x)\) and

\[
(M_i f)(x) g(x) - f(x) (M_i g)(x) = A_i(x) (f(x - 2\mu_i)g(x) - f(x)g(x - 2\mu_i))
\]

\[
+ A_i(-x) (f(x + 2\mu_i)g(x) - f(x)g(x + 2\mu_i))
\]

it follows that \([f, g](\cdot)\) is 2\(\mu_i\)-translation invariant if \(f\) and \(g\) are eigenfunctions of \(M_i\) with the same eigenvalue. Using the asymptotic expansion of \(\Phi_i(\cdot, \xi)\) we thus get

\[
[\Phi_i(\cdot, \xi), \Phi_i(\cdot, s_i, \xi)](x) = \lim_{m \to \infty} [\Phi_i(\cdot, \xi), \Phi_i(\cdot, s_i, \xi)](x - 2m\mu_i)
\]

\[
= (q^{-\tilde{\alpha}_i, (\xi)} - q^{\tilde{\alpha}_i, (\xi)}) \frac{W_i(x, \tilde{\alpha}_i, (\xi)) W_i(x, -\tilde{\alpha}_i, (\xi)) \Gamma_0(\xi) \Gamma_0(s, \xi) \theta(q^{2s}; q^2)}{S(\xi) S(s_i, \xi) \theta(a_i q^x, b_i q^x, c_i q^x, d_i q^x; q^2)}.
\]
Write \( \Phi_1^-(x, \xi) := \Phi_1(-x, \xi) \). From (4.19) we now conclude that

\[
\begin{align*}
\eta_1^s(x, \xi) &= \frac{[\Phi_1^-(\cdot, \xi), \Phi_1(\cdot, s_\xi)](x)}{[\Phi_1(\cdot, \xi), \Phi_1(\cdot, s_\xi)](x)} \\
\eta_2^s(x, \xi) &= -\frac{[\Phi_1^-(\cdot, \xi), \Phi_1(\cdot, \xi)](x)}{[\Phi_1(\cdot, \xi), \Phi_1(\cdot, s_\xi)](x)}
\end{align*}
\]

as meromorphic functions in \((x, \xi) \in \mathbb{C} \times E_C\). On the other hand, since \( m_s^t, \gamma, \kappa(\cdot, \xi) \) is \(\tilde{\Lambda}\)-translation invariant and \(a_c(\tilde{\alpha}_c) = 2\mu_i\), we have by Proposition 4.2(1),

\[
m_s^t, \gamma, \kappa(z, \xi) = \frac{[\Phi_1^-(\cdot, \xi), \Phi_1(\cdot, s_\xi)](\alpha_i(z))}{[\Phi_1(\cdot, \xi), \Phi_1(\cdot, s_\xi)](\alpha_i(z))},
\]

\[
m_s^t, \gamma, \kappa(z, \xi) = -\frac{[\Phi_1^-(\cdot, \xi), \Phi_1(\cdot, \xi)](\alpha_i(z))}{[\Phi_1(\cdot, \xi), \Phi_1(\cdot, s_\xi)](\alpha_i(z))}
\]

as meromorphic functions in \((z, \xi) \in E_C \times E_C\). Hence

\[
m_s^t, \gamma, \kappa(z, \xi) = n_1^s(\alpha_i(z), \xi), \quad m_s^t, \gamma, \kappa(z, \xi) = n_2^s(\alpha_i(z), \xi)
\]

as meromorphic functions in \((z, \xi) \in E_C \times E_C\), which completes the proof of the theorem. \(\square\)

### 5. Higher Rank Addition Formula for Theta Functions

Recall from Subsection 1.5 that the basic hypergeometric function \(E_{sph}\) is a distinguished Weyl group invariant solution \(E_{sph} \in SO_{RMKC}^{W_0 \times W_0}\) of the bispectral problem of the RMKC operators in case the root system datum is of the form \(D = (R_0, \Delta_0, t, \Lambda, \Lambda)\) and \(\kappa_a > 0\) for all \(a \in R\). It has an explicit \(c\)-function expansion (see (1.13)) with quantum \(c\)-function \(e_{sph}\) as defined in Definition 1.7. In particular, the quantum \(e_{sph}\) satisfies (1.15). In this section we investigate the equations (1.15) directly.

**Remark 5.1.** The notations in [59] are matched to the present ones as follows: t, \(\gamma, \kappa_0, \gamma_0\) in [59] correspond to \(q^\pm, q^{-\xi}, q^{\alpha_\kappa}, q^{-\rho}\) (where \(q^\pm\) is viewed as element of the complex algebraic torus \(Hom(\Lambda, \mathbb{C}^*)\)) by \(\lambda \mapsto q^{(\Lambda, \lambda)}\), and the basic Harish-Chandra series \(\Phi(t, \gamma)\) in [59] 4.6 matches with our renormalized Harish-Chandra series

\[
(\sum_{w \in W_0} q^{2\kappa_0 w})^{-1} W(z, \xi)^{-1} \frac{\partial_N(z + w_0 \xi)}{\partial_N(\rho - \xi)} \Phi(z, \xi).
\]

Let \((D, \kappa, q)\) be an arbitrary choice of initial datum. Let \(\Xi \in \mathcal{M}\) and write

\[
e_\Xi(z, \xi) := \frac{\Xi(z, \xi)}{W(z, \xi)} \prod_{\alpha \in R_0^+} \frac{\theta(\tilde{a}_\alpha q^{\alpha(z)}; q^{\alpha(z)})}{\theta(q^{\alpha(z)}; q^{\alpha(z)})}. \]

Using \(q^{(\Lambda, \tilde{\alpha})} = \prod_{\alpha \in R_0^+} a_\alpha^{(\Lambda, \alpha)}\) and the functional equation (1.8), we have \(e_\Xi \in \mathcal{F}\) if and only if

\[
\Xi(z + \mu, \xi) = q^{(\mu - \xi, w_0 \mu)} \Xi(z, \xi) \quad \forall \mu \in \tilde{\Lambda},
\]

\[
\Xi(z, \xi + \lambda) = q^{(\lambda, \xi - w_0 \lambda)} \Xi(z, \xi) \quad \forall \lambda \in \Lambda
\]

(5.1) (use for the theta function factors of \(e_\Xi\) corresponding to \(\alpha \in R_0^+\) with \((\Lambda, \alpha) = \mathbb{Z}\) the fact that \(\tilde{c}_\alpha = q_\alpha \tilde{a}_\alpha\) and \(\tilde{a}_\alpha = q_\alpha \tilde{b}_\alpha\)).
Remark 5.2. In the twisted equal lattice case $D = (R_0, \Delta, t, \Lambda, \Lambda)$ we have $c_{sph} = c_{\Xi_{sph}}$ with

\begin{equation}
\Xi_{sph}(z, \xi) := \frac{\vartheta_\Lambda(\rho + (\kappa_{2a_0} - \kappa_0)\delta^\vee_s + z + w_0\xi)}{\vartheta_\Lambda((\kappa_{2a_0} - \kappa_0)\delta^\vee_s + z)} \vartheta_\Lambda((\kappa_{2a_0} - \kappa_20)\delta^\vee_s - \xi).
\end{equation}

The quasi-invariance properties (5.1) for $\Xi_{sph}$ can be directly checked using the functional equations

\begin{equation}
\vartheta_\Lambda(z + \lambda) = q^{-\frac{|\lambda|^2}{2}} q^{-(\lambda, z)} \vartheta_\Lambda(z) \quad \forall \lambda \in \Lambda
\end{equation}

and noting that $\rho + (\kappa_{2a_0} - \kappa_0)\delta^\vee_s = \tilde{\rho}$.

Lemma 5.3. Let $i \in \{1, \ldots, n\}$ such that $(\Lambda, \alpha^\vee_i) = Z = (\tilde{\Lambda}, \tilde{\alpha}^\vee_i)$. Suppose $\Xi \in \mathcal{M}$ satisfies (5.1). Then $c_\Xi \in \mathcal{F}$ satisfies (1.15), i.e.

\begin{equation}
c_\Xi(z, \xi) = m^{e_i}_{e, c}(z, \xi)c_\Xi(s_i z, \xi) + m^{s_i}_{s_i, c}(z, s_i \xi)c_\Xi(s_i z, s_i \xi),
\end{equation}

if and only if

\begin{equation}
\vartheta(q^{2\kappa_i - \alpha_i}(z); q_i)\Xi(z, \xi) = \vartheta(q^{2\kappa_i - \alpha_i}(z); q_i)\Xi(s_i z, \xi)
- q^{\tilde{\alpha}_i}(z)\vartheta(q^{2\kappa_i - \tilde{\alpha}_i}(z); q_i)\Xi(s_i z, s_i \xi).
\end{equation}

Proof. This follows from a straightforward computation using the expressions of the connection coefficients $m^{e_i}_{e, c}$ and $m^{s_i}_{s_i, c}$ from Proposition 4.6 and using that

\begin{equation}
(\tilde{a}_i, b_i, \tilde{c}_i, d_i) = (q^{2\kappa_i}, -1, q, q^{2\kappa_i}, -q_i).
\end{equation}

Lemma 5.3 leads to the root system analog (Proposition 1.8) of the addition formula (1.10) for Jacobi theta functions.

Proofs of Proposition 1.8

1. It suffices to show that the equation (1.15) for $\Xi_{sph}$ implies that (1.16) is valid with $\rho$ replaced by $\rho + (\kappa_{2a_0} - \kappa_0)\delta^\vee_s$. This follows immediately from the explicit expression (5.2) of $\Xi_{sph}$ and the $W_0$-invariance of $\vartheta_\Lambda$ if $(\Lambda, \alpha^\vee) = Z$ for all $\alpha \in R_0$. If the condition $(\Lambda, \alpha^\vee) = Z$ is not valid for all $\alpha \in R_0$, then $(\Lambda, \alpha^\vee) = 2Z$ for short roots $\alpha \in R_0$ (see [59]). Hence $\alpha_i$ has to be a long root, consequently $s_i(\delta^\vee_s) = \delta^\vee_s$. The result follows again by the explicit expression (5.2) of $\Xi_{sph}$ and the $W_0$-invariance of $\vartheta_\Lambda$.

2. This second proof is by direct analytical methods. Fix generic $\xi \in E_C$ and write $g_{\xi}(z)$ for the right hand side of (1.16). It is a holomorphic function in $z \in E_C$ which vanishes if

\begin{equation}
\alpha_i(z) = 2\kappa_i + \mu_i k + \frac{2\pi \sqrt{-1}}{\tau} m \quad (k, m \in \mathbb{Z}).
\end{equation}

Hence

\begin{equation}
f_{\xi}(z) := \frac{g_{\xi}(z)}{\vartheta(q^{2\kappa_i - \alpha_i}(z); q_i)}
\end{equation}

is holomorphic in $z \in E_C$. Let $\Lambda^\vee$ be the dual lattice of $\Lambda$ in $E$ with respect to the scalar product $\langle \cdot, \cdot \rangle$. By a direct computation it follows that $f_{\xi}$ satisfies

\begin{equation}
f_{\xi}(z + \nu) = f_{\xi}(z), \quad \forall \nu \in \frac{2\pi \sqrt{-1}}{\tau} \Lambda^\vee,
\end{equation}

\begin{equation}
f_{\xi}(z + \lambda) = q^{-|\lambda|^2} q^{-(\lambda, \rho + z + w_0\xi)} f_{\xi}(z), \quad \forall \lambda \in \Lambda.
\end{equation}
This implies that \( f_ξ(z) = \text{cst}_ξ \theta_3(\rho + z + w_0ξ) \) for some constant \( \text{cst}_ξ \). Setting \( z = 0 \) shows that \( \text{cst}_ξ = \theta(q^{\tilde{\alpha}_i}(\xi)) \).

**Remark 5.4.** To see how the addition formula \((5.10)\) for the Jacobi theta function can be recovered from \((1.10)\), take \( D = (\mathbb{R}_0, \Delta, t, Q, Q) \) with the root system \( \mathbb{R}_0 \) of type \( B_n \) \((n \geq 2)\) realized within the Euclidean space \( E = \mathbb{R}^n \) with orthonormal basis \( \{e_i\}_{i=1}^n \) as ordered basis of \( E_0 \). We take \( \Delta_0 = (e_1 - e_2, \ldots, e_{n-1} - e_n, e_n) \) as ordered basis of \( \mathbb{R}_0 \). Since \( Q = \bigoplus_{i=1}^n \mathbb{Z}e_i \), the Jacobi triple product identity implies that

\[
\theta_Q(z) = (q; q)_\infty \prod_{j=1}^n \theta(-q^{\frac{1}{2}}z_j; q),
\]

where \( z_j := e_j(z) \). Then \((1.10)\) for \( 1 \leq i < n \) is easily seen to reduce to \((1.10)\).

The following lemma gives yet another reformulation of \((1.15)\).

**Lemma 5.5.** Suppose that \( \Xi \in \mathcal{M} \) satisfying \((5.1)\). Then \( c_\Xi \in \mathcal{F} \) satisfies \((5.3)\) if and only if

\[
\frac{\theta(\tilde{d}_i q^{\tilde{\alpha}_i}(\xi), d_i q^{-\alpha_i}(z); q_i^2)}{\theta(\frac{d_i}{\alpha_i} q^{\tilde{\alpha}_i}(\xi) - \alpha_i(z); q_i^2)} \Xi(z, \xi) = \frac{\theta(\tilde{d}_i q^{\tilde{\alpha}_i}(\xi), d_i q^{\alpha_i}(z); q_i^2)}{\theta(\frac{d_i}{\alpha_i} q^{\tilde{\alpha}_i}(\xi) + \alpha_i(z); q_i^2)} \Xi(s_i z, \xi) = \frac{\theta(\tilde{d}_i q^{\tilde{\alpha}_i}(\xi), d_i q^{-\alpha_i}(z); q_i^2)}{\theta(\frac{d_i}{\alpha_i} q^{\tilde{\alpha}_i}(\xi) + \alpha_i(z); q_i^2)} \Xi(s_i z, \xi, \xi).
\]

**Proof.** By \([61, \text{Cor. 2.8}]\),

\[
m_{\alpha, \xi}^{s_i}(z, \xi) = \frac{\epsilon_{\alpha_i}(\alpha_i(z), \tilde{\alpha}_i(\xi)) - \epsilon_{\alpha_i}(-\alpha_i(z), -\tilde{\alpha}_i(\xi))m_{\alpha, \xi}^{s_i}(z, s_i \xi)}{\epsilon_{\alpha_i}(-\alpha_i(z), \tilde{\alpha}_i(\xi))}
\]

(has identity boils down to the \( W_0 \)-invariance of the Askey-Wilson function, which is the basic hypergeometric function associated to \( D = (\mathbb{R}_0, \Delta_0, t, Q, Q) \) with \( \mathbb{R}_0 \) of type \( A_1 \)). Combined with the explicit expression of the connection coefficient \( m_{\alpha, c}^{s_i} \) (see Theorem 1.5), we conclude that \((5.3)\) is equivalent to

\[
c_\Xi(z, \xi) = \frac{\epsilon_{\alpha_i}(\alpha_i(z), \tilde{\alpha}_i(\xi))}{\epsilon_{\alpha_i}(-\alpha_i(z), \tilde{\alpha}_i(\xi))} c_\Xi(s_i z, \xi) = \frac{\epsilon_{\alpha_i}(\alpha_i(z), \tilde{\alpha}_i(\xi))}{\epsilon_{\alpha_i}(-\alpha_i(z), \tilde{\alpha}_i(\xi))} \left( c_\Xi(s_i z, s_i \xi) - \epsilon_{\alpha_i}(-\alpha_i(z), \tilde{\alpha}_i(\xi)) c_\Xi(s_i z, \xi) \right)
\]

Straightforward simplification now complete the proof of the lemma.

**Corollary 5.6.** If \( \Xi \in \mathcal{M} \) satisfies \((5.1)\) and

\[
\Xi(s_i z, \xi) = \frac{\theta(\frac{d_i}{\alpha_i} q^{\tilde{\alpha}_i}(\xi) + \alpha_i(z), d_i q^{-\alpha_i}(z); q_i^2)}{\theta(\frac{d_i}{\alpha_i} q^{\tilde{\alpha}_i}(\xi) - \alpha_i(z), d_i q^{\alpha_i}(z); q_i^2)} \Xi(z, \xi),
\]

\[
\Xi(s_i z, s_i \xi) = \frac{\theta(\frac{d_i}{\alpha_i} q^{\tilde{\alpha}_i}(\xi) - \alpha_i(z), d_i q^{\alpha_i}(z); q_i^2)}{\theta(\frac{d_i}{\alpha_i} q^{\tilde{\alpha}_i}(\xi) + \alpha_i(z), d_i q^{-\alpha_i}(z); q_i^2)} \Xi(z, \xi),
\]

then \( c_\Xi \in \mathcal{F} \) satisfies \((5.3)\).
For $D = (R_0, \Delta_0, t, \Lambda, \Lambda)$ and $i \in \{1, \ldots, n\}$ such that $(\Lambda, \alpha_i^\vee) = \mathbb{Z}$, the solution $\Xi_{\text{sph}}$ (see (5.2)) of (5.3) does not satisfy (5.5); in that case, one has to resort to the root system analog (1.11) of the addition formula for the Jacobi theta function for a direct proof of (5.3). On the other hand, for the Koornwinder root system datum (see Remark 5.4), it can be directly checked that $\Xi_{\text{sph}}$ satisfies (5.5) for $i = n$ (here we use the conventions from Remark 5.4, i.e. $\alpha_n \in \Delta_0$ is the short simple root), thus providing a direct proof that $\Xi_{\text{sph}}$ satisfies (5.3) for $i = n$.

References


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