The partition function is defined as the corresponding continuum Hamiltonian and finally in way. In Sec. II we review the solution of 2D lattice gauge theory. The rest of this article is organized in the following results from these studies to solve CDT coupled to gauge theories. Despite the existence of a matrix formulation \[8\]. Here we theory coupled to a well-defined continuum matter theory yet been possible to provide exact solutions of the gravity dynamics, involving a finite number of degrees of freedom. In the CDT case we consider space-time with the topology of a cylinder, space being compactified to \(S^1\), and we thus have nontrivial dynamics associated with the holonomies of \(S^1\). This has been studied in great detail in flat space-time (see [12] and references therein). We will use the results from these studies to solve CDT coupled to gauge theory. The rest of this article is organized in the following way. In Sec. II we review the solution of 2D lattice gauge theory. In Sec. III we find the lattice transfer matrix and the corresponding continuum Hamiltonian and finally in Sec. IV we discuss “cosmological” applications.

II. 2D GAUGE THEORIES ON A CYLINDER

Let us briefly review 2D gauge theory on a fixed lattice. The partition function is defined as

\[
Z(g) = \int \prod_{\ell} dU_\ell \prod_{\text{plaquettes}} Z_P[U_P].
\]  

where we to each link \(\ell\) associate a \(U_\ell \in G\), and \(U_P\) is the product of the \(U_\ell\)'s around the plaquette. One has a large choice for \(Z_P[U_P]\), but for the purpose of extracting the Hamiltonian it is convenient to use the so-called heat kernel action,

\[
Z_P[U_P] = \langle U_P | e^{-i_v^2 A_P \Delta_G} | U \rangle = \sum_R d_R x_R(U_P) e^{-i_v^2 A_P C_2(R)},
\]  

where \(A_P = a_1 a_s\) denotes the area of the plaquette with spatial lattice link length \(a_s\) and timelike link length \(a_t\) (we will usually think of \(a_s = a_t\)), \(\mathcal{I}\) denotes the identity element in \(G\) and \(\Delta_G\) the Laplace-Beltrami operator on \(G\). The convenient property of the heat kernel action in 2D is that it is additive, i.e. if we integrate over a link in (1) the action is unchanged: write \(U_{P_1} = U_4 U_3 U_2 U_1\) and \(U_{P_2} = U_4^{-1} U_3 U_2 U_1\); then

\[
\int dU_4 Z_P[U_{P_1}] Z_P[U_{P_2}] = Z_{P_1 + P_2}^P[U_{P_1 + P_2}],
\]  

where \(U_{P_1 + P_2} = U_3 U_2 U_1\); see Fig. 1.

Let us now consider a lattice with \(r\) links in the time direction and \(l\) links in the spatial direction. We have two boundaries, with gauge field configurations \(\{U_\ell\}\) and \(\{\bar{U}_\ell\}\), which we choose to keep fixed [Dirichlet-like boundary conditions]. We can then write

\[
Z(g, \{U_\ell\}, \{\bar{U}_\ell\}) = \langle \{U_\ell\} | \tilde{T} | \{\bar{U}_\ell\} \rangle, \quad \tilde{T} = e^{-a_{I}}
\]  

where \(\tilde{T}\) is the transfer matrix, giving us the transition amplitude between link configurations at neighboring time slices. However in 2D we can restrict \(\tilde{T}\) to be an operator only acting on the holonomies since we can use (3) to integrate out the temporal links \(U_\ell^{(2)}\) which connect two time slices. We obtain

\[
\langle U'^{r} | \tilde{T} | U \rangle = \langle U'| \tilde{P} e^{-a_I} | \tilde{P}| U \rangle = \langle U'| \tilde{P} e^{-a_I} | U \rangle
\]  

where the projection operator \(\tilde{P}\) is defined by

\[
\tilde{P}| U \rangle = \int dG |GUG^{-1}\rangle,
\]  

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1To be precise, CDT has been solved when coupled to some “nonstandard” hard dimer models \([9,10]\), but it is unknown if these dimer models have an interesting continuum limit. Also, “generalized CDT” models coupled to ordinary hard dimer models have been solved \([9,11]\), using matrix models.
and it appears in (5) as the result of integration over the last temporal link connecting the two time slices.

Denote the length of the lattice \( L = a_l \). From (4) and (5) it follows that

\[
\hat{H} = \frac{1}{2} g^2 L \Delta_G
\]

if we restrict to the gauge invariant subspace (i.e. the subspace of class functions) projected out by \( \tilde{P} \).

**III. COUPLING TO GEOMETRY**

The covariant version of the Yang-Mills theory is

\[
S_{YM} = \frac{1}{4} \int d^2 x \sqrt{g} (F_{\mu \nu}^a F^{\mu \nu})^a.
\]

We want a path integral formulation which includes also the integration over geometries. Here the CDT formulation is natural: one is summing over geometries which have cylindrical geometry and a time foliation, each geometry being defined by a triangulation and the sum over geometries in the path integral being performed by summing over all triangulations with topology of the cylinder and a time foliation. The coupling of gauge fields to a geometry via dynamical triangulations (where the length of a link is \( a \)) is well known [13]: One uses as plaquettes the triangles. Thus the 2D partition function becomes

\[
Z(\Lambda, g, l', l, \{U_{l'}\}, \{U_l\}) = \sum_{\mathcal{T}} e^{-4 \nu_{\mathcal{T}} \Lambda^2 g \Delta^2} Z_G^\mathcal{T}(\beta),
\]

where the summation is over CDT triangulations \( \mathcal{T} \), with an “entrance” boundary consisting of \( l \) links and an “exit” boundary consisting of \( l' \) links, \( \Lambda \) is the lattice cosmological constant, \( N_{\mathcal{T}} \) the number of triangulations in \( \mathcal{T} \), and the gauge partition function for a given triangulation \( \mathcal{T} \) is defined as

\[
Z_G^\mathcal{T}(g, \{U_{l'}\}, \{U_l\}) = \int \prod_{\ell} dU_{\ell} \prod_{P} Z_P[U_P].
\]

The integration is over all lattice links except the boundary links and \( \prod_{P} \) is the product over plaquettes (here triangles) in \( \mathcal{T} \). For the plaquette action defining \( Z_P[U_P] \) we have again many choices, and for convenience we will use the heat kernel action (2).

We can introduce a transfer matrix \( \hat{T} \), which connects geometry and fields at time label \( l' \) to geometry and fields at time label \( l' + 1 \), and if the (discretized) universe has \( t + 1 \) time labels we can write

\[
Z(\Lambda, g, l', l, \{U_{l'}\}, \{U_l\}) = \langle \{U_{l'}\}, l' | T | \{U_l\}, l \rangle, \quad T = e^{-a_{\hat{H}}}. \tag{11}
\]

The one-dimensional geometry at \( l' \) is characterized by the number \( l \) of links (each of length \( a \)), and on these links we have field configurations \( \{U_l\} \). Similarly the geometry at \( l' + 1 \) has \( l' \) links and field configurations \( \{U_{l'}\} \). For fixed \( l \) and \( l' \) the number of plaquettes (triples) in the spacetime cylinder “slab” between \( l' \) and \( l' + 1 \) is \( l + l' \) and the number of temporal links \( l + l' \). There is a number of possible triangulations of the slab for fixed \( l \) and \( l' \), namely,

\[
N(l', l) = \frac{1}{l + l'} \binom{l + l'}{l, l'}, \tag{12}
\]

For each of these triangulations we can integrate over the \( l + l' \) temporal link variables \( U_{l'}^{(0)} \), as we did for a fixed lattice and we obtain as in that case

\[
\langle U' | \hat{P} e^{-a_{l'} \frac{L^2}{8} \Delta_0} \hat{P} | U \rangle, \tag{13}
\]

where \( U' \) and \( U \) are the holonomies corresponding to \( \{U_{l'}\} \) and \( \{U_l\} \), respectively, and \( \hat{P} \) is the projection operator (6) to class functions coming from the last integration over a temporal link \( U_0 \). The factor \( \sqrt{3}/8 \) rather than the factor \( 1/2 \) appears because we are using equilateral triangles rather than squares as in Sec. II. In order to have unified formulas we make a redefinition \( g^2 \sqrt{3}/4 \rightarrow g^2 \) and thus we have the matrix element,

\[
\langle U' | \hat{P} e^{-a(l + l') \frac{L^2}{8} \Delta_0} \hat{P} | U \rangle. \tag{14}
\]

If we did not have the matter fields the transfer matrix would be

\[
\langle l' | \hat{T}_{\text{geometry}} | l \rangle = N(l', l) e^{-a(l + l') \frac{L^2}{8} \Lambda}, \tag{15}
\]

where we have made a redefinition \( \Lambda \sqrt{3}/4 \rightarrow \Lambda \), similar to the one made for \( g^2 \), in order to be in accordance with notations in other articles. The limit where \( a \rightarrow 0 \) and

FIG. 1. Integrating out the link \( U_4 \) using the heat kernel action. The graphic notation is such that one has cyclic matrix multiplication on loops and if an arrow is reversed (oriented link \( t \rightarrow -t \)) then \( U_{-t} = U_t^{-1} \).
\[ L' = aL' \text{ and } L = al \] are kept fixed has been studied \cite{14} and one finds
\[ \hat{T}_{\text{geometry}} = e^{-a(R_{\alpha\beta} + \theta_{\alpha\beta})}, \quad \hat{H}_{\text{cdt}} = -\frac{d^2}{dL^2}L + \Lambda L. \] (16)

From the definition (11) of \( \hat{H} \) and (14) it follows that
\[ \hat{H} = \hat{H}_{\text{cdt}} + \frac{1}{2} g^2 L \Delta_G, \] (17)
acting on the Hilbert space which is the tensor product of the Hilbert space of square integrable class functions on \( G \) and the Hilbert space of the square integrable functions on \( R_+ \) with measure \( d\mu(L) = LdL. \)

Since the eigenfunctions of \( \Delta_G \) after projection with \( \hat{P} \) are just the characters \( \chi_{\alpha}(U) \) on \( G \) and they have eigenvalues \( C_\alpha(R) \), we can solve the eigenvalue equation for \( \hat{H} \) by writing \( \Psi(L, U) = \psi(U) \chi_{\alpha}(U) \). For \( \hat{H}_{\text{cdt}} \) we have \cite{14,15}
\[ \hat{H}_{\text{cdt}} \psi_n(L, \Lambda) = e_n \psi_n(L, \Lambda), \quad e_n = 2n\sqrt{\Lambda}, \quad n > 0, \] (18)
where the eigenfunctions are of the form \( \Lambda p_n(L\sqrt{\Lambda}) e^{-L\sqrt{\Lambda}}, \) \( p_n(x) \) being a polynomial of degree \( n - 1 \). The corresponding solution for \( \psi(U) \) is obtained by the substitution
\[ \Lambda \rightarrow \Lambda_R = \Lambda + \frac{1}{2} g^2 C_2(R), \] (19)
i.e.
\[ \hat{H} \Psi_{n,R} = E(n, R) \Psi_{n,R}, \quad E(n, R) = 2n\sqrt{\Lambda_R}, \quad n > 0, \] (20)
\[ \Psi_{n,R}(L, U) = \Lambda_R p_n(L\sqrt{\Lambda_R}) e^{-L\sqrt{\Lambda_R} \chi_{\alpha}(U)}, \] (21)
with the reservation that the correct variable is not really the group variable \( U \) but rather the conjugacy class corresponding to \( U \). In the simplest case of \( SU(2) \) the group manifold can be identified with \( S^3 \) and \( \Delta_G \) is the Laplace-Beltrami operator on \( S^3 \). The conjugacy classes are labeled by the geodesic distance \( \theta \) to the north pole and the representations are labeled by \( R = j \) and we have\footnote{Using the lattice we have effectively performed a quantization using the fact that \( SU(2) \) is a compact group. However, there are subtleties associated with the quantization, more precisely whether one chooses first to project to the algebra and quantize there, or first to quantize using the group variables and then project to the holonomies. We refer to \cite{12} for a detailed discussion.}
\[ C_j = j(j + 1), \quad \chi_j(\theta) = \frac{\sin(j + \frac{1}{2} \theta)}{\sin \frac{1}{2} \theta}, \quad j = 0, 1, 2, \ldots, \ldots. \] (22)

The above results are also valid in simpler cases. If \( G = U(1) \) where one has

\[ U(\theta) = e^{i\theta}, \quad \Delta_G = -\frac{d^2}{d\theta^2}. \] (23)

\[ C_n = n^2, \quad \chi_n(\theta) = e^{in\theta}, \quad n = 0, \pm 1, \pm 2, \ldots, \] (24)
and if \( G = Z_N \), the discrete cyclic group of order \( N \),
\[ U(k) = e^{2\pi k/N}, \quad (\Delta_G)_{k,k'} = \delta_{k,k'+1} + \delta_{k,k'-1} - 2\delta_{k,k'}, \]
\[ k = 0, \ldots, N - 1, \] (25)
\[ C_n = 2(1 - \cos \left( \frac{2\pi}{N} n \right)), \quad \chi_n(k) = e^{2\pi n k}, \]
\[ n = 0, 1, \ldots, N - 1. \] (26)

### IV. THE GROUND STATE OF THE UNIVERSE

In CDT the disk amplitude is defined as
\[ W_A(L) = \int_0^\infty dt \langle L | e^{-it\hat{H}_{\text{cdt}}} | L' \rangle \rightarrow 0. \] (27)
It is a version of the Hartle-Hawking wave function. One can calculate \( W_A(L) \) \cite{1}:
\[ W_A(L) = \frac{e^{-\sqrt{\Lambda}L}}{L}. \] (28)
This function satisfies
\[ \hat{H}_{\text{cdt}} W_A(L) = 0, \] (29)
and one can view (29) as the Wheeler-deWitt equation. Formally \( W_A(L) \propto \psi_0(L) \) in the notation used in Eq. (18), but it was not included as an eigenfunction in the listing in (18) since it does not belong to the Hilbert space \( L^2(R_+) \) with measure \( LdL. \)

If we couple the theory of fluctuating geometries to gauge fields as above, we have to decide what kind of boundary condition to impose in the limit \( L' \rightarrow 0 \) in (27). A possible interpretation of this “singularity” in the discrete setting is that all the vertices of the first time slice at time \( t' = 1 \) have additional temporal links joining a single vertex at time \( t' = 0 \) (see Fig. 2). We can view this as an explicit, discretized, realization of the matter part of the Hartle-Hawking boundary condition.

Denote by \( \{ U_0 \}, \ell = 1, \ldots, l \) the gauge fields on these temporal links and by \( \{ U_1 \}, \ell = 1, \ldots, l \) the gauge fields on the spatial links constituting the first loop at time \( t' = 1 \) and denote by \( U(1) \) the corresponding holonomy at time \( t' = 1 \). The contribution to the matter partition function coming from this first “big bang” part of the universe is then
f gauge fields as we just saw. Combining the two we might define the natural to say that the universe starts out in the matter state 1, or expanded in characters:

\[ \langle U(1) \rangle = \langle e^{-i\gamma^2 L^2} |U(1)\rangle, \quad \langle U(2) \rangle = \langle e^{-i\gamma^2 L^2} |U(2)\rangle, \ldots, \langle U(3) \rangle = \langle e^{-i\gamma^2 L^2} |U(3)\rangle, \]

where we have integrated over the temporal links \( \{U_\ell^{(0)}\} \). The matter partition function can now be written (after integrating out the temporal links in the rest of the lattice too) as the integral over \( t \) holonomies \( U(1), U(2), \ldots, U(t) \),

\[
\int \prod_{\ell=1}^t dU_\ell^{(0)} \prod_{k=1}^l Z_{P_k}[U_{P_k}] = Z_{disk}[U(1)],
\]

where \( U(0) = I \) and \( l_0 = 0 \). From this expression it is natural to say that the universe starts out in the matter state \( |I\rangle \), or expanded in characters:

\[ \langle U|I\rangle = \delta(U - I) = \sum_R d_R \chi_R(U). \]

This wave function is not normalizable if the group has infinitely many representations, but neither is \( W_A(L) \) as we just saw. Combining the two we might define the Hartle-Hawking wave function for 2D CDT coupled to gauge fields as

\[
W(L, U) = \int_0^\infty dT (L, U) e^{-T\bar{H}} |L = 0, U = I\rangle = \sum_R d_R \chi_R(U) W_A(L),
\]

where \( \Lambda_R \) is defined in Eq. (19). We have explicitly:

\[
W(L, k) = \sum_{r} e^{i2\pi k r} \exp \left( -L \left( \sqrt{\Lambda + g^2 \left[ 1 - \cos(2\pi r/n) \right]} \right) \right) \frac{1}{L},
\]

for the \( Z_N \) theory.

**V. CONCLUSION**

We have tried to define the initial matter state \( |I\rangle \) in the Hartle-Hawking spirit as coming from “no boundary” conditions by closing the universe into a disk. Even if the “initial” (big bang) state is then a simple tensor product \( |L = 0\rangle \otimes |I\rangle \), the corresponding Hartle-Hawking wave function is the result of a nontrivial interaction between matter and geometry. However, we cannot claim that the model points to such a no boundary condition in a really compelling way. From a continuum point of view it should not make a difference if we, rather than implementing the continuum statement \( L' \to 0 \) by adding a little cap, had implemented it by insisting that the first time slice had \( i = 2 \) or \( i = 3 \), say. The calculation of \( W_A(L) \) is insensitive to such details. However, if our universe really started with such a microscopic loop, there is no reason that we should not choose the matter ground state, i.e. the trivial, constant, character as the initial state. In this case absolutely nothing happens with matter during the time evolution of the universe. It just stays in this state and the state does not influence the geometry. Clearly the state \( |I\rangle \) is much more interesting and more in accordance with the picture we have of the big bang of the real 4D world where matter and geometry have interacted. Even if the argument for the state \( |I\rangle \) is not compelling, as just mentioned, it is nevertheless encouraging that the “natural” Hartle-Hawking like boundary condition leads to a nontrivial interaction between geometry and matter.
gauge field at the big bang. With a boundary condition similar to the one proposed by Hartle and Hawking we find that the matter and gauge degrees of freedom become entangled.

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