Unruh effect in vacua with anisotropic scaling: Applications to multilayer graphene

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Abstract

We extend the calculation of the Unruh effect to the universality classes of quantum vacua obeying topologically protected invariance under anisotropic scaling $\mathbf{r} \to b\mathbf{r}$, $t \to b^z t$. Two situations are considered. The first one is related to the accelerated detector which detects the electron - hole pairs. The second one is related to the system in external electric field, when the electron - hole pairs are created due to the Schwinger process. As distinct from the Unruh effect in relativistic systems (where $z = 1$) the calculated radiation is not thermal, but has properties of systems in the vicinity of quantum criticality. The vacuum obeying anisotropic scaling can be realized, in particular, in multilayer graphene with the rhombohedral stacking. Opportunities of the experimental realization of Unruh effect in this situation are discussed.

1. Introduction

Relativistic physics can be considered as emergent phenomenon, which occurs in the low-energy limit in some universality classes of quantum vacua with topologically protected nodes in the energy spectrum. The general topological approach to classification of gapless vacua, which is based on connection to the K-theory, has been developed by Hořava [1]. Hořava demonstrated that the reason for the emergence of Lorentz invariance is the topological theorem, called the Atiyah-Bott-Shapiro construction.

The Lorentz invariance emerges only for the particular classes of topological vacua, where nodes in energy spectrum are characterized by the elementary topological charges, $N = \pm 1$. For the higher values of the winding number, the Lorentz invariance can be supported by symmetry between the fermionic species. This symmetry makes the vacuum to be effectively equivalent to system of fermions, each having elementary topological charges $N = \pm 1$, see [2]. If such symmetry is absent but the splitting of the nodes in the energy spectrum into the elementary nodes is still forbidden, then in the quantum vacua with higher values of topological number $|N| > 1$ the low-energy fermionic spectrum obeys the invariance under anisotropic scale transformation. An example of the anisotropic scale transformation is $\mathbf{r} \to b\mathbf{r}$, $t \to b^z t$ with $z \neq 1$, i.e. one has different scale transformation for space and time and thus contrary to the relativistic systems with $z = 1$ space and time cannot be united into space-time. Systems with anisotropic scaling in 3+1 and 2+1 vacua are discussed in Ref. [2] and in Refs. [3, 4, 5, 6, 7, 8], respectively. In a simple case of 2+1 systems, such as multilayered graphene [9], the scaling parameter z coincides with topological charge of the node in the energy spectrum, $z = |N|$, in other words, the anisotropic scaling is topologically protected.

Physics obeying the topologically protected anisotropic scaling is fundamentally different from the relativistic physics. It emerges in the vacua, which belong to different topological classes, and thus cannot overlap with the relativistic physics. The transition between different topological classes cannot occur adiabatically. That is why the transfer from the vacuum with emergent relativistic physics to the anisotropic vacuum with a given scaling parameter $z \neq 1$ occurs only by discontinuous jump – by quantum phase transition. The

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same refers to the quantum transitions between anisotropic vacua with different z. Thus, the topologically protected quantum vacua with anisotropic scaling open the new natural area for application of quantum field theory. For example, while traditional relativistic QED was considered as the function of the space dimension D [10], the general QED must be considered as function of two parameters, D and N [8]. The QED with the anisotropic scaling can be realized in the multilayered graphene, and it is essentially richer than the relativistic QED with $z = 1$. The modified Heisenberg-Euler action describing the polarization of the anisotropic quantum vacuum with $z \neq 1$ has been discussed in Refs. [8, 11, 12]. In particular, this effective action has different scaling laws for electric and magnetic fields.

The anisotropic scaling can be useful also to quantum gravity. For example, the vacua with topologically protected anisotropic scaling can host the Hořava gravity, which is based just on the idea of anisotropic scaling [13, 14, 15]. The advantage of the Hořava extension of general relativity is that for $z = 3$ the quantum gravity can be ultraviolet complete.

The natural question is how the other quantum phenomena, such as Hawking radiation [16] and Unruh effect [17], which are intrinsic to the quantum vacuum, are modified compared to the relativistic case. The related phenomenon – Schwinger pair creation in electric field in anisotropic vacuua – has been considered in Refs. [8, 11, 12]. In the relativistic vacuum, the Bekenstein conjecture of the generalized second law of the black hole thermodynamics [18] and Hawking conjecture of the black hole radiance [16] indicated the close connection between general relativity (GR) and thermodynamics of the quantum vacuum. This was followed by the Unruh conjecture [17] that the detector moving in the Lorentz invariant vacuum with constant acceleration views the quantum vacuum as a thermal background. The latter indicated the close connection between the thermodynamics and the vacuum in special relativity. Then the thermal behavior of the quantum vacuum has been extended to the expanding Universe [19]. At the same time, it was found that the vacuum of Lorentz invariant quantum electrodynamics (QED) in an exactly constant electric field $\mathbf{E}$ can be also understood as a thermal background characterized by the temperature parameter [20] (for recent papers see [21]).

All these phenomena occurring in the relativistic quantum vacuum are interrelated and have similar properties. In particular, the power spectrum of the vacuum noise (or the detector-response function) seen by a uniformly accelerated observer in flat spacetimes depends on space dimension D, and for even D it exhibits the phenomenon of the apparent inversion of statistics [23]. A similar phenomenon of the inversion of spin-statistics relation occurs for the vacuum polarization in a constant electric field [20, 25]. As another example, it is argued that the same back reaction effect of particle creation influences the vacuum of QED in electric field and the quantum vacuum in the de Sitter space-time [26]. This implies the instability of the de Sitter vacuum towards decay, which remains the issue of controversy [27, 28, 29, 30, 31]. Extension of the stability analysis to de Sitter vacuum with different $z \neq 1$ has been considered in Ref. [32].

Here we discuss the modification of the thermal properties of the quantum vacuum as compared to the Lorentz invariant vacua. We consider the particle production in electric field and the Unruh effect in the anisotropic scaling system, such as multilayered graphene. In general anisotropic case the energy distribution of electron-hole pairs in electric field and the distribution detected by the accelerated observer do not look as thermal. Nevertheless the distributions depend on the same dimensionless quantity E/T, where T is some effective parameter of dimension of energy, which is determined by electric field or acceleration correspondingly and by the integer parameter z. This indicates the new type of quantum criticality [33, 34, 35, 36, 37, 38], emerging in the anisotropic vacua with $z \neq 1$. We will also discuss the opportunities to study these effects in the multilayer graphene experimentally.

2. Unruh temperature in multilayer graphene

In this section we extend the notion of Unruh temperature T_U to the field theory of multilayer graphene (with ABC stacking). Namely, we introduce it as the typical energy of processes that take place in the presence of external force F acting on the quasiparticles. (In the case of external electric field $\mathcal{E}$ we have $F = e\mathcal{E}$.) We consider classical motion of quasiparticles, introduce the generalized acceleration that characterizes this motion, and relate it to the Unruh temperature. We give qualitative estimate of the "typical" energy in graphene in the presence of external force. This estimate is considered as the definition of Unruh temperature and will be given in Eq. (7). It will be shown in the following sections that this temperature enters the considered distributions over energy ω of the electron - hole pairs in a number of situations. Namely these distributions depend on the ratio ω/T_U.

2.1. Classical motion of quasiparticles

We deal with the two-component spinors placed in the external electric field directed along the x-axis. We introduce the external force F acting on the fermion excitation adding the potential energy Fx to the one-particle Hamiltonian. Then the Hamiltonian for low-energy fermionic quasiparticles propagating in the system with anisotropic scaling $z = J$ has the form

$$H = H_0 + Fx, \quad H_0 = \begin{pmatrix} 0 & v\Big(\hat{p}_x - i\hat{p}_y\Big)^J \\ v\Big(\hat{p}_x + i\hat{p}_y\Big)^J & 0\,. \end{pmatrix} \tag{1}$$

Here $v = v_F$ is the Fermi velocity for the case of monolayer ($J = 1$) and $v = 1/2m$ for the case of bilayer ($J = 2$), where $m \approx 0.028 m_e$ [39] and m_e is the free-electron mass. In general case $v \propto t_\perp^{1-J} v_F^J$, where $t_\perp$ is the interlayer hopping parameter. Here and further in the text we work in the system of units with $\hbar = 1$. Schrödinger equation has the usual form

$$i\partial_t \Psi = H\Psi\,. \tag{2}$$

We consider the fermion excitations with fixed value of the transverse momentum p_y: $\Psi(t, x, y) = e^{ip_y y}\psi(t, x)$. The conserved transverse momentum p_y plays the role of mass of the one-dimensional fermion (multiplied by v_F instead of the speed of light). We denote $p_y = M = \mathcal{M} v_F$. The corresponding energy spectrum is $E = \pm v\Big({p_x}^2 + M^2\Big)^{J/2} = \pm t_\perp^{1-J} v_F^J \Big({p_x}^2 + \mathcal{M}^2 v_F^2\Big)^{J/2}$. Since p_y plays the role of mass, it is natural to identify the generalized acceleration as $a = F/\mathcal{M} = v_F F/p_y$. To justify this choice, let us apply classical approximation to $\psi(t, x)$ with the corresponding classical Hamiltonian

$$H_{cl}(p_x, x) = Fx + v\Big({p_x}^2 + p_y^2\Big)^{J/2}\,. \tag{3}$$

Hamilton equations of motion give classical trajectory that corresponds to $\dot{x}(0) = 0$:

$$x(t) = vF^{J-1}\Big(t^2 + \frac{v_F^2}{a^2}\Big)^{J/2} - vF^{J-1}\frac{v_F^J}{a^J} + x(0) \quad , \quad a = \frac{F}{\mathcal{M}} = v_F\frac{F}{p_y}\,. \tag{4}$$

This is the generalization of the relativistic $J = 1$ case, where Eq.(4) describes the hyperbolic motion with correctly determined linear acceleration a. The particular case $p_y = 0$ corresponds to the trajectory

$$x(t) = vF^{J-1}|t|^J + x(0) \tag{5}$$

2.2. Unruh temperature

As it was mentioned above we define the Unruh temperature T_U as the “typical” energy of processes taking place in the system under the action of the external force. We estimate this energy semiclassically using the above mentioned classical trajectory for the particular case $p_y = 0$. Hamiltonian equations relate classical momentum to time as $p_{cl} = Ft$. In Eq. (3) both terms of the Hamiltonian are typically of the same order. So, we can estimate the energy scale as $T_U \sim v p_{cl}^J \sim v t^J F^J$. Here typical values of time are related to energy by the uncertainty relation $tT_U \sim 1$. Thus we get

$$T_U \sim \Big(vF^J\Big)^{\frac{1}{J+1}} \tag{6}$$

For $J = 1$ we shall typically use the definition $T_U = \Big(v_F F\Big)^{\frac{1}{2}}/(2\pi)$. For $J \geq 2$ we shall define $T_U = \Big(vF^J\Big)^{\frac{1}{J+1}}$. For the completeness let us rewrite the expression for the Unruh temperature at $J \geq 2$ in the usual system of units (with the dimensional Fermi velocity for the single-layer graphene v_F and the Plank constant $\hbar$ restored):

$$T_U = t_\perp\Big(\frac{v_F F\hbar}{t_\perp^2}\Big)^{\frac{J}{J+1}} \tag{7}$$

with $t_\perp \approx 0.4$ eV, and $v_F \approx c/300$, where c is the speed of light. Here it is used that $v = t_\perp^{1-J} v_F^J$.

2.3. Generalized acceleration

The identification of Eq. (4) for the generalized acceleration does not work for the case of Eq. (5), i.e. for $p_y = 0$. This is the case that was used above for the estimate of the "typical" energy T_U. Unruh temperature in (quasi) relativistic models is related to the acceleration as $T_U = a/(2\pi v_F)$. This allows us to estimate the "typical" acceleration in the presence of the external force F for the multilayer graphene. For $J \geq 1$ we define the acceleration as equal to the Unruh temperature of Eq. (7) multiplied by the Fermi velocity v_F.

The same result is obtained when we define the generalized acceleration as F/M_{eff}, where the effective mass $M_{\rm eff}$ is played by vacuum fluctuations of the p_x projection of momentum, $\langle |p_x|\rangle$. The latter can be found from the Hamiltonian (3) at $p_y = 0$. The contribution of two terms to the energy of vacuum fluctuations is of the same order, while the fluctuations $\langle |x|\rangle$ and $\langle |p_x|\rangle$ are related by the Heisenberg uncertainty relation: $\langle |x|\rangle \sim 1/\langle |p_x|\rangle$. Then one has $F/\langle |p_x|\rangle = v\langle |p_x|\rangle^J$ which gives $M_{\rm eff} \sim \langle |p_x|\rangle/v_F = (F/v)^{1/(J+1)}/v_F$ and the generalized acceleration $a = F/M_{\rm eff}$.

In the special case $J = 2$ the equation $a = F/M_{\rm eff}$ is applicable also for massive case $M \neq 0$. The reason for that is that the trajectory (4) for $J = 2$

$$x(t) = vFt^2 + x(0) = \frac{Ft^2}{2m} + x(0)\,, \tag{8}$$

does not depend on M and thus the generalized acceleration should be the same as for $M = 0$. The trajectory (8) is similar to the trajectory of a Galilean particle with mass m, and at first glance the properly determined acceleration is the Galilean acceleration $a = F/m$. However, the considered particles are essentially non-Galilean, since they have positive and negative branches of energy spectrum $E = \pm v\left({p_x}^2 + {p_y}^2\right)$, which correspond to the gapless energy spectrum with parabolic touching. That is why instead of Galilean acceleraton $a = F/m$, the characteristic acceleration is determined by the quantum fluctuation of p_x on the trajectory, as in the case of $M = 0$, i.e. $a = F/M_{\rm eff} \sim v_F F(1/mF)^{1/3}$.

So, in what follows the generalized acceleration in 2+1 anisotropic scaling systems can be defined as

$$a = v_F F/|p_y| \quad \text{for} \quad J \neq 2\ ,\ p_y \neq 0\,, \tag{9}$$

$$a = v_F F\left(\frac{v}{F}\right)^{1/(J+1)} = v_F T_U \quad \text{for} \quad J = 2\ , \quad \text{or} \quad \text{for} \quad J > 2\ ,\ p_y = 0\,. \tag{10}$$

3. Accelerated observer interacting with quantum fluctuations

In this section we consider the effect of the vacuum fluctuations on the observer accelerated by the constant external force in the 2+1 quantum vacuum with anisotropic scaling. As an example, we shall use the multilayer graphene with rhombohedral (ABC...) stacking [9], in which the topologically protected scaling exponent z coincides with the number of layers, $z = J$ [4, 8].

3.1. Trajectory of the detector, the interaction between the detector and the fermionic excitations, and the click rate

In principle, one can use any trajectory of the detector. In this section we shall consider for $J \geq 2$ the trajectory given by Eq. (5). For the special case $J = 1$ we consider different trajectory because Eq. (5) at $J = 1$ would lead to the singularity.

In this section we assume that the detector interacts with the fermion excitations via the term

$$V_{interaction} \sim \psi^+ O_A \psi \tag{11}$$

with some operators O_A. Different choices of these operators correspond to different practical realizations of Unruh detector. In real graphene various experimental situations may be suggested corresponding to different practical realizations of the Unruh detector. In subsection 3.7 we shall briefly consider one of the realizations, - the Raman scattering from the spot moving along the graphene sheet. In this case rough approximation gives $V_{interaction} \sim \psi^+\sigma^a\sigma^b\psi$ $(a, b = 1, 2)$ if the scattered light is not polarized and its direction is orthogonal to the graphene sheet. It is worth mentioning that in this particular situation as well as in the other possible experiments the microscopic description of the Unruh detector is rather complicated. However, the properties of the spectra become independent on the practical realization, and, in particular, on the form of $V_{interaction}$ for the energies much larger than T_U, when the semiclassical approximation works. Therefore, expressions obtained semiclassically in subsection 3.6 are universal. Nevertheless, for the completeness we consider several cases, when the problem can be solved exactly. The considered cases correspond to idealized Unruh detectors corresponding to different trajectories and different interaction terms Eq. (11).

The response function responsible for the transition of the detector to the excited state is (Eq. (3) of [45]):

$$\dot{F}_t(\omega) = 2\mathrm{Re}\int_0^{\infty} df e^{-i\omega f}\, W(t, t-f) \tag{12}$$

In Eq. (12) we encounter the Wightman function $W(t_1,t_2)$ of two currents $\psi^+ O_A \psi$. However, for $t_1 > t_2$ it is equal to the T-ordered two-particle Green function. We represent it as

$$\begin{aligned} W(t_1,t_2) &= \langle T\psi^+(t_1,x[t_1])O_A\psi(t_1,x[t_1])\psi^+(t_2,x[t_2])O_A^+\psi(t_2,x[t_2])\rangle \\ &\sim \mathrm{Tr}\, O_A \mathcal{G}(t_1-t_2,x[t_1]-x[t_2])\, O_A^+ \mathcal{G}(t_2-t_1,x[t_2]-x[t_1]), \quad t_1>t_2 \end{aligned} \tag{13}$$

where $\mathcal{G}$ is the single-fermion Green function. It is worth mentioning that the expression for the click rate written in this form resembles the conventional expression for the cross - section of the deep inelastic processes like $e^+ + e^- \to$ hadrons. Both quantities are related to the imaginary parts of the corresponding polarization operators. In fact, the derivations of these expressions are similar. In Eq. (13) we should consider carefully the limit $t_1 - t_2 \to 0$ (see discussion at the end of this subsection).

The summation over A is assumed in Eq. (13). Below we concern several particular choices of O_A. In pure relativistic $2+1$ theory there are the following two natural choices: $V_{interaction} \sim \bar{\psi}\psi = \psi^+\sigma^3\psi$ and $V_{interaction} \sim \bar{\psi}\sigma^\mu\psi = \psi^+\sigma^3\sigma^\mu\psi, \mu = 1,2,3$.

In non - relativistic models of graphene the natural choices are: $V_{interaction} \sim \bar{\psi}\psi = \psi^+\sigma^3\psi$, $V_{interaction} \sim \psi^+\psi$, $V_{interaction} \sim \psi^+\sigma^a\psi, a = 1,2$, and $V_{interaction} \sim \psi^+\sigma^a\sigma^b\psi, a,b = 1,2$. The latter case corresponds to Raman scattering from the moving spot for the nonpolarized light incoming (and outgoing) within the beam orthogonal to the graphene plane (see subsection 3.7). In the relativistic case in $4D$ the Unruh detectors interacting with the electron-hole pairs were considered in [47].

The response function is expressed as follows:

$$\dot{F}_0(\omega) \sim \mathrm{Re}\int_0^{\infty} dt e^{-i\omega t}\, \mathrm{Tr}\, O_A \mathcal{G}(-t,-x[t]) O_A^+\, \mathcal{G}(t,x[t]) \tag{14}$$

We need to substitute into this expression the particular trajectory of the detector and the particular operators O_A corresponding to the interaction of electron - hole pairs with the internal degrees of freedom of the detector. The integral in Eq. (14) may be divergent at $t \to 0$. Let us consider the polarization operator

$$\Pi(t,x) = i\mathrm{Tr}\, O_A \mathcal{G}(-t,-x) O_A^+\, \mathcal{G}(t,x) \tag{15}$$

The expression $Q(\omega,x) = \mathrm{Im}\int_0^{+\infty} dt \exp\Big(-i\omega t\Big)\,\Pi(t,x)$ gives the probability that the detector at rest ($x = const$) absorbs ($\omega > 0$) or emits ($\omega < 0$) the electron - hole pair. Correspondingly, $Q(\omega,x)$ should vanish for $\omega > 0$ because there are no free electron - hole pairs in vacuum. We consider the analytical continuation of $\Pi(t,x)$ to negative values of t, and extend integration to the whole real axis of t. Now we already must remember, that we deal with the Wightman functions. According to the general properties of Wightman functions, $\Pi(-t,x) = -\Pi^*(t,x)$. We close the contour in the lower half of the complex plane. The resulting integral vanishes only if there are no poles within this contour. That is, $\Pi(t,x)$ is analytical in the lower half of the complex t - plane.

If the singularities appear at $t \in R$, we should go around them from the bottom. This gives the rules of going around poles. These rules are equivalent to the existence of infinitely small imaginary contribution to t. We are to substitute everywhere $\Pi(t-i\epsilon,x)$ instead of $\Pi(t,x)$. Thus we shift in Eq. (14) the integration to $t - i\epsilon$:

$$\dot{F}_0(\omega) \sim \mathrm{Re}\int_0^{\infty} dt e^{-i\omega t}\, \mathrm{Tr}\, O_A \mathcal{G}(-t+i\epsilon,-x[t]) O_A^+\, \mathcal{G}(t-i\epsilon,x[t]) \tag{16}$$

This is the standard way of the regularization of Wightman functions (see, for example, [22, 24]). We adopt it throughout the text of the present paper when the nonrelativistic results are concerned. However, for the true relativistic case considered in section 3.4 there is a certain complication [45, 46].

3.2. The two point Green function

The two point Green function can be expressed as

$$\mathcal{G}(t,x) \sim \int d\epsilon d^2 p e^{-i\epsilon t + ip_x x}\mathcal{G}(\epsilon, p) \tag{17}$$

Here $\mathcal{G}(\epsilon, p)$ is the fermion Green function in momentum space:

$$\begin{aligned} \mathcal{G}(\epsilon, p) &= \frac{i}{\epsilon - H_0[p]} = R[p]^+ \frac{i}{\epsilon - v|p|^J \sigma^3} R[p], \\ & R[p] = \frac{1}{\sqrt{2}} \left(\begin{array}{cc} 1 & \left(\frac{p^*}{|p|}\right)^J \\ -\left(\frac{p}{|p|}\right)^J & 1 \end{array} \right), \quad p = p_1 + i p_2 \end{aligned} \tag{18}$$

In order to go around the poles correctly in the integral over ϵ we consider the Wick rotated Green functions. We introduce the Euclidean time z: $t = -iz$. Integration over the Euclidean frequency ω ($\epsilon = -i\omega$) gives for $z > 0$:

$$\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega z}}{\omega - iv|p|^J \sigma^3} = 2\pi i \left(\begin{array}{cc} e^{-v|p|^J z} & 0 \\ 0 & 0 \end{array} \right) \tag{19}$$

One can see, that for $z > 0$ the Green function describes the propagation of the excitation with positive energy (electron). As a result

$$\begin{aligned} \mathcal{G}(iz,x) &= -\pi i \int d^2 p \, e^{i(p,x) - vz|p|^J} \left(\begin{array}{cc} 1 & \left(\frac{p^*}{|p|}\right)^J \\ \left(\frac{p}{|p|}\right)^J & 1 \end{array} \right) \\ &= \left(\begin{array}{cc} \mathcal{G}_{11} & \mathcal{G}_{12} \\ \mathcal{G}_{12} & \mathcal{G}_{11} \end{array} \right) \\ \mathcal{G}_{11}(iz,x) &= -2\pi^2 i \int pdp \, J_0(|p|x) \exp(-vz|p|^J) \\ \mathcal{G}_{12}(iz,x) &= -2\pi^2 i \int pdp \, i^J J_J(|p|x) \exp(-vz|p|^J) \end{aligned} \tag{20}$$

At $z < 0$ we have

$$\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega z}}{\omega - iv|p|^J \sigma^3} = -2\pi i \left(\begin{array}{cc} 0 & 0 \\ 0 & e^{-v|p|^J |z|} \end{array} \right) \tag{21}$$

This corresponds to the propagation of the excitation with negative energy (hole). For the elements of the Green function we have at $z < 0$:

$$\begin{aligned} \mathcal{G}_{11}(iz,x) &= 2\pi^2 i \int pdp \, J_0(|p|x) \exp(-v|z||p|^J) \\ \mathcal{G}_{12}(iz,x) &= -2\pi^2 i \int pdp \, i^J J_J(|p|x) \exp(-v|z||p|^J) \end{aligned} \tag{22}$$

Next, we rotate back the Green function to real time $t = iz$, and get:

$$\begin{aligned} \mathcal{G}_{11}(t,x) &= -2 \, \mathrm{sign}(t) \, \pi^2 i \int pdp \, J_0(|p|x) \exp(-iv|t||p|^J) \\ \mathcal{G}_{12}(t,x) &= -2\pi^2 i \int pdp \, i^J J_J(|p|x) \exp(-iv|t||p|^J) \end{aligned} \tag{23}$$

Below we shall describe several particular cases.

3.3. The case $J = 1$

In this subsection we consider the (emergent) relativistic invariant interaction term corresponding to $O_A = \sigma_3$ that gives

$$\dot{F}_0(\omega) \sim 2\,\mathrm{Re} \int_0^\infty dt e^{-i\omega t} \frac{1}{(v^2t^2 - x^2(t))^2}, \tag{24}$$

and the case $O_A = \sigma_a\sigma_b$ corresponding to Raman scattering. In the latter case

$$\dot{F}_0(\omega) \sim 2\,\mathrm{Re} \int_0^\infty dt e^{-i\omega t} \frac{t^2}{(v^2t^2 - x^2(t))^3} \tag{25}$$

Substituting here the trajectory Eq. (5) with $x(0) = 0$ we get the singularity. Therefore, we consider the motion along the following trajectory, which experiences the thermal-like radiation [49]. This is

$$x[t] = tu_0 \mathrm{th} \frac{at}{u_0} \tag{26}$$

with the limiting velocity which is smaller than the effective speed of light, $u_0 < v$. The correlation function for the electron - hole pairs has the poles at $t = 0$ and at

$$t = \pm \frac{u_0}{a}[i\pi/2 \pm (1/2)\log\frac{v + u_0}{v - u_0}] \tag{27}$$

(compare this with Eqs. A.2, A.7 in [49].) Therefore, both expressions for the click rate Eq. (24) and Eq. (25) contain Boltzmann exponent. Expressions Eq. (24), (25) are divergent at $t \to 0$. As it was explained at the end of section 3.1, we apply the conventional regularization of the Wightman functions shifting the argument t to $t - i\epsilon$ with small ϵ (see also [45]). In addition, the real part of $2\int_0^\infty$ is equal to $\int_{-\infty}^\infty$. The contour is closed either in the lower half of the complex plane or in the upper half of the complex plane depending on the sign of ω.

For the case of the emission of the electron - hole pair the dominant nonexponential term $\sim \omega^3$ is given by the pole at $t = 0$. For the case of the absorbtion of the pair the pole at $t = 0$ does not enter the expression for the click rate, and the term corresponding to the second pole dominates. The corresponding tunneling exponent is given by

$$\dot{F}_0(\omega) \sim \exp\Big(-\frac{\omega}{T}\Big), \quad \omega > 0 \tag{28}$$

with the temperature $T = \frac{a}{2\pi u_0}$. This is similar to Unruh temperature but with the limiting speed u_0 on the trajectory instead of the effective "speed of light" v.

3.4. $J = 1$: true relativistic case

In this subsection we consider the situation that cannot be realized in real graphene. However, it resembles the true relativistic case and, therefore, deserves the consideration. Namely, we suppose, that the internal degrees of freedom of the detector are invariant under the Lorentz symmetry. This occurs in true relativistic $2+1$ model.

We consider the hyperbolic motion

$$t(s) = \frac{v}{a}\,\mathrm{sh}\,(as/v), \quad x(s) = \frac{v^2}{a}\,\mathrm{ch}\,(as/v) - \frac{v^2}{a} \tag{29}$$

with the linear acceleration a. Instead of Eq. (12) we use

$$\dot{F}_t(\omega) = 2\mathrm{Re} \int_0^\infty dS[f] e^{-i\omega S[f]}\, W(t, t - f) \tag{30}$$

where S is the internal time of the detector, and

$$\dot{F}_0(\omega) \sim \mathrm{Re} \int_0^\infty dS[t] e^{-i\omega S[t]}\, \mathrm{Tr}\,\mathcal{G}(-t, -x)\,\mathcal{G}(t, x) \tag{31}$$

instead of Eq. (14). We have

$$\dot{F}_0(\omega) \sim \mathrm{Re} \int_0^\infty dse^{-i\omega s} \frac{1}{(v^2t^2(s) - x^2(s))^2} \sim \int_{-\infty}^\infty dse^{-i\omega s} \frac{1}{\mathrm{sh}^4 \frac{as}{2v}} \tag{32}$$

The regularization described in section 3.1 applied here would lead to the values of $\dot{F}_t(\omega)$ depending on t. Fortunately, for $t = 0$ the answer is obtained that coincides with the result obtained in Refs. [45, 23], when the finite small size of the detector is taken into account. We suggest the following resolution of this puzzle. Since the interaction between the detector and the electron - hole pairs in this case is completely Lorentz invariant we *should* consider the Wightman functions that describe emission/absorbtion in the reference frame moving together with the detector. The arguments of section 3.1 then lead us to the conclusion that the shift is to be applied to variable s (time in the moving reference frame) instead of t. Then the regularized Wightman function $W(t[s_1 - i\epsilon], t[s_2 + i\epsilon])$ coincides with the Wightman function regularized as in Eq. (20) of [45].

Thus, we assume that the integration contour is shifted to $s - i\epsilon$ with small ϵ. Then the integral is equal to the sum over poles:

$$\dot{F}_0(\omega) \sim \left(\begin{array}{c} \omega^3 \sum_{k>0} e^{-\frac{\omega}{T_U}k} \sim \frac{\omega^3}{e^{\frac{\omega}{T_U}}-1}, \quad \omega > 0 \\ \omega^3 \sum_{k\geq 0} e^{-\frac{|\omega|}{T_U}k} \sim \frac{\omega^3}{e^{\frac{|\omega|}{T_U}}-1} e^{\frac{|\omega|}{T_U}}, \quad \omega < 0 \end{array} \right) \tag{33}$$

Here $T_U = a/(2\pi v)$ is the Unruh temperature. Thus the ratio of the absorption probability to the emission probability is $\sim e^{-\omega/T_U}$.

3.5. The Dirac-Galilean case $J = 2$, the interaction term corresponds to Raman scattering

In this subsection we consider again the interaction term $V_{interaction} \sim \psi^+\sigma^a\sigma^b\psi$ that corresponds to Raman scattering. For the parabolic trajectory Eq. (8) the problem can be solved exactly. Here

$$\begin{aligned} \mathcal{G}_{11}(iz, x) &= -\mathrm{sign}(z)\, 2\pi^2 \frac{i}{2v|z|} e^{-\frac{1}{4v|z|}x^2} \\ \mathcal{G}_{12}(iz, x) &= -2\pi^2 \frac{i}{2v|z|x^2}(-4v|z| + e^{-\frac{1}{4v|z|}x^2}x^2 + 4e^{-\frac{1}{4v|z|}x^2}v|z|) \end{aligned} \tag{34}$$

and

$$W(iz) = \mathcal{G}_{11}(iz, x)\mathcal{G}_{11}(-iz, -x) \sim \frac{\pi^4}{v^2z^2} e^{-\frac{1}{2vz}x^2} \tag{35}$$

Next, we substitute $z = it$, and the trajectory $x[t]$ from Eq. (8) with $x[0] = 0$:

$$W(t) \sim -\frac{\pi^4}{v^2t^2} e^{\frac{i}{2}vF^2t^3} \tag{36}$$

The click rate (response function) has the form

$$\dot{F}_0(\omega) \sim -\int_{-\infty}^\infty dse^{-i\omega s/T_U^{(2)}} \frac{1}{s^2} e^{\frac{i}{2}s^3}. \tag{37}$$

Here $T_U^{(2)}$ is the Unruh temperature for vacua with $J = 2$, it is determined by the generalized acceleration in Eq.(10) and is given by Eq. (7):

$$T_U^{(2)} = \left(\frac{\hbar^2F^2}{2m}\right)^{1/3} \tag{38}$$

(in this expression we have restored the Planck constant).

Expression Eq. (37) is linearly divergent at $s \to 0$. The regularization described in section 3.1 effectively results in the shift of the integration contour to $s - i\epsilon$ with small ϵ. This is equivalent to the following regular expression for the click rate:

$$\dot{F}_0(\omega) \sim \pi(1 - \mathrm{sign}(\omega))\frac{|\omega|}{T_U^{(2)}} - \int_{-\infty}^\infty dse^{-i\omega s/T_U^{(2)}} \left(\frac{1}{s^2}e^{\frac{i}{2}s^3} - \frac{1}{s^2}\right). \tag{39}$$

In principle, the integral can be taken and expressed through the generalized hypergeometric functions [53]. However, we do not present here the corresponding lengthy result since the regular expression of Eq. (39) itself can be used effectively for the calculation of the values of the distribution and for the drawing of the plot. This distribution is highly asymmetric. For $\omega \to -\infty$ it tends to infinity, while for $\omega \to +\infty$ it tends to zero.

Alternatively, this integral can be taken using the stationary phase approximation. At $\omega > 0$ and $\omega \gg T_U^{(2)}$ there are the stationary points $s = \pm\sqrt{\frac{2\omega}{3T_U^{(2)}}}$. The asymptotic expansion at large ω may be obtained similar to that of the Airy function [48]:

$$\dot{F}_0(\omega) \quad \sim \quad \Big(\frac{3T_U^{(2)}}{2\omega}\Big)^{5/4} \cos\Big[\Big(\frac{2\omega}{3T_U^{(2)}}\Big)^{3/2}\Big], \quad \omega \gg T_U^{(2)}\,, \tag{40}$$

The derivation of this result admits the following interpretation. If we disregard the pseudospin structure, in the reference frame moving together with the detector the one - particle hamiltonian has the form $\mathcal{H} = -2vFp_x\tilde{t} + vp_x^2 + vp_y^2$. It depends on time $\tilde{t}$, and the one - particle problem becomes non - stationary. In the above expressions the motion starts at $\tilde{t} = 0$ and ends at $\tilde{t} = t$. The momentum p does not depend on time but it depends on t at the end of the time interval. The "classical" motion corresponds to the extremum of $\int_0^t \mathcal{H}(t)d\tilde{t} = -vFp_xt^2 + vp_x^2t + vp_y^2t$ with respect to p_x, p_y, and gives $p_{cl,y} = 0, p_{cl,x} = \frac{F}{2}t$. The corresponding energy as a function of time satisfies $\int_0^t E(\tilde{t})d\tilde{t} = -\frac{vF^2}{4}t^3$. In semiclassical approximation the wave function of the electrons/holes contains the factor $\exp(-i\int_0^t E(\tilde{t})d\tilde{t}) \sim \exp(i\frac{vF^2}{4}t^3)$. Because of the interaction with the detector (the interaction potential is given by Eq. (11)) the hole may be transformed to electron at any time. Therefore, the process is allowed, when first, the pair is created, and second, it is annihilated. The amplitude is proportional to $\exp(-2i\int_0^t E(\tilde{t})d\tilde{t}) \sim \exp(i\frac{vF^2}{2}t^3)$. The amplitude of the process with the two changes of the energy of the detector by $\pm\omega$ is given by the Fourier transformation

$$\mathcal{A} \sim \int dt(...) \exp(-i\omega t - 2i\int_0^t E(\tilde{t})d\tilde{t}) = \int dt(...) \exp(-i\omega t + i\frac{vF^2}{2}t^3) \tag{41}$$

By unitarity the total probability that the detector emits/absorbs the pair is proportional to the imaginary part of $\mathcal{A}$.

Signs in Eq. (41) are important. This is also important that positive values of ω correspond to the absorption of the electron - hole pairs. Those signs can be checked as follows. Eq. (41) is similar to the quantum mechanical expression for the amplitude of the excitation of the quantum level with energy $2E$ by the absorption of the light with frequency $-\omega > 0$. In the simplest case when E is positive and does not depend on time the integral over t gives the delta - function of $(\omega + 2E)$. This delta - function may be nonzero for the excitation of the quantum level ($\omega < 0$). It vanishes for the inverse process ($\omega > 0$) because the system is assumed to be in the ground state.

At negative values of ω corresponding to the emission of pairs and at $|\omega| \gg T_U^{(2)}$ the dominant term corresponds to the Loran expansion at $s = 0$. This gives the power - like behavior:

$$\dot{F}_0(\omega) \quad \sim \quad 2\pi\,\frac{|\omega|}{T_U^{(2)}}, \quad \omega < 0. \tag{42}$$

Equations (37) and (40) are similar to equations which determine thermal effects in systems at quantum criticality, where the thermodynamic and kinetic parameters depend on frequency and temperature through their ratio ω/T. This generalizes the Hawking and Unruh effects to the vacua with anisotropic scaling. This consideration clearly demonstrates an intimate relation between Lorentz invariance of the field theory and Boltzmann distribution in statistical physics. For the theory at $J = 1$ with emergent Lorentz invariance the trajectory of the detector Eq. (4) being substituted to the expression for the click would give the singularity. Therefore, we considered the alternative trajectory Eq. (26) that leads to the Boltzmann distribution of the pairs absorbed by the detector Eq. (28). The true Lorentz invariant theory leads to the Boltzmann distribution as well. For non-Lorentz invariant theory with $J = 2$, there is no Boltzmann distribution for absorbed particles. Instead, we have the law of Eq. (40).

3.6. Arbitrary $J \geq 2$, the semiclassical consideration

For arbitrary $J \geq 2$ we have (in the case of Raman scattering with $O_A = \sigma_a \sigma_b$):

$$\begin{aligned}
\dot{F}_0(\omega) \sim \dot{\sigma}_0(\omega) \quad &\sim \quad \int dt e^{-i\omega t} \mathcal{G}_{11}(t,x) \mathcal{G}_{11}(-t,-x) \\
&= \quad 4\,\pi^4 \int_{0-i0}^{\infty - i0} dt \int_0^\infty p dp \int_0^\infty p' dp'\, J_0(px[t])\, J_0(p'x[t]) \exp(-i\omega t - ivt(p^J + [p']^J), \\
& \quad\; x[t] = vF^{J-1} t^J
\end{aligned} \tag{43}$$

(For the definition of $\sigma(\omega)$ see subsection 3.7.)

Already for $J = 3$ the elements of the two - point Green function are represented as a rather complicated composition of generalized hypergeometric functions. Therefore, for $J \geq 3$ we apply the semiclassical approximation that works at $|\omega| \gg T_U$. One of the advantages of the semiclassical consideration is that the pseudospin degrees of freedom of the electrons and holes are neglected. Therefore, the practical choice of operators O_A entering the interaction term Eq. (11) does not matter.

Let us consider the trajectory Eq. (5) for $J \geq 2$. The dimensionality analysis demonstrates that the general form of the click rate is

$$\dot{F}_0(\omega) \quad \sim \quad \int_{-\infty}^{\infty} ds e^{-i\omega s / T_U^{(J)}} g_J(s) \tag{44}$$

with some function $g_J(t)$, and $T_U^{(J)}$ given by Eq.(7):

$$T_U^{(J)} = (F^J v)^{1/(J+1)} = a/v_F\,. \tag{45}$$

At $\omega \gg T_U$ (absorption) in semiclassical approximation we have

$$\mathcal{G}(t, x[t]) \quad \sim \quad \sum_{N=0,1;K=0..J-2} \frac{C_{NK}}{t^{J-1}} \exp\Big(i\, \frac{(J-1)}{J^{J/(J-1)}}\, vF^J t^{J+1} e^{i\pi \frac{2K+NJ}{J-1}}\Big), \tag{46}$$

where C_{NK} are constants, and

$$\dot{F}_0(\omega) \sim \mathrm{Re} \int_0^\infty dt e^{-i\omega t} \mathcal{G}^2(t, x[t]) \tag{47}$$

There are several stationary points in the complex plane. The dominant contribution corresponds to the terms with $N = K = 0$ in Eq. (46). Stationary phase approximation gives the following asymptotic behavior (the consideration is similar to that of the Airy function [48]):

$$\begin{aligned}
\dot{F}_0(\omega) \quad &\sim \quad \Big(\frac{T_U^{(J)}}{\omega}\Big)^{\frac{5(J-1)}{2J}} \cos\Big[\beta\, \Big(\frac{\omega}{T_U^{(J)}}\Big)^{1+1/J} + \mathrm{const}\Big], \\
\beta &= \frac{J^{J/(J-1)}}{[2(J^2-1)]^{1/J}(J+1)}
\end{aligned} \tag{48}$$

This result corresponds to the absorption of the electron - hole pairs by the detector. Emission of the pairs by the detector corresponds to the power - like behavior that can be extracted from the Loran expansion of the integrand in Eq. (47). Since the integration contour should be shifted to $t - i0$ we can calculate the residue of the integrand at $t = 0$. The result gives the power - like dependence on ω:

$$\dot{F}_0(\omega) \quad \sim \quad \Big(\frac{|\omega|}{T_U^{(J)}}\Big)^{2J-3}, \quad |\omega| >> T_U^{(J)}, \quad \omega < 0. \tag{49}$$

Applying the same technique to Eq. (32) we can obtain the dominant power - like term for the emission of the pairs in the relativistic case $\sim |\omega|^3$ that can also be seen in the exact result Eq. (33). This behavior is in some sense similar to that of considered in [50] and in Sec. 31.4 of [2], where the power-law asymptote for radiation from rotating black hole is obtained by tunneling method.

3.7. Discussion of possible experiments with the accelerated detector of electron - hole pairs

Experiments for the case of "accelerating detector" look difficult but not impossible: one should create a laser spot moving along the graphene surface with acceleration and measure the Raman scattering from such a spot. The emission (adsorption) of electron-hole pairs by the moving spot corresponds to Stokes (anti-Stokes) Raman scattering, with the lower (higher) frequency of the scattered light in comparison with the incident one. In order to check the distributions obtained in this section this spot should move along the corresponding trajectories given by Eq. (5). One should expect a huge asymmetry, with the oscillating factor for the dependence on ω for the absorption versus the power-law for the emission. Unruh effect corresponds to the absorption of the electron - hole pairs by the accelerated detector, or, better to say, the Unruh effect probability is the ratio of the absorption probability to the emission probability. In order to model the Unruh detector the absorbtion of the incoming photon and emission of the secondary photons should occur in the small vicinity of the spot and without the essential delay. Otherwise the spot cannot be considered as the point - like detector moving along the given trajectory.

The detailed consideration of this process, and the questions related to the delays and the correlation length of Raman scattering from the spot is out of the scope of the present paper. Nevertheless, in order to understand, what plays the role of the operator O_A of Eq. (11) in this case, let us consider the simplest diagram that describes the Raman scattering. The incoming photon is transformed to the electron - hole pair. Next, the electron and the hole scatter elastically on the atoms of the crystal lattice. During this scattering process the momentum of the incoming photon (minus the momentum of the outgoing photon) is absorbed by the crystal lattice. After that the scattered electron emits the outgoing photon[2]. The initial state is vacuum, the final state includes the created electron - hole pair.

For the cross section of the Raman scattering from the moving spot we have

$$\sigma_T(\omega) \quad \sim \quad \int_{-\infty}^{T} dt \int_{-\infty}^{T} dt' e^{-i\omega(t-t')} \sum_{a,b} \langle 0|\psi^+(x[t'])\sigma_b\sigma_a\psi(x[t'])\psi^+(x[t])\sigma_a\sigma_b\psi(x[t])|0\rangle \tag{50}$$

Here the sum is over the polarizations $a, b = 1, 2$ of incoming and outgoing photons. The so - called click rate [45] is given by the derivative of σ with respect to T in the integral over t:

$$\dot{\sigma}_0(\omega) \quad \sim \quad 2\,\mathrm{Re} \int_{-\infty}^{0} dt e^{-i\omega t} \sum_{a,b} \langle 0|\psi^+(x[t])\sigma_b\sigma_a\psi(x[t])\psi^+(x[0])\sigma_a\sigma_b\psi(x[0])|0\rangle \tag{51}$$

We come to Eq. (12) calculated for the interaction term Eq. (11) with $O_A = \sigma_a\sigma_b$. Eq. (51) was derived for the emission of electron - hole pairs. This corresponds to negative values of ω. For the spot at rest $\dot{\sigma}(\omega)$ vanishes for $\omega > 0$ since there are no free electron - hole pairs in vacuum. However, for the accelerated spot $\dot{\sigma}(\omega)$ already does not vanish for positive ω. This means that the energy of the scattered photon may be larger than the energy of the incoming photon. This can be interpreted as the detection of the electron - hole pairs by the Unruh detector.

It is worthwhile to note that the moving laser beams were successfully used to study quantum phenomena in ultracold gases [51]. The laser spot moving along the circular trajectory in BEC has been suggested [52] to study the analog of Zel'dovich-Starobinsky effect [50], which is similar to the effects discussed here.

4. The system in the presence of electric field

In the presence of external electric field electron - hole pairs are created due to the Schwinger mechanism. These pairs may be detected in some way by the detector at rest. This situation is somehow similar to that of the accelerated detector considered in the previous section. Namely, free electrons and holes would move with the acceleration (in opposite directions). In some sense we may speak on the "accelerated vacuum" in this situation.

[2] Actually, the hole may emit the photon, and this may happen before the scattering on the crystal lattice. Also, the photon may first scatter elastically on the lattice, and then be transformed to the electron - hole pair. But the description of the corresponding processes will lead to the same final answer for the operator O_A.

4.1. Distribution of electron-hole pairs as a function of energy

The one-particle Hamiltonian for $2D$ massive fermions with anisotropic scaling $E^2 = v^2(p_x^2 + p_y^2)^J$ in electric field $\mathcal{E}$ has the form

$$H = \begin{pmatrix} \mathcal{E}x & v\Big(\hat{p}_x - i\hat{p}_y\Big)^J \\ v\Big(\hat{p}_x + i\hat{p}_y\Big)^J & \mathcal{E}x \end{pmatrix} \tag{52}$$

Here $\hat{p}_j = -i\partial_j$; we use the units $\hbar = e = 1$.

Integration over the classically forbidden region gives us the pair production probability (probability of the Schwinger effect) [8]

$$\begin{aligned} |\eta_0|^2 &= \exp\Big(-\alpha\frac{v}{\mathcal{E}}p_y^{J+1}\Big), \\ &\alpha = 2B\Big(\frac{1}{2}, \frac{J+2}{2}\Big) \end{aligned} \tag{53}$$

The energy of the electron-hole pair is $\omega = 2v(p_x^2 + p_y^2)^{J/2}$. The distribution of the created pairs as a function of energy has the form:

$$\begin{aligned} f(\omega) &= \int dp_x dp_y e^{-\alpha\frac{v}{\mathcal{E}}p_y^{J+1}}\delta(\omega - 2v(p_x^2 + p_y^2)^{J/2}) \\ &\sim \int_{-\left(\frac{\omega}{2T}\right)^{1/J}}^{\left(\frac{\omega}{2T}\right)^{1/J}} dq \frac{\exp\Big(-\alpha\Big(\Big(\frac{\omega}{2T}\Big)^{2/J} - q^2\Big)^{(J+1)/2}\Big)}{\omega^{1-2/J}\Big(\Big(\frac{\omega}{2T}\Big)^{2/J} - q^2\Big)^{1/2}} \\ &\sim \int_{-1}^{1} dx \frac{\exp\Big(-\alpha\Big(\frac{\omega}{2T}\Big)^{(J+1)/J}\Big(1 - x^2\Big)^{(J+1)/2}\Big)}{\omega^{1-1/J}\Big(1 - x^2\Big)^{1/2}} \\ &\sim \frac{1}{\omega^{1-1/J}}\int_0^1 dz \exp\Big(-\alpha\Big(\frac{\omega}{2T}\Big)^{(J+1)/J}z\Big)z^{1/(J+1)-1}\Big(1 - z^{2/(J+1)}\Big)^{1/2-1} \end{aligned} \tag{54}$$

Let us introduce the Unruh temperature $T = T_U^{(J)}$ in correspondence with Eq. (7):

$$T_U^{(J)} = (\mathcal{E}^J v)^{1/(J+1)} \tag{55}$$

(we use here $\mathcal{E}$ instead of F.)

4.2. Particular cases

For $J = 1$ we arrive at

$$\begin{aligned} f(\omega) &\sim \int_0^1 dz \exp^{-\alpha\left(\frac{\omega}{2T}\right)^2 z} z^{1/2-1}\Big(1 - z\Big)^{1/2-1} \\ &= \pi\exp\Big(-\frac{\alpha}{2}\Big(\frac{\omega}{2T}\Big)^2\Big)I_0\left(\frac{\alpha}{2}\Big(\frac{\omega}{2T}\Big)^2\right) \end{aligned} \tag{56}$$

This distribution was also derived in Ref. [54], where the spontaneous recombination of electrons and holes created due to the Schwinger effect in monolayer graphene was considered.

For $J \geq 2$ the function $f(\omega)$ is expressed through the generalized hypergeometric functions (for their definition see [53]). In general case $\omega^2 f(\omega)$ is the analytical function of $\zeta = \alpha\Big(\frac{\omega}{2T}\Big)^{(J+1)/J}$:

$$\omega^2 f(\omega) \sim \zeta\int_0^1 dz \exp\Big(-\zeta z\Big)z^{1/(J+1)-1}\Big(1 - z^{2/(J+1)}\Big)^{1/2-1} \tag{57}$$

This integral is convergent for any values of J. It can be represented as a composition of the generalized hypergeometric functions. For example, for $J = 2$:

$$\omega^2 f(\omega) \sim -\frac{2}{3}\zeta^2 \, {}_3F_4\Big[\begin{array}{c} \frac{2}{3}, 1, \frac{4}{3}; \\ \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \frac{3}{2} \end{array} \; \frac{\zeta^2}{4} \Big] + \frac{\pi}{2}\zeta \, {}_2F_3\Big[\begin{array}{c} \frac{1}{6}, \frac{5}{6}; \\ \frac{1}{3}, \frac{2}{3}, 1 \end{array} \; \frac{\zeta^2}{4} \Big] \tag{58}$$

In this expression the functions ${}_pF_q$ at small values of the argument are given by [53]

$${}_pF_q\Big[\begin{array}{c} a_1, ..., a_p; \\ b_1, ..., b_q \end{array} \; z \Big] = \sum_{n=0}^{\infty} \frac{(a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{z^n}{n!}, \quad (a)_n = a(a+1)...(a+n-1) \tag{59}$$

4.3. Asymptotic expansions

The expansion of Eq. (59) works in a vicinity of $z = 0$ and does not work for $z \to \infty$. In order to evaluate the distributions at $\zeta \to \infty$ we use integral representation Eq. (57). We obtain:

$$\begin{aligned} f(\omega) & \sim \zeta^{\frac{1-J}{J+1}} \Gamma(1/(J+1)) \frac{1}{\zeta^{1/(J+1)}} \\ & \sim \frac{2T_U}{\omega}, \quad \omega \gg T_U \end{aligned} \tag{60}$$

One can see that the dominant term is power - like. Among the other terms there are also the subdominant terms that contain the tunneling exponent $e^{-\zeta}$ (this may be obtained as an expansion near $z = 1$ in the integral of Eq. (57)). It is instructive to compare this with the behavior of the distributions of electron - hole pairs detected/emitted by the moving Unruh detector. Eqs. (57),(60) are to be compared with Eq (48). In both cases there is the dependence on $\zeta = \left(\frac{\omega}{2T_U^{(J)}}\right)^{1/J+1}$. This indicates that the two phenomena may indeed be similar, but there is no direct correspondence. In the distribution of the electron - hole pairs created in external Electric field the leading term is power - like. This is somewhat similar to the distribution of pairs absorbed by the accelerated detector Eq. (48). However, the powers are different, and in the latter case the distribution contains also the oscillating factor.

4.4. $f(\omega)$ expressed through the Green functions

Expression Eq. (54) can be rewritten as

$$f(\omega) = \int d\tau e^{i\omega\tau} \int dp_x dp_y e^{-\alpha \frac{v}{\mathcal{E}} p_y^{J+1} - i\tau 2v(p_x^2+p_y^2)^{J/2}} \sim \int d\tau e^{i\omega\tau} W(\tau) \tag{61}$$

Recall that the expression $Q = -\alpha \frac{v}{\mathcal{E}} p_y^{J+1}$ has its origin at

$$Q = 2\,\mathrm{Im} \int \mathcal{P} dx, \tag{62}$$

where

$$\mathcal{P} = \sqrt{\Big((E - \mathcal{E}x)/v\Big)^{2/J} - p_y^2} \tag{63}$$

is the classical momentum. This, in turn, may be obtained as a semi-classical approximation to the representation

$$\begin{aligned} f(\omega) & = \mathrm{Re} \int d\tau e^{i\omega\tau} \langle \bar{\psi}(0,0)\psi(0,0)\bar{\psi}(\tau,x)\psi(\tau,x)\rangle dx \\ & \sim \mathrm{Re} \int d\tau e^{i\omega\tau} \mathrm{Tr}\, \mathcal{G}_{\mathcal{E}}(\tau,x)\, \mathcal{G}_{\mathcal{E}}(-\tau,-x) dx \end{aligned} \tag{64}$$

Here fermion Green functions in external electric field are present. This expression is to be compared with Eq. (14).

4.5. Discussion of possible experiments with the "accelerated vacuum" of graphene

Relevant energies of electron excitations for this case lie in the infrared, which allows us to suggest for the detection an efficient optical tool, namely, the electron Raman scattering (see, e.g., [57]). One should probe the dependence of the Raman spectra on electric field. This will correspond to our case of "accelerating vacuum". Another way to probe the distribution of the electron - hole pairs on energy is to consider the spontaneous recombination of electrons and holes forming the electron - hole plasma [54] which may be in principle probed via luminescence.

Of course, estimation of the Unruh temperature is crucially important for the choice of the proper experimental technique. First, notice that graphene is very robust material with respect to electric breakdown; the current densities such as $(3-4)\times 10^4$ A/m are reachable (for recent references, see [55, 56]) which corresponds to the electric fields $\mathcal{E}_c \approx 10^8 - 10^9$ V/m. According to Eq. (38), for the case of bilayer graphene it corresponds to Unruh temperature as large as 0.5 eV $\approx$ 6000 K. So strong electric fields produce a strong heating [56] and this, of course, can be a serious problem but even for the orders of magnitude weaker fields the Unruh temperature can be still quite high (it is about 10 meV for $\mathcal{E} \approx 3\times 10^6$ V/m). Relevant energies of electron excitations for this case lie in the infrared, which allows us to suggest for the detection an efficient optical tool, namely, the electron Raman scattering (see, e.g., [57]).

One should keep in mind, however, that below the energy of the order of 10 meV the effects of trigonal warping and electron-electron interactions become important [9, 39] with the reconstruction of parabolic touching point to several conical touching points at low energies. This means that for the electric fields $\mathcal{E} \approx\ll 10^6$ V/m) we will probe these conical points rather than the parabolic point, and the Unruh spectra should be similar to those for the single-layer graphene. However, the Fermi velocity for these new conical points is much smaller than in the single-layer graphene (1.4×10^5 m/s and 9×10^5 m/s, respectively [39]) which will lead to an essential increase of the Unruh temperature (inversely proportional to the Fermi velocity), the effect which looks very interesting by itself.

Alternative types of detection of the created electron-hole pairs and their spectroscopy are emission Mössbauer spectroscopy (one should put on graphene, e.g., radioactive ^{57}Co) [58] or electron spin resonance [59] of magnetic adatoms on graphene. In both cases the spectra are modulated by the fluctuating electron density. However, optical methods (Raman spectroscopy) look the most promising and realistic.

5. Massive 1D fermions

In this section we consider the dynamics of the one - dimensional system with the Hamiltonian given by Eq. (1), where p_y is fixed and is equal to M playing the role of effective mass of the fermionic excitations. The multilayer graphene itself (at ABC stacking) can be considered as a collection of such $1D$ systems corresponding to different values of M. The one - dimensional system considered in this section is naturally realized in graphene nanoribbons that is the thin strip of the multilayer graphene. The values p_y are quantized there [9], and the subsystems with different values of $M = p_y$ can be distinguished from each other (see, e.g., the discussion of conductance quantization in Section 5.5 of the book [9]).

5.1. Müller temperature for 1+1 massive fermions in electric field

In the presence of constant external electric field we deal with the "accelerated vacuum". We consider the Hamiltonian Eq. (1) in the case when p_y is fixed and equal to M. Then $p = p_x + iM$, and one-particle Hamiltonian for 1D massive fermions with anisotropic scaling $E^2 = v^2(p^2+M^2)^{J/2}$ in electric field $\mathcal{E}$ has the form

$$H = \begin{pmatrix} \mathcal{E}x & v\Big(\hat{p}_x - iM\Big)^J \\ v\Big(\hat{p}_x + iM\Big)^J & \mathcal{E}x \end{pmatrix} \tag{65}$$

Integration over the classically forbidden region gives us the pair production probability [8]

$$|\eta_0|^2 \quad = \quad e^{-2JB\left(\frac{3}{2},\frac{J}{2}\right)vM^{J+1}/\mathcal{E}}, . \tag{66}$$

With the definition of generalized acceleration $a = v_F\mathcal{E}/M$ of Eq.(9) this expression resembles the thermal distribution. Namely, we may represent

$$|\eta_0|^2 = \exp\left(-\frac{vM^J}{T_{\rm M}}\right),$$

$$T_{\rm M} = \gamma^{-1}\hbar a/v_F \quad , \quad a = v_F\frac{\dot{p}}{M} = v_F\frac{\mathcal{E}}{M} \quad , \quad \gamma = \frac{2J}{\pi}B\Big(\frac{3}{2},\frac{J}{2}\Big)\,. \tag{67}$$

with the analogue of Müller temperature [20] T_M. (In the last expression we restore Plank constant $\hbar$.) In the relativistic systems ($J = 1$) the Müller temperature $T_{\rm M} = \hbar a/\pi c$ is twice the Unruh temperature $T_{\rm U} = \hbar a/2\pi c$ [17], see e.g. recent paper [21]. The only difference from the relativistic system is that in anisotropic scaling system the pre-factor depends on J.

5.2. Unruh effect at $|\omega| \gg T_U$

In this subsection we consider the situation, when the detector moves in the given $1D$ system along the trajectory given by Eq. (4). With $p = p_x + iM$ one has the Green's function of $1D$ quasiparticles

$$\begin{aligned}
\tilde{\mathcal{G}}(it,0) &= 2\pi i\int dp_x\, e^{ip_x x - v|t||p|^J}\begin{pmatrix} 1 & \mathrm{sign}(t)\left(\frac{p^*}{|p|}\right)^J \\ \mathrm{sign}(t)\left(\frac{p}{|p|}\right)^J & 1\end{pmatrix} \\
&= \begin{pmatrix} G_{11}(it) & G_{12}(it) \\ G_{21}(it) & G_{11}(it)\end{pmatrix} \\
G_{11}(it) &= (2\pi)i\int dp_x \exp(ip_x x - v|t|(p_x^2+M^2)^{J/2}) \\
G_{12}(it) &= \mathrm{sign}(t)\,(2\pi)i\int dp_x \exp(ip_x x - v|t|(p_x^2+M^2)^{J/2})\Big(\frac{p_x - iM}{(p_x^2+M^2)^{1/2}}\Big)^J \\
G_{21}(it) &= \mathrm{sign}(t)\,(2\pi)i\int dp_x \exp(ip_x x - v|t|(p_x^2+M^2)^{J/2})\Big(\frac{p_x + iM}{(p_x^2+M^2)^{1/2}}\Big)^J
\end{aligned} \tag{68}$$

We evaluate the Green function in the semi - classical approximation. Then

$$\mathcal{G}(t,x[t]) \sim \int dp_x\, e^{ip_x x - iv|t||p|^J} \tag{69}$$

Stationary phase conditions give

$$x = v|t|Jp_x(p_x^2+M^2)^{J/2-1} \tag{70}$$

Below we consider the two particular cases ($J = 1,2$) of Unruh radiation of massive particles with the energy near to the threshold $2E(p=0) = 2vM^J$.

1. $J = 1$.
   In this case we have

$$\mathcal{G}(t,x[t]) \sim \frac{e^{i\pi/4}}{v|t|^{1/2}}\exp\Big(-ivM\sqrt{1-\Big(\frac{x}{vt}\Big)^2}\,|t|\Big) \tag{71}$$

   As a result

$$\dot{F}_0(\omega) \sim \mathrm{Re}\int_0^\infty dt\frac{i}{vt}\exp\Big(-i\omega t - 2ivM\sqrt{1-\Big(\frac{x}{vt}\Big)^2}\,t\Big) \sim \int_{-\infty}^\infty dt\frac{i}{vt}\exp\Big(-i\omega t - 2ivM\sqrt{1-\Big(\frac{x}{vt}\Big)^2}\,t\Big) \tag{72}$$

   Here the trajectory of the detector $x(t)$ is given by Eq. (4):

$$x(t) = v\Big(t^2 + \frac{v^2}{a^2}\Big)^{1/2} - v\frac{v}{a} \tag{73}$$

We also use the following parametrization of this trajectory:

$$t(s) = \frac{v}{a}\,\mathrm{sh}\,(as/v), \quad x(s) = \frac{v^2}{a}\,\mathrm{ch}\,(as/v) - \frac{v^2}{a} \tag{74}$$

We come to

$$\dot{F}_0(\omega) \sim i \int_{-\infty}^{\infty} ds \frac{1}{\mathrm{th}\,(as/v)} \exp\Big(-\frac{iv}{a}\Big[\omega\,\mathrm{sh}\,(as/v) + 4vM\,\mathrm{sh}\frac{as}{2v}\Big]\Big) \tag{75}$$

At $\omega \sim 2vM \gg T_U = \frac{a^2}{v^3 M}$ we can apply the stationary phase approximation. For $0 < \omega < 2vM$ and $\frac{|\omega - 2vM|}{2vM} << 1$ there are no stationary points at real s but there is the stationary point at imaginary values of s in the lower half of the complex plane. Therefore, the absorbtion of the electron - hole pairs is described by the tunneling exponent. One can define the effective chemical potential $\mu_{eff} = 2vM$ and the effective Unruh temperature $T_U = \frac{a^2}{v^3 M} = \frac{F^2}{vM^3}$. Then

$$\dot{F}_0(\omega) \sim e^{-4\sqrt{6}\Big(\frac{|\omega - \mu_{eff}|}{T_U}\Big)^{1/2}}, \quad |\omega - 2vM| \ll 2vM, \quad \omega \gg a/v, \quad \omega < 2vM \tag{76}$$

It is worth mentioning that in this case the Unruh temperature is not given by Eq. (7). Instead we have the relation that involves both mass parameter M and the acceleration a. For $\omega > 2vM$ there is the stationary point for the real value of s that results in the appearance of the oscillating factor in $\dot{F}_0(\omega)$:

$$\dot{F}_0(\omega) \sim \sin\Big(4\sqrt{6}\Big(\frac{|\omega - \mu_{eff}|}{T_U}\Big)^{1/2}\Big), \quad |\omega - 2vM| \ll 2vM, \quad \omega \gg a/v, \quad \omega > 2vM \tag{77}$$

2. $J = 2$.
In this case

$$\mathcal{G}(t, x[t]) \quad \sim \quad \frac{e^{i\pi/4}}{|t|^{1/2}} \exp\Big(\frac{i}{4} v|t|(Ft - M)(Ft + M)\Big) \tag{78}$$

As a result

$$\dot{F}_0(\omega) \sim \int_{-\infty}^{\infty} dt \frac{i}{vt} \exp\Big(-i\omega t + \frac{i}{2} vt(Ft - M)(Ft + M)\Big) \tag{79}$$

Next, we use the stationary phase approximation. It is valid at $|\omega| \gg T_U = (vF^2)^{1/3}$. Positive values of ω correspond to the absorbtion of the electron - hole pairs. The Unruh temperature for this case $T_U \sim (vF^2)^{1/3}$ is equal to that of defined by Eq. (7) with the generalized acceleration of Eq.(10) as expected for the $J = 2$ case, when the generalized acceleration does not depend on M. We denote $\mu_{eff} = 2vM^2$, and get similar to the case of the Airy function [48]:

$$\dot{F}_0(\omega) \sim \sin\Big(\frac{2\sqrt{6}}{9}\Big(\frac{\omega + \mu_{eff}}{T_U}\Big)^{3/2}\Big), \quad \omega \gg T_U, \quad \frac{|\omega - 2vM^2|}{2vM^2} \ll 1 \tag{80}$$

5.3. Suggested experiments

To probe the distributions obtained in this section we suggest the same methods as suggested in Sections III and IV but for the graphene nanoribbons. Namely, Raman scattering and spontaneous recombination can be used to probe the spectrum of electrons and holes created during the Schwinger process. The Raman scattering from the moving spot can be used to probe the Unruh effect itself.

6. Conclusions

Vacua with anisotropic scaling, such as $\mathbf{r} \to b\mathbf{r}$, $t \to b^z t$, represent new topological classes of quantum vacua, where the scale parameter z is determined by the topological charge of the quantum vacuum. The relativistic invariance emerges only in one of the the universality classes of vacua, where topology supports the isotropic scaling $z = 1$. Anisotropic scaling naturally emerges in many condensed matter systems, including multilayer graphene. That is why it is instructive to consider general properties of such quantum vacua. Here we discussed the quantum effects, such as Unruh radiation and Schwinger pair production, in vacua which belong to universality classes with topologically protected anisotropic scaling. Though these effects do not possess thermal properties relevant for the relativistic class of quantum vacua with $z = 1$, some features are proved to be universal. Both effects, Unruh radiation and Schwinger pair production, are characterized by the modification of correspondingly Unruh and Müller temperatures. This characteristic temperature is proportional to the acceleration properly generalized to systems with the anisotropic scaling in Eqs. (9) and (10). As a rule such temperature does not describe any thermal distribution of radiated excitations. Instead it acts as the temperature, which enters the thermodynamic and kinetic parameters of the many-body system in the vicinity of the quantum phase transition, where the vacuum experiences the properties of quantum criticality.

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