

Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification

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Abstract

The assumption that space-time is a noncommutative space formed as a product of a continuous four dimensional manifold times a finite space predicts, almost uniquely, the Standard Model with all its fermions, gauge fields, Higgs field and their representations. A strong restriction on the noncommutative space results from the first order condition which came from the requirement that the Dirac operator is a differential operator of order one. Without this restriction, invariance under inner automorphisms requires the inner fluctuations of the Dirac operator to contain a quadratic piece expressed in terms of the linear part. We apply the classification of product noncommutative spaces without the first order condition and show that this leads immediately to a Pati-Salam $SU(2)_R \times SU(2)_L \times SU(4)$ type model which unifies leptons and quarks in four colors. Besides the gauge fields, there are 16 fermions in the $(2, 2, 4)$ representation, fundamental Higgs fields in the $(2, 2, 1)$, $(2, 1, 4)$ and $(1, 1, 1 + 15)$ representations. Interestingly we encounter a new phenomena where the Higgs fields in the high energy sector are composite and depend quadratically on the fundamental Higgs fields. The Pati-Salam symmetries are broken spontaneously at high energies to those of the Standard Model.

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I. INTRODUCTION

Noncommutative geometry was shown to provide a promising framework for unification of all fundamental interactions including gravity [3], [5], [6], [12], [10]. Historically, the search to identify the structure of the noncommutative space followed the bottom-up approach where the known spectrum of the fermionic particles was used to determine the geometric data that defines the space. This bottom-up approach involved an interesting interplay with experiments. While at first the experimental evidence of neutrino oscillations contradicted the first attempt [6], it was realized several years later in 2006 ([12]) that the obstruction to get neutrino oscillations was naturally eliminated by dropping the equality between the metric dimension of space-time (which is equal to 4 as far as we know) and its KO -dimension which is only defined modulo 8. When the latter is set equal to 2 modulo 8 [2], [4] (using the freedom to adjust the geometry of the finite space encoding the fine structure of space-time) everything works fine, the neutrino oscillations are there as well as the see-saw mechanism which appears for free as an unexpected bonus. Incidentally, this also solved the fermionic doubling problem by allowing a simultaneous Weyl-Majorana condition on the fermions to halve the degrees of freedom.

The second interplay with experiments occurred a bit later when it became clear that the mass of the Brout-Englert-Higgs boson would not comply with the restriction (that $m_H \succeq 170$ Gev) imposed by the validity of the Standard Model up to the unification scale. This obstruction to lower m_H was overcome in [11] simply by taking into account a scalar field which was already present in the full model which we had computed previously in [10]. One lesson which we learned on that occasion is that we have to take all the fields of the noncommutative spectral model seriously, without making assumptions not backed up by valid analysis, especially because of the almost uniqueness of the Standard Model (SM) in the noncommutative setting.

The SM continues to conform to all experimental data. The question remains whether this model will continue to hold at much higher energies, or whether there is a unified theory whose low-energy limit is the SM. One indication that there must be a new higher scale that effects the low energy sector is the small mass of the neutrinos which is explained through the see-saw mechanism with a Majorana mass of at least of the order of 10^{11} Gev. In addition and as noted above, a scalar field which acquires a vev generating that mass scale

can stabilize the Higgs coupling and prevent it from becoming negative at higher energies and thus make it consistent with the low Higgs mass of 126 Gev [11]. Another indication of the need to modify the SM at high energies is the failure (by few percent) of the three gauge couplings to be unified at some high scale which indicates that it may be necessary to add other matter couplings to change the slopes of the running of the RG equations.

This leads us to address the issue of the breaking from the natural algebra \mathcal{A} which results from the classification of irreducible finite geometries of KO -dimension 6 (modulo 8) performed in [9], to the algebra corresponding to the SM. This breaking was effected in [9], [8] using the requirement of the first order condition on the Dirac operator. The first order condition is the requirement that the Dirac operator is a derivation of the algebra \mathcal{A} into the commutant of $\hat{\mathcal{A}} = J\mathcal{A}J^{-1}$ where J is the charge conjugation operator. This in turn guarantees the gauge invariance and linearity of the inner fluctuations [7] under the action of the gauge group given by the unitaries $U = uJuJ^{-1}$ for any unitary $u \in \mathcal{A}$. This condition was used as a mathematical requirement to select the maximal subalgebra

$$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \subset \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

which is compatible with the first order condition and is the main reason behind the unique selection of the SM.

The existence of examples of noncommutative spaces where the first order condition is not satisfied such as quantum groups and quantum spheres provides a motive to remove this condition from the classification of noncommutative spaces compatible with unification [14], [15], [16], [17]. This study was undertaken in a companion paper [13] where it was shown that in the general case the inner fluctuations of D with respect to inner automorphisms of the form $U = uJuJ^{-1}$ are given by

$$D_A = D + A_{(1)} + \tilde{A}_{(1)} + A_{(2)}$$

where

$$\begin{aligned} A_{(1)} &= \sum_i a_i [D, b_i] \\ \tilde{A}_{(1)} &= \sum_i \hat{a}_i [D, \hat{b}_i], \quad \hat{a}_i = Ja_iJ^{-1}, \quad \hat{b}_i = Jb_iJ^{-1} \\ A_{(2)} &= \sum_{i,j} \hat{a}_i a_j [[D, b_j], \hat{b}_i] = \sum_{i,j} \hat{a}_i [A_{(1)}, \hat{b}_i]. \end{aligned}$$

Clearly $A_{(2)}$ which depends quadratically on the fields in $A_{(1)}$ vanishes when the first order condition is satisfied.

The plan of this paper is as follows. In section 2 we review the classification of irreducible finite geometries (without the first order condition) and show that the resultant algebra is, almost uniquely, given by $\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$. In section 3 we compute the inner fluctuations of the Dirac operator on the above algebra and determine the field content. In section 4 we evaluate the spectral action using a cutoff function and the heat kernel expansion method, where we show that the resultant model is the Pati-Salam [21] $SU(2)_R \times SU(2)_L \times SU(4)$ type model with all the appropriate Higgs fields necessary to break the symmetry to $U(1)_{\text{em}} \times SU(3)_c$. In section 5 we show that this model truncates correctly to the SM. In section 6 we analyze the potential and possible symmetry breaking, noting in particular the novel feature where some of the Higgs fields are fundamental while others are made of quadratic products of the fundamental ones. Section 7 is the appendix where all details of the calculation are given and where we illustrate the power and precision of noncommutative geometric methods by showing how all the physical fields arise. This is done to the benefit of researchers interested in becoming practitioners in the field.

II. CLASSIFICATION OF FINITE GEOMETRIES WITHOUT FIRST ORDER CONDITION

Some time ago the question of classifying finite noncommutative spaces was carried out in [9]. The main restriction came from requiring that spinors which belong to the product of the continuous four dimensional space, times the finite space must be such that the conjugate spinor is not an independent field, in order to avoid doubling the fermions. This could only be achieved when the spinors satisfy both the Majorana and Weyl conditions, which implies that the KO -dimension of the finite space be $6 \pmod{8}$. Consistency with the zeroth order condition

$$[a, b^\circ] = 0, \quad b^\circ = Jb^*J^{-1}, \quad \forall a, b \in \mathcal{A}$$

(since \mathcal{A} is an involutive algebra this condition is the same if one replaces b° by $\hat{b} = JbJ^{-1}$) restricts the center of the complexified algebra to be $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$. The dimension of the Hilbert space is then restricted to be the square of an integer. The algebra is then of the

form

$$M_k(\mathbb{C}) \oplus M_k(\mathbb{C}).$$

A symplectic symmetry imposed on the first algebra forces k to be even $k = 2a$ and the algebra to be of quaternionic matrices of the form $M_a(\mathbb{H})$. The existence of the chirality operator breaks $M_a(\mathbb{H})$ and further restricts the integer a to be even, and thus the number of fundamental fermions must be of the form $4a^2$ where a is an even integer. This shows that the first possible realistic case is the finite space with $k = 4$ to be based on the algebra

$$\mathcal{A} = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}). \quad (1)$$

A further restriction arises from the first order condition requiring the commutation of the commutator $[D, a]$ where D is the Dirac operator and $a \in \mathcal{A}$ with elements $b^\circ, b \in \mathcal{A}$,

$$[[D, a], b^\circ] = 0, \quad a, b \in \mathcal{A}, \quad b^\circ = Jb^*J^{-1}$$

(since \mathcal{A} is an involutive algebra this condition is the same if one replaces b° by $\hat{b} = JbJ^{-1}$)

This condition, together with the requirement that the neutrinos must acquire a Majorana mass restricts the above algebra further to the subalgebra

$$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}). \quad (2)$$

The question is whether the first order condition is an essential requirement for noncommutative spaces. There are known examples of noncommutative spaces where the first order condition is not satisfied such as the quantum group $SU(2)_q$ ([16], [17]). The main novelty of not imposing the first order condition is that the fluctuations of the Dirac operator (gauge and Higgs fields) will not be linear anymore and part of it $A_{(2)}$ will depend quadratically on the fields appearing in $A_{(1)}$. In this work we shall study the resulting noncommutative space without imposing the first order condition on the Dirac operator. Our starting point, however, will be an initial Dirac operator (without fluctuations) satisfying the first order condition relative to the subalgebra (2), but inner fluctuations would spoil this property.

The noncommutative geometric setting provided answers to some of the basic questions about the SM, such as the number of fermions in one family, the nature of the gauge symmetries and their fields, the fermionic representations, the Higgs fields as gauge fields along discrete directions, the phenomena of spontaneous symmetry breaking as well many other explanations [10]. In other words, noncommutative geometry successfully gave a geometric

setting for the SM. The dynamics of the model was then determined by the spectral action principle which is based on the idea that all the geometric invariants of the space can be found in the spectrum of the Dirac operator of the associated space. Indeed it was shown that the spectral action, which is a function of the Dirac operator, can be computed and gives the action of the SM coupled to gravity valid at some high energy scale. When the couplings appearing in this action are calculated at low energies by running the RG equations one finds excellent agreement with all known results to within few percents.

The first order condition is what restricted a more general gauge symmetry based on the algebra $\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$ to the subalgebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. It is thus essential to understand the physical significance of such a requirement. In what follows we shall examine the more general algebra allowed without the first order condition, and shall show that the number of fundamental fermions is still dictated to be 16. We determine the inner automorphisms of the algebra \mathcal{A} and show that the resulting gauge symmetry is a Pati-Salam type left-right model

$$SU(2)_R \times SU(2)_L \times SU(4)$$

where $SU(4)$ is the color group with the lepton number as the fourth color. In addition we discover a novel phenomena where the Higgs fields appearing in $A_{(2)}$ are composite and depend quadratically on those appearing in $A_{(1)}$. The representations of the Higgs fields are $(2_R, 2_L, 1)$, $(2_R, 1_L, 4)$ and $(1_R, 1_L, 1 + 15)$ with respect to $SU(2)_R \times SU(2)_L \times SU(4)$. The last of which will be absent if the Yukawa coupling of the up quark is equated with that of the neutrino and of the down quark equated with that of the electron. In addition the $1 + 15$ of $SU(4)$ decouple if we assume that at unification scale there is exact $SU(4)$ symmetry between the quarks and leptons. The resulting model is very similar to the one considered by Marshak and Mohapatra [20].

III. DIRAC OPERATOR AND INNER FLUCTUATIONS ON $\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$

When one considers inner fluctuations of the Dirac operator one finds that the gauge transformation takes the form

$$D_A \rightarrow U D_A U^*, \quad U = u J u J^{-1}, \quad u \in \mathcal{U}(\mathcal{A})$$

which implies that

$$A \rightarrow u A u^* + u \delta (u^*).$$

This in turn gives

$$\begin{aligned} A_{(1)} &\rightarrow u A_{(1)} u^* + u [D, u^*] \in \Omega_D^1(\mathcal{A}) \\ A_{(2)} &\rightarrow J u J^{-1} A_{(2)} J u^* J^{-1} + J u J^{-1} [u [D, u^*], J u^* J^{-1}] \end{aligned}$$

where the $A_{(2)}$ in the right hand side is computed using the gauge transformed $A_{(1)}$. Thus $A_{(1)}$ is a one-form and behaves like the usual gauge transformations. On the other hand $A_{(2)}$ transforms non-linearly and includes terms with quadratic dependence on the gauge transformations.

In this work we shall examine the hypothesis that the first order condition on the sub-algebra (2) arises as the vacuum solution of the spectral action. In this way the first order condition becomes a dynamical requirement instead of being imposed from outside.

Fortunately, in a previous work a compact tensorial notation was developed to simplify the structure of the Dirac operator, its fluctuations and the computation of the spectral action. This will come in handy in this case because the quadratic dependence of $A_{(2)}$ is too cumbersome to write otherwise. We shall therefore follow closely the presentation in [10] and use the same notation.

An element of the Hilbert space $\Psi \in \mathcal{H}$ is represented by

$$\Psi_M = \begin{pmatrix} \psi_A \\ \psi_{A'} \end{pmatrix}, \quad \psi_{A'} = \psi_A^c = C \psi_A^*$$

where ψ_A^c is the conjugate spinor to ψ_A . It is acted on by both the left algebra $M_2(\mathbb{H})$ and the right algebra $M_4(\mathbb{C})$. Therefore the index A can take 16 values and is represented by

$$A = \alpha I$$

where the index α is acted on by the quaternionic matrices and the index I by the $M_4(\mathbb{C})$ matrices. Moreover, when grading breaks $M_2(\mathbb{H})$ into $\mathbb{H}_R \oplus \mathbb{H}_L$ the index α is decomposed to $\alpha = \dot{a}, a$ where $\dot{a} = \dot{1}, \dot{2}$ is acted on by the right quaternionic algebra \mathbb{H}_R and $a = 1, 2$ is acted on by the left quaternionic algebra \mathbb{H}_L . Therefore the various components of the

spinor ψ_A are

$$\begin{aligned}\psi_{\alpha I} &= \begin{pmatrix} \nu_R & u_{iR} & \nu_L & u_{iL} \\ e_R & d_{iR} & e_L & d_{iL} \end{pmatrix} \\ &= (\psi_{\dot{a}1}, \psi_{\dot{a}i}, \psi_{a1}, \psi_{ai}), \quad a = 1, 2, \quad i = 1, 2, 3.\end{aligned}$$

The Dirac action then takes the form

$$\Psi_M^* D_M{}^N \Psi_N$$

which we can expand to give

$$\begin{aligned}\psi_A^* D_A{}^B \psi_B + \psi_{A'}^* D_{A'}{}^B \psi_B + \psi_A^* D_A{}^{B'} \psi_{B'} + \psi_{A'}^* D_{A'}{}^{B'} \psi_{B'} \\ = \psi_A^* D_A{}^B \psi_B + \psi_A C D^{AB} \psi_B + \text{h.c}\end{aligned}$$

where C is the charge conjugation matrix, and we have denoted $D_{A'}{}^B = D^{AB}$. The Dirac operator can be written in matrix form

$$(D)_M^N = \begin{pmatrix} D_A{}^B & D_A{}^{B'} \\ D_{A'}{}^B & D_{A'}{}^{B'} \end{pmatrix},$$

where

$$\begin{aligned}A &= \alpha I, \quad \alpha = (\dot{a}, a), \quad \dot{a} = \dot{1}, \dot{2}, \quad a = 1, 2, \quad I = 1, \dots, 4 \\ A' &= \alpha' I', \quad \alpha' = (\dot{a}', a'), \quad I' = 1', \dots, 4'.\end{aligned}$$

Thus $D_A{}^B = D_{\alpha I}{}^{\beta J}$. We also note that the property that $DJ = JD$ implies that

$$D_{A'}{}^{B'} = \hat{D}_A{}^B.$$

Starting with $a \in M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$ we write

$$a = \begin{pmatrix} X_\alpha^\beta \delta_I^J & 0 \\ 0 & \delta_{\alpha'}^{\beta'} Y_{I'}^{J'} \end{pmatrix}$$

where $X_\alpha^\beta \in M_4(\mathbb{C})$ and $Y_{I'}^{J'} \in M_4(\mathbb{C})$. We further impose the condition of symplectic isometry on the first $M_4(\mathbb{C})$

$$(\sigma_2 \otimes 1) (\bar{a}) (\sigma_2 \otimes 1) = a, \quad a \in M_4(\mathbb{C})$$

which reduces $M_4(\mathbb{C})$ to $M_2(\mathbb{H})$. From the property of commutation of the grading operator G_α^β with $M_2(\mathbb{H})$

$$[G, X] = 0$$

where $G_\alpha^\beta = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$, reduces the algebra $M_2(\mathbb{H})$ to $\mathbb{H}_R \oplus \mathbb{H}_L$. Thus we now have

$$X_\alpha^\beta = \begin{pmatrix} X_a^b & 0 \\ 0 & X_a^b \end{pmatrix}, \quad X_a^b = \begin{pmatrix} X_1^1 & X_1^2 \\ -\overline{X_1^2} & \overline{X_1^1} \end{pmatrix} \in \mathbb{H}_L$$

and similarly for $X_a^b \in \mathbb{H}_R$. The anti-linear isometry J is represented by

$$J = \begin{pmatrix} 0 & \delta_\alpha^{\beta'} \delta_I^{J'} \\ \delta_{\alpha'}^\beta \delta_I^J & 0 \end{pmatrix} \times \text{complex conjugation}$$

and satisfies $J^2 = 1$ and

$$J\Psi = \Psi$$

In this form

$$a^\circ = Ja^*J^{-1} = \begin{pmatrix} \delta_\alpha^\beta Y_I^{tJ} & 0 \\ 0 & X_{\alpha'}^{t\beta'} \delta_{I'}^{J'} \end{pmatrix}$$

where the superscript t denotes the transpose matrix. This clearly satisfies the commutation relation

$$[a, b^\circ] = 0$$

which is simply the statement that the right action and left action commute. In matrix form the operator D_F has the sub-matrices [10]

$$D_{\alpha 1}^{\beta 1} = \begin{pmatrix} 0 & D_{a1}^{b1} \\ D_{\dot{a}1}^{b1} & 0 \end{pmatrix}, \quad D_{a1}^{b1} = (D_{\dot{a}1}^{b1})^* \equiv D_{a(l)}^b$$

$$D_{\alpha i}^{\beta j} = \begin{pmatrix} 0 & D_{a(q)}^b \delta_i^j \\ D_{\dot{a}(q)}^b \delta_i^j & 0 \end{pmatrix}, \quad D_{a(q)}^b = (D_{\dot{a}(q)}^b)^*$$

where

$$D_{a1}^{b1} = D_{a(l)}^b = \begin{pmatrix} k^{*\nu} & 0 \\ 0 & k^{*e} \end{pmatrix}, \quad a = 1, 2, \quad b = \dot{1}, \dot{2}$$

and

$$D_{a(q)}^b = \begin{pmatrix} k^{*u} & 0 \\ 0 & k^{*d} \end{pmatrix}.$$

The Yukawa couplings k^ν , k^e , k^u , k^d are 3×3 matrices in generation space. Notice that this structure gives Dirac masses to all the fermions, but Majorana masses only for the right-handed neutrinos. This was shown in [9] to be the unique possibility consistent with the first order condition on the subalgebra (2). We can summarize all the information about the finite space Dirac operator without fluctuations, in the tensorial equation

$$\begin{aligned} (D_F)_{\alpha I}^{\beta J} &= \left(\delta_\alpha^1 \delta_1^\beta k^{*\nu} + \delta_\alpha^1 \delta_1^\beta k^\nu + \delta_\alpha^2 \delta_2^\beta k^{*e} + \delta_\alpha^2 \delta_2^\beta k^e \right) \delta_I^1 \delta_1^J \\ &\quad + \left(\delta_\alpha^1 \delta_1^\beta k^{*u} + \delta_\alpha^1 \delta_1^\beta k^u + \delta_\alpha^2 \delta_2^\beta k^{*d} + \delta_\alpha^2 \delta_2^\beta k^d \right) \delta_I^i \delta_j^J \delta_i^j \\ (D_F)_{\alpha I}^{\beta' K'} &= \delta_\alpha^1 \delta_{i'}^{\beta'} \delta_I^1 \delta_{1'}^{K'} k^{*\nu R} \end{aligned}$$

where $k^{\nu R}$ are Yukawa couplings for the right-handed neutrinos. One can also consider the special case of lepton and quark unification by equating

$$k^\nu = k^u, \quad k^e = k^d$$

where we expect some simplifications. From the previous discussion, it will be clear that the Dirac operator D_A including inner fluctuations $U = u J u J^{-1}$, $u \in \mathcal{U}(\mathcal{A})$ would not obey the first order condition. We now proceed to compute the Dirac operator on the product space $M \times F$. The initial operator is given by

$$D = \gamma^\mu D_\mu \otimes 1 + \gamma_5 D_F$$

where $\gamma^\mu D_\mu = \gamma^\mu \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \right)$ is the Dirac operator on the four dimensional spin manifold. Then the Dirac operator including inner fluctuations is given by

$$D_A = D + A_{(1)} + J A_{(1)} J^{-1} + A_{(2)}$$

$$A_{(1)} = \sum a [D, b]$$

$$A_{(2)} = \sum a [J A_{(1)} J^{-1}, b].$$

The computation is very involved thus for clarity we shall collect all the details in the appendix and only quote the results in what follows. The different components of the operator D_A are then given by

$$\begin{aligned} (D_A)_{\dot{a}I}^{\dot{b}J} &= \gamma^\mu \left(D_\mu \delta_a^{\dot{b}} \delta_I^J - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_a^{\dot{b}} \delta_I^J - \delta_a^{\dot{b}} \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J + \frac{i}{2} g V_\mu \delta_I^J \right) \right) \\ (D_A)_{aI}^{bJ} &= \gamma^\mu \left(D_\mu \delta_a^b \delta_I^J - \frac{i}{2} g_L W_{\mu L}^\alpha (\sigma^\alpha)_a^b \delta_I^J - \delta_a^b \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J + \frac{i}{2} g V_\mu \delta_I^J \right) \right) \end{aligned}$$

where the fifteen 4×4 matrices $(\lambda^m)_I^J$ are traceless and generate the group $SU(4)$ and $W_{\mu R}^\alpha$, $W_{\mu L}^\alpha$, V_μ^m are the gauge fields of $SU(2)_R$, $SU(2)_L$, and $SU(4)$. The requirement that A is unimodular implies that

$$\text{Tr}(A) = 0$$

which gives the condition

$$V_\mu = 0.$$

In addition we have

$$\begin{aligned} (D_A)_{\dot{a}I}^{bJ} &= \gamma_5 \left(\left(k^\nu \phi_a^b + k^e \tilde{\phi}_a^b \right) \Sigma_I^J + \left(k^u \phi_a^b + k^d \tilde{\phi}_a^b \right) (\delta_I^J - \Sigma_I^J) \right) \equiv \gamma_5 \Sigma_{\dot{a}I}^{bJ} \\ (D_A)_{\dot{a}I}^{\dot{b}'J'} &= \gamma_5 k^{*\nu R} \Delta_{\dot{a}J} \Delta_{\dot{b}'I} \equiv \gamma_5 H_{\dot{a}I \dot{b}'J} \end{aligned}$$

where the Higgs field ϕ_a^b is in the $(2_R, \bar{2}_L, 1)$ of the product gauge group $SU(2)_R \times SU(2)_L \times SU(4)$, and $\Delta_{\dot{a}J}$ is in the $(2_R, 1_L, 4)$ representation while Σ_I^J is in the $(1_R, 1_L, 1 + 15)$ representation. The field $\tilde{\phi}_a^b$ is not an independent field and is given by

$$\tilde{\phi}_a^b = \tau_2 \bar{\phi}_a^b \tau_2.$$

Note that the field Σ_I^J decouples (and set to $\delta_I^1 \delta_1^J$) in the special case when there is lepton and quark unification of the couplings

$$k^\nu = k^u, \quad k^e = k^d.$$

The important point to notice is the novel phenomena of the appearance of composite Higgs field as is apparent in the above formulas where the Higgs field $\Sigma_{\dot{a}I}^{bJ}$ is formed out of the products of the fields ϕ_a^b and Σ_I^J while the Higgs field $H_{\dot{a}I \dot{b}'J}$ is made from the product of $\Delta_{\dot{a}J} \Delta_{\dot{b}'I}$. This composite structure is a result of the quadratic dependence of the gauge fields $A_{(2)}$ on those appearing in $A_{(1)}$. The importance of this point should not be underestimated. The reason is that the main disadvantage of grand unified theories is the need for complicated Higgs representations with arbitrary potentials. In the noncommutative geometric setting, this problem is now solved by having minimal representations of the Higgs fields allowing for (quadratic) products of these representations. We also note that a very close model to the one deduced here is the one considered by Marshak and Mohapatra where the $U(1)$ of the left-right model is identified with the $B - L$ symmetry. They proposed the same Higgs fields $(2_R, 2_L, 1)$, $(2_R, 1, 4)$ and $(1, 1, 15)$ we have, but also in addition the field $(1, 2_L, 4)$.

However, they assumed that this Higgs fields does not get a vev, and thus does not effect the symmetry breaking. Although the broken generators of the $SU(4)$ gauge fields can mediate lepto-quark interactions leading to proton decay, it was shown that in all such types of models with partial unification, the proton is stable. In addition this type of model arises in the first phase of breaking of $SO(10)$ to $SU(2)_R \times SU(2)_L \times SU(4)$ and these have been extensively studied [1]. The recent work in [18] considers noncommutative grand unification based on the $k = 8$ algebra $M_4(\mathbb{H}) \oplus M_8(\mathbb{C})$ keeping the first order condition.

IV. THE SPECTRAL ACTION FOR THE $SU(2)_R \times SU(2)_L \times SU(4)$ MODEL

Having determined the Dirac operator acting on the Hilbert space of spinors in terms of the gauge fields of $SU(2)_R \times SU(2)_L \times SU(4)$ and Higgs fields, some of which are fundamental while others are composite, the next step is to study the dynamics of these fields as governed by the spectral action principle. The geometric invariants of the noncommutative space are encoded in the spectrum of the Dirac operator D_A . The bosonic action is given by

$$\text{Trace} (f(D_A/\Lambda))$$

where Λ is some cutoff scale and the function f is restricted to be even and positive. Using heat kernel methods the trace can be expressed in terms of Seeley-de Witt coefficients a_n :

$$\text{Trace} f(D_A/\Lambda) = \sum_{n=0}^{\infty} F_{4-n} \Lambda^{4-n} a_n$$

where the function F is defined by $F(u) = f(v)$ where $u = v^2$, thus $F(D^2) = f(D)$. We define

$$f_k = \int_0^{\infty} f(v) v^{k-1} dv, \quad k > 0$$

then

$$\begin{aligned}
F_4 &= \int_0^\infty F(u)u du = 2 \int_0^\infty f(v)v^3 dv = 2f_4 \\
F_2 &= \int_0^\infty F(u)du = 2 \int_0^\infty f(v)v dv = 2f_2 \\
F_0 &= F(0) = f(0) = f_0 \\
F_{-2n} &= (-1)^n F^{(n)}(0) = \left[(-1)^n \left(\frac{1}{2v} \frac{d}{dv} \right)^n f \right] (0) \quad n \geq 1.
\end{aligned}$$

Using the same notation and formulas as in reference [10], the first Seeley-de Witt coefficient is

$$\begin{aligned}
a_0 &= \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} (1) \\
&= \frac{1}{16\pi^2} (4) (32) (3) \int d^4x \sqrt{g} \\
&= \frac{24}{\pi^2} \int d^4x \sqrt{g}
\end{aligned}$$

where the numerical factors come, respectively, from the traces on the Clifford algebra, the dimensions of the Hilbert space and number of generations. The second coefficient is

$$a_2 = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} \left(E + \frac{1}{6} R \right)$$

where E is a 384×384 matrix over Hilbert space of three generations of spinors, whose components are derived and listed in the appendix. Taking the various traces we get

$$\begin{aligned}
a_2 &= \frac{1}{16\pi^2} \int d^4x \sqrt{g} \left((R(-96 + 64) - 8 \left(H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + 2\Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{a}I} \right) \right) \\
&= -\frac{2}{\pi^2} \int d^4x \sqrt{g} \left(R + \frac{1}{4} \left(H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + 2\Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{a}I} \right) \right).
\end{aligned}$$

It should be understood in the above formula and in what follows, that whenever the matrices k^ν, k^u, k^e, k^d and $k^{\nu R}$ appear in an action, one must take the trace over generation space.

The mass terms can be expressed in terms of the fundamental Higgs field to give

$$H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} = |k^{\nu R}|^2 \left(\Delta_{\dot{a}K} \bar{\Delta}^{\dot{a}K} \right)^2$$

and

$$\begin{aligned}
2\Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{a}I} &= 2 \left(\left((k^\nu - k^u) \phi_a^c + (k^e - k^d) \tilde{\phi}_a^c \right) \Sigma_I^K + \left(k^u \phi_a^c + k^d \tilde{\phi}_a^c \right) \delta_I^K \right) \\
&\quad \left(\left((k^{*\nu} - k^{*u}) \phi_c^{\dot{a}} + (k^{*e} - k^{*d}) \tilde{\phi}_c^{\dot{a}} \right) \Sigma_K^I + \left(k^{*u} \phi_c^{\dot{a}} + k^{*d} \tilde{\phi}_c^{\dot{a}} \right) \delta_K^I \right).
\end{aligned}$$

The next coefficient is

$$a_4 = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} \left(\frac{1}{360} (5R^2 - 2R_{\mu\nu}^2 + 2R_{\mu\nu\rho\sigma}^2) 1 + \frac{1}{2} \left(E^2 + \frac{1}{3} RE + \frac{1}{6} \Omega_{\mu\nu}^2 \right) \right)$$

where $\Omega_{\mu\nu}$ is the 384×384 curvature matrix of the connection ω_μ . Using the expressions for the matrices E and $\Omega_{\mu\nu}$ derived in the appendix, and taking the traces, we get

$$\begin{aligned} a_4 = \frac{1}{2\pi^2} \int d^4x \sqrt{g} \left[-\frac{3}{5} C_{\mu\nu\rho\sigma}^2 + \frac{11}{30} R^* R^* + g_L^2 (W_{\mu\nu L}^\alpha)^2 + g_R^2 (W_{\mu\nu R}^\alpha)^2 + g^2 (V_{\mu\nu}^m)^2 \right. \\ \left. + \nabla_\mu \Sigma_{aI}^{\dot{c}K} \nabla^\mu \Sigma_{\dot{c}K}^{aI} + \frac{1}{2} \nabla_\mu H_{\dot{a}I\dot{b}J} \nabla^\mu H^{\dot{a}I\dot{b}J} + \frac{1}{12} R \left(H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + 2\Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{a}I} \right) \right. \\ \left. + \frac{1}{2} \left| H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{b}J} \right|^2 + 2H_{\dot{a}I\dot{c}K} \Sigma_{\dot{b}J}^{\dot{c}K} H^{\dot{a}I\dot{d}L} \Sigma_{\dot{d}L}^{\dot{b}J} + \Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{b}J} \Sigma_{\dot{b}J}^{\dot{d}L} \Sigma_{\dot{d}L}^{\dot{a}I} \right] \end{aligned}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. Thus the bosonic spectral action to second order is given by

$$S = F_4 \Lambda^4 a_0 + F_2 \Lambda^2 a_2 + F_0 a_4 + \dots$$

which finally gives

$$\begin{aligned} S_b = \frac{24}{\pi^2} F_4 \Lambda^4 \int d^4x \sqrt{g} \\ - \frac{2}{\pi^2} F_2 \Lambda^2 \int d^4x \sqrt{g} \left(R + \frac{1}{4} \left(H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + 2\Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{a}I} \right) \right) \\ + \frac{1}{2\pi^2} F_0 \int d^4x \sqrt{g} \left[\frac{1}{30} (-18C_{\mu\nu\rho\sigma}^2 + 11R^* R^*) + g_L^2 (W_{\mu\nu L}^\alpha)^2 + g_R^2 (W_{\mu\nu R}^\alpha)^2 + g^2 (V_{\mu\nu}^m)^2 \right. \\ \left. + \nabla_\mu \Sigma_{aI}^{\dot{c}K} \nabla^\mu \Sigma_{\dot{c}K}^{aI} + \frac{1}{2} \nabla_\mu H_{\dot{a}I\dot{b}J} \nabla^\mu H^{\dot{a}I\dot{b}J} + \frac{1}{12} R \left(H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + 2\Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{a}I} \right) \right. \\ \left. + \frac{1}{2} \left| H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{b}J} \right|^2 + 2H_{\dot{a}I\dot{c}K} \Sigma_{\dot{b}J}^{\dot{c}K} H^{\dot{a}I\dot{d}L} \Sigma_{\dot{d}L}^{\dot{b}J} + \Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{b}J} \Sigma_{\dot{b}J}^{\dot{d}L} \Sigma_{\dot{d}L}^{\dot{a}I} \right]. \end{aligned}$$

The physical content of this action is a cosmological constant term, the Einstein Hilbert term R , a Weyl tensor square term $C_{\mu\nu\rho\sigma}^2$, kinetic terms for the $SU(2)_R \times SU(2)_L \times SU(4)$ gauge fields, kinetic terms for the composite Higgs fields $H_{\dot{a}I\dot{b}J}$ and $\Sigma_{\dot{b}J}^{\dot{c}K}$ as well as mass terms and quartic terms for the Higgs fields. This is a grand unified Pati-Salam type model with a completely fixed Higgs structure which we expect to spontaneously break at very high energies to the $U(1) \times SU(2) \times SU(3)$ symmetry of the SM. We also notice that this action gives the gauge coupling unification

$$g_R = g_L = g.$$

A test of this model is to check whether this relation when run using RG equations would give values consistent with the values of the gauge couplings for electromagnetic, weak and

strong interactions at the scale of the Z -boson mass. Having determined the full Dirac operators, including fluctuations, we can write all the fermionic interactions including the ones with the gauge vectors and Higgs scalars. It is given by

$$\begin{aligned}
& \int d^4x \sqrt{g} \left\{ \psi_{\dot{a}I}^* \gamma^\mu \left(D_\mu \delta_a^b \delta_I^J - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_a^b \delta_I^J - \delta_a^b \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J + \frac{i}{2} g V_\mu \delta_I^J \right) \right) \psi_{bJ} \right. \\
& + \psi_{aI}^* \gamma^\mu \left(D_\mu \delta_a^b \delta_I^J - \frac{i}{2} g_L W_{\mu L}^\alpha (\sigma^\alpha)_a^b \delta_I^J - \delta_a^b \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J + \frac{i}{2} g V_\mu \delta_I^J \right) \right) \psi_{bJ} \\
& + \psi_{\dot{a}I}^* \gamma_5 \left(\left(k^\nu \phi_a^b + k^e \tilde{\phi}_a^b \right) \Sigma_I^J + \left(k^u \phi_a^b + k^d \tilde{\phi}_a^b \right) (\delta_I^J - \Sigma_I^J) \right) \psi_{bJ} \\
& + \psi_{aI}^* \gamma_5 \left(\left(k^{*\nu} \phi_a^b + k^{*e} \tilde{\phi}_a^b \right) \Sigma_I^J + \left(k^{*u} \phi_a^b + k^{*d} \tilde{\phi}_a^b \right) (\delta_I^J - \Sigma_I^J) \right) \psi_{bJ} \\
& \left. + C \psi_{\dot{a}I} \gamma_5 k^{\nu R} \bar{\Delta}^{\dot{a}J} \bar{\Delta}^{\dot{b}I} \psi_{bJ} + \text{h.c} \right\}
\end{aligned}$$

V. TRUNCATION TO THE STANDARD MODEL.

It is easy to see that this model truncates to the Standard Model. The Higgs field $\phi_a^b = (2_R, 2_L, 1)$ must be truncated to the Higgs doublet H by writing

$$\phi_a^b = \delta_a^1 \epsilon^{bc} H_c.$$

The other Higgs field $\Delta_{aI} = (2_R, 1, 4)$ is truncated to a real singlet scalar field

$$\Delta_{aI} = \delta_a^1 \delta_I^1 \sqrt{\sigma}.$$

These then imply the relations

$$\begin{aligned}
\Sigma_{\dot{a}I}^{bJ} &= \left(\delta_a^1 k^\nu \epsilon^{bc} H_c + \delta_a^2 \bar{H}^b k^e \right) \delta_I^1 \delta_1^J + \left(\delta_a^1 k^u \epsilon^{bc} H_c + \delta_a^2 k^d \bar{H}^b \right) \delta_I^i \delta_j^J \delta_i^j \\
H_{\dot{a}I b J} &= \delta_a^1 \delta_b^1 k^{\nu R} \delta_I^1 \delta_1^J \sigma \\
g_R W_{\mu R}^3 &= g_1 B_\mu, \quad W_{\mu R}^\pm = 0 \\
\sqrt{\frac{3}{2}} g V_\mu^{15} &= -g_1 B_\mu \quad (V_\mu)_1^i = 0
\end{aligned}$$

where V_μ^{15} is the $SU(4)$ gauge field corresponding to the generator

$$\lambda^{15} = \frac{1}{\sqrt{6}} \text{diag} (3, -1, -1, -1)$$

which could be identified with the $B - L$ generator. In particular the components $(D_A)_{11}^{\dot{1}1}$ and $(D_A)_{\dot{2}1}^{\dot{2}1}$ of the Dirac operator simplify to

$$\begin{aligned} (D_A)_{11}^{\dot{1}1} &= \gamma^\mu \left(D_\mu - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_1^{\dot{1}} - \left(\frac{i}{2} g V_\mu^m (\lambda^m)_1^1 \right) \right) \\ &= \gamma^\mu \left(D_\mu - \frac{i}{2} g_R W_{\mu R}^3 - \left(\frac{i}{2} g V_\mu^{15} \sqrt{\frac{3}{2}} \right) \right) \\ &= \gamma^\mu D_\mu \end{aligned}$$

$$\begin{aligned} (D_A)_{\dot{2}1}^{\dot{2}1} &= \gamma^\mu \left(D_\mu - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_2^{\dot{2}} - \left(\frac{i}{2} g V_\mu^m (\lambda^m)_1^1 \right) \right) \\ &= \gamma^\mu \left(D_\mu + \frac{i}{2} g_R W_{\mu R}^3 - \left(\frac{i}{2} g V_\mu^{15} \sqrt{\frac{3}{2}} \right) \right) \\ &= \gamma^\mu (D_\mu + i g_1 B_\mu) \end{aligned}$$

which are identified with the Dirac operators acting on the right-handed neutrino and right-handed electron. Similar substitutions give the action of the Dirac operators on the remaining fermions and give the expected results. We now compute the various terms in the spectral action. First for the mass terms we have

$$\begin{aligned} \frac{1}{4} H_{\dot{a}I bJ} H^{\dot{b}J \dot{a}I} &= \frac{1}{4} \left(\delta_a^1 \delta_b^1 k^{\nu R} \delta_I^1 \delta_1^J \sigma \right) \left(\delta_a^{\dot{1}} \delta_b^{\dot{1}} \delta_I^{\dot{1}} \delta_1^J k^{*\nu R} \sigma \right) \\ &= \frac{1}{4} \text{tr} |k^{\nu R}|^2 \sigma^2 = \frac{1}{4} c \sigma^2 \\ \frac{1}{2} \Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{a}I} &= \frac{1}{2} \left| \left(\delta_a^1 k^\nu \epsilon^{bc} H_c + \delta_a^2 \overline{H}^b k^e \right) \delta_I^1 \delta_1^J + \left(\delta_a^1 k^u \epsilon^{bc} H_c + \delta_a^2 k^d \overline{H}^b \right) \delta_I^i \delta_j^J \delta_i^j \right|^2 \\ &= \frac{1}{2} a \overline{H} H \end{aligned}$$

where

$$\begin{aligned} a &= \text{tr} \left(k^{*\nu} k^\nu + k^{*e} k^e + 3 \left(k^{*u} k^u + k^{*d} k^d \right) \right) \\ c &= \text{tr} \left(k^{*\nu R} k^{\nu R} \right) \end{aligned}$$

Next for the a_4 term, starting with the gauge kinetic energies we have

$$g_L^2 (W_{\mu\nu L}^\alpha)^2 + g_R^2 (W_{\mu\nu R}^\alpha)^2 + g^2 (V_{\mu\nu}^m)^2 \rightarrow g_L^2 (W_{\mu\nu L}^\alpha)^2 + \frac{5}{3} g_1^2 B_{\mu\nu}^2 + g_3^2 (V_{\mu\nu}^m)^2$$

where $m = 1, \dots, 8$ for $V_{\mu\nu}^m$ restricted to the $SU(3)$ gauge group. Next for the Higgs kinetic and quartic terms we have

$$\begin{aligned}
& \nabla_\mu \Sigma_{aI}^{\dot{c}K} \nabla^\mu \Sigma_{\dot{c}K}^{aI} \rightarrow a \nabla_\mu \bar{H} \nabla^\mu H \\
& \frac{1}{2} \nabla_\mu H_{\dot{a}IbJ} \nabla^\mu H^{\dot{a}IbJ} \rightarrow \frac{1}{2} c \partial_\mu \sigma \partial^\mu \sigma \\
& \frac{1}{12} R \left(H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + 2 \Sigma_{\dot{a}I}^{cK} \Sigma_{\dot{c}K}^{\dot{a}I} \right) \rightarrow \frac{1}{12} R (2a \bar{H} H + c \sigma^2) \\
& \frac{1}{2} \left| H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{b}J} \right|^2 \rightarrow \frac{1}{2} d \sigma^4 \\
& 2 H_{\dot{a}I\dot{c}K} \Sigma_{\dot{b}J}^{\dot{c}K} H^{\dot{a}I\dot{d}L} \Sigma_{\dot{d}L}^{bJ} \rightarrow 2e \bar{H} H \sigma^2 \\
& \Sigma_{aI}^{\dot{c}K} \Sigma_{\dot{c}K}^{bJ} \Sigma_{\dot{b}J}^{\dot{d}L} \Sigma_{\dot{d}L}^{aI} \rightarrow b (\bar{H} H)^2
\end{aligned}$$

Collecting all terms we end up with the bosonic action for the Standard Model:

$$\begin{aligned}
S_b = & \frac{24}{\pi^2} F_4 \Lambda^4 \int d^4 x \sqrt{g} \\
& - \frac{2}{\pi^2} F_2 \Lambda^2 \int d^4 x \sqrt{g} \left(R + \frac{1}{2} a \bar{H} H + \frac{1}{4} c \sigma^2 \right) \\
& + \frac{1}{2\pi^2} F_0 \int d^4 x \sqrt{g} \left[\frac{1}{30} (-18 C_{\mu\nu\rho\sigma}^2 + 11 R^* R^*) + \frac{5}{3} g_1^2 B_{\mu\nu}^2 + g_2^2 (W_{\mu\nu}^\alpha)^2 + g_3^2 (V_{\mu\nu}^m)^2 \right. \\
& \left. + \frac{1}{6} a R \bar{H} H + b (\bar{H} H)^2 + a |\nabla_\mu H_a|^2 + 2e \bar{H} H \sigma^2 + \frac{1}{2} d \sigma^4 + \frac{1}{12} c R \sigma^2 + \frac{1}{2} c (\partial_\mu \sigma)^2 \right]
\end{aligned}$$

where

$$\begin{aligned}
b &= \text{tr} \left((k^{*\nu} k^\nu)^2 + (k^{*e} k^e)^2 + 3 \left((k^{*u} k^u)^2 + (k^{*d} k^d)^2 \right) \right) \\
d &= \text{tr} \left((k^{*\nu R} k^{\nu R})^2 \right) \\
e &= \text{tr} \left(k^{*\nu} k^\nu k^{*\nu R} k^{\nu R} \right).
\end{aligned}$$

This action completely agrees with the results in reference [10].

VI. THE POTENTIAL AND SYMMETRY BREAKING

We now study the resulting potential and try to investigate the possible minima:

$$\begin{aligned}
V = & \frac{F_0}{2\pi^2} \left(\frac{1}{2} \left| H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{b}J} \right|^2 + 2 H_{\dot{a}I\dot{c}K} \Sigma_{\dot{b}J}^{\dot{c}K} H^{\dot{a}I\dot{d}L} \Sigma_{\dot{d}L}^{bJ} + \Sigma_{aI}^{\dot{c}K} \Sigma_{\dot{c}K}^{bJ} \Sigma_{\dot{b}J}^{\dot{d}L} \Sigma_{\dot{d}L}^{aI} \right) \\
& - \frac{F_2}{2\pi^2} \left(H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + 2 \Sigma_{\dot{a}I}^{cK} \Sigma_{\dot{c}K}^{\dot{a}I} \right).
\end{aligned}$$

However, the Higgs field here are not fundamental and we have to express the potential in terms of the fundamental Higgs fields ϕ_a^c , Δ_{aK} and Σ_K^I . Expanding the composite Higgs

fields in terms of the fundamental ones, we have for the quartic terms

$$\frac{1}{2} \left| H_{aI\dot{c}K} H^{\dot{c}K\dot{b}J} \right|^2 = \frac{1}{2} |k^{\nu R}|^4 \left(\Delta_{\dot{a}K} \bar{\Delta}^{\dot{a}L} \Delta_{\dot{b}L} \bar{\Delta}^{\dot{b}K} \right)^2$$

$$\begin{aligned} \Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{b}J} \Sigma_{\dot{b}J}^{\dot{d}L} \Sigma_{\dot{d}L}^{\dot{a}I} &= \left(\left((k^{*\nu} - k^{*u}) \phi_{\dot{a}}^{\dot{c}} + (k^{*e} - k^{*d}) \tilde{\phi}_{\dot{a}}^{\dot{c}} \right) \Sigma_I^K + \left(k^{*u} \phi_{\dot{a}}^{\dot{c}} + k^{*d} \tilde{\phi}_{\dot{a}}^{\dot{c}} \right) \delta_I^K \right) \\ &\quad \left(\left((k^\nu - k^u) \phi_{\dot{c}}^{\dot{b}} + (k^e - k^d) \tilde{\phi}_{\dot{c}}^{\dot{b}} \right) \Sigma_K^J + \left(k^u \phi_{\dot{c}}^{\dot{b}} + k^d \tilde{\phi}_{\dot{c}}^{\dot{b}} \right) \delta_K^J \right) \\ &\quad \left(\left((k^{*\nu} - k^{*u}) \phi_{\dot{b}}^{\dot{d}} + (k^{*e} - k^{*d}) \tilde{\phi}_{\dot{b}}^{\dot{d}} \right) \Sigma_J^L + \left(k^{*u} \phi_{\dot{b}}^{\dot{d}} + k^{*d} \tilde{\phi}_{\dot{b}}^{\dot{d}} \right) \delta_J^L \right) \\ &\quad \left(\left((k^\nu - k^u) \phi_{\dot{d}}^{\dot{a}} + (k^e - k^d) \tilde{\phi}_{\dot{d}}^{\dot{a}} \right) \Sigma_L^I + \left(k^u \phi_{\dot{d}}^{\dot{a}} + k^d \tilde{\phi}_{\dot{d}}^{\dot{a}} \right) \delta_L^I \right) \end{aligned}$$

$$\begin{aligned} 2H_{\dot{a}I\dot{c}K} \Sigma_{\dot{b}J}^{\dot{c}K} H^{\dot{a}I\dot{d}L} \Sigma_{\dot{d}L}^{\dot{b}J} &= 2 |k^{\nu R}|^2 \left(\Delta_{\dot{a}K} \bar{\Delta}^{\dot{a}L} \Delta_{\dot{c}I} \bar{\Delta}^{\dot{c}I} \right) \\ &\quad \left(\left((k^{*\nu} - k^{*u}) \phi_{\dot{b}}^{\dot{c}} + (k^{*e} - k^{*d}) \tilde{\phi}_{\dot{b}}^{\dot{c}} \right) \Sigma_J^K + \left(k^{*u} \phi_{\dot{b}}^{\dot{c}} + k^{*d} \tilde{\phi}_{\dot{b}}^{\dot{c}} \right) \delta_J^K \right) \\ &\quad \left(\left((k^\nu - k^u) \phi_{\dot{d}}^{\dot{b}} + (k^e - k^d) \tilde{\phi}_{\dot{d}}^{\dot{b}} \right) \Sigma_L^J + \left(k^u \phi_{\dot{d}}^{\dot{b}} + k^d \tilde{\phi}_{\dot{d}}^{\dot{b}} \right) \delta_L^J \right). \end{aligned}$$

Next we have the mass terms

$$H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} = |k^{\nu R}|^2 \left(\Delta_{\dot{a}K} \bar{\Delta}^{\dot{a}K} \right)^2$$

and

$$\begin{aligned} 2\Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{a}I} &= 2 \left(\left((k^\nu - k^u) \phi_{\dot{a}}^{\dot{c}} + (k^e - k^d) \tilde{\phi}_{\dot{a}}^{\dot{c}} \right) \Sigma_I^K + \left(k^u \phi_{\dot{a}}^{\dot{c}} + k^d \tilde{\phi}_{\dot{a}}^{\dot{c}} \right) \delta_I^K \right) \\ &\quad \left(\left((k^{*\nu} - k^{*u}) \phi_{\dot{c}}^{\dot{a}} + (k^{*e} - k^{*d}) \tilde{\phi}_{\dot{c}}^{\dot{a}} \right) \Sigma_K^I + \left(k^{*u} \phi_{\dot{c}}^{\dot{a}} + k^{*d} \tilde{\phi}_{\dot{c}}^{\dot{a}} \right) \delta_K^I \right). \end{aligned}$$

The potential must be analyzed to determine all the possible minima that breaks the symmetry $SU(2)_R \times SU(2)_L \times SU(4)$. In this respect it is useful to determine whether the symmetries of this model break correctly at high energies to the Standard Model.

Needless to say that it is difficult to determine all allowed vacua of this potential, especially since there is dependence of order eight on the fields. It is possible, however, to expand this potential around the vacuum that we started with which breaks the gauge symmetry directly from $SU(2)_R \times SU(2)_L \times SU(4)$ to $U(1)_{\text{em}} \times SU(3)_c$. Explicitly, this vacuum is given by

$$\langle \phi_{\dot{a}}^{\dot{b}} \rangle = v \delta_{\dot{a}}^1 \delta_{\dot{b}}^1 \quad \langle \Sigma_{\dot{a}J}^{\dot{a}I} \rangle = u \delta_{\dot{a}}^1 \delta_{\dot{b}}^1 \quad \langle \Delta_{\dot{a}J} \rangle = w \delta_{\dot{a}}^1 \delta_{\dot{b}}^1. \quad (3)$$

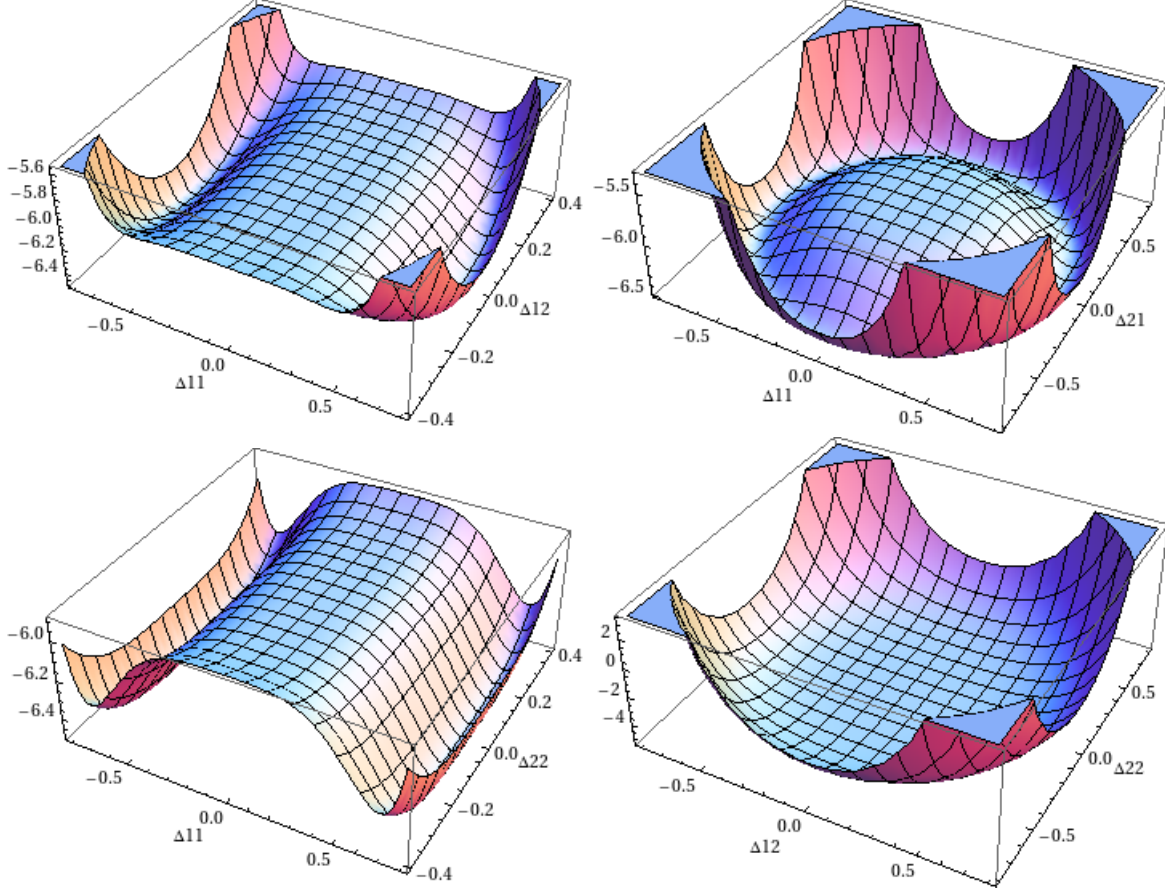


FIG. 1: The scalar potential in some of the $\Delta_{\dot{a}J}$ -directions, with all other fields at their SM-vevs as in Equation (3). We have put $k^\nu = k^e = 1$ and $k^{\nu R} = k^u = k^d = 2$. With these choices, the Standard Model vacuum corresponds to $\Delta_{11} = \frac{1}{\sqrt{2}}$, $\Sigma_1^1 = 2$, $\phi_1^1 = \frac{1}{2}$ and all other fields are zero. At this point the Hessian in the Δ -directions is nonnegative.

We have included several plots of the scalar potential in the $\Delta_{\dot{a}J}$ -directions in Figure 1. A computation of the Hessian in the Δ -directions shows that the SM-vev is indeed a local minimum.

The first order condition now arises as a vacuum solution of the spectral action as follows. We let the Δ -fields take their vev according to the scalar potential, *i.e.* $\Delta_{\dot{a}J} = w\delta_{\dot{a}}^1\delta_J^1$. Since $\Delta_{\dot{a}J}$ is in the $(2_R, 1_L, 4)$ representation of $SU(2)_R \times SU(2)_L \times SU(4)$, this vacuum solution is only invariant under the subgroup

$$\left\{ \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, u_L, \lambda \oplus \lambda^{-1/3}u \right) : \lambda \in U(1), u_L \in SU(2), u \in SU(3) \right\} \subset SU(2)_R \times SU(2)_L \times SU(4).$$

This is the spontaneous symmetry breaking to $U(1) \times SU(2)_L \times SU(3)_c$, thus selecting the subalgebra (2). Note that unimodularity on $\mathcal{U}(\mathcal{A})$ naturally induces unimodularity of the

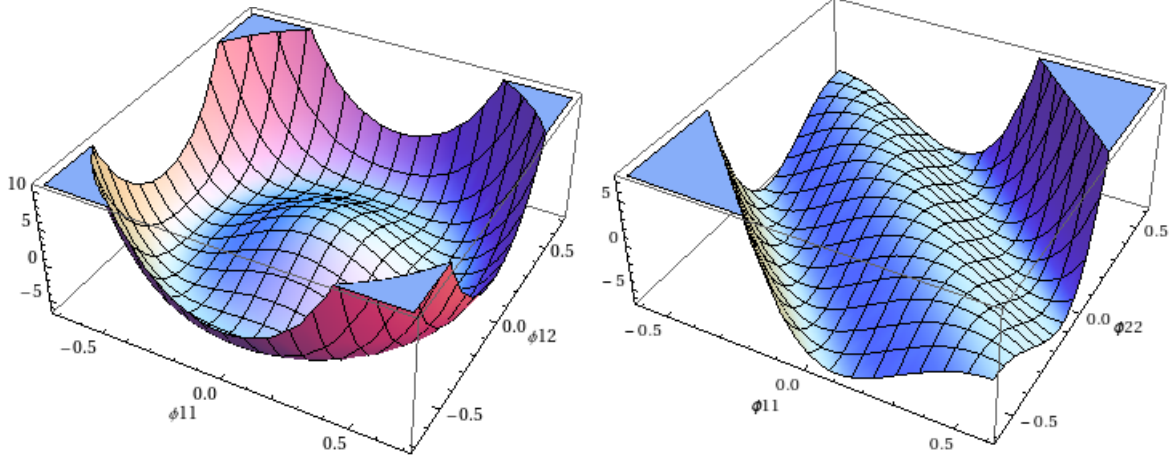


FIG. 2: The scalar potential in the ϕ_a^b -directions, after the Σ and Δ -fields have acquired their SM-vevs as in Equation (3). Again, we have put $k^\nu = k^e = 1$ and $k^{\nu R} = k^u = k^d = 2$.

spectral Standard Model, hence it generates the correct hypercharges for the fermions.

After the Δ and Σ -fields have acquired their vevs, there is a remaining scalar potential for the ϕ -fields, which is depicted in Figure 2. As with the Standard Model Higgs sector, the selection of a minimum further breaks the symmetry from $U(1) \times SU(2)_L \times SU(3)_c$ to $U(1)_{\text{em}} \times SU(3)_c$. The plot on the right in Figure 2 suggests that, instead of the SM-vacuum, the vevs of the ϕ -fields can also be taken of the form

$$\langle \phi_a^b \rangle = v \delta_a^1 \delta_1^b + v' \delta_a^2 \delta_2^b.$$

Let us see which of the gauge fields acquire non-zero mass after spontaneous symmetry breaking, by expanding around the Standard Model vacuum

$$\begin{aligned} \phi_a^b &= v \delta_a^1 \delta_1^b + H_a^b \\ \Sigma_J^I &= u \delta_1^I \delta_J^1 + M_I^J \\ \Delta_{aJ} &= w \delta_a^1 \delta_J^1 + N_{aJ} \end{aligned}$$

and keep only terms of up to order 4. First we look at the kinetic term

$$\begin{aligned} \nabla_\mu H_{aIbJ} &= \partial_\mu H_{aIbJ} - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_a^c H_{cIbJ} - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_b^c H_{aIcJ} \\ &\quad - \frac{i}{2} g V_\mu^m (\lambda^m)_I^K H_{aKbJ} - \frac{i}{2} g V_\mu^m (\lambda^m)_J^K H_{aIbK}. \end{aligned}$$

To lowest orders we have

$$\begin{aligned} H_{aIbJ} &= (k^{*\nu_R})^2 \left(w\delta_a^i \delta_J^1 + N_{aJ} \right) \left(w\delta_b^i \delta_I^1 + N_{bI} \right) \\ &= (k^{*\nu_R})^2 \left(w^2 \delta_a^i \delta_J^1 \delta_b^i \delta_I^1 + w\delta_a^i \delta_J^1 N_{bI} + w\delta_b^i \delta_I^1 N_{aJ} + N_{aJ} N_{bI} \right) \end{aligned}$$

and so

$$\begin{aligned} \nabla_\mu H_{aIbJ} &= (k^{*\nu_R})^2 w \left(\delta_a^i \delta_J^1 \partial_\mu N_{bI} + \delta_b^i \delta_I^1 \partial_\mu N_{aJ} - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_a^i w \delta_J^1 \delta_b^i \delta_I^1 - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_b^i w \delta_a^i \delta_J^1 \delta_I^1 \right. \\ &\quad \left. - \frac{i}{2} g V_\mu^m (\lambda^m)_I^1 w \delta_a^i \delta_J^1 \delta_b^i - \frac{i}{2} g V_\mu^m (\lambda^m)_J^1 w \delta_a^i \delta_b^i \delta_I^1 \right) \\ &= (k^{*\nu_R})^2 w \left(2 \delta_a^i \delta_J^1 \delta_b^i \delta_I^1 \partial_\mu N_{i1} + \delta_a^i \delta_J^1 \delta_b^i \delta_I^2 \partial_\mu N_{2i} + \delta_a^i \delta_J^1 \delta_b^i \delta_I^1 \partial_\mu N_{i1} + \delta_a^i \delta_J^1 \delta_b^i \delta_I^2 \partial_\mu N_{2i} \right. \\ &\quad + \delta_b^i \delta_I^1 \delta_a^j \delta_J^1 \partial_\mu N_{ij} + \delta_b^i \delta_I^1 \delta_a^j \delta_J^2 \partial_\mu N_{2j} + \delta_b^i \delta_I^1 \delta_a^j \delta_J^1 \partial_\mu N_{2j} - i g_R W_{\mu R}^3 w \delta_a^i \delta_J^1 \delta_b^i \delta_I^1 \\ &\quad - \frac{i}{2} g_R W_{\mu R}^- w \delta_a^2 \delta_J^1 \delta_b^i \delta_I^1 - \frac{i}{2} g_R W_{\mu R}^- \delta_b^2 w \delta_a^i \delta_J^1 \delta_I^1 \\ &\quad \left. - i \left(g_R W_{\mu R}^3 + g \sqrt{\frac{3}{2}} V_\mu^{15} \right) \delta_a^i \delta_J^1 \delta_b^i \delta_I^1 - \frac{i}{2} g V_\mu^m (\lambda^m)_i^1 w \delta_a^i \delta_J^1 \delta_b^i \delta_I^1 \right) \end{aligned}$$

from which it is clear that if we write

$$\begin{aligned} g_R W_{\mu R}^3 &= g_1 B_\mu + g'_1 Z'_\mu \\ g \sqrt{\frac{3}{2}} V_\mu^{15} &= -g_1 B_\mu + g'_1 Z'_\mu \end{aligned}$$

then the vector B_μ will not get a mass term while the fields $W_{\mu R}^\pm$, Z'_μ , $V_\mu^m (\lambda^m)_i^1$ (these are the fields in the coset of $\frac{SU(2)_R \times SU(4)}{SU(3) \times U(1)}$) will all become massive, with mass of order w^2 as can be seen from the kinetic term

$$\begin{aligned} \nabla_\mu \Sigma_{aI}^{bJ} &= \partial_\mu \Sigma_{aI}^{bJ} - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_a^c \Sigma_{cI}^{bJ} + \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_c^b \Sigma_{aI}^{cJ} \\ &\quad - \frac{i}{2} g V_\mu^m (\lambda^m)_I^K \Sigma_{aK}^{bJ} + \frac{i}{2} g V_\mu^m (\lambda^m)_K^J \Sigma_{aI}^{bK}. \end{aligned}$$

To lowest orders we have

$$\begin{aligned} \Sigma_{aI}^{bJ} &= \left(\left((k^\nu - k^u) \phi_a^b + (k^e - k^d) \tilde{\phi}_a^b \right) \Sigma_I^J + \left(k^u \phi_a^b + k^d \tilde{\phi}_a^b \right) \delta_I^J \right) \\ &= \left((k^\nu - k^u) \left(v \delta_a^1 \delta_1^b + H_a^b \right) + (k^e - k^d) \left(v \delta_a^2 \delta_2^b + \tilde{H}_a^b \right) \right) \left(u \delta_1^J \delta_I^1 + M_I^J \right) \\ &\quad + \left(k^u \left(v \delta_a^1 \delta_1^b + H_a^b \right) + k^d \left(v \delta_a^2 \delta_2^b + \tilde{H}_a^b \right) \right) \delta_I^J \\ &= v \left(\left((k^\nu - k^u) \delta_a^1 \delta_1^b + (k^e - k^d) \delta_a^2 \delta_2^b \right) u \delta_1^J \delta_I^1 + \left(k^u \delta_a^1 \delta_1^b + k^d \delta_a^2 \delta_2^b \right) \delta_I^J \right) \\ &\quad + \left((k^\nu - k^u) H_a^b + (k^e - k^d) \tilde{H}_a^b \right) u \delta_1^J \delta_I^1 + \left(k^u H_a^b + k^d \tilde{H}_a^b \right) \delta_I^J \\ &\quad + v \left((k^\nu - k^u) \delta_a^1 \delta_1^b + (k^e - k^d) \delta_a^2 \delta_2^b \right) M_I^J \end{aligned}$$

$$\begin{aligned}
\nabla_\mu \Sigma_{\dot{a}I}^{bJ} &= \left((k^\nu - k^u) \partial_\mu H_a^b + (k^e - k^d) \partial_\mu \tilde{H}_a^b \right) u \delta_1^J \delta_I^1 + \left(k^u \partial_\mu H_a^b + k^d \partial_\mu \tilde{H}_a^b \right) \delta_I^J \\
&+ v \left((k^\nu - k^u) \delta_a^1 \delta_1^b + (k^e - k^d) \delta_a^2 \delta_2^b \right) \partial_\mu M_I^J \\
&- \frac{i}{2} v g_R W_{\mu R}^\alpha (\sigma^\alpha)_a^{\dot{c}} \left((k^\nu - k^u) \delta_c^1 \delta_1^b + (k^e - k^d) \delta_c^2 \delta_2^b \right) u \delta_1^J \delta_I^1 + \left(k^u \delta_c^1 \delta_1^b + k^d \delta_c^2 \delta_2^b \right) \delta_I^J \\
&+ \frac{i}{2} v g_L W_{\mu L}^\alpha (\sigma^\alpha)_c^b \left((k^\nu - k^u) \delta_a^1 \delta_1^c + (k^e - k^d) \delta_a^2 \delta_2^c \right) u \delta_1^J \delta_I^1 + \left(k^u \delta_a^1 \delta_1^c + k^d \delta_a^2 \delta_2^c \right) \delta_I^J \\
&- \frac{i}{2} v g V_\mu^m (\lambda^m)_I^K \left((k^\nu - k^u) \delta_a^1 \delta_1^b + (k^e - k^d) \delta_a^2 \delta_2^b \right) u \delta_1^J \delta_I^K + \left(k^u \delta_a^1 \delta_1^b + k^d \delta_a^2 \delta_2^b \right) \delta_I^K \\
&+ \frac{i}{2} g v V_\mu^m (\lambda^m)_K^J \left((k^\nu - k^u) \delta_a^1 \delta_1^b + (k^e - k^d) \delta_a^2 \delta_2^b \right) u \delta_1^K \delta_I^1 + \left(k^u \delta_a^1 \delta_1^b + k^d \delta_a^2 \delta_2^b \right) \delta_I^K.
\end{aligned}$$

For simplicity we will set $u = 1$. Isolating the gauge dependent part

$$\begin{aligned}
\nabla_\mu \Sigma_{11}^{11} &\supset -\frac{i}{2} v (g_R W_{\mu R}^3 - g_L W_{\mu L}^3) k^\nu \\
\nabla_\mu \Sigma_{21}^{21} &\supset \frac{i}{2} v (g_R W_{\mu R}^3 - g_L W_{\mu L}^3) k^e \\
\nabla_\mu \Sigma_{1i}^{11} &\supset -\frac{i}{2} v g V_\mu^m (\lambda^m)_i^1 (k^\nu - k^u) \\
\nabla_\mu \Sigma_{1i}^{1j} &\supset -\frac{i}{2} v (g_R W_{\mu R}^3 - g_L W_{\mu L}^3) k^u \delta_i^j \\
\nabla_\mu \Sigma_{2i}^{2j} &\supset \frac{i}{2} v (g_R W_{\mu R}^3 - g_L W_{\mu L}^3) k^d \delta_i^j \\
\nabla_\mu \Sigma_{11}^{21} &\supset -\frac{i}{2} v (g_R W_{\mu R}^- - g_L W_{\mu L}^-) k^\nu \\
\nabla_\mu \Sigma_{21}^{11} &\supset -\frac{i}{2} v (g_R W_{\mu R}^+ - g_L W_{\mu L}^+) k^e \\
\nabla_\mu \Sigma_{2i}^{11} &\supset 0.
\end{aligned}$$

Noticing that $g_R W_{\mu R}^3 - g_L W_{\mu L}^3 = (g_1 B_\mu - g_L W_{\mu L}^3) + g_1' Z'_\mu$ shows that the Z_μ vector gets a mass of order of the weak scale gv while the $W_{\mu R}^\pm$ and Z'_μ will get a small correction to its mass of order gw . Thus we get the correct gauge breaking pattern with the gauge fields $W_{\mu L}$ and Z of the Standard model having masses of the order of the electroweak scale. It is important, however, to see explicitly that the mixing between the Z and Z' vectors and W_L^\pm, W_R^\pm are suppressed.

It remains to minimize the potential to determine all possible minima as well as studying the unified model and check whether it allows for unification of coupling constants

$$g_R = g_L = g$$

in addition to determining the top quark mass and Higgs mass. Obviously, this model deserves careful analysis, which will be the subject of future work.

We conclude that the study of noncommutative spaces based on a product of a continuous four dimensional manifold times a finite space of KO -dimension 6, without the first order condition gives rise to almost unique possibility in the form of a Pati-Salam type model. This provides a setting for unification avoiding the desert and which goes beyond the SM. In addition one of the vacua of the Higgs fields gives rise at low energies to a Dirac operator satisfying the first order condition. In this way, the first order condition arises as a spontaneously broken phase of higher symmetry and is not imposed from outside.

VII. APPENDIX: DETAILED CALCULATIONS FOR THE PRACTITIONER

For the benefit of the reader, we shall present in this appendix a detailed derivation of the Dirac operator and the spectral action for the noncommutative space on $\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$.

For $A_{(1)}$ we have the definition

$$(A_{(1)})_M^N = \sum a_M^P [D, b]_P^N$$

where

$$a_M^N = \begin{pmatrix} X'_\alpha{}^\beta \delta_I^J & 0 \\ 0 & \delta_{\alpha'}^{\beta'} Y_{I'}^{J'} \end{pmatrix}$$

which in terms of components give

$$\begin{aligned} (A_{(1)})_{\alpha I}^{\beta J} &= \sum a_{\alpha I}^{\gamma K} \left(D_{\gamma K}^{\delta L} b_{\delta L}^{\beta J} - b_{\gamma K}^{\delta L} D_{\delta L}^{\beta J} \right) \\ &= \sum X'_\alpha{}^\gamma \left(D_{\gamma I}^{\delta J} X_\delta^\beta - X_\gamma^\delta D_{\delta I}^{\beta J} \right) \end{aligned}$$

where we use the notation for b_M^N to be the same as that of a_M^N without primes (i.e. $X' \rightarrow X$, $Y' \rightarrow Y$). Since $D_{\alpha I}^{\beta J}$ is non vanishing when connecting a dotted index \dot{a} to a , we have the non-vanishing components

$$\begin{aligned} (A_{(1)})_{\dot{a} I}^{b J} &= \sum X'_{\dot{a}}{}^c \left(D_{c I}^{d J} X_d^b - X_{\dot{c}}^d D_{d I}^{b J} \right) \\ &= \delta_I^1 \delta_1^J \left(\sum X'_{\dot{a}}{}^c \left(\left(\delta_{\dot{a}}^1 \delta_1^d k^\nu + \delta_{\dot{a}}^2 \delta_2^d k^e \right) X_d^b \right) - X_{\dot{c}}^d \left(\delta_{\dot{a}}^1 \delta_1^b k^\nu + \delta_{\dot{a}}^2 \delta_2^b k^e \right) \right) \\ &+ \delta_I^i \delta_j^J \delta_i^j \left(\sum X'_{\dot{a}}{}^c \left(\left(\delta_{\dot{a}}^1 \delta_1^b k^u + \delta_{\dot{a}}^2 \delta_2^b k^d \right) X_d^b \right) - X_{\dot{c}}^d \left(\delta_{\dot{a}}^1 \delta_1^b k^u + \delta_{\dot{a}}^2 \delta_2^b k^d \right) \right) \\ &= \delta_I^1 \delta_1^J \left(k^\nu \phi_{\dot{a}}^b + k^e \tilde{\phi}_{\dot{a}}^b \right) + \delta_I^i \delta_j^J \delta_i^j \left(k^u \phi_{\dot{a}}^b + k^d \tilde{\phi}_{\dot{a}}^b \right) \end{aligned}$$

where

$$\begin{aligned}\phi_a^b &= \sum X_a'^i X_1^b - X_a'^c X_c^i \delta_1^b \\ \tilde{\phi}_a^b &= \sum X_a'^2 X_2^b - X_a'^c X_c^2 \delta_2^b.\end{aligned}$$

We can check that

$$\tilde{\phi}_a^b = \tau_2 \overline{\phi}_a^b \tau_2.$$

For example

$$\begin{aligned}\tilde{\phi}_1^1 &= \sum X_1'^2 X_2^1 \\ &= \sum \overline{X}_2^1 \overline{X}_1^2 \\ &= \overline{\phi}_2^2\end{aligned}$$

using the quaternionic property of the X . Note that ϕ_a^b is in the $(2_R, 2_L, 1)$ representation of $SU(2)_R \times SU(2)_L \times SU(4)$.

Similarly we have

$$A_{aI}^{bJ} = \left(A_{bJ}^{aI} \right)^*.$$

(In reality one obtains an expression for A_{aI}^{bJ} in terms of $\phi_a^{b'}$ which is expressed in terms of the X , but the hermiticity of the Dirac operator forces the above relation and imposes a constraint on the X .)

Next we have

$$\begin{aligned}(A_{(1)})_{\alpha I}^{\beta' J'} &= \sum a_{\alpha I}^{\gamma K} \left(D_{\gamma K}^{\delta' L'} b_{\delta' L'}^{\beta' J'} - b_{\gamma K}^{\delta L} D_{\delta L}^{\beta' J'} \right) \\ &= \sum X_{\alpha}^{\gamma} \left(D_{\gamma I}^{\beta' L'} Y_{L'}^{J'} - X_{\gamma}^{\delta} D_{\delta I}^{\beta' J'} \right) \\ &= k^{*\nu_R} \sum X_{\alpha}^{\gamma} \left(\left(\delta_{\gamma}^i \delta_{i'}^{\beta'} \delta_I^1 \delta_{1'}^{L'} \right) Y_{L'}^{J'} - X_{\gamma}^{\delta} \left(\delta_{\delta}^1 \delta_{i'}^{\beta'} \delta_I^1 \delta_{1'}^{J'} \right) \right) \\ &= k^{*\nu_R} \delta_{\alpha}^i \delta_{i'}^{\beta'} \delta_I^1 \sum \left(X_a'^i Y_{1'}^{J'} - X_a'^c X_c^i \delta_{1'}^{J'} \right) \\ (A_{(1)})_{\dot{a} I}^{\dot{b}' J'} &= k^{*\nu_R} \delta_{i'}^{\dot{b}'} \delta_I^1 \Delta_{\dot{a}}^{J'}\end{aligned}$$

where

$$\Delta_{\dot{a}}^{J'} = \sum \left(X_a'^i Y_{1'}^{J'} - X_a'^c X_c^i \delta_{1'}^{J'} \right) \equiv \Delta_{\dot{a} J}$$

which is in the $(2_R, 1_L, 4)$ representation of $SU(2)_R \times SU(2)_L \times SU(4)$. Again, we can compute $A_{\alpha' I'}^{\beta J}$, which gives a similar expression, but using hermiticity we write

$$A_{\alpha' I'}^{\beta J} = \left(A_{\beta J}^{\alpha' I'} \right)^* = k^{\nu_R} \delta_b^{\beta} \delta_{\alpha'}^1 \delta_1^J \Delta_{I'}^{\dot{b}}$$

In the conjugate space we have

$$\begin{aligned} (A_{(1)})_{\alpha' I'}^{\beta' J'} &= \sum a_{\alpha' I'}^{\gamma' K'} \left(D_{\gamma' K'}^{\delta' L'} b_{\delta' L'}^{\beta' J'} - b_{\gamma' K'}^{\delta' L'} D_{\delta' L'}^{\beta' J'} \right) \\ &= \sum Y_{I'}^{\gamma' K'} \left(D_{\alpha' K'}^{\beta' L'} Y_{L'}^{J'} - Y_{K'}^{L'} D_{\alpha' L'}^{\beta' J'} \right). \end{aligned}$$

The only non-vanishing expression would involve a D with mixed a' and b'

$$\begin{aligned} (A_{(1)})_{\dot{a}' I'}^{b' J'} &= \sum Y_{I'}^{\gamma' K'} \left(D_{\dot{a}' K'}^{b' L'} Y_{L'}^{J'} - Y_{K'}^{L'} D_{\dot{a}' L'}^{b' J'} \right) \\ &= \sum Y_{I'}^{\gamma' K'} \left(\left(\delta_{K'}^{1'} \delta_{1'}^{L'} \left(\delta_{\dot{a}'}^{1'} \delta_{1'}^{b'} \bar{k}^\nu + \delta_{\dot{a}'}^{2'} \delta_{2'}^{b'} \bar{k}^e \right) + \delta_{K'}^{k'} \delta_{l'}^{L'} \delta_{k'}^{l'} \left(\delta_{\dot{a}'}^{1'} \delta_{1'}^{b'} \bar{k}^u + \delta_{\dot{a}'}^{2'} \delta_{2'}^{b'} \bar{k}^d \right) \right) Y_{L'}^{J'} \right. \\ &\quad \left. - Y_{K'}^{L'} \left(\left(\delta_{L'}^{1'} \delta_{1'}^{J'} \left(\delta_{\dot{a}'}^{1'} \delta_{1'}^{b'} \bar{k}^\nu + \delta_{\dot{a}'}^{2'} \delta_{2'}^{b'} \bar{k}^e \right) + \delta_{L'}^{l'} \delta_{j'}^{J'} \delta_{l'}^{j'} \left(\delta_{\dot{a}'}^{2'} \delta_{2'}^{b'} \bar{k}^u + \delta_{\dot{a}'}^{2'} \delta_{2'}^{b'} \bar{k}^d \right) \right) \right) \right) \\ &= \left(\left(\bar{k}^\nu - \bar{k}^u \right) \delta_{\dot{a}'}^{1'} \delta_{1'}^{b'} + \left(\bar{k}^e - \bar{k}^d \right) \delta_{\dot{a}'}^{2'} \delta_{2'}^{b'} \right) \Sigma_{I'}^{J'} \end{aligned}$$

where

$$\Sigma_{I'}^{J'} = \sum Y_{I'}^{\alpha' I} Y_{I'}^{J'} - Y_{I'}^{\alpha' I} Y_{I'}^{J'} \delta_{I'}^{J'}.$$

Notice that if $k^\nu = k^u$ and $k^e = k^d$ which is consistent with the picture of having the lepton number as the fourth color then Σ_I^J will decouple. We also have

$$(A_{(1)})_{\dot{a}' I'}^{b' J'} = \left(\left((k^\nu)^t - (k^u)^t \right) \delta_{\dot{a}'}^{1'} \delta_{1'}^{b'} + \left((k^e)^t - (k^d)^t \right) \delta_{\dot{a}'}^{2'} \delta_{2'}^{b'} \right) \Sigma_{I'}^{J'}$$

which implies by the hermiticity of

$$A_{\dot{a}' I'}^{b' J'} = \left(A_{b' J'}^{\dot{a}' I'} \right)^*$$

that

$$\Sigma_I^J = \left(\Sigma_I^J \right)^*$$

and thus belong to the $1 + 15$ representation of $SU(4)$. There is no indication that the singlet which is equal to the trace Σ_I^I should be absent as there is no apparent identity that equates this trace to zero. In this case we can write

$$\Sigma_I^J = \tilde{\Sigma}_I^J + \frac{1}{4} \delta_I^J \Sigma, \quad \Sigma = \Sigma_I^I, \quad \tilde{\Sigma}_I^I = 0.$$

Thus at first order we have the Higgs fields $\phi_a^{\dot{b}}$ and $\Delta_{\dot{a} I}$. In addition if the Yukawa couplings of the leptons are different from the corresponding quarks (and thus requiring the breaking of the lepton number as the fourth color) then an additional Higgs field Σ_I^J is also generated.

Next it is straightforward to evaluate various components of $JA_{(1)}J^{-1}$ which are given by

$$\begin{aligned}(JAJ^{-1})_A^B &= \overline{A}_{A'}^{B'} \\ (JAJ^{-1})_A^{B'} &= \overline{A}_{A'}^B \\ (JAJ^{-1})_{A'}^{B'} &= \overline{A}_A^B \\ (JAJ^{-1})_{A'}^B &= \overline{A}_A^{B'}.\end{aligned}$$

In particular

$$\begin{aligned}(JA_{(1)}J^{-1})_{\dot{a}I}^{bJ} &= (\overline{A}_{(1)})_{\dot{a}'I'}^{b'J'} \\ &= \left((k^\nu - k^u) \delta_a^{\dot{1}} \delta_1^b + (k^e - k^d) \delta_a^{\dot{2}} \delta_2^b \right) \Sigma_I^J\end{aligned}$$

$$\begin{aligned}(JA_{(1)}J^{-1})_{\dot{a}I}^{\dot{b}J'} &= (\overline{A}_{(1)})_{\dot{a}'I'}^{\dot{b}J} \\ &= \overline{k}^{\nu R} \delta_a^{\dot{1}} \delta_{1'}^{J'} \overline{\Delta}^{\dot{b}}{}_I \\ &\equiv \overline{k}^{\nu R} \delta_a^{\dot{1}} \delta_{1'}^{J'} \Delta_{\dot{b}I}.\end{aligned}$$

We now evaluate

$$(A_{(2)})_M^N = \sum a_M^P [JA_{(1)}J^{-1}, b]_P^N.$$

First we have

$$\begin{aligned}(A_{(2)})_{\alpha I}^{\beta J} &= \sum a_{\alpha I}^{\gamma K} \left((JA_{(1)}J^{-1})_{\gamma K}^{\delta L} b_{\delta L}^{\beta J} - b_{\gamma K}^{\delta L} (JA_{(1)}J^{-1})_{\delta L}^{\beta J} \right) \\ &= \sum X_{\alpha}^{\gamma} \left((JA_{(1)}J^{-1})_{\gamma I}^{\delta J} X_{\delta}^{\beta} - X_{\gamma}^{\delta} (JA_{(1)}J^{-1})_{\delta I}^{\beta J} \right).\end{aligned}$$

Thus

$$\begin{aligned}(A_{(2)})_{\dot{a}I}^{bJ} &= \sum X_{\dot{a}}^{\dot{c}} \left((JA_{(1)}J^{-1})_{\dot{c}I}^{dJ} X_d^b - X_{\dot{c}}^d (JA_{(1)}J^{-1})_{\dot{a}I}^{bJ} \right) \\ &= \sum X_{\dot{a}}^{\dot{c}} \left(\left((k^\nu - k^u) \delta_{\dot{c}}^{\dot{1}} \delta_1^d + (k^e - k^d) \delta_{\dot{c}}^{\dot{2}} \delta_2^d \right) X_d^b - \right. \\ &\quad \left. - X_{\dot{c}}^d \left((k^\nu - k^u) \delta_a^{\dot{1}} \delta_1^b + (k^e - k^d) \delta_a^{\dot{2}} \delta_2^b \right) \right) \Sigma_I^J \\ &= \left((k^\nu - k^u) \left(\sum X_{\dot{a}}^{\dot{1}} X_1^b - X_{\dot{a}}^{\dot{c}} X_{\dot{c}}^{\dot{1}} \delta_1^b \right) + (k^e - k^d) \left(\sum X_{\dot{a}}^{\dot{2}} X_2^b - X_{\dot{a}}^{\dot{c}} X_{\dot{c}}^{\dot{2}} \delta_2^b \right) \right) \Sigma_I^J \\ &= \left((k^\nu - k^u) \phi_{\dot{a}}^b + (k^e - k^d) \tilde{\phi}_{\dot{a}}^b \right) \Sigma_I^J.\end{aligned}$$

Similarly $(A_{(2)})_{\dot{a}I}^{\dot{b}J}$ is the Hermitian conjugate of $(A_{(2)})_{\dot{b}I}^{aJ}$. Next we have

$$\begin{aligned}
(A_{(2)})_{\dot{a}I}^{\dot{b}'J'} &= \sum X_{\dot{a}}^{\dot{c}} \left((JA_{(1)}J^{-1})_{\dot{c}I}^{\dot{b}'L'} Y_{L'}^{J'} - X_{\dot{c}}^{\dot{d}} (JA_{(1)}J^{-1})_{\dot{d}I}^{\dot{b}'J'} \right) \\
&= \bar{k}^{\nu R} \sum \left(X_{\dot{a}}^{\dot{c}} Y_{\dot{c}}^{J'} - X_{\dot{a}}^{\dot{c}} X_{\dot{c}}^{\dot{d}} \delta_{\dot{d}}^{J'} \right) \bar{\Delta}_{\dot{a}I}^{\dot{b}'} \\
&= \bar{k}^{\nu R} \Delta_{\dot{a}}^{J'} \bar{\Delta}_{\dot{a}I}^{\dot{b}'} \\
&= k^{*\nu R} \Delta_{\dot{a}J} \Delta_{\dot{b}I}.
\end{aligned}$$

Collecting all terms we get

$$\begin{aligned}
(D_A)_{\dot{a}I}^{bJ} &= \left(\delta_{\dot{a}}^{\dot{c}} \delta_1^b k^\nu + \delta_{\dot{a}}^{\dot{c}} \delta_2^b k^e \right) \delta_I^1 \delta_1^J + \left(\delta_{\dot{a}}^{\dot{c}} \delta_1^b k^u + \delta_{\dot{a}}^{\dot{c}} \delta_2^b k^d \right) \delta_I^i \delta_j^J \delta_i^j \\
&+ \delta_I^1 \delta_1^J \left(k^\nu \phi_{\dot{a}}^b + k^e \tilde{\phi}_{\dot{a}}^b \right) + \delta_I^i \delta_j^J \delta_i^j \left(k^u \phi_{\dot{a}}^b + k^d \tilde{\phi}_{\dot{a}}^b \right) \\
&+ \left((k^\nu - k^u) \delta_{\dot{a}}^{\dot{c}} \delta_1^b + (k^e - k^d) \delta_{\dot{a}}^{\dot{c}} \delta_2^b \right) \Sigma_I^J \\
&+ \left((k^\nu - k^u) \phi_{\dot{a}}^b + (k^e - k^d) \tilde{\phi}_{\dot{a}}^b \right) \Sigma_I^J \\
&= \left(k^\nu \left(\delta_{\dot{a}}^{\dot{c}} \delta_1^b + \phi_{\dot{a}}^b \right) + k^e \left(\delta_{\dot{a}}^{\dot{c}} \delta_2^b + \tilde{\phi}_{\dot{a}}^b \right) \right) \left(\delta_I^1 \delta_1^J + \Sigma_I^J \right) \\
&+ \left(k^u \left(\delta_{\dot{a}}^{\dot{c}} \delta_1^b + \phi_{\dot{a}}^b \right) + k^d \left(\delta_{\dot{a}}^{\dot{c}} \delta_2^b + \tilde{\phi}_{\dot{a}}^b \right) \right) \left(\delta_I^i \delta_j^J \delta_i^j - \Sigma_I^J \right).
\end{aligned}$$

The other non-vanishing term is

$$\begin{aligned}
(D_A)_{\dot{a}I}^{\dot{b}'J'} &= k^{*\nu R} \left(\delta_{\dot{a}}^{\dot{c}} \delta_{\dot{c}}^{\dot{b}'} \delta_I^1 \delta_{\dot{c}}^{J'} + \delta_{\dot{c}}^{\dot{b}'} \delta_I^1 \Delta_{\dot{a}}^{J'} + \delta_{\dot{a}}^{\dot{c}} \delta_{\dot{c}}^{J'} \bar{\Delta}_{\dot{a}I}^{\dot{b}'} + \Delta_{\dot{a}}^{J'} \bar{\Delta}_{\dot{a}I}^{\dot{b}'} \right) \\
&= k^{*\nu R} \left(\delta_{\dot{a}}^{\dot{c}} \delta_{\dot{c}}^{J'} + \Delta_{\dot{a}}^{J'} \right) \left(\delta_{\dot{c}}^{\dot{b}'} \delta_I^1 + \bar{\Delta}_{\dot{a}I}^{\dot{b}'} \right) \\
&\equiv k^{*\nu R} \left(\delta_{\dot{a}}^{\dot{c}} \delta_{\dot{c}}^1 + \Delta_{\dot{a}J} \right) \left(\delta_{\dot{c}}^{\dot{b}'} \delta_I^1 + \Delta_{\dot{b}I} \right) \\
&\equiv (D_A)_{\dot{a}I\dot{b}J}.
\end{aligned}$$

All other non-vanishing terms are related to the above two by Hermitian conjugation.

Note that $D_{\dot{a}I}^{bJ}$ gives, after spontaneous breaking, the Dirac masses while $D_{\dot{a}I}^{\dot{b}'J'}$ gives the Majorana masses. The Higgs fields are composite, the fundamental ones being of similar form to those formed from the fermion bilinears.

It is possible to absorb the constant terms (vacuum expectation values) by redefining the

fields

$$\begin{aligned}\delta_a^1 \delta_1^b + \phi_a^b &\rightarrow \phi_a^b \\ \delta_a^1 \delta_J^1 + \Delta_{aJ} &\rightarrow \Delta_{aJ} \\ \delta_I^1 \delta_1^J + \Sigma_I^J &\rightarrow \Sigma_I^J\end{aligned}$$

so that when the potential of the spectral action is minimized one will get

$$\begin{aligned}\langle \phi_a^b \rangle &= \delta_a^1 \delta_1^b \\ \langle \Delta_{aJ} \rangle &= \delta_a^1 \delta_J^1 \\ \langle \Sigma_I^J \rangle &= \delta_I^1 \delta_1^J.\end{aligned}$$

Thus

$$\begin{aligned}(D_A)_{aI}^{bJ} &= \gamma_5 \left(\left(k^\nu \phi_a^b + k^e \tilde{\phi}_a^b \right) \Sigma_I^J + \left(k^u \phi_a^b + k^d \tilde{\phi}_a^b \right) (\delta_I^J - \Sigma_I^J) \right) \equiv \gamma_5 \Sigma_{aI}^{bJ} \\ (D_A)_{aI}^{\dot{b}J'} &= \gamma_5 k^{*\nu R} \Delta_{aJ} \Delta_{bI} \equiv \gamma_5 H_{aIbJ}\end{aligned}$$

and the fundamental Higgs fields are $(2_R, 2_L, 1)$, $(2_R, 1_L, 4)$, $(1_R, 1_L, 1 + 15)$. The last of which Σ_I^J drops out in the case when we take the lepton and quark Yukawa couplings to be identical. This is a realistic possibility and has the advantage that the Higgs sector becomes minimal.

The full Dirac operator on the product space $M \times F$ is

$$(D_A) = \gamma^\mu D_\mu \otimes 1 + \gamma_5 D_F.$$

This gives the gauge fields

$$A_{\alpha I}^{\beta J} = \gamma^\mu \sum X_\alpha^{\prime\gamma} \partial_\mu X_\gamma^\beta \delta_I^J$$

and in particular

$$\begin{aligned}A_{aI}^{\dot{b}J} &= \gamma^\mu \sum X_a^{\prime c} \partial_\mu X_c^{\dot{b}} \delta_I^J \\ &= \gamma^\mu \left(-\frac{i}{2} g_R W_{\mu R}^\alpha \right) (\sigma^\alpha)_a^{\dot{b}} \delta_I^J\end{aligned}$$

which is the gauge field of $SU(2)_R$. Notice that $W_{\mu R}^\alpha$ are $SU(2)_R$ and not $U(2)$ gauge fields because $X_a^{\prime c} \partial_\mu X_c^{\dot{b}}$ depend on quaternionic elements. Similarly

$$\begin{aligned}A_{aI}^{bJ} &= \gamma^\mu \sum X_a^{\prime c} \partial_\mu X_c^b \delta_I^J \\ &= \gamma^\mu \left(-\frac{i}{2} g_L W_{\mu L}^\alpha \right) (\sigma^\alpha)_a^b \delta_I^J\end{aligned}$$

where the $W_{\mu L}^\alpha$ are $SU(2)_L$ gauge fields. In the conjugate sector we have

$$\begin{aligned} A_{\alpha' I'}^{\beta' J'} &= \gamma^\mu \delta_{\alpha'}^{\beta'} \sum Y_{I'}^{K'} \partial_\mu Y_{K'}^{J'} \\ &= \gamma^\mu \delta_{\alpha'}^{\beta'} \left(\frac{i}{2} g V_\mu^m (\lambda^m)_{I'}^{J'} + \frac{i}{2} g V_\mu \delta_{I'}^{J'} \right) \end{aligned}$$

where V_μ^m and V_μ are the $U(4)$ gauge fields. This implies that

$$\begin{aligned} (JAJ^{-1})_{\dot{a}I}^{bJ} &= -\gamma^\mu \delta_{\dot{a}}^b \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J + \frac{i}{2} g V_\mu \delta_I^J \right) \\ (JAJ^{-1})_{aI}^{bJ} &= -\gamma^\mu \delta_a^b \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J + \frac{i}{2} g V_\mu \delta_I^J \right) \end{aligned}$$

where λ^m are the generators of the group $SU(4)$ satisfying $\text{Tr}(\lambda^m) = 0$. We deduce that we get new contributions to

$$\begin{aligned} (D_A)_{\dot{a}I}^{bJ} &= \gamma^\mu \left(D_\mu \delta_{\dot{a}}^b \delta_I^J - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_{\dot{a}}^b \delta_I^J - \delta_{\dot{a}}^b \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J + \frac{i}{2} g V_\mu \delta_I^J \right) \right) \\ (D_A)_{aI}^{bJ} &= \gamma^\mu \left(D_\mu \delta_a^b \delta_I^J - \frac{i}{2} g_L W_{\mu L}^\alpha (\sigma^\alpha)_a^b \delta_I^J - \delta_a^b \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J + \frac{i}{2} g V_\mu \delta_I^J \right) \right). \end{aligned}$$

The requirement that A is unimodular implies that

$$\text{Tr}(A) = 0$$

which gives the condition

$$V_\mu = 0$$

and thus the gauge group of this space is

$$SU(2)_R \times SU(2)_L \times SU(4).$$

Summarizing, we have

$$\begin{aligned} (D_A)_{\dot{a}I}^{bJ} &= \gamma^\mu \left(D_\mu \delta_{\dot{a}}^b \delta_I^J - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_{\dot{a}}^b \delta_I^J - \delta_{\dot{a}}^b \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J \right) \right) \otimes 1_3 \\ (D_A)_{aI}^{bJ} &= \gamma^\mu \left(D_\mu \delta_a^b \delta_I^J - \frac{i}{2} g_L W_{\mu L}^\alpha (\sigma^\alpha)_a^b \delta_I^J - \delta_a^b \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J \right) \right) \otimes 1_3 \\ (D_A)_{\dot{a}I}^{bJ} &= \gamma_5 \left(\left(k^\nu \phi_a^b + k^e \tilde{\phi}_a^b \right) \Sigma_I^J + \left(k^u \phi_a^b + k^d \tilde{\phi}_a^b \right) (\delta_I^J - \Sigma_I^J) \right) \equiv \gamma_5 \Sigma_{\dot{a}I}^{bJ} \\ (D_A)_{\dot{a}I}^{\dot{b}' J'} &= \gamma_5 k^{*\nu R} \Delta_{\dot{a}J} \Delta_{\dot{b}' I} \equiv \gamma_5 H_{\dot{a}I \dot{b}' J} \end{aligned}$$

where 1_3 is for generations and

$$D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{cd}(e)\gamma_{cd}$$

and other components are related to the ones above by

$$D_{A'}^{B'} = \overline{D}_A^B, \quad D_{A'}^B = \overline{D}_A^{B'}, \quad D_A^{B'} = \overline{D}_{A'}^B.$$

We now proceed to calculate $(D_A)^2$. The first step is to expand D^2 into the form

$$(D_A)^2 = -(g^{\mu\nu}\partial_\mu\partial_\nu + \mathcal{A}^\mu\partial_\mu + B)$$

and from this extract the connection ω_μ

$$(D_A)^2 = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + E) \tag{4}$$

where

$$\nabla_\mu = \partial_\mu + \omega_\mu.$$

This gives

$$\begin{aligned} \omega_\mu &= \frac{1}{2}g_{\mu\nu}(\mathcal{A}^\nu + \Gamma^\nu) \\ E &= B - g^{\mu\nu}(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu - \Gamma_{\mu\nu}^\rho\omega_\rho) \\ \Omega_{\mu\nu} &= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu] \end{aligned}$$

where $\Gamma^\nu = g^{\rho\sigma}\Gamma_{\rho\sigma}^\nu$ and $\Gamma_{\mu\nu}^\rho$ is the Christoffel connection of the metric $g_{\mu\nu}$. We now proceed to evaluate the various components of D^2 :

$$\begin{aligned} ((D_A)^2)_{aI}^{bJ} &= (D_A)_{aI}^{\dot{c}K} (D_A)_{\dot{c}K}^{bJ} + (D_A)_{aI}^{cK} (D_A)_{cK}^{bJ} \\ &= \Sigma_{aI}^{\dot{c}K} \Sigma_{\dot{c}K}^{bJ} \\ &+ \left[\gamma^\mu \left(D_\mu \delta_a^{\dot{c}} \delta_I^K - \frac{i}{2} g_L W_{\mu L}^\alpha (\sigma^\alpha)_a^{\dot{c}} \delta_I^K + \delta_a^{\dot{c}} \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^K \right) \right) \right. \\ &\left. \gamma^\nu \left(D_\nu \delta_c^{\dot{b}} \delta_K^J - \frac{i}{2} g_L W_{\nu L}^\alpha (\sigma^\alpha)_c^{\dot{b}} \delta_K^J + \delta_c^{\dot{b}} \left(\frac{i}{2} g V_\nu^m (\lambda^m)_K^J \right) \right) \right] 1_3 \\ ((D_A)^2)_{\dot{a}I}^{\dot{b}J} &= (D_A)_{\dot{a}I}^{\dot{c}K} (D_A)_{\dot{c}K}^{\dot{b}J} + (D_A)_{\dot{a}I}^{cK} (D_A)_{cK}^{\dot{b}J} + (D_A)_{\dot{a}I}^{\dot{c}K'} (D_A)_{\dot{c}K'}^{\dot{b}J} \\ &= H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{b}J} + \Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{b}J} \\ &+ \left[\gamma^\mu \left(D_\mu \delta_{\dot{a}}^{\dot{c}} \delta_I^K - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_{\dot{a}}^{\dot{c}} \delta_I^K + \delta_{\dot{a}}^{\dot{c}} \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^K \right) \right) \right. \\ &\left. \gamma^\nu \left(D_\nu \delta_c^{\dot{b}} \delta_K^J - \frac{i}{2} g_R W_{\nu R}^\alpha (\sigma^\alpha)_c^{\dot{b}} \delta_K^J + \delta_c^{\dot{b}} \left(\frac{i}{2} g V_\nu^m (\lambda^m)_K^J \right) \right) \right] \end{aligned}$$

$$\begin{aligned}
((D_A)^2)_{\dot{a}I}^{bJ} &= (D_A)_{\dot{a}I}^{cK} (D_A)_{cK}^{bJ} + (D_A)_{\dot{a}I}^{\dot{c}K} (D_A)_{\dot{c}K}^{bJ} \\
&= \gamma_5 \gamma^\mu \Sigma_{\dot{a}I}^{cK} \left(D_\mu \delta_c^b \delta_K^J - \frac{i}{2} g_L W_{\mu L}^\alpha (\sigma^\alpha)_c^b \delta_K^J + \delta_c^b \left(\frac{i}{2} g V_\mu^m (\lambda^m)_K^J \right) \right) \\
&\quad - \gamma_5 \gamma^\mu \left(D_\mu \delta_{\dot{a}}^{\dot{c}} \delta_I^K - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_{\dot{a}}^{\dot{c}} \delta_I^K + \delta_{\dot{a}}^{\dot{c}} \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^K \right) \right) \Sigma_{\dot{c}K}^{bJ} \\
&= \gamma^\mu \gamma_5 \nabla_\mu \Sigma_{\dot{a}I}^{bJ}
\end{aligned}$$

where the covariant derivative $\nabla_\mu \Sigma_{\dot{a}I}^{bJ}$ is taken with respect to the gauge group $SU(2)_R \times SU(2)_L \times SU(4)$. We also have

$$\begin{aligned}
((D_A)^2)_{\dot{a}I}^{\dot{b}'J'} &= (D_A)_{\dot{a}I}^{\dot{c}K} (D_A)_{\dot{c}K}^{\dot{b}'J'} + (D_A)_{\dot{a}I}^{\dot{c}'K'} (D_A)_{\dot{c}'K'}^{\dot{b}'J'} \\
&= \gamma^\mu \gamma_5 \left(D_\mu \delta_{\dot{a}}^{\dot{c}} \delta_I^K - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_{\dot{a}}^{\dot{c}} \delta_I^K + \delta_{\dot{a}}^{\dot{c}} \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^K \right) \right) H_{cK\dot{b}J} \\
&\quad - \gamma^\mu \gamma_5 H_{\dot{a}I\dot{c}K} \overline{\left(D_\mu \delta_c^b \delta_K^J - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_c^b \delta_K^J + \delta_c^b \left(\frac{i}{2} g V_\mu^m (\lambda^m)_K^J \right) \right)} \\
&= \gamma^\mu \gamma_5 \nabla_\mu H_{\dot{a}I\dot{b}J}
\end{aligned}$$

where the covariant derivative now will be with respect to $SU(2)_R \times SU(4)$. Next we have

$$\begin{aligned}
(D^2)_{\dot{a}I}^{b'J'} &= D_{\dot{a}I}^{\dot{c}'K'} D_{\dot{c}'K'}^{b'J'} \\
&= H_{\dot{a}I\dot{c}K} \Sigma_{\dot{b}J}^{\dot{c}K}
\end{aligned}$$

and finally

$$\begin{aligned}
((D_A)^2)_{\dot{a}I}^{\dot{b}'J'} &= (D_A)_{\dot{a}I}^{\dot{c}K} (D_A)_{\dot{c}K}^{\dot{b}'J'} \\
&= \Sigma_{\dot{a}I}^{\dot{c}K} H_{\dot{c}K\dot{b}J}.
\end{aligned}$$

We then list the entries of the matrices $(\omega_\mu)_M^N$, $(E)_M^N$ which are deduced from the form of the operator $(D_A)^2$. First we have

$$\begin{aligned}
(\omega_\mu)_{aI}^{bJ} &= \left(\left(\frac{1}{4} \omega_\mu^{cd} (e) \gamma_{cd} \right) \delta_a^b \delta_I^J - \frac{i}{2} g_L W_{\mu L}^\alpha (\sigma^\alpha)_a^b \delta_I^J - \frac{i}{2} g V_\mu^m (\lambda^m)_I^J \delta_a^b \right) \otimes 1_3 \\
(\omega_\mu)_{\dot{a}I}^{\dot{b}J} &= \left(\left(\frac{1}{4} \omega_\mu^{cd} (e) \gamma_{cd} \right) \delta_{\dot{a}}^{\dot{b}} \delta_I^J - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_{\dot{a}}^{\dot{b}} \delta_I^J - \frac{i}{2} g \delta_{\dot{a}}^{\dot{b}} V_\mu^m (\lambda^m)_I^J \right) \otimes 1_3 \\
(\omega_\mu)_{A'}^{B'} &= (\bar{\omega}_\mu)_A^B.
\end{aligned}$$

This in turn implies that the components of the curvature

$$\Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu]$$

are given by

$$\begin{aligned}
(\Omega_{\mu\nu})_{aI}^{bJ} &= \left(\left(\frac{1}{4} R_{\mu\nu}^{cd} \gamma_{cd} \right) \delta_a^b \delta_I^J - \frac{i}{2} g_L W_{\mu\nu L}^\alpha (\sigma^\alpha)_a^b \delta_I^J - \frac{i}{2} g V_{\mu\nu}^m (\lambda^m)_I^J \delta_a^b \right) \otimes 1_3 \\
(\Omega_{\mu\nu})_{\dot{a}I}^{bJ} &= \left(\left(\frac{1}{4} R_{\mu\nu}^{cd} \gamma_{cd} \right) \delta_{\dot{a}}^b \delta_I^J - \frac{i}{2} g_R W_{\mu\nu R}^\alpha (\sigma^\alpha)_{\dot{a}}^b \delta_I^J - \frac{i}{2} g V_{\mu\nu}^m (\lambda^m)_I^J \delta_{\dot{a}}^b \right) \otimes 1_3 \\
(\Omega_{\mu\nu})_{A'}^{B'} &= (\bar{\Omega}_{\mu\nu})_A^B.
\end{aligned}$$

Comparing with equation (4) we deduce that

$$\begin{aligned}
-(E)_{aI}^{bJ} &= \left(\left(\frac{1}{4} R \delta_a^b \delta_I^J + \frac{1}{2} \gamma^{\mu\nu} \left(-\frac{i}{2} g_L W_{\mu\nu L}^\alpha (\sigma^\alpha)_a^b \delta_I^J - \frac{i}{2} g V_{\mu\nu}^m (\lambda^m)_I^J \delta_a^b \right) \right) 1_3 + \Sigma_{aI}^{\dot{c}K} \Sigma_{\dot{c}K}^{bJ} \right) \\
(-E)_{\dot{a}I}^{bJ} &= \left(\left(\frac{1}{4} R \delta_{\dot{a}}^b \delta_I^J + \frac{1}{2} \gamma^{\mu\nu} \left(-\frac{i}{2} g_R W_{\mu\nu R}^\alpha (\sigma^\alpha)_{\dot{a}}^b \delta_I^J - \frac{i}{2} g V_{\mu\nu}^m (\lambda^m)_I^J \delta_{\dot{a}}^b \right) \right) 1_3 \right. \\
&\quad \left. + H_{\dot{a}I\dot{c}K} H^{\dot{c}KbJ} + \Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{bJ} \right) \\
-(E)_{\dot{a}I}^{bJ} &= \gamma^\mu \gamma_5 \nabla_\mu \Sigma_{\dot{a}I}^{bJ} \\
-(E)_{\dot{a}I}^{\dot{b}J'} &= \gamma^\mu \gamma_5 \nabla_\mu H_{\dot{a}I\dot{b}J} \\
(-E)_{\dot{a}I}^{b'J'} &= H_{\dot{a}I\dot{c}K} \Sigma_{b'J'}^{\dot{c}K} \\
(-E)_{\dot{a}I}^{\dot{b}'J'} &= \Sigma_{\dot{a}I}^{\dot{c}K} H_{\dot{c}K\dot{b}'J'}.
\end{aligned}$$

Evaluating the various traces of the 384×384 matrices on spinor and generation space, we get

$$\begin{aligned}
\text{Tr} (E) &= \text{tr} \left(E_A^A + E_{A'}^{A'} \right) = \text{tr} \left(E_A^A + \bar{E}_A^A \right) \\
-\text{tr} (E)_{aI}^{aI} &= 4 \left[\frac{3}{4} R(2)(4) + H_{aI}^{\dot{c}K} H_{\dot{c}K}^{aI} \right] \\
-\text{tr} (E)_{\dot{a}I}^{\dot{a}I} &= 4 \left[\frac{3}{4} R(2)(4) + H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + \Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{a}I} \right] \\
-\frac{1}{2} \text{Tr} (E) &= 4 \left(12R + H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + 2\Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{a}I} \right).
\end{aligned}$$

Next

$$\begin{aligned}
\text{Tr} (\Omega_{\mu\nu}^2)_M^M &= 2\text{Tr} (\Omega_{\mu\nu}^2)_A^A \\
&= 2\text{Tr} \left((\Omega_{\mu\nu}^2)_{\dot{a}I}^{\dot{a}I} + (\Omega_{\mu\nu}^2)_{aI}^{aI} \right)
\end{aligned}$$

$$\begin{aligned}
\text{Tr} (\Omega_{\mu\nu}^2)_{aI} &= \text{Tr} \left(\left(\left(\frac{1}{4} R_{\mu\nu}^{cd} \gamma_{cd} \right) \delta_a^b \delta_I^J - \frac{i}{2} g_L W_{\mu\nu L} (\sigma^\alpha)_a^b \delta_I^J - \frac{i}{2} g V_{\mu\nu}^m (\lambda^m)_i^j \delta_a^b \right) \otimes 1_3 \right)^2 \\
&= 4 \left[-\frac{1}{8} R_{\mu\nu\rho\sigma}^2 (4) (2) (3) - \frac{1}{4} g_L^2 (W_{\mu\nu L}^\alpha)^2 (4) (2) (3) - \frac{1}{4} g^2 (V_{\mu\nu}^m)^2 (3) (2) (2) \right] \\
&= 4 \left[-3R_{\mu\nu\rho\sigma}^2 - 6g_L^2 (W_{\mu\nu L}^\alpha)^2 - 3g^2 (V_{\mu\nu}^m)^2 \right] \\
\text{Tr} (\Omega_{\mu\nu}^2)_{\dot{a}I} &= \text{Tr} \left(\left(\left(\frac{1}{4} R_{\mu\nu}^{cd} \gamma_{cd} \right) \delta_{\dot{a}}^{\dot{b}} \delta_I^J - \frac{i}{2} g_R W_{\mu\nu R} (\sigma^\alpha)_{\dot{a}}^{\dot{b}} \delta_I^J - \frac{i}{2} g V_{\mu\nu}^m (\lambda^m)_I^J \delta_{\dot{a}}^{\dot{b}} \right) \otimes 1_3 \right)^2 \\
&= 4 \left[-3R_{\mu\nu\rho\sigma}^2 - 6g_R^2 (W_{\mu\nu R}^\alpha)^2 - 3g^2 (V_{\mu\nu}^m)^2 \right].
\end{aligned}$$

Therefore

$$\frac{1}{2} \text{Tr} (\Omega_{\mu\nu}^2)_M = 24 \left[-R_{\mu\nu\rho\sigma}^2 - g_L^2 (W_{\mu\nu L}^\alpha)^2 - g_R^2 (W_{\mu\nu R}^\alpha)^2 - g^2 (V_{\mu\nu}^m)^2 \right].$$

Next we compute

$$(E^2)_A^B = E_A^C E_C^B + E_A^{C'} E_{C'}^B$$

and listing the components of this matrix we get

$$\begin{aligned}
(E^2)_{aI}^{bJ} &= E_{aI}^{cK} E_{cK}^{bJ} + E_{aI}^{\dot{c}K} E_{\dot{c}K}^{bJ} + E_{aI}^{c'K'} E_{c'K'}^{bJ} \\
(E^2)_{\dot{a}I}^{bJ} &= E_{\dot{a}I}^{\dot{c}K} E_{\dot{c}K}^{bJ} + E_{\dot{a}I}^{cK} E_{cK}^{bJ} + E_{\dot{a}I}^{c'K'} E_{c'K'}^{bJ} + E_{\dot{a}I}^{\dot{c}'K'} E_{\dot{c}'K'}^{bJ}.
\end{aligned}$$

Collecting terms and tracing we obtain for the right-handed components

$$\begin{aligned}
\text{tr} (E^2)_{\dot{a}I}^{\dot{a}I} &= \text{tr} \left\{ \left(\gamma^\mu \gamma_5 \nabla_\mu \Sigma_{\dot{a}I}^{bJ} \gamma^\nu \gamma_5 \nabla_\nu \Sigma_{bJ}^{\dot{a}I} \right) + \left(\gamma^\mu \gamma_5 \nabla_\mu H_{\dot{a}I \dot{b}J} \gamma^\nu \gamma_5 \nabla_\nu H^{\dot{a}I \dot{b}J} \right) \right. \\
&\quad + \left(\left[\frac{1}{4} R \delta_{\dot{a}}^{\dot{b}} \delta_I^J + \frac{1}{2} \gamma^{\mu\nu} \left(-\frac{i}{2} g_R W_{\mu\nu R} (\sigma^\alpha)_{\dot{a}}^{\dot{b}} \delta_I^J - \frac{i}{2} g V_{\mu\nu}^m (\lambda^m)_I^J \delta_{\dot{a}}^{\dot{b}} \right) 1_3 \right. \right. \\
&\quad \left. \left. + H_{\dot{a}I \dot{c}K} H^{\dot{c}K \dot{b}J} + \Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{b}J} \right]^2 + H_{\dot{a}I \dot{c}K} \Sigma_{\dot{b}J}^{\dot{c}K} H^{\dot{a}I \dot{d}L} \Sigma_{\dot{d}L}^{\dot{b}J} \right\} \\
&= 4 \left[\frac{1}{4} (-2) \left(-\frac{1}{4} g_R^2 (W_{\mu\nu R}^\alpha)^2 (2) (4) (3) - \frac{1}{4} g^2 (V_{\mu\nu}^m)^2 (2) (2) (3) \right) + \frac{1}{16} R^2 (2) (4) (3) \right. \\
&\quad + \frac{1}{2} R \left(H_{\dot{a}I \dot{c}K} H^{\dot{c}K \dot{a}I} + \Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{a}I} \right) + \nabla_\mu H_{\dot{a}I \dot{b}J} \nabla^\mu H^{\dot{a}I \dot{b}J} + \nabla_\mu \Sigma_{\dot{a}I}^{bJ} \nabla^\mu \Sigma_{bJ}^{\dot{a}I} \\
&\quad \left. + H_{\dot{a}I \dot{c}K} \Sigma_{\dot{b}J}^{\dot{c}K} H^{\dot{a}I \dot{d}L} \Sigma_{\dot{d}L}^{\dot{b}J} + \left| H_{\dot{a}I \dot{c}K} H^{\dot{c}K \dot{b}J} + \Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{b}J} \right|^2 \right] \\
&= 4 \left[\frac{3}{2} \left(2g_R^2 (W_{\mu\nu R}^\alpha)^2 + g^2 (V_{\mu\nu}^m)^2 \right) + \frac{3}{2} R^2 + \nabla_\mu H_{\dot{a}I \dot{b}J} \nabla^\mu H^{\dot{a}I \dot{b}J} + \nabla_\mu \Sigma_{\dot{a}I}^{bJ} \nabla^\mu \Sigma_{bJ}^{\dot{a}I} + \right. \\
&\quad \left. + \frac{1}{2} R \left(H_{\dot{a}I \dot{c}K} H^{\dot{c}K \dot{a}I} + \Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{a}I} \right) + H_{\dot{a}I \dot{c}K} \Sigma_{\dot{b}J}^{\dot{c}K} H^{\dot{a}I \dot{d}L} \Sigma_{\dot{d}L}^{\dot{b}J} + \left| H_{\dot{a}I \dot{c}K} H^{\dot{c}K \dot{b}J} + \Sigma_{\dot{a}I}^{cK} \Sigma_{cK}^{\dot{b}J} \right|^2 \right]
\end{aligned}$$

and for the left-handed components

$$\begin{aligned}
\text{tr} (E^2)_{aI} &= \text{tr} \left\{ \left(\left(\frac{R}{4} \delta_a^b \delta_I^J + \frac{1}{2} \gamma^{\mu\nu} \left(-\frac{i}{2} g_L W_{\mu\nu L}^\alpha (\sigma^\alpha)_a^b \delta_I^J - \frac{i}{2} g V_{\mu\nu}^m (\lambda^m)_I^J \right) \delta_a^b \right) 1_3 + \Sigma_{aI}^{cK} \Sigma_{cK}^{bJ} \right)^2 \right. \\
&\quad \left. + \gamma^\mu \gamma_5 \nabla_\mu \Sigma_{aI}^{cK} \gamma^\nu \gamma_5 \nabla_\nu \Sigma_{cK}^{aI} + \left| \Sigma_{aI}^{cK} H_{cKbJ} \right|^2 \right\} \\
&= 4 \left[\frac{1}{4} (-2) \left(-\frac{1}{4} g_L^2 (W_{\mu\nu L}^\alpha)^2 (2) (4) (3) - \frac{1}{4} g_3^2 (V_{\mu\nu}^m)^2 (2) (2) (3) \right) + \frac{1}{16} R^2 (2) (4) (3) \right. \\
&\quad \left. + \frac{1}{2} R \Sigma_{aI}^{cK} \Sigma_{cK}^{aI} + \nabla_\mu \Sigma_{aI}^{cK} \nabla^\mu \Sigma_{cK}^{aI} + \Sigma_{aI}^{cK} \Sigma_{cK}^{bJ} \Sigma_{bJ}^{dL} \Sigma_{dL}^{aI} + \left| \Sigma_{aI}^{cK} H_{cIbJ} \right|^2 \right] \\
&= 4 \left[\frac{3}{2} \left(2g_L^2 (W_{\mu\nu L}^\alpha)^2 + g_3^2 (V_{\mu\nu}^m)^2 \right) + \frac{3}{2} R^2 + \nabla_\mu \Sigma_{aI}^{cK} \nabla^\mu \Sigma_{cK}^{aI} \right. \\
&\quad \left. + \frac{1}{2} R \Sigma_{aI}^{cK} \Sigma_{cK}^{aI} + \Sigma_{aI}^{cK} \Sigma_{cK}^{bJ} \Sigma_{bJ}^{dL} \Sigma_{dL}^{aI} + \left| \Sigma_{aI}^{cK} H_{cKbJ} \right|^2 \right].
\end{aligned}$$

Collecting all terms we finally get

$$\begin{aligned}
\frac{1}{2} \text{tr} (E^2) &= 4 \left[3 \left(g_L^2 (W_{\mu\nu L}^\alpha)^2 + g^2 (V_{\mu\nu}^m)^2 + g_R^2 (W_{\mu\nu R}^\alpha)^2 + R^2 \right) \right. \\
&\quad \left. + 2 \nabla_\mu \Sigma_{aI}^{cK} \nabla^\mu \Sigma_{cK}^{aI} + \nabla_\mu H_{aIbJ} \nabla^\mu H^{aIbJ} + \frac{1}{2} R \left(H_{aIcK} H^{cK\dot{a}I} + 2 \Sigma_{aI}^{cK} \Sigma_{cK}^{\dot{a}I} \right) \right. \\
&\quad \left. + 2 \Sigma_{aI}^{cK} \Sigma_{cK}^{bJ} \Sigma_{bJ}^{dL} \Sigma_{dL}^{aI} + 4 H_{aIcK} \Sigma_{bJ}^{cK} H^{\dot{a}I\dot{d}L} \Sigma_{dL}^{bJ} + \left| H_{aIcK} H^{cKbJ} \right|^2 \right].
\end{aligned}$$

The first two Seely-de Witt coefficients are, first for a_0

$$\begin{aligned}
a_0 &= \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} (1) \\
&= \frac{1}{16\pi^2} (4) (32) (3) \int d^4x \sqrt{g} \\
&= \frac{24}{\pi^2} \int d^4x \sqrt{g}
\end{aligned}$$

then for a_2 :

$$\begin{aligned}
a_2 &= \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} \left(E + \frac{1}{6} R \right) \\
&= \frac{1}{16\pi^2} \int d^4x \sqrt{g} \left((R(-96 + 64)) - 8 \left(H_{aIcK} H^{cK\dot{a}I} + 2 \Sigma_{aI}^{cK} \Sigma_{cK}^{\dot{a}I} \right) \right) \\
&= -\frac{2}{\pi^2} \int d^4x \sqrt{g} \left(R + \frac{1}{4} \left(H_{aIcK} H^{cK\dot{a}I} + 2 \Sigma_{aI}^{cK} \Sigma_{cK}^{\dot{a}I} \right) \right).
\end{aligned}$$

With all the above information we can now compute the Seeley-de Witt coefficient a_4 :

$$a_4 = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} \left(\frac{1}{360} (5R^2 - 2R_{\mu\nu}^2 + 2R_{\mu\nu\rho\sigma}^2) 1 + \frac{1}{2} \left(E^2 + \frac{1}{3} RE + \frac{1}{6} \Omega_{\mu\nu}^2 \right) \right)$$

and where we have omitted the surface terms. Thus

$$\begin{aligned}
& \frac{1}{2} \text{Tr} \left(E^2 + \frac{1}{3} RE + \frac{1}{6} \Omega_{\mu\nu}^2 \right) \\
&= 4 \left[3 \left(g_L^2 (W_{\mu\nu L}^\alpha)^2 + g^2 (V_{\mu\nu}^m)^2 + g_R^2 (W_{\mu\nu R}^\alpha)^2 + R^2 \right) + 2 \nabla_\mu \Sigma_{aI}^{\dot{c}K} \nabla^\mu \Sigma_{\dot{c}K}^{aI} \right. \\
&+ \nabla_\mu H_{\dot{a}I\dot{b}J} \nabla^\mu H^{\dot{a}I\dot{b}J} + \frac{1}{2} R \left(H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + 2 \Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{a}I} \right) \\
&+ 4 H_{\dot{a}I\dot{c}K} \Sigma_{\dot{b}J}^{\dot{c}K} H^{\dot{a}I\dot{d}L} \Sigma_{\dot{d}L}^{\dot{b}J} + 2 \Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{b}J} \Sigma_{\dot{b}J}^{\dot{d}L} \Sigma_{\dot{d}L}^{\dot{a}I} + \left| H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{b}J} \right|^2 \\
&- \frac{1}{3} R \left(12R + H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + 2 \Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{a}I} \right) \\
&- R_{\mu\nu\rho\sigma}^2 - g_L^2 (W_{\mu\nu L}^\alpha)^2 - g_R^2 (W_{\mu\nu R}^\alpha)^2 - g^2 (V_{\mu\nu}^m)^2 \left. \right] \\
&= 4 \left[-R_{\mu\nu\rho\sigma}^2 - R^2 + 2g_L^2 (W_{\mu\nu L}^\alpha)^2 + 2g_R^2 (W_{\mu\nu R}^\alpha)^2 + 2g^2 (V_{\mu\nu}^m)^2 \right. \\
&+ 2 \nabla_\mu \Sigma_{aI}^{\dot{c}K} \nabla^\mu \Sigma_{\dot{c}K}^{aI} + \nabla_\mu H_{\dot{a}I\dot{b}J} \nabla^\mu H^{\dot{a}I\dot{b}J} + \frac{1}{6} R \left(H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + 2 \Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{a}I} \right) \\
&+ \left. \left| H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{b}J} \right|^2 + 4 H_{\dot{a}I\dot{c}K} \Sigma_{\dot{b}J}^{\dot{c}K} H^{\dot{a}I\dot{d}L} \Sigma_{\dot{d}L}^{\dot{b}J} + 2 \Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{b}J} \Sigma_{\dot{b}J}^{\dot{d}L} \Sigma_{\dot{d}L}^{\dot{a}I} \right].
\end{aligned}$$

Collecting terms we get

$$\begin{aligned}
a_4 &= \frac{1}{2\pi^2} \int d^4x \sqrt{g} \left[\frac{1}{30} (5R^2 - 8R_{\mu\nu}^2 - 7R_{\mu\nu\rho\sigma}^2) + g_L^2 (W_{\mu\nu L}^\alpha)^2 + g_R^2 (W_{\mu\nu R}^\alpha)^2 + g^2 (V_{\mu\nu}^m)^2 \right. \\
&+ \nabla_\mu \Sigma_{aI}^{\dot{c}K} \nabla^\mu \Sigma_{\dot{c}K}^{aI} + \frac{1}{2} \nabla_\mu H_{\dot{a}I\dot{b}J} \nabla^\mu H^{\dot{a}I\dot{b}J} + \frac{1}{12} R \left(H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + \Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{a}I} + H_{\dot{a}I}^{\dot{c}K} H_{\dot{c}K}^{\dot{a}I} \right) \\
&+ \left. \frac{1}{2} \left| H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{b}J} \right|^2 + 2 H_{\dot{a}I\dot{c}K} \Sigma_{\dot{b}J}^{\dot{c}K} H^{\dot{a}I\dot{d}L} \Sigma_{\dot{d}L}^{\dot{b}J} + \Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{b}J} \Sigma_{\dot{b}J}^{\dot{d}L} \Sigma_{\dot{d}L}^{\dot{a}I} \right].
\end{aligned}$$

Using the identities

$$\begin{aligned}
R_{\mu\nu\rho\sigma}^2 &= 2C_{\mu\nu\rho\sigma}^2 + \frac{1}{3}R^2 - R^*R^* \\
R_{\mu\nu}^2 &= \frac{1}{2}C_{\mu\nu\rho\sigma}^2 + \frac{1}{3}R^2 - \frac{1}{2}R^*R^*
\end{aligned}$$

where $R^*R^* = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}R_{\mu\nu}^{\alpha\beta}R_{\rho\sigma}^{\gamma\delta}$.

$$\begin{aligned}
\frac{1}{30} (5R^2 - 8R_{\mu\nu}^2 - 7R_{\mu\nu\rho\sigma}^2) &= R^2 \frac{1}{30} \left(5 - \frac{8}{3} - \frac{7}{3} \right) + \frac{1}{30} C_{\mu\nu\rho\sigma}^2 (-4 - 14) + \frac{1}{30} R^*R^* (4 + 7) \\
&= -\frac{3}{5} C_{\mu\nu\rho\sigma}^2 + \frac{11}{30} R^*R^*
\end{aligned}$$

Then a_4 simplifies to

$$\begin{aligned}
a_4 &= \frac{1}{2\pi^2} \int d^4x \sqrt{g} \left[-\frac{3}{5} C_{\mu\nu\rho\sigma}^2 + \frac{11}{30} R^*R^* + g_L^2 (W_{\mu\nu L}^\alpha)^2 + g_R^2 (W_{\mu\nu R}^\alpha)^2 + g^2 (V_{\mu\nu}^m)^2 \right. \\
&+ \nabla_\mu \Sigma_{aI}^{\dot{c}K} \nabla^\mu \Sigma_{\dot{c}K}^{aI} + \frac{1}{2} \nabla_\mu H_{\dot{a}I\dot{b}J} \nabla^\mu H^{\dot{a}I\dot{b}J} + \frac{1}{12} R \left(H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{a}I} + 2 \Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{a}I} \right) \\
&+ \left. \frac{1}{2} \left| H_{\dot{a}I\dot{c}K} H^{\dot{c}K\dot{b}J} \right|^2 + 2 H_{\dot{a}I\dot{c}K} \Sigma_{\dot{b}J}^{\dot{c}K} H^{\dot{a}I\dot{d}L} \Sigma_{\dot{d}L}^{\dot{b}J} + \Sigma_{\dot{a}I}^{\dot{c}K} \Sigma_{\dot{c}K}^{\dot{b}J} \Sigma_{\dot{b}J}^{\dot{d}L} \Sigma_{\dot{d}L}^{\dot{a}I} \right]
\end{aligned}$$

At this point, it will be useful to write explicitly all the fermionic interactions obtained by substituting the full Dirac operator, including fluctuations. This will give the fermionic kinetic terms, the gauge vectors couplings, and the Higgs scalars terms. We give first the non-Majorana type terms

$$\begin{aligned}\psi_{\alpha I}^* (D_A)_{\alpha I}^{\beta J} \psi_{\beta J} &= \psi_{\dot{a} I}^* (D_A)_{\dot{a} I}^{\dot{b} J} \psi_{\dot{b} J} + \psi_{\dot{a} I}^* (D_A)_{\dot{a} I}^{\dot{b} J} \psi_{\dot{b} J} \\ &+ \psi_{a I}^* (D_A)_{a I}^{\dot{b} J} \psi_{\dot{b} J} + \psi_{a I}^* (D_A)_{a I}^{\dot{b} J} \psi_{\dot{b} J}\end{aligned}$$

Substituting the evaluated values of D_A , we get

$$\begin{aligned}&\psi_{\dot{a} I}^* \gamma^\mu \left(D_\mu \delta_a^{\dot{b}} \delta_I^J - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_{\dot{a}}^{\dot{b}} \delta_I^J - \delta_a^{\dot{b}} \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J + \frac{i}{2} g V_\mu \delta_I^J \right) \right) \psi_{\dot{b} J} \\ &+ \psi_{a I}^* \gamma^\mu \left(D_\mu \delta_a^{\dot{b}} \delta_I^J - \frac{i}{2} g_L W_{\mu L}^\alpha (\sigma^\alpha)_a^{\dot{b}} \delta_I^J - \delta_a^{\dot{b}} \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J + \frac{i}{2} g V_\mu \delta_I^J \right) \right) \psi_{\dot{b} J} \\ &+ \left\{ \psi_{\dot{a} I}^* \gamma_5 \left(\left(k^\nu \phi_a^{\dot{b}} + k^e \tilde{\phi}_a^{\dot{b}} \right) \Sigma_I^J + \left(k^u \phi_a^{\dot{b}} + k^d \tilde{\phi}_a^{\dot{b}} \right) (\delta_I^J - \Sigma_I^J) \right) \psi_{\dot{b} J} + \text{h.c} \right\}\end{aligned}$$

These can be expanded further into quarks and leptons terms by allowing $I = 1, i$ and $J = 1, j$ where i, j are $SU(3)$ color indices and the 1 to leptonic index. Next, we have the Majorana type terms

$$\begin{aligned}\psi_{\alpha I}^* (D_A)_{\alpha I}^{\beta' J'} \psi_{\beta' J'} &= \psi_{\dot{a} I}^* (D_A)_{\dot{a} I}^{\dot{b}' J'} \psi_{\dot{b}' J'} \\ &= C \psi_{\dot{a} I}^* \gamma_5 k^{*\nu R} \Delta_{\dot{a} J} \Delta_{\dot{b} I} \psi_{\dot{b} J}^* \\ \psi_{\alpha' I'}^* (D_A)_{\alpha' I'}^{\beta J} \psi_{\beta J} &= \psi_{\dot{a}' I'}^* (D_A)_{\dot{a}' I'}^{\dot{b} J} \psi_{\dot{b} J} \\ &= C \psi_{\dot{a}' I'}^* \gamma_5 k^{\nu R} \bar{\Delta}^{\dot{a} J} \bar{\Delta}^{\dot{b} I} \psi_{\dot{b} J}\end{aligned}$$

We summarize, the full fermionic interactions are given by

$$\begin{aligned}&\int d^4x \sqrt{g} \left\{ \psi_{\dot{a} I}^* \gamma^\mu \left(D_\mu \delta_a^{\dot{b}} \delta_I^J - \frac{i}{2} g_R W_{\mu R}^\alpha (\sigma^\alpha)_{\dot{a}}^{\dot{b}} \delta_I^J - \delta_a^{\dot{b}} \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J + \frac{i}{2} g V_\mu \delta_I^J \right) \right) \psi_{\dot{b} J} \right. \\ &+ \psi_{a I}^* \gamma^\mu \left(D_\mu \delta_a^{\dot{b}} \delta_I^J - \frac{i}{2} g_L W_{\mu L}^\alpha (\sigma^\alpha)_a^{\dot{b}} \delta_I^J - \delta_a^{\dot{b}} \left(\frac{i}{2} g V_\mu^m (\lambda^m)_I^J + \frac{i}{2} g V_\mu \delta_I^J \right) \right) \psi_{\dot{b} J} \\ &+ \psi_{\dot{a} I}^* \gamma_5 \left(\left(k^\nu \phi_a^{\dot{b}} + k^e \tilde{\phi}_a^{\dot{b}} \right) \Sigma_I^J + \left(k^u \phi_a^{\dot{b}} + k^d \tilde{\phi}_a^{\dot{b}} \right) (\delta_I^J - \Sigma_I^J) \right) \psi_{\dot{b} J} \\ &+ \psi_{a I}^* \gamma_5 \left(\left(k^{*\nu} \phi_a^{\dot{b}} + k^{*e} \tilde{\phi}_a^{\dot{b}} \right) \Sigma_I^J + \left(k^{*u} \phi_a^{\dot{b}} + k^{*d} \tilde{\phi}_a^{\dot{b}} \right) (\delta_I^J - \Sigma_I^J) \right) \psi_{\dot{b} J} \\ &\left. + C \psi_{\dot{a} I}^* \gamma_5 k^{\nu R} \bar{\Delta}^{\dot{a} J} \bar{\Delta}^{\dot{b} I} \psi_{\dot{b} J} + \text{h.c} \right\}\end{aligned}$$

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