From Kleisli Categories to Commutative C^*-Algebras: Probabilistic Gelfand Duality

Robert Furber and Bart Jacobs

Institute for Computing and Information Sciences (iCIS),
Radboud University Nijmegen, The Netherlands
www.cs.ru.nl/B.Jacobs

Abstract. C^*-algebras form rather general and rich mathematical structures that can be studied with different morphisms (preserving multiplication, or not), and with different properties (commutative, or not). These various options can be used to incorporate various styles of computation (set-theoretic, probabilistic, quantum) inside categories of C^*-algebras. This paper concentrates on the commutative case and shows that there are functors from several Kleisli categories, of monads that are relevant to model probabilistic computations, to categories of C^*-algebras. This yields a new probabilistic version of Gelfand duality, involving the "Radon" monad on the category of compact Hausdorff spaces. We also show that a commutative C^*-algebra is isomorphic to the space of convex continuous functionals from its state space to the complex numbers. This allows us to obtain an appropriately commuting state-and-effect triangle for commutative C^*-algebras.

1 Introduction

There are several notions of computation. We have the classical notion of computation, probabilistic computation, where a computer may make random choices, and quantum computation, which uses quantum mechanical interference and measurement. Normally we would consider classical computation to be done on sets, probabilistic computation on spaces with a measure, and quantum computation on Hilbert spaces. We can instead use categories with C^*-algebras as objects and a choice of either *-homomorphisms (called MIU-map below) or positive unital maps as the morphisms. The general outline is represented in this table.

	set-theoretic	**probabilistic**	**quantum**
C^*-algebras	commutative	commutative	non-commutative
maps preserve	multiplication involution unit	positivity unit	positivity unit
maps abbreviation	MIU	PU	PU

R. Heckel and S. Milius (Eds.): CALCO 2013, LNCS 8089, pp. 141–157, 2013.

While the quantum case is an important source of motivation, we will be concerned with the classical and probabilistic cases in this article. In particular, we will relate the alternative method of representing probabilistic computation, using monads, to the C^*-algebraic approach.

In recent years the methods and tools of category theory have been applied to Hilbert spaces — see *e.g.* [1] and the references there — and also to C^*-algebras, see for instance [23,21]. In this paper we show that clearly distinguishing different types of homomorphisms of C^*-algebras already brings quite some clarity. Moreover, we demonstrate the relevance of monads (and their Kleisli and Eilenberg-Moore categories) in this field. The aforementioned paper [23] concerns itself with only the *-homomorphisms (*i.e.* with the MIU-maps in our terminology).

Giry [10, I.4] described how we can consider a stochastic process as being a diagram in the Kleisli category of the Giry monad on measure spaces. By using the Radon monad on compact spaces instead, we can get a different category of stochastic processes on compact spaces as diagrams in the (opposite of the) category of *commutative* C^*-algebras with PU-maps. This allows the quantum generalization to taking diagrams in the category of *non-commutative* C^*-algebras. The relationship to quantum computation is that $B(\mathcal{H})$, the algebra of all bounded operators on a Hilbert space is a C^*-algebra, and for every C^*-algebra A, there is a Hilbert space $\mathcal{H}$ such that A is isomorphic to a norm-closed *-subalgebra of $B(\mathcal{H})$. The category of C^*-algebras allows us to represent measurement with maps from a commutative C^*-algebra to $B(\mathcal{H})$. We can also represent composite systems that are partly quantum and partly classical. Girard also used certain special C^*-algebras, von Neumann algebras, for his Geometry of Interaction [9].

2 Preliminaries on C^*-Algebras

We write $\mathbf{Vect} = \mathbf{Vect}_{\mathbb{C}}$ for the category of vector spaces over the complex numbers $\mathbb{C}$. This category has direct product $V \oplus W$, forming a biproduct (both a product and a coproduct) and tensors $V \otimes W$, which distribute over $\oplus$. The tensor unit is the space $\mathbb{C}$ of complex numbers. The unit for $\oplus$ is the singleton (null) space 0. We write $\overline{V}$ for the vector space with the same vectors/elements as V, but with conjugate scalar product: $z \bullet_{\overline{V}} v = \overline{z} \bullet_V v$. This makes **Vect** an involutive category, see [14].

A **-algebra* is an involutive monoid A in the category **Vect**. Thus, A is itself a vector space, carries a multiplication $\cdot\colon A \otimes A \to A$, linear in each argument, and has a unit $1 \in A$. Moreover, there is an involution map $(-)^*\colon \overline{A} \to A$, preserving 0 and $+$ and satisfying:

$$1^* = 1 \qquad (x \cdot y)^* = y^* \cdot x^* \qquad x^{**} = x \qquad (z \bullet x)^* = \overline{z} \bullet x^*.$$

Here we have written a fat dot $\bullet$ for scalar multiplication, to distinguish it from the algebra's multiplication $\cdot$. For $z = a + bi \in \mathbb{C}$ we have the conjugate $\overline{z} = a - bi$. Often we omit the multiplication dot $\cdot$ and simply write xy for $x \cdot y$. Similarly, the scalar multiplication $\bullet$ is often omitted. We then rely on the context to distinguish the two multiplications.

A C^**-algebra* is a *-algebra A with a norm $\|-\|\colon A \to \mathbb{R}_{\geq 0}$ in which it is complete. This norm satisfies $\|x\| = 0$ iff $x = 0$ and:

$$\begin{aligned} \|x+y\| &\leq \|x\| + \|y\| & \qquad \|z \bullet x\| &= |z| \cdot \|x\| \\ \|x \cdot y\| &\leq \|x\| \cdot \|y\| & \qquad \|x^* \cdot x\| &= \|x\|^2. \end{aligned}$$

Especially this last equation $\|x^* \cdot x\| = \|x\|^2$ is vital for C^*-algebras, as it distinguishes them from Banach *-algebras. In the current setting, each C^*-algebra is unital, *i.e.* has a (multiplicative) unit 1. A C^*-algebra is called *commutative* if its multiplication is commutative, and *finite-dimensional* is it has finite dimension when considered as a vector space.

An element x in a C^*-algebra A is called *positive* if it can be written in the form $x = y^* \cdot y$. We write $A^+ \subseteq A$ for the subset of positive elements in A. This subset is a cone, which is to say it is closed under addition and scalar multiplication with positive real numbers. The multiplication $x \cdot y$ of two positive elements need not be positive in general (think of matrices). The square $x^2 = x \cdot x$ of a self-adjoint element $x = x^*$, however, is obviously positive. In a *commutative* C^*-algebra the positive elements are closed under multiplication. A cone A^+ in a vector space defines a partial order as follows.

$$x \leq y \Leftrightarrow y - x \in P \tag{1}$$

This is defines an order on every C^*-algebra.

There are mainly two options when it comes to maps between C^*-algebras. The difference between them plays an important role in this paper.

Definition 1. *We define two categories* ***Cstar***$_{MIU}$ *and* ***Cstar***$_{PU}$ *with* C^**-algebras as objects, but with different morphisms.*

1. *A morphism* $f\colon A \to B$ *in* ***Cstar***$_{MIU}$ *is a linear map preserving multiplication (M), involution (I), and unit (U). Explicitly, this means for all* $x, y \in A$,

$$f(x \cdot y) = f(x) \cdot f(y) \qquad f(x^*) = f(x)^* \qquad f(1) = 1.$$

   *Often such "MIU" maps are called *-homomorphisms.*
2. *A morphism* $f\colon A \to B$ *in* ***Cstar***$_{PU}$ *is a linear map that preserves positive elements and the unit. This means that* f *restricts to a function* $A^+ \to B^+$*. Alternatively, for each* $x \in A$ *there is an* $y \in B$ *with* $f(x^*x) = y^*y$.

For both $X = MIU$ *and* $X = PU$ *there are obvious full subcategories of commutative and/or finite-dimensional* C^**-algebras, as described in:*

$$\begin{array}{ccccc} & & \textbf{\textit{CCstar}}_X & & \\ & \nearrow & & \searrow & \\ \textbf{\textit{FdCCstar}}_X & & & & \textbf{\textit{Cstar}}_X \\ & \searrow & & \nearrow & \\ & & \textbf{\textit{FdCstar}}_X & & \end{array}$$

Clearly, each "MIU" map is also a "PU" map, so that we have inclusions $\mathbf{Cstar}_{\mathrm{MIU}} \hookrightarrow \mathbf{Cstar}_{\mathrm{PU}}$, also for the various subcategories. A map that preserves positive elements is called positive itself; and a unit preserving map is called unital.

For a category $\mathbf{B}$ one often writes $\mathbf{B}(X,Y)$ or $\mathit{Hom}(X,Y)$ for the "homset" of morphisms $X \to Y$ in $\mathbf{B}$. For C^*-algebras A,B we write $\mathit{Hom}_{\mathrm{MIU}}(A,B) = \mathbf{Cstar}_{\mathrm{MIU}}(A,B)$ and $\mathit{Hom}_{\mathrm{PU}}(A,B) = \mathbf{Cstar}_{\mathrm{PU}}(A,B)$ for the homsets of MIU- and PU-maps. There is also the commonly used notion of completely positive maps, which is a stronger condition than positivity but weaker than being MIU. These maps are important when defining the tensor of C^*-algebras as a functor, as the tensor of positive maps need not be positive. They are also widely considered to represent the physically realizable transformations. Positive, but non-completely positive maps of C^*-algebras also have their uses, as entanglement witnesses for example[12, theorem2]. Since we mainly consider the commutative case, where positive and completely positive coincide, we do not consider the category of C^*-algebras with completely positive maps any further in this paper.

We collect some basic (standard) properties of PU-morphisms between C^*-algebras (see *e.g.* [25,3]).

Lemma 1. *A PU-map, i.e. a morphism in the category* $\mathbf{Cstar}_{PU}$*, commutes with involution* $(-)^*$*, and preserves the partial order* $\leq$ *on self-adjoint elements given by* (1).

Moreover, a PU-map f *satisfies* $\|f(x)\| \leq 4\|x\|$*, so that* $\|f(x) - f(y)\| \leq 4\|x-y\|$*, making* f *continuous.*

Proof. An element x is called self-adjoint if $x^* = x$. Each self-adjoint x can be written as difference $x = x_p - x_n$ of positive elements x_p, x_n, with $\|x_p\|, \|x_n\| \leq \|x\|$, see [5, 1.5.7]; as a result $f(x^*) = f(x) = f(x)^*$, for a PU-map f. Next, an arbitrary element y can be written as $y = y_r + iy_i$ for self-adjoint elements $y_r = \frac{1}{2}(y+y^*)$, $y_i = \frac{1}{2i}(y-y^*)$, so that $\|y_r\|, \|y_i\| \leq \|y\|$. Then $f(y^*) = f(y)^*$. Preservation of the order is trivial.

For positive x we have $x \leq \|x\| \bullet 1$, and thus $f(x) \leq \|x\| \bullet 1$, which gives $\|f(x)\| \leq \|x\|$. An arbitrary element x can be written as linear combination of four positive elements x_i, as in $x = x_1 - x_2 + ix_3 - ix_4$, with $\|x_i\| \leq \|x\|$. Finally, $\|f(x)\| = \|f(x_1) - f(x_2) + if(x_3) - if(x_4)\| \leq \sum_i \|f(x_i)\| \leq \sum_i \|x_i\| \leq 4\|x\|$. □

The following famous result is known as Gelfand duality, relating compact Hausdorff spaces and *commutative* C^*-algebras. Notice that this result involves the "MIU" maps.

Theorem 1. *Let* $\mathbf{CH}$ *be the category of compact Hausdorff spaces, with continuous maps between them. Sending* $X \in \mathbf{CH}$ *to the algebra of continuous functions* $X \to \mathbb{C}$ *yields an equivalence of categories* $C(-) = \mathit{Cont}(-,\mathbb{C}) \colon \mathbf{CH} \xrightarrow{\simeq} (\mathbf{CCstar}_{MIU})^{op}$. □

The inverse to the functor $C(-)$ sends a commutative C^*-algebra A to its spectrum $\mathrm{Spec}(A)$, given by the MIU-maps $A \to \mathbb{C}$, or equivalently, by the so-called pure states (see below).

Corollary 1. *For each finite-dimensional commutative* C^**-algebra* A *there is an* $n \in \mathbb{N}$ *with* $A \cong \mathbb{C}^n$ *in* $\mathbf{FdCCstar}_{MIU}$.

Proof. By the previous theorem there is a compact Hausdorff space X such that A is MIU-isomorphic to the algebra of continuous maps $X \to \mathbb{C}$. This X must be finite, and since a finite Hausdorff space is discrete, all maps $X \to \mathbb{C}$ are continuous. Let $n \in \mathbb{N}$ be the number of elements in X; then we have an isomorphism $A \cong \mathbb{C}^n$. □

As we can already see in the above theorem, it is the *opposite* of a category of C^*-algebras that provides the most natural setting for computations. This is in line with what is often called Heisenberg's picture. In a logical setting it corresponds to computation of weakest preconditions, going backwards. The situation may be compared to the category of complete Heyting algebras, which is most usefully known in opposite form, as the category of locales, see [18].

For a C^*-algebra A a *state* is positive unital map $A \to \mathbb{C}$. The set $\mathit{Hom}_{\mathrm{PU}}(A, \mathbb{C})$ of such states can be equipped with the weak *-topology, which is the coarsest (smallest) topology in which all maps $\mathit{ev}_x = \lambda s.\, s(x) \colon \mathit{Hom}_{\mathrm{PU}}(A, \mathbb{C}) \to \mathbb{C}$, for $x \in A$, are continuous.

Proposition 1. *For each C^*-algebra A, the set of states $\mathit{Hom}_{PU}(A, \mathbb{C})$ is convex, and compact Hausdorff in the weak-* topology. Each PU-map $f \colon A \to B$ yields an affine continuous function $(-) \circ f \colon \mathit{Hom}_{PU}(B, \mathbb{C}) \to \mathit{Hom}_{PU}(A, \mathbb{C})$.*

We recall that a function (between convex sets) is called *affine* if it preserves convex sums. We will see shortly that such affine maps are homomorphisms of Eilenberg-Moore algebras for the distribution monad $\mathcal{D}$.

Proof. For each finite set $h_i \in \mathit{Hom}_{\mathrm{PU}}(A, \mathbb{C})$ with $r_i \in [0, 1]$ satisfying $\sum_i r_i = 1$, the function $h = \sum_i r_i h_i$ is again a state. Moreover, such convex sums are preserved by precomposition, making the maps $(-) \circ f$ affine.

If states $h, k \in \mathit{Hom}_{\mathrm{PU}}(A, \mathbb{C})$ are not equal, say $h(x) \neq k(x)$, then either the real or imaginary parts of $h(x)$ and $k(x)$ differ. Let's consider the former. Then there is a real number r with $\mathrm{re}(h(x)) < r < \mathrm{re}(k(x))$. The two open subsets $\mathit{ev}_x^{-1}(\{z \in \mathbb{C} \mid \mathrm{re}(z) < r\})$ and $\mathit{ev}_x^{-1}(\{z \in \mathbb{C} \mid \mathrm{re}(z) > r\})$ then separate h, k. Hence $\mathit{Hom}_{\mathrm{PU}}(A, \mathbb{C})$ is Hausdorff. It is compact by Alaoglu's Theorem.

Precomposition $(-) \circ f$ is continuous, since for $x \in A$ and $U \subseteq \mathbb{C}$ open we get an open subset $\big((-) \circ f\big)^{-1}(\mathit{ev}_x^{-1}(U)) = \{h \mid \mathit{ev}_x(h \circ f) \in U\} = \mathit{ev}_{f(x)}^{-1}(U)$. □

2.1 Effect Modules

Effect algebras have been introduced in mathematical physics [7], in the investigation of quantum probability, see [6] for an overview. An *effect algebra* is a partial commutative monoid $(M, 0, \ovee)$ with an orthocomplement $(-)^\perp$. One writes $x \perp y$ if $x \ovee y$ is defined. The formulation of the commutativity and associativity requirements is a bit involved, but essentially straightforward. The orthocomplement satisfies $x^{\perp\perp} = x$ and $x \ovee x^\perp = 1$, where $1 = 0^\perp$. There is always a partial order, given by $x \leq y$ iff $x \ovee z = y$, for some z. The main example is the unit interval $[0, 1] \subseteq \mathbb{R}$, where addition $+$ is obviously partial, commutative, associative, and has 0 as unit; moreover, the orthocomplement is $r^\perp = 1 - r$. We write **EA** for the category of effect algebras, with morphism preserving $\ovee$ and 1 — and thus all other structure.

For each set X, the set $[0, 1]^X$ of fuzzy predicates on X is an effect algebra, via pointwise operations. Each Boolean algebra B is an effect algebra with $x \perp y$ iff $x \wedge y = \perp$;

then $x \ovee y = x \vee y$. In a quantum setting, the main example is the set of effects $\mathcal{E}f(H) = \{E\colon H \to H \mid 0 \leq E \leq I\}$ on a Hilbert space H, see *e.g.* [6,11].

An *effect module* is an "effect" version of a vector space. It involves an effect algebra M with a scalar multiplication $s \bullet x \in M$, where $s \in [0,1]$ and $x \in M$. This scalar multiplication is required to be a suitable homomorphism in each variable separately. The algebras $[0,1]^X$ and $\mathcal{E}f(H)$ are clearly such effect modules. In the subcategory **EMod** $\hookrightarrow$ **EA** maps additionally commute with scalar multiplication.

For a C^*-algebra A the subset $A^+ \hookrightarrow A$ of positive elements carries a partial order $\leq$ defined on self-adjoint elements in (1). We write $[0,1]_A \subseteq A^+ \subseteq A$ for the subset of positive elements below the unit. The elements in $[0,1]_A$ will be called effects (or sometimes also: predicates). For instance, for the C^*-algebra $B(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$ the unit interval $[0,1]_{B(\mathcal{H})} \subseteq B(H)$ contains the effects $\mathcal{E}f(\mathcal{H}) = \{A \in B(\mathcal{H}) \mid 0 \leq A \leq \mathrm{id}\}$ on $\mathcal{H}$.

We claim that $[0,1]_A$ is an *effect algebra* and carries a $[0,1] \subseteq \mathbb{R}$ scalar multiplication, thus making it an *effect module*.

- Since A with $0, +$ is a partially ordered Abelian group, $[0,1]_A$ is a so-called interval effect algebra, with $x \perp y$ iff $x + y \leq 1$, and in that case $x \ovee y = x + y$. The ortocomplement $x^\perp$ is given by $1 - x$.
- For $r \in [0,1]$ and $x \in [0,1]_A$ the scalar multiplications rx and $(1-r)x$ are positive, and their sum is $x \leq 1$. Hence $rx \leq 1$ and thus $rx \in [0,1]_A$.

Each map of C^*-algebras $f\colon A \to B$ preserves $\leq$ and thus restricts to $[0,1]_A \to [0,1]_B$. This restriction is a map of effect modules. Hence we get a "predicate" functor $\mathbf{Cstar}_{\mathrm{PU}} \to \mathbf{EMod}$.

Lemma 2. *The functor* $[0,1]_{(-)}\colon$ ***Cstar***$_{PU} \to$ ***EMod*** *is full and faithful.*

Proof. Any PU-map $f\colon A \to B$ is completely determined (and defined by) its action on $[0,1]_A$: for a non-zero positive element $x \in A$ we use $x \leq \|x\|\,1$ and thus $\frac{1}{\|x\|}\,x \in [0,1]_A$ to see that $f(x) = \|x\|\,f(\frac{1}{\|x\|}\,x)$. An arbitrary element $y \in A$ can be written as linear sum of four positive elements (see Lemma 1), determining $f(y)$. □

The (finite, discrete probability) distribution monad $\mathcal{D}\colon$ **Sets** $\to$ **Sets** sends a set X to the set $\mathcal{D}(X) = \{\varphi\colon X \to [0,1] \mid \mathit{supp}(\varphi) \text{ is finite, and } \sum_x \varphi(x) = 1\}$, where $\mathit{supp}(\varphi) = \{x \mid \varphi(x) \neq 0\}$. Such an element $\varphi \in \mathcal{D}(X)$ may be identified with a finite, formal convex sum $\sum_i r_i x_i$ with $x_i \in X$ and $r_i \in [0,1]$ satisfying $\sum_i r_i = 1$. The unit $\eta\colon X \to \mathcal{D}(X)$ and multiplication $\mu\colon \mathcal{D}^2(X) \to \mathcal{D}(X)$ of this monad are given by singleton/Dirac convex sum and by matrix multiplication:

$$\eta(x) = 1x \qquad \mu(\Phi)(x) = \textstyle\sum_\varphi \Phi(\varphi) \cdot \varphi(x).$$

A *convex* set is an Eilenberg-Moore algebra of this monad: it consists of a carrier set X in which actual sums $\sum_i r_i x_i \in X$ exist for all convex combinations. We write $\mathbf{Conv} = \mathcal{EM}(\mathcal{D})$ for the category of convex sets, with "affine" functions preserving convex sums.

Effect modules and convex sets are related via a basic adjunction [17], obtained by "homming into $[0,1]$", as in:

$$\mathbf{EMod}^{\mathrm{op}} \underset{\mathbf{Conv}(-,[0,1])}{\overset{\mathbf{EMod}(-,[0,1])}{\rightleftarrows}} \mathbf{Conv} \quad (\top) \tag{2}$$

3 Set-Theoretic Computations in C^*-Algebras

For a set X, a function $f\colon X \to \mathbb{C}$ is called *bounded* if $|f(x)| \leq s$, for some $s \in \mathbb{R}_{\geq 0}$. We write $\ell^\infty(X)$ for the set of such bounded functions. Notice that if X is finite, any function $X \to \mathbb{C}$ is bounded, so that $\ell^\infty(X) = \mathbb{C}^X$.

Each $\ell^\infty(X)$ is a commutative C^*-algebra, with pointwise addition, multiplication and involution, and with the supremum norm $\|f\| = \inf\{s \in \mathbb{R}_{\geq 0} \mid \forall x.\, |f(x)| \leq s\}$. In fact it is a typical example of a commutative W^*-algebra, but we do not require this fact. This yields a functor $\ell^\infty\colon \mathbf{Sets} \to (\mathbf{CCstar}_{\mathrm{MIU}})^{\mathrm{op}}$, where for $h\colon X \to Y$ we have $\ell^\infty(h) = (-) \circ h\colon \ell^\infty(Y) \to \ell^\infty(X)$; it preserves the (pointwise) operations. When we restrict to the full subcategory $\mathbf{FinSets} \hookrightarrow \mathbf{Sets}$ we obtain a functor $\ell^\infty = \mathbb{C}^{(-)}\colon \mathbf{FinSets} \to (\mathbf{FdCCstar}_{\mathrm{MIU}})^{\mathrm{op}}$. The next result is then a well-known special case of Gelfand duality (Theorem 1). We elaborate the proof in some detail because it is important to see where the preservation of multiplication plays a role.

Proposition 2. *The functor $\mathbb{C}^{(-)}\colon$ **FinSets** $\to$ $(\mathbf{FdCCstar}_{MIU})^{op}$ is an equivalence of categories.*

Proof. It is easy to see that the functor $\mathbb{C}^{(-)}$ is faithful. The crucial part is to see that it is full. So assume we have two finite sets, seen as natural numbers n, m, and a MIU-homomorphism $h\colon \mathbb{C}^m \to \mathbb{C}^n$. For $j \in m$, let $|j\rangle \in \mathbb{C}^m$ be the standard base vector with 1 at the j-th position and 0 elsewhere. Since this $|j\rangle$ is positive, so is $h(|j\rangle)$, and thus we may write it as $h(|j\rangle) = (r_{1j}, \ldots, r_{nj})$, with $r_{ij} \in \mathbb{R}_{\geq 0}$. Because $|j\rangle \cdot |j\rangle = |j\rangle$, and h preserves multiplication, we get $h(|j\rangle)\cdot h(|j\rangle) = h(|j\rangle)$, and thus $r_{ij}^2 = r_{ij}$. This means $r_{ij} \in \{0,1\}$, so that h is a (binary) Boolean matrix. But h is also unital, and so:

$$1 = h(1) = h(|1\rangle + \cdots + |m\rangle) = h(|1\rangle) + \cdots + h(|m\rangle). \tag{3}$$

For each $i \in n$ there is thus precisely one $j \in m$ with $r_{ij} = 1$ — so that h is a "functional" Boolean matrix. This yields the required function $f\colon n \to m$ with $\mathbb{C}^f = h$.

Corollary 1 says that the functor $\mathbb{C}^{(-)}\colon \mathbf{FinSets} \to (\mathbf{FdCCstar}_{\mathrm{MIU}})^{\mathrm{op}}$ is essentially surjective on objects, and thus an equivalence. □

This proof demonstrates that preservation of multiplication, as required for "MIU" maps, is a rather strong condition. We make this more explicit.

Corollary 2. *For $n \in \mathbb{N}$ we have $Hom_{MIU}(\mathbb{C}^n, \mathbb{C}) \cong n$.*

Proof. When we identify $n \in \mathbb{N}$ with the n-element set $n = \{0, 1, \ldots, n-1\} \in$ **FinSets**, we get by Proposition 2, $Hom_{\mathrm{MIU}}(\mathbb{C}^n, \mathbb{C}) \cong \mathbf{FinSets}(1, n) \cong n$. □

4 Discrete Probabilistic Computations in C^*-Algebras

We turn to probabilistic computations and will see that we remain in the world of commutative C^*-algebras, but with PU-maps (positive unital) instead of MIU-maps. Recall that states of a C^*-algebra A are the PU-maps $A \to \mathbb{C}$.

Lemma 3. *Sending a set X to the set of states of the C^*-algebra $\ell^\infty(X)$ yields the expectation monad $\mathcal{E}$ from [16]: the mapping $X \mapsto Hom_{PU}(\ell^\infty(X), \mathbb{C})$ is isomorphic to the expectation monad $\mathcal{E}\colon$ **Sets** $\to$ **Sets**, defined in [16] via effect module homomorphisms: $\mathcal{E}(X) = \textbf{\textit{EMod}}\big([0,1]^X, [0,1]\big)$.*

As a result, $Hom_{PU}(\mathbb{C}^n, \mathbb{C}) \cong \mathcal{D}(n)$, for $n \in \mathbb{N}$, where $\mathcal{D}(n)$ is the standard n-simplex.

Proof. The predicate/effect functor $[0,1]_{(-)}\colon \mathbf{Cstar}_{\mathrm{PU}} \to \mathbf{EMod}$ is full and faithful by Lemma 2, and so:

$$Hom_{\mathrm{PU}}\big(\ell^\infty(X), \mathbb{C}\big) \cong \mathbf{EMod}\big([0,1]_{\ell^\infty(X)}, [0,1]_{\mathbb{C}}\big) = \mathbf{EMod}\big([0,1]^X, [0,1]\big).$$

The isomorphism $\alpha\colon Hom_{\mathrm{PU}}(\mathbb{C}^n, \mathbb{C}) \xrightarrow{\cong} \mathcal{D}(n)$ follows because the expectation and distribution monad coincide on finite sets, see [16]. Explicitly, it is given by $\alpha(h) = \lambda i \in n.\, h(|\,i\,\rangle)$ and $\alpha^{-1}(\varphi)(v) = \sum_i \varphi(i) \cdot v(i)$. $\square$

The unit η and multiplication μ structure on $\mathcal{E}(X) \cong Hom_{\mathrm{PU}}(\ell^\infty(X), \mathbb{C})$ is very much like for "continuation" or "double dual" monads, see [19,22,13], with:

$$X \xrightarrow{\eta} Hom_{\mathrm{PU}}(\ell^\infty(X), \mathbb{C}) \qquad Hom_{\mathrm{PU}}\Big(\ell^\infty\big(Hom_{\mathrm{PU}}(\mathbb{C}^X, \mathbb{C})\big), \mathbb{C}\Big) \xrightarrow{\mu} Hom_{\mathrm{PU}}(\ell^\infty(X), \mathbb{C})$$

$$x \longmapsto \lambda v.\, v(x) \qquad\qquad g \longmapsto \lambda v.\, g\big(\lambda h.\, h(v)\big).$$

For an arbitrary monad $T = (T, \eta, \mu)$ on a category $\mathbf{B}$ we write $\mathcal{K}\ell(T)$ for the Kleisli category of T. Its objects are the same as those of $\mathbf{B}$, but its maps $X \to Y$ are the maps $X \to T(Y)$ in $\mathbf{B}$. The unit $\eta\colon X \to T(X)$ is the identity map $X \to X$ in $\mathcal{K}\ell(T)$; and composition of $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{K}\ell(T)$ is given by $g \odot f = \mu \circ T(g) \circ f$. Maps in such a Kleisli category are understood as computations with outcomes of type T, see [22]. For a monad $T\colon \mathbf{Sets} \to \mathbf{Sets}$ we write $\mathcal{K}\ell_{\mathbb{N}}(T) \hookrightarrow \mathcal{K}\ell(T)$ for the full subcategory with numbers $n \in \mathbb{N}$ as objects, considered as n-element sets.

Proposition 3. *The expectation monad $\mathcal{E}(X) \cong Hom_{PU}(\ell^\infty(X), \mathbb{C})$ gives rise to a full and faithful functor:*

$$\begin{array}{ccc} \mathcal{K}\ell(\mathcal{E}) & \xrightarrow{\quad \mathcal{C}_{\mathcal{E}} \quad} & (\textbf{\textit{CCstar}}_{PU})^{op} \\ X & \longmapsto & \ell^\infty(X) \\ \big(X \xrightarrow{f} \mathcal{E}(Y)\big) & \longmapsto & \lambda v \in \ell^\infty(Y).\, \lambda x \in X.\, f(x)(v). \end{array} \tag{4}$$

Proof. First we need to see that $\mathcal{C}_{\mathcal{E}}(f)$ is well-defined: the function $\mathcal{C}_{\mathcal{E}}(f)(v)\colon X \to \mathbb{C}$ must be bounded. We can apply Lemma 1 to the function $f(x) \in Hom_{\mathrm{PU}}(\ell^\infty(Y), \mathbb{C})$;

it yields $\|f(x)(v)\| \leq 4\|v\|$. This holds for each $x \in X$, so that $|\mathcal{C}_\mathcal{E}(f)(v)(x)| = |f(x)(v)|$ is bounded by $4\|v\|$. Next, the map $\mathcal{C}_\mathcal{E}(f)$ is a PU-map of C^*-algebras via the pointwise definitions of the relevant constructions.

We check that $\mathcal{C}_\mathcal{E}$ preserves (Kleisli) identities and composition:

$$\begin{aligned}
\mathcal{C}_\mathcal{E}(\mathrm{id})(v)(x) &= \mathcal{C}_\mathcal{E}(\eta)(v)(x) \\
&= \eta(x)(v) \\
&= v(x) \\
\mathcal{C}_\mathcal{E}(g \odot f)(v)(x) &= (g \odot f)(x)(v) \\
&= \mu\Big(\mathcal{E}(g)(f(x))\Big)(v) \\
&= \mathcal{E}(g)(f(x))\big(\lambda w.\, w(v)\big) \\
&= f(x)\big((\lambda w.\, w(v)) \circ g\big) \\
&= f(x)\big(\lambda y.\, g(y)(v)\big) \\
&= f(x)\big(\mathcal{C}_\mathcal{E}(g)(v)\big) \\
&= \mathcal{C}_\mathcal{E}(f)\big(\mathcal{C}_\mathcal{E}(g)(v)\big)(x) \\
&= \big(\mathcal{C}_\mathcal{E}(f) \circ \mathcal{C}_\mathcal{E}(g)\big)(v)(x).
\end{aligned}$$

Further, $\mathcal{C}_\mathcal{E}$ is obviously faithful, and it is full since for $h\colon \ell^\infty(Y) \to \ell^\infty(X)$ in $\mathbf{CCstar}_{\mathrm{PU}}$ we can define $f\colon X \to \mathit{Hom}_{\mathrm{PU}}(\ell^\infty(Y), \mathbb{C})$ by $f(x)(v) = h(v)(x)$. Then each $f(x)$ is a PU-map of C^*-algebras. □

We turn to the finite case, like in the previous section. We do so by considering the Kleisli category $\mathcal{K}\ell_\mathbb{N}(\mathcal{E})$ obtained by restricting to objects $n \in \mathbb{N}$. Since the expectation monad $\mathcal{E}$ and the distribution monad $\mathcal{D}$ coincide on finite sets, we have $\mathcal{K}\ell_\mathbb{N}(\mathcal{E}) \cong \mathcal{K}\ell_\mathbb{N}(\mathcal{D})$. Maps $n \to m$ in this category are probabilistic transition matrices $n \to \mathcal{D}(m)$. The following equivalence is known, see *e.g.* [20], although possibly not in this categorical form.

Proposition 4. *The functor $\mathcal{C}_\mathcal{E}$ from* (4) *restricts in the finite case to an equivalence of categories:*

$$\mathcal{K}\ell_\mathbb{N}(\mathcal{D}) \xrightarrow[\simeq]{\mathcal{C}_\mathcal{D}} (\boldsymbol{FdCCstar}_{PU})^{op} \tag{5}$$

It is given by $\mathcal{C}_\mathcal{D}(n) = \mathbb{C}^n$ *and* $\mathcal{C}_\mathcal{D}\big(n \stackrel{f}{\to} \mathcal{D}(m)\big) = \lambda v \in \mathbb{C}^m.\, \lambda i \in n.\, \sum_{j \in m} f(i)(j) \cdot v(j)$.

This equivalence (5) may be read as: the category $\mathbf{FdCCstar}_{\mathrm{PU}}$ of finite-dimensional commutative C^*-algebras, with positive unital maps, is the *Lawvere theory* of the distribution monad $\mathcal{D}$.

Proof. Fullness and faithfulness of the functor $\mathcal{C}_\mathcal{D}$ follow from Proposition 3, using the isomorphism $\mathit{Hom}_{\mathrm{PU}}(\mathbb{C}^n, \mathbb{C}) \cong \mathcal{D}(n)$ from Lemma 3. This functor $\mathcal{C}_\mathcal{D}$ is essentially surjective on objects by Corollary 1, using the fact that a MIU-map is a PU-map. □

5 Continuous Probabilistic Computations

The question arises if the full and faithful functor $\mathcal{K}\ell(\mathcal{E}) \to (\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$ from Proposition 3 can be turned into an equivalence of categories, but not just for the finite case like in Proposition 4. In order to make this work we have to lift the expectation monad $\mathcal{E}$ on **Sets** to the category **CH** of compact Hausdorff spaces. As lifting we use what we call the *Radon* monad $\mathcal{R}$, defined on $X \in \mathbf{CH}$ as:

$$\mathcal{R}(X) = \mathit{Hom}_{\mathrm{PU}}\big(C(X), \mathbb{C}\big), \tag{6}$$

where, as usual, $C(X) = \{f \colon X \to \mathbb{C} \mid f \text{ is continuous}\}$; notice that the functions $f \in C(X)$ are automatically bounded, since X is compact. These are related to measures in the following way. If μ is a probability measure on the Borel sets of X, integration of continuous functions with respect to μ gives $\int_X \text{-} d\mu \in \mathcal{R}(X)$. A Radon probability measure, or an inner regular probability measure, is one such that $\mu(S) = \sup_{K \subseteq S} \mu(K)$ where K ranges over compact sets. The map from measures to elements of $\bar{R}(X)$ is a bijection[24, Thm. 2.14], and accordinly we shall sometimes refer to elements of $\mathcal{R}(X)$ as measures. Therefore the Radon monad can be considered as a variant of the Giry monad. It differs in that it uses the topology of a space, and that in the case of a non-Polish space there can be non-Radon measures[8, 434K (d), page 192].

This Radon monad $\mathcal{R}$ is known, it first occurs implicitly in [27] as the monad of an adjunction ("probability measure" is used to mean "Radon probability measure" in that article).

From Proposition 1 it is immediate that $\mathcal{R}(X)$ is again a compact Hausdorff space. The unit $\eta \colon X \to \mathcal{R}(X)$ and multiplication $\mu \colon \mathcal{R}^2(X) \to \mathcal{R}(X)$ are defined as for the expectation monad, namely as $\eta(x)(v) = v(x)$ and $\mu(g)(v) = g\big(\lambda h.\, h(v)\big)$. We check that η is continuous. Recall from the proof of Proposition 1 that a basic open in $\mathcal{R}(X)$ is of the form $ev_s^{-1}(U) = \{h \in \mathcal{R}(X) \mid h(s) \in U\}$, where $s \in C(X)$ and $U \subseteq \mathbb{C}$ is open. Then:

$$\eta^{-1}\big(\mathrm{ev}_s^{-1}(U)\big) = \{x \in X \mid \eta(x)(s) \in U\} = \{x \in X \mid s(x) \in U\} = s^{-1}(U).$$

The latter is an open subset of X since $s \colon X \to \mathbb{C}$ is a continuous function.

We are now ready to state our main, new duality result. It is may be understood as a probabilistic version of Gelfand duality.

Theorem 2. *The Radon monad* (6) *yields an equivalence of categories:*

$$\mathcal{K}\ell(\mathcal{R}) \simeq (\textbf{\textit{CCstar}}_{PU})^{op}.$$

Proof. We define a functor $\mathcal{C}_{\mathcal{R}} \colon \mathcal{K}\ell(\mathcal{R}) \to (\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$ like in (4), namely by:

$$\mathcal{C}_{\mathcal{R}}(X) = C(X) \qquad \mathcal{C}_{\mathcal{R}}(f) = \lambda v.\, \lambda x.\, f(x)(v).$$

Since $f \colon X \to \mathcal{R}(Y)$ is itself continuous, so is $f(-)(v) \colon X \to \mathbb{C}$.

The fact that $\mathcal{C}_{\mathcal{R}}$ is a full and faithful functor follows as in the proof of Proposition 3. This functor is essentially surjective on objects by ordinary Gelfand duality (Theorem 1). □

We investigate the Radon monad $\mathcal{R}$ a bit further, in particular its relation to the distribution monad $\mathcal{D}$ on **Sets**.

Lemma 4. *There is a map of monads* $(U, \tau)\colon \mathcal{R} \to \mathcal{D}$ *in:*

$$\mathcal{R} \circlearrowright \textbf{\textit{CH}} \xrightarrow{\;U\;} \textbf{\textit{Sets}} \circlearrowleft \mathcal{D} \qquad\qquad \mathcal{D}U \stackrel{\tau}{\Longrightarrow} U\mathcal{R}$$

where U is the forgetful functor and τ commutes appropriately with the units and multiplications of the monads $\mathcal{D}$ and $\mathcal{R}$. (Such a map is called a "monad functor" in [26, §1].)

As a result the forgetful functor lifts to the associated categories of Eilenberg-Moore algebras:

$$\begin{array}{ccc} \mathcal{E}\mathcal{M}(\mathcal{R}) & \longrightarrow & \mathcal{E}\mathcal{M}(\mathcal{D}) = \textbf{\textit{Conv}} \\ \big(\mathcal{R}(X) \stackrel{\alpha}{\to} X\big) & \longmapsto & \big(\mathcal{D}(UX) \stackrel{\tau}{\to} U\mathcal{R}(X) \stackrel{U\alpha}{\to} UX\big) \end{array}$$

Hence the carrier of a $\mathcal{R}$-algebra is a convex compact Hausdorff space, and every algebra map is an affine function.

Proof. For $X \in$ **CH** and $\varphi \in \mathcal{D}(UX)$, that is for $\varphi\colon UX \to [0, 1]$ with finite support and $\sum_x \varphi(x) = 1$, we define $\tau(\varphi) \in U\mathcal{R}(X)$ on $h \in C(X)$ as:

$$\tau(\varphi)(h) \;=\; \textstyle\sum_x \varphi(x) \cdot h(x) \;\in\; \mathbb{C}. \qquad (7)$$

It is easy to see that τ is a linear map $C(X) \to \mathbb{C}$ that preserves positive elements and the unit. Moreover, it commutes appropriately with the units and multiplications. For instance:

$$\big(\tau_X \circ \eta^{\mathcal{D}}_{UX}\big)(x)(h) \;=\; \tau_X(1x)(h) \;=\; h(x) \;=\; U(\eta^{\mathcal{R}}_X)(x)(h). \qquad \square$$

The dual space of $C(X)$ can be ordered using (1), by taking the positive cone to be those linear functionals that map positive functions to positive numbers.

Definition 2. *A state $\phi \in \mathcal{R}(X) = Hom_{PU}(C(X), \mathbb{C})$ is a* pure state *if for for each positive linear functional $\psi \leq \phi$ there exists an $\alpha \in [0, 1]$ such that $\psi = \alpha\phi$.*

Lemma 5. *For a compact Hausdorff space X, the subset of unit (or Dirac) measures $\{\eta(x) \mid x \in X\} \subseteq \mathcal{R}(X)$ is the set of extreme points of the set of Radon measures $\mathcal{R}(X)$ — where $\eta(x) = \eta^{\mathcal{R}}(x) = ev_x = \lambda h.\, h(x)$ is the unit of the monad $\mathcal{R}$.*

Proof. We rely on the basic fact, see [5, 2.5.2, page 43], that Dirac measures $\eta(x) \in \mathcal{R}(X)$ are "pure" states. We prove the above lemma by showing that the pure states are precisely the extreme points of the convex set $\mathcal{R}(X)$.

– If $\phi \in \mathcal{R}(X)$ is a pure state, suppose $\phi = \alpha_1\phi_1 + \alpha_2\phi_2$, a convex combination of two states $\phi_i \in \mathcal{R}(X)$ with $\alpha_i \in [0, 1]$ satisfying $\alpha_1 + \alpha_2 = 1$, where no two elements of $\{\phi, \phi_1, \phi_2\}$ are the same. Then $\phi \geq \alpha_1\phi_1$, since for a positive function $f \in C(X)$ one has $(\phi - \alpha_1\phi_1)(f) = \alpha_2\phi_2(f) \geq 0$. Thus $\alpha_1\phi_1 = \alpha\phi$, for some $\alpha \in [0, 1]$, since ϕ is pure. Then $\alpha_1 = \alpha_1\phi_1(1) = \alpha\phi(1) = \alpha$. If $\alpha_1 = 0$, then $\alpha_2 = 1$ and so $\phi = \phi_2$. If $\alpha_1 > 0$, then $\phi = \phi_1$. Hence ϕ is an extreme point.

– Suppose ϕ is an extreme point of $\mathcal{R}(X)$, *i.e.* that $\phi = \alpha_1\phi_1 + \alpha_2\phi_2$ implies ϕ_1 or $\phi_2 = \phi$. Then if there is a positive linear functional $\psi \leq \phi$, we may take $\alpha_1 = \psi(1) \geq 0$; since $\alpha_1 = \psi(1) \leq \phi(1) = 1$, we get $\alpha_1 \in [0, 1]$. If $\alpha_1 = 0$, then since $\|\psi\| = \psi(1) = 0$ we get $\psi = 0$ and $\psi = 0 \cdot \phi$. If $\alpha_1 = 1$, then $(\phi - \psi)(1) = 0$, which since $\phi - \psi$ was assumed to be positive implies $\phi - \psi = 0$ and hence $\psi = 1 \cdot \phi$. Having dealt with those cases, we have that $\alpha_1 \in (0, 1)$, and so we have a state $\phi_1 = \frac{1}{\alpha_1}\psi$. We may take $\alpha_2 = 1 - \alpha_1 \in (0, 1)$ and obtain a second state $\phi_2 = \frac{1}{\alpha_2}(\phi - \psi)$. By construction we have a convex decomposition of $\phi = \alpha_1\phi_1 + \alpha_2\phi_2$. Therefore either $\phi = \phi_1 = \frac{1}{\alpha_1}\psi$ or $\phi = \phi_2 = \frac{1}{\alpha_2}(\phi - \psi)$. In the first case, $\psi = \alpha_1\phi$, making ϕ pure. But also in the second case ϕ is pure, since we have $\alpha_2\phi = \phi - \psi$ and thus $\psi = (1 - \alpha_2)\phi$. □

Lemma 6. *Let X be a compact Hausdorff space.*

1. *The maps $\tau_X : \mathcal{D}(UX) \to U\mathcal{R}(X)$ from* (7) *are injective; as a result, the unit/Dirac maps $\eta : X \to \mathcal{R}(X)$ are also injective.*
2. *The maps $\tau_X : \mathcal{D}(UX) \rightarrowtail U\mathcal{R}(X)$ are dense.*

Proof. For the first point, assume $\varphi, \psi \in \mathcal{D}(UX)$ satisfying $\tau(\varphi) = \tau(\psi)$. We first show that the finite support sets are equal: $supp(\varphi) = supp(\psi)$. Since X is Hausdorff, singletons are closed, and hence finite subsets too. Suppose $supp(\varphi) \not\subseteq supp(\psi)$, so that $S = supp(\varphi) - supp(\psi)$ is non-empty. Since S and $supp(\psi)$ are disjoint closed subsets, there is by Urysohn's lemma a continuous function $f : X \to [0, 1]$ with $f(x) = 1$ for $x \in S$ and $f(x) = 0$ for $x \in supp(\psi)$. But then $\tau(\psi)(f) = 0$, whereas $\tau(\varphi)(f) \neq 0$.

Now that we know $supp(\varphi) = supp(\psi)$, assume $\varphi(x) \neq \psi(x)$, for some $x \in supp(\varphi)$. The closed subsets $\{x\}$ and $supp(\varphi) - \{x\}$ are disjoint, so there is, again by Urysohn's lemma a continuous function $f : X \to [0, 1]$ with $f(x) = 1$ and $f(y) = 0$ for all $y \in supp(\varphi)$. But then $\varphi(x) = \tau(\varphi)(f) = \tau(\psi)(f) = \psi(x)$, contradicting the assumption.

We can conclude that the unit $X \to \mathcal{R}(X)$ is also injective, since its underlying function can be written as composite $U(\eta) = \tau \circ \eta : UX \rightarrowtail \mathcal{D}(UX) \rightarrowtail U\mathcal{R}(X)$, because τ is a map of monads.

To show that the image of τ_X is dense, we proceed as follows. By Lemmas 5 and 4, the extreme points of $\mathcal{R}(X)$ are

$$\{\eta^{\mathcal{R}}(x) \mid x \in X\} = \{\tau\big(\eta^{\mathcal{D}}(x)\big) \mid x \in X\}$$

and are thus in the image of $\tau : \mathcal{D}(UX) \rightarrowtail U\mathcal{R}(X)$. Since every convex combination of $\eta^{\mathcal{R}}(x)$ comes from a formal convex sum $\varphi \in \mathcal{D}(UX)$, all convex combinations of extreme points are in the image of τ_X. As $\mathcal{R}(X)$ is a compact convex subset of $C(X)^{w*}$ (*i.e.* with the weak-* topology), a locally convex space, we may apply the Krein-Milman theorem [4, Proposition 7.4, page 142] to conclude the set of convex combinations of extreme points is dense. □

Lemma 7. *Let X, Y be compact Hausdorff spaces. Each Eilenberg-Moore algebra $\alpha : \mathcal{R}(X) \to X$ is an affine function. For each continuous map $f : X \to Y$, the function $\mathcal{R}(f) : \mathcal{R}(X) \to \mathcal{R}(Y)$ is affine.*

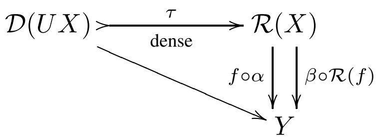
Proof. This follows from naturality of $\tau\colon \mathcal{D}U \Rightarrow U\mathcal{R}$. □

Proposition 5. *Let $\alpha\colon \mathcal{R}(X) \to X$ and $\beta\colon \mathcal{R}(Y) \to Y$ be two Eilenberg-Moore algebras of the Radon monad $\mathcal{R}$. A function $f\colon X \to Y$ is an algebra homomorphism if and only f is both continuous and affine.*

*As a result, the functor $\mathcal{EM}(\mathcal{R}) \to \mathcal{EM}(\mathcal{D}) =$ **Conv** from Lemma 4 is full and faithful.*

We shall follow the convention of writing $\mathcal{A}(X,Y)$ for the homset of continuous and affine functions $X \to Y$.

Proof. Clearly, each algebra map is both continuous and affine. For the converse, if $f\colon X \to Y$ is continuous, it is a map in the category **CH** of compact Hausdorff spaces. Since it is affine, both triangles commute in:



Since Y is Hausdorff, there is at most one such map. □

The category $\mathcal{EM}(\mathcal{R})$ of Eilenberg-Moore algebras of the Radon monad may thus be understood as a suitable category of convex compact Hausdorff spaces, with affine continuous maps between them. We side-step its precise characterisation — in order to avoid "observability" issues like in [16] — and will proceed with $\mathcal{EM}(\mathcal{R})$ as such. Details will be elaborated in an extended version of this paper.

6 States and Effects

We start with a simple observation.

Lemma 8. *The unit interval $[0,1]$ is obviously compact and Hausdorff. It carries a $\mathcal{R}$-algebra structure $\mathcal{R}([0,1]) \to [0,1]$, given by $h \mapsto h\big([0,1] \hookrightarrow \mathbb{C}\big)$.*

For an arbitrary $\mathcal{R}$-algebra X, the homset of algebra maps:

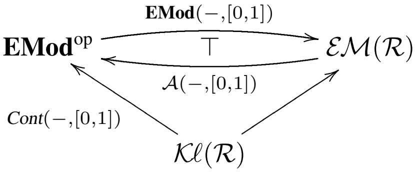
$$\mathcal{EM}(\mathcal{R})\big(X,[0,1]\big) = \mathcal{A}(X,[0,1])$$

*is an effect module, via pointwise constructions. Recall from Proposition 5 that this homset contains the affine and continuous functions $X \to [0,1]$. In this way we get a functor $\mathcal{A}(-,[0,1])\colon \mathcal{EM}(\mathcal{R}) \to$ **EMod**^{op}.* □

In [16] it is shown that for an effect module M, the homset $\mathbf{EMod}(M,[0,1])$ is a convex compact Hausdorff space. In fact, it carries an $\mathcal{R}$-algebra structure:

$$\begin{array}{rcl} \mathcal{R}\big(\mathbf{EMod}(M,[0,1])\big) & \xrightarrow{\quad \alpha_M \quad} & \mathbf{EMod}(M,[0,1]) \\ h & \longmapsto & \lambda x \in M.\, h(\mathrm{ev}_x) \end{array}$$

where $ev_x = \lambda v.\, v(x) \colon C\big(\mathbf{EMod}(M, [0,1])\big) \to \mathbb{C}$. For each map of effect modules $f \colon M \to M'$ one obtains a map of $\mathcal{R}$-algebras $(-) \circ f \colon \mathbf{EMod}(M', [0,1]) \to \mathbf{EMod}(M, [0,1])$. We thus obtain the following situation:



(8)

Such diagrams appear in [13] as a categorical representation of the duality between states and effects, with Schrödinger's picture on the right, and Heisenberg's picture on the left (see also [15]). In this diagram:

- The map $\mathcal{K}\ell(\mathcal{R}) \to \mathbf{EMod}^{\mathrm{op}}$ on the left is the "predicate" functor, sending a space X to the predicates on X, given by the effect module $\mathit{Cont}(X, [0,1])$ of continuous functions $X \to [0,1]$, or equivalently by the effects $[0,1]_{C(X)}$ on the C^*-algebra $C(X)$. This functor is full and faithful by Lemma 2 and Theorem 2:

$$\begin{aligned} \mathbf{EMod}\big(\mathit{Cont}(Y,[0,1]), \mathit{Cont}(X,[0,1])\big) &= \mathbf{EMod}\big([0,1]_{C(Y)}, [0,1]_{C(X)}\big) \\ &\cong \mathit{Hom}_{\mathrm{PU}}\big(C(Y), C(X)\big) \\ &\cong \mathcal{K}\ell(\mathcal{R})\big(X, Y\big). \end{aligned}$$

- The "state" functor $\mathcal{K}\ell(\mathcal{R}) \to \mathcal{EM}(\mathcal{R})$ is the standard full and faithful "comparison" functor from a Kleisli category to a category of Eilenberg-Moore algebras.
- The diagram (8) commutes in one direction:

$$\begin{aligned} \mathbf{EMod}\big(\mathit{Cont}(X,[0,1]), [0,1]\big) &= \mathbf{EMod}\big([0,1]_{C(X)}, [0,1]_{\mathbb{C}}\big) \\ &\cong \mathit{Hom}_{\mathrm{PU}}\big(C(X), \mathbb{C}\big) = \mathcal{R}(X). \end{aligned}$$

- The remainder of this section will be devoted to proving that the diagram also commutes in the other direction, *i.e.* $\mathcal{A}(\mathcal{R}(X), [0,1]) \cong \mathit{Cont}(X, [0,1])$.

There is an evaluation map ζ from $C(X)$ to $\mathcal{R}(X) \to \mathbb{C}$ defined as follows:

$$\zeta(f) = ev_f = \lambda\phi.\, \phi(f).$$

Lemma 9. *For $f \in C(X)$, this $\zeta(f)$ is affine and continuous, so $\zeta(f) \in \mathcal{A}(\mathcal{R}(X), \mathbb{C})$.*

Proof. The map $\zeta(f)$ is continuous because for an open $U \subseteq \mathbb{C}$ the inverse image $\zeta(f)^{-1}(U) = ev_f^{-1}(U)$ is by definition a basic open of the weak-* topology on $\mathcal{R}(X)$. It is also affine, since it sends convex sums in $\mathcal{R}(X)$ to convex sums in $\mathbb{C}$:

$$\zeta(f)\big(\textstyle\sum_i r_i\phi_i\big) = \zeta(f)\big(\lambda g.\, \sum_i r_i\phi_i(g)\big) = \sum_i r_i\phi_i(f) = \sum_i r_i\zeta(f)(\phi_i). \quad \square$$

The following lemma is a special case of the complexification of [2, proposition 2.2]. However, a simpler proof is possible in this special case by applying Gelfand duality, which we include here.

Lemma 10. *The set $\mathcal{A}(\mathcal{R}(X), \mathbb{C})$ can be given (pointwise) the structure of an ordered vector space with unit.*

Moreover, this space is isomorphic to $C(X)$, with one direction given by ζ.

Proof. The complex numbers are an ordered vector space with positive cone $[0, \infty) \subseteq \mathbb{C}$ and unit $1 \in \mathbb{C}$. By treating $U(\mathcal{R}(X))$ as a set, we obtain an ordered unital vector space structure on $\mathbb{C}^{U(\mathcal{R}(X))}$ as an infinite product. Therefore we need only show that the affine and continuous maps are closed under linear combinations. The latter holds because addition and scalar multiplication are continuous on $\mathbb{C}$, so we reduce to the former. If we consider a finite linear combination of maps $f = \sum_i \alpha_i f_i$ with each $f_i \in \mathcal{A}(\mathcal{R}(X), \mathbb{C})$, we may use the fact that all summations involved are finite and the associativity of addition to get that f is affine from the fact that each f_i is.

We proceed to the second part of the statement. By lemma 9 we have that ζ maps into $\mathcal{A}(\mathcal{R}(X), \mathbb{C})$. We need to show that it is a linear map preserving the positive cone and unit. To do this we use the fact that ζ is an evaluation map and that each ϕ is linear and preserves the positive cone and unit, being a state.

At this point we note that if $\zeta(f)(\phi) \in [0, \infty)$ for all $\phi \in \mathcal{R}(X)$ then f is positive. To see this, assume f has this property. Then in particular $\zeta(f)(\phi) \geq 0$ for all pure states ϕ in the spectrum of $C(X)$. Since ζ is evalution, $f(x) \geq 0$ at each point.

This shows that if ζ has an inverse, it is a positive map. We also have that if ζ has an inverse, it would preserve the unit, so to show that ζ is an isomorphism of ordered vector spaces with unit we only have to show that it is a bijection.

To show ζ is injective, assume that there are $f, g \in C(X)$ such that $\zeta(f) = \zeta(g)$. Then $\zeta(f)$ agrees with $\zeta(g)$ at each pure state in the spectrum of $C(X)$, *i.e.* f agrees with g at each point $x \in X$, and so $f = g$.

To show ζ is surjective, let $f \in \mathcal{A}(\mathcal{R}(X), \mathbb{C})$. By restriction we obtain a continuous function $\mathrm{Spec}(C(X)) \hookrightarrow \mathcal{R}(X) \to \mathbb{C}$; it corresponds to a unique element $g \in C(X)$, *i.e.* to a map $g \colon X \to \mathbb{C}$, by Gelfand's isomorphism. We need to show that $\zeta(g) = f$. We know $\zeta(g)$ agrees with f on the spectrum, and by affineness they must agree on all convex combinations of these. Using Lemma 6 (2) we see that they agree on a dense set, so $\zeta(g) = f$ by continuity. □

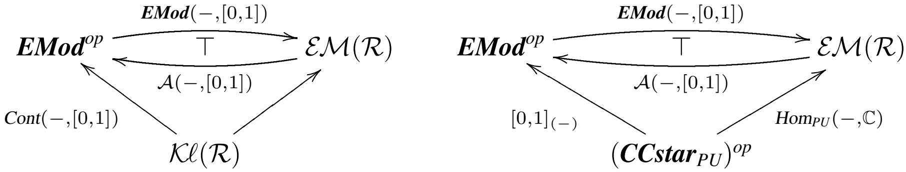
Corollary 3. *For each commutative C^*-algebra A there is an isomorphism of ordered vector spaces with unit: $A \cong \mathcal{A}\big(\mathit{Hom}_{PU}(A, \mathbb{C}), \mathbb{C}\big)$.* □

Proof. Using Gelfand duality, we extend the above to all commutative C^*-algebras, via $A \cong C(\mathrm{Spec}(A))$. □

Proposition 6. *The $\mathcal{R}$-algebra maps $\mathcal{A}(\mathcal{R}(X), [0, 1])$, with their pointwise effect module structure, are isomorphic to predicates $\mathit{Cont}(X, [0, 1])$ on X, i.e. to the effects $[0, 1]_{C(X)}$ on $C(X)$.*

Proof. By Lemma 10, there is an isomorphism $C(X) \cong \mathcal{A}(\mathcal{R}(X)), \mathbb{C})$ of unital ordered vector spaces. Restriction to their intervals from zero to the unit then yields an isomorphism $[0, 1]_{C(X)} \cong \mathcal{A}(\mathcal{R}(X)), [0, 1])$. Recalling Proposition 5, we see the latter maps are exactly the $\mathcal{R}$-algebra maps. □

Theorem 3. *There are commuting "state-and-effect" triangles:*



Proof. The triangle on the left is the diagram (8), in which the missing commutation result is given by Proposition 6. The diagram on the right follows from the equivalence $\mathcal{K}\ell(\mathcal{R}) \simeq (\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$ from Theorem 2. □

Final Remarks

The main contribution of this article lies in establishing a connection between two different worlds, namely the world of theoretical computer scientists using program language semantics (and logic) via monads, and the world of mathematicians and theoretical physicists using C^*-algebras. This connection involves the distribution monad $\mathcal{D}$ on **Sets**, which is heavily used for modeling discrete probabilistic systems (Markov chains), in the finite-dimensional case (see Proposition 4) and the less familiar Radon monad $\mathcal{R}$ on compact Hausdorff spaces (see Theorem 2). These results apply to *commutative* C^*-algebras. Follow-up research will concentrate on the non-commutative case.

Acknowledgements. The authors wish to thank Hans Maassen and Jorik Mandemaker for helpful discussions.

References

1. Abramsky, S., Coecke, B.: A categorical semantics of quantum protocols. In: Engesser, K., Gabbai, D.M., Lehmann, D. (eds.) Handbook of Quantum Logic and Quantum Structures, pp. 261–323. North Holland, Elsevier, Computer Science Press (2009)
2. Alfsen, E.M., Shultz, F.W.: State Spaces of Operator Algebras. Birkhaäuser (2001)
3. Arveson, W.: An Invitation to C^*-Algebra. Springer (1981)
4. Conway, J.B.: A Course In Functional Analysis, 2nd edn. Graduate Texts in Mathematics, vol. 96. Springer (1990)
5. Dixmier, J.: C^*-Algebras. North-Holland Mathematical Library, vol. 15. North-Holland Publishing Company (1977)
6. Dvurečenskij, A., Pulmannová, S.: New Trends in Quantum Structures. Kluwer Acad. Publ., Dordrecht (2000)
7. Foulis, D.J., Bennett, M.K.: Effect algebras and unsharp quantum logics. Found. Physics 24(10), 1331–1352 (1994)
8. Fremlin, D.H.: Measure Theory, vol. 4 (2003), `http://www.essex.ac.uk/maths/people/fremlin/mt.htm`
9. Girard, J.-Y.: Geometry of Interaction V: Logic in the hyperfinite factor. Theor. Comput. Sci. 412(20), 1860–1883 (2011)

10. Giry, M.: A categorical approach to probability theory. In: Banaschewski, B. (ed.) Categorical Aspects of Topology and Analysis. Lecture Notes in Mathematics, vol. 915, pp. 68–85. Springer, Heidelberg (1982)
11. Heinosaari, T., Ziman, M.: The Mathematical Language of Quantum Theory. From Uncertainty to Entanglement. Cambridge Univ. Press (2012)
12. Horodecki, M., Horodecki, P., Horodecki, R.: Separability of Mixed States: Necessary and Sufficient Conditions. Physics Letters A 223(12), 1–8 (1996)
13. Jacobs, B.: Introduction to Coalgebra. Towards Mathematics of States and Observations, Book, in preparation; version 2.0 (2012), http://www.cs.ru.nl/B.Jacobs/CLG/JacobsCoalgebraIntro.pdf
14. Jacobs, B.: Involutive categories and monoids, with a GNS-correspondence. Found. of Physics 42(7), 874–895 (2012)
15. Jacobs, B.: Measurable spaces and their effect logic. In: Logic in Computer Science. IEEE Computer Science Press (2013)
16. Jacobs, B., Mandemaker, J.: The expectation monad in quantum foundations. In: Jacobs, B., Selinger, P., Spitters, B. (eds.) Quantum Physics and Logic (QPL) 2011. Elect. Proc. in Theor. Comp. Sci., vol. 95, pp. 143–182 (2012)
17. Jacobs, B., Mandemaker, J.: Relating operator spaces via adjunctions. In: Chubb Reimann, J., Harizanov, V., Eskandarian, A. (eds.) Logic and Algebraic Structures in Quantum Computing and Information. Lect. Notes in Logic, Cambridge Univ. Press (2013), See arxiv.org/abs/1201.1272
18. Johnstone, P.: Stone Spaces. Studies in Advanced Mathematics, vol. 3. Cambridge Univ. Press (1982)
19. Kock, A.: On double dualization monads. Math. Scand. 27, 151–165 (1970)
20. Maassen, H.: Quantum probability and quantum information theory. In: Benatti, F., Fannes, M., Floreanini, R., Petritis, D. (eds.) Quantum Information, Computation and Cryptography. Lect. Notes Physics, vol. 808, pp. 65–108. Springer, Berlin (2010)
21. Mislove, M., Ouaknine, J., Pavlovic, D., Worrell, J.: Duality for labelled Markov processes. In: Walukiewicz, I. (ed.) FOSSACS 2004. LNCS, vol. 2987, pp. 393–407. Springer, Heidelberg (2004)
22. Moggi, E.: Notions of computation and monads. Inf. & Comp. 93(1), 55–92 (1991)
23. Wick Pelletier, J., Rosický, J.: On the Equational Theory of C^*-algebras. Algebra Universalis 30, 275–284 (1993)
24. Rudin, W.: Functional Analysis, 3rd edn. McGraw-Hill Book Company (1987)
25. Sakai, S.: C^*-algebras and W^*-algebras. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 60. Springer (1971)
26. Street, R.: The formal theory of monads. Journ. of Pure & Appl. Algebra 2, 149–169 (1972)
27. Swirszcz, T.: Monadic Functors and Convexity. Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Math. Astr. et Phys. 22(1), 39–42 (1974)