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COOPERATIVE GAME THEORY
AND
APPLICATIONS

IMMA CURIEL

COOPERATIVE GAME THEORY
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COOPERATIVE GAME THEORY AND APPLICATIONS

een wetenschappelijke proeve op het gebied van de
wiskunde en natuurwetenschappen

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PREFACE AND SUMMARY

The systematic study of cooperative games started with their treatment in the well known book "Theory of games and economic behaviour" by von Neumann and Morgenstern (1944). Cooperative games are well suited to model situations of cooperation among a group of people. In this monograph cooperative games in characteristic function form are studied. Such a game is an ordered pair consisting of a finite set, the set of players and a function, the characteristic function which assigns to each subset of players a real number, describing the worth of that subset in the game. In a situation modeled by such a game the players can decide to work alone or in subgroups called coalitions or all together in the grand coalition. The characteristic function describes the gain that they make in all these cases. A question to which much attention has been paid in cooperative game theory is the following. Suppose that all the players decide to work together. How should the reward obtained by taking this course of action be divided among them? A desirable property for such a division to satisfy is that of securing stability when this is profitable. What is meant by this? Suppose that it is profitable for all the players to work together because the worth of the grand coalition is more than the sum of the worths of the coalitions in any partition of the group. Yet, for certain divisions of the reward it will be more profitable for a subgroup to leave the others and work on its own since it will gain more in this way. Such a division makes the formation of the grand coalition unstable, even when it is profitable. In such a situation a division which secures stability is desirable. However, such a division need not always exist while in other cases more than one division with this property may exist. This inspired the definition of division rules based on other properties than stability. In chapter 1 attention will be paid to the existence of stable divisions and the definition of some of these other division rules. This monograph consists of five chapters. Chapter 1 is an introductory chapter. Cooperative games are defined and various of their properties are treated. Some well known division rules are discussed.

Chapter 2 is dedicated to the study of simple games. These are special cooperative games in which the characteristic function takes only the values 0 and 1. These games can be used to model situations of committee control and of voting. The characteristic function of a simple game is regarded as describing the power of the coalitions rather than the gains they can achieve. Instead of a rule describing how to divide the gains, what is important here is an index describing the power of each player in the game. In chapter 2

various power indices from the literature are mentioned and a new one is introduced and characterized in two ways. Other properties of this index which are not needed for the characterizations are treated as well.

Chapter 3 is devoted to the study of cooperative games with the worth of a coalition equal to the value of a linear programming problem which depends on the coalition. These games are called linear programming games. They turn out to be balanced, i.e. a division securing stability exists for them. Two examples of linear programming games, namely, flow games and linear production games are discussed. With the aid of these two examples the notion of committee control in linear programming games is introduced. The simple games of chapter 2 are used to model this committee control. Linear programming games with committee control need not be balanced. If all the simple games describing the committee control in a linear programming game are balanced, then the game itself is balanced. It is proved that every non-negative balanced game is a linear programming game with committee control. For non-balanced linear programming games two methods to divide the gains are described. A variation on linear production games involving claims is studied. For this purpose claim games are introduced which will appear again in chapter 5 under the name bankruptcy games and will be studied extensively there.

While in chapter 3 a linear programming problem determined the worth of a coalition, in chapter 4 this is done by a combinatorial programming problem. Thus, a combinatorial game arises. In section 4.2, 4.3, 4.4, and 4.5 matching and permutation situations are studied. In section 4.3 combinatorial games arising from these situations are treated. In general these games need not be balanced but various subclasses containing only balanced games are given. Section 4.4 treats economies with indivisibilities arising from permutation situations. In section 4.5 ordinal matching situations are handled. Section 4.6 is dedicated to another combinatorial game, the sequencing game. This game turns out to have nice properties. One of them is balancedness. For two division rules from cooperative game theory, the Shapley-value and the τ -value, simple expressions can be found in the case of sequencing games. A division rule for sequencing situations defined without game-theoretic considerations is shown to have the property that it secures stability of the grand coalition. This rule is characterized axiomatically. In section 4.7, 4.8 and 4.9 combinatorial games which are concerned with costs rather than with rewards are treated. Section 4.7 deals with traveling salesman games. In general traveling salesman games need not be balanced. However, subclasses that contain only balanced games are given. Routing games which

are closely related to traveling salesman games are defined. Contrary to traveling salesman games they are balanced. Section 4.8 deals with minimum cost spanning tree games. Both these games as well as the location games studied in 4.9 are balanced.

As already mentioned before, chapter 5 deals with bankruptcy games. These games arise from bankruptcy problems. Three division rules for bankruptcy problems are treated. It is proved that they correspond to three well known division rules from cooperative game theory.

A more detailed summary of the contents of each chapter is given in the first section of that chapter.

NOTATION

2^N	power set of the set N
$x(S)$	$\sum_{i \in S} x_i$ where $x \in \mathbb{R}^n$ and S is a finite set
$ A $	number of elements of the set A
Π_N	set of permutations of the set N
$P(\pi, i)$	set of predecessors of i according to π i.e. $j \in P(\pi, i) \Leftrightarrow \pi(j) < \pi(i)$
a_+	maximum of a and 0 where $a \in \mathbb{R}$
$\vee$	maximum
$\wedge$	minimum
$A * B$	$\sum_{i \in A} \sum_{j \in B} a_{ij} b_{ij}$ where A and B are $m \times n$ -matrices
A°	interior of the set A
$\square$	end of a proof

In each chapter the numbered formulas start with (1).

COOPERATIVE GAMES

1.1 Introduction

This chapter contains basic notions and fundamental definitions from cooperative game theory needed throughout this monograph. In section 2 cooperative games in characteristic function form and various of their properties are introduced. In section 3 three solution concepts for cooperative games, the core, the core cover and the Weber set are discussed. Section 4 deals with the Shapley-value, a one point solution concept. Section 5 is dedicated to the (pre-)kernel and the (pre-)nucleolus and their properties. In section 6 the τ -value is studied. All the notions in these five sections are introduced in terms of the rewards that a group of people can achieve by working together. In section 7 it is shown that all these notions can be adapted to situations of costs also, where a group of people can decrease their costs by cooperating.

1.2 Cooperative games in characteristic function form

Cooperative games in characteristic function form are well suited to model situations of cooperation among a group of people. This cooperation may have as a goal to obtain more revenues than in the case that they work separately.

1.2.1 Definition *A cooperative game in characteristic function form is an ordered pair $\langle N, v \rangle$ where N is a finite set, the set of players and the characteristic function v is a function from 2^N to $\mathbb{R}$ with $v(\emptyset) = 0$.*

A subset S of N is called a *coalition*. The number $v(S)$ can be regarded as the worth of coalition S in the game v . In the remainder of this work cooperative games in characteristic function form will be called shortly cooperative games. The cooperative game $\langle N, v \rangle$ will be identified with its characteristic function v whenever there can be no confusion about the player set N . The set of all cooperative games with player set N is denoted by G^N . In general the set N is taken to be equal to $\{1, 2, \dots, n\}$. For a coalition $S = \{i_1, i_2, \dots, i_k\}$, $v(S)$ will be denoted by $v(i_1, i_2, \dots, i_k)$ instead

of $v(\{i_1, i_2, \dots, i_k\})$.

1.2.2 Definition A cooperative game is said to be superadditive if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for all } S, T \in 2^N \text{ with } S \cap T = \emptyset.$$

The game v is called subadditive if the reverse inequality holds and additive if equality holds.

In the context of revenues superadditivity of a game is important while in the context of costs subadditivity of a game is important.

1.2.3 Definition A cooperative game is said to be convex if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for all } S, T \in 2^N.$$

The game v is called concave if the reverse inequality holds.

Shapley (1971) proved that a game v is convex iff

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \text{ for all } i \in N \text{ and all } S \subset T \subset N \setminus \{i\}.$$

It follows that v is concave if the reverse inequality holds.

1.2.4 Definition A cooperative game v is called 0-normalized if $v(i) = 0$ for all $i \in N$, v is called (0,1)-normalized if $v(i) = 0$ for all $i \in N$ and $v(N) = 1$.

1.2.5 Definition Two cooperative games u and v are said to be strategically equivalent if there exist a $k > 0$ and $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$v(S) = ku(S) + \sum_{i \in S} \alpha_i \text{ for all } S \in 2^N.$$

Every cooperative game v with $v(N) > \sum_{i \in N} v(i)$ is strategic equivalent

to exactly one $(0,1)$ -normalized game.

1.2.6 Definition A cooperative game v is monotonic if $v(S) \geq v(T)$ whenever $S \supset T$.

1.2.7 Definition A cooperative game v is called 0-monotonic if v is strategically equivalent to a 0-normalized monotonic game.

A game v is 0-monotonic if and only if

$$v(T) \leq v(S) - \sum_{i \in S \setminus T} v(i) \text{ whenever } S \supset T.$$

1.2.8 Definition The dual game v^* of a cooperative game v is defined by $v^*(S) := v(N) - v(N \setminus S)$ for all $S \in 2^N$.

Note that the dual game of a convex game is concave and vice versa and that $v^{**} = v$ for every game v .

The set G^N of all cooperative games with player set N is a $(2^n - 1)$ -dimensional linear space.

1.2.9 Definition For each $T \in 2^N$ the T -unanimity game u_T is defined by

$$u_T(S) = \begin{cases} 1 & \text{if } S \subset T \\ 0 & \text{otherwise.} \end{cases}$$

The set $\{u_T \mid T \in 2^N \setminus \{\emptyset\}\}$ is a basis of the linear space G^N . Any game $v \in G^N$ can be written as $\sum_{\emptyset \neq T \subset N} c_T(v) u_T$ where $c_T(v) = \sum_{R \subset T} (-1)^{|T|-|R|} v(R)$.

1.2.10 Definition A collection $\mathcal{B}$ of non-empty subsets of N is called a

balanced collection if there exist positive numbers λ_S for all $S \in \mathcal{B}$ such that $\sum_{S \in \mathcal{B}} \lambda_S 1_S = 1_N$.

The numbers λ_S are called *weights* of the elements of $\mathcal{B}$. A balanced collection containing no proper balanced subcollection is called a *minimal balanced collection*.

1.2.11 Definition A cooperative game v is called a *balanced game* if for every balanced collection $\mathcal{B}$ with weights $\{\lambda_S\}_{S \in \mathcal{B}}$ the following holds

$$v(N) \geq \sum_{S \in \mathcal{B}} \lambda_S v(S). \quad (1)$$

Let v be a cooperative game. For each $S \in 2^N$ the *subgame* $\langle S, v_S \rangle$ is defined by $v_S(T) = v(T)$ for all $T \subset S$.

1.2.12 Definition A cooperative game v is called *totally balanced* if for each coalition S the subgame v_S is balanced.

1.3 The core, the core cover and the Weber set

If in a game v all the players decide to work together then the question arises how $v(N)$ should be divided among them. A distribution of $v(N)$ can be represented by a payoff vector $x \in \mathbb{R}^n$ where x_i denotes the amount assigned to player $i \in N$. An $x \in \mathbb{R}^n$ with $x(N) = v(N)$ is called *efficient*.

1.3.1 Definition A function f which assigns to every cooperative game v a, possibly empty, subset $f(v)$ of $\mathbb{R}^n$ is called a *solution concept*.

1.3.2 Definition A solution concept f is called *invariant under strategic equivalence* if for every game v , every $k > 0$ and every additive game α , $f(kv + \alpha) = kf(v) + \alpha$.

A solution concept f is a *one-point solution concept* if $|f(v)| = 1$ for every cooperative game v . If f is a one-point solution concept then $f(v)$ is denoted as an element of $\mathbb{R}^n$ rather than as a subset of $\mathbb{R}^n$, i.e. $f(v) = x$ instead of $f(v) = \{x\}$.

1.3.3 Definition Let v be a cooperative game. Then the set of pre-imputations $PI(v)$ of the game v is the subset of $\mathbb{R}^n$ containing all efficient vectors for the game v .

A vector $x \in \mathbb{R}^n$ is said to be *individual rational* for a game v if $x_i \geq v(i)$ for all $i \in N$.

1.3.4 Definition The imputation set $I(v)$ of a game v is defined by

$$I(v) := \{x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x_i \geq v(i) \text{ for all } i \in N\}.$$

An imputation of a game v represents a distribution of $v(N)$ in such a way that no player can do better by working alone. However, it may still be possible for a group of players to improve its situation by leaving the grand coalition and working on its own. This is not the case when x belongs to the *core* of v .

1.3.5 Definition Let v be a cooperative game. The core $C(v)$ of v is defined by

$$C(v) := \{x \in \mathbb{R}^n \mid x(N) = v(N), x(S) \geq v(S) \text{ for all } S \in 2^N\}.$$

The core of a game v consists of those vectors in $\mathbb{R}^n$ representing distributions which can not be improved upon by any coalition. The core of a game v is a convex, compact subset of $\mathbb{R}^n$ and can be empty. The following well-known theorem, stated here without a proof, gives a characterization of games with a non-empty core.

1.3.6 Theorem (Bondareva (1961)-Shapley (1967)) *A cooperative game v has a non-empty core if and only if v is balanced.*

From theorem 1.3.6 and definition 1.2.12 it follows that a game v is totally balanced if and only if each subgame of v has a non-empty core.

In fact it can be proved that a game v has a non-empty core if and only if (1) holds for all minimal balanced collections $\mathcal{B}$. Cf. Shapley (1967). For superadditive games with three players this leads to the following proposition.

1.3.7 Proposition *Let v be a superadditive game with set of players $\{1, 2, 3\}$. Then $C(v) \neq \emptyset$ if and only if $v(1, 2) + v(1, 3) + v(2, 3) \leq 2v(1, 2, 3)$.*

Proof. Apart from the partitions of $\{1, 2, 3\}$ there is only one minimal balanced collection, namely, $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ with weight $\frac{1}{2}$ for all elements of $\mathcal{B}$. Because of the superadditivity of v (1) holds for all partitions of $\{1, 2, 3\}$. Therefore $C(v) \neq \emptyset$ if and only if (1) holds for $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ and this is the case if and only if $v(1, 2) + v(1, 3) + v(2, 3) \leq 2v(1, 2, 3)$. $\square$

1.3.8 Definition *Let v be a cooperative game. Then the upper vector M^v of v is defined by*

$$M_i^v := v(N) - v(N \setminus \{i\}) \text{ for all } i \in N.$$

The lower vector μ^v of v is defined by

$$\mu_i^v := \max_{S: i \in S} (v(S) - \sum_{j \in S \setminus \{i\}} M_j^v).$$

M_i^v is the marginal contribution of player i to the grand coalition. If player i asks for more than M_i^v it is better for the other players to work without him. Therefore, M_i^v can be seen as the highest payoff that player i can expect to obtain if N forms. The following argument shows that μ_i^v can be regarded as the minimum payoff that player i will consent to receive. Player i will argue that he can obtain μ_i^v by forming a coalition for which the maximum in definition 1.3.8 is achieved and giving to the other members of the

coalition their highest payoff. Thus, μ_i^v can be seen as the *minimal right* of player i in the game v .

1.3.9 Definition The core cover $CC(v)$ of a cooperative game v is defined by

$$CC(v) := \{x \in I(v) \mid \mu^v \leq x \leq M^v\}.$$

1.3.10 Theorem (Tijs and Lipperts (1982)) Let v be a cooperative game. Then $C(v) \subset CC(v)$.

Proof. Let $x \in C(v)$, then $x_i = x(N) - x(N \setminus \{i\}) \leq v(N) - v(N \setminus \{i\}) = M_i^v$ for each $i \in N$. Further, for each S with $i \in S$

$$x_i = x(S) - x(S \setminus \{i\}) \geq v(S) - \sum_{j \in S \setminus \{i\}} M_j^v.$$

Hence

$$x_i \geq \max_{S: S \ni i} (v(S) - \sum_{j \in S \setminus \{i\}} M_j^v) = \mu_i^v$$

and it follows that $x \in CC(v)$. □

Tijs and Lipperts (1982) gave some classes of games with $C(v) = CC(v)$. In chapter 5 a new class of games for which the core and the core cover coincide is given.

1.3.11 Definition Let v be a cooperative game and let π be a permutation of v . Then the marginal vector $m^\pi(v)$ of v corresponding to π is defined by

$$m_i^\pi(v) := v(P(\pi, i) \cup \{i\}) - v(P(\pi, i))$$

where $P(\pi, i) := \{j \in N \mid \pi(j) < \pi(i)\}$ is the set of predecessors of i with respect to π .

1.3.12 Definition The Weber set $W(v)$ of a cooperative game v is the convex hull of the $n!$ marginal vectors.

The following theorems are stated without proofs.

1.3.13 Theorem (Weber (1978)) *Let v be a cooperative game. Then $C(v) \subset W(v)$.*

1.3.14 Theorem (Shapley (1971)) *Let v be a convex game. Then $C(v) = W(v)$.*

1.3.15 Theorem (Ichiishi (1981)) *Let v be a cooperative game such that $C(v) = W(v)$. Then v is convex.*

Since the Weber set of a game is never empty it follows that convex games are balanced. Because any subgame of a convex game is again convex it follows that convex games are totally balanced.

1.4 The Shapley-value

Shapley (1953) used an axiomatic approach to introduce a one-point solution concept. In the following a one-point solution concept will also be called a solution function. Shapley posted a set of properties a solution concept should fulfil and proved that there exists a unique solution concept which satisfies these properties.

1.4.1 Definition *A player in a cooperative game v is called a dummy player if $v(S \cup \{i\}) - v(S) = v(i)$ for all $S \subset N \setminus \{i\}$.*

1.4.2 Definition *A solution function f is said to satisfy the dummy player property if $f_i(v) = v(i)$ whenever i is a dummy player in the game v .*

1.4.3 Definition *A solution function f is said to be efficient if $f(v)$ is*

efficient for every game v .

1.4.4 Definition A solution function f satisfies the symmetry property if $f_{\pi(i)}(\pi v) = f_i(v)$ for every game v , each player $i \in N$ and each permutation $\pi \in \Pi_N$. Here the game πv is defined by $\pi v(\pi(S)) := v(S)$ for all $S \in 2^N$.

1.4.5 Definition A solution function is said to be additive if $f(u + v) = f(u) + f(v)$ for all cooperative games u, v .

The following theorem is stated without a proof.

1.4.6 Theorem (Shapley (1953)) There exists a unique solution concept φ which satisfies the dummy player property, the efficiency property, the symmetry property and the additivity property. Moreover, φ is given by

$$\varphi_i(v) = \sum_{S \subset N \setminus \{i\}} \frac{|S|!(n - |S|)!}{n!} (v(S \cup \{i\}) - v(S)).$$

The solution function φ from theorem 1.4.6 is called the Shapley-value. Another way to describe the Shapley-value is with the aid of the marginal vectors. The Shapley-value of a game v is the average of the marginal vectors of the game, i.e.

$$\varphi(v) = \frac{1}{n!} \sum_{\pi \in \Pi_N} m^{\pi}(v). \quad (2)$$

From (2) and theorem 1.3.13 it follows that the Shapley-value of a convex game lies in the barycenter of the core of the game. If v is a superadditive game then $\varphi(v)$ is individual rational. For a general game v this need not be the case. The Shapley-value of a non-convex game need not be an element of the core of the game.

Young (1985a) gave a different characterization of the Shapley-value by replacing the dummy player property and the additivity property with a strong

monotonicity property. Driessen (1985a) characterized the Shapley-value by replacing the symmetry property and the dummy player property with a weak dummy player property and a restricted equality property. Hart and Mas-Colell (1985) provided an interpretation of the Shapley-value in terms of the potential function of a game. They characterized the Shapley-value with the aid of a *consistency* property. Here this consistency property and the *reduced game* needed for its definition will be called the HM-consistency property and the HM-reduced game in order to distinguish them from the consistency property and the reduced game which will be treated in section 1.5.

1.4.7 Definition Let v be a game and let f be a solution function. For any $S \subset N$ the HM-reduced game $\langle N \setminus S, \bar{v}_f^S \rangle$ of v with respect to S and f is defined by

$$\bar{v}_f^S(T) = v(T \cup S) - \sum_{i \in S} f_i(T \cup S, v) \text{ for all } T \subset N \setminus S$$

where $f_i(T \cup S, v)$ is the payoff to player i according to the solution function f in the game $\langle T \cup S, v \rangle$.

1.4.8 Definition A solution concept f is HM-consistent if for every game v and each balanced collection $S \subset N$, one has

$$f_j(N \setminus S, \bar{v}_f^S) = f_j(v) \text{ for all } j \in N \setminus S$$

where $f_j(N \setminus S, \bar{v}_f^S)$ is the payoff to player j according to the solution function f in the game $\langle N \setminus S, \bar{v}_f^S \rangle$.

The interpretation of HM-consistency is as follows. Consider a game v and a coalition $S \subset N$. The members of a coalition $T \subset N \setminus S$ look at the remainder after the members of S have received a payoff according to the solution function f . Considering this, the worth of T in the new situation is taken to be equal to the original worth of $S \cup T$ minus the payoffs of the members of S in the subgame $\langle S \cup T, v \rangle$ according to f . For the solution function f to be HM-consistent f has to appoint to each member of T in the HM-reduced game the same payoff as in the original game v . The following

result of Hart and Mas-Colell is stated without a proof.

1.4.9 Theorem *There is a unique HM-consistent solution function f with $f_i(v) = v(i) + \frac{1}{2}(v(i, j) - v(i) - v(j))$ and $f_j(v) = v(j) + \frac{1}{2}(v(i, j) - v(i) - v(j))$ for all cooperative games v with two players i and j . Moreover, f is equal to the Shapley-value φ .*

Owen (1972) defined the multilinear extension of a game and described a way to calculate the Shapley-value of a game by means of the multilinear extension of the game.

1.5 The (pre-)kernel and the (pre-)nucleolus

The kernel of a cooperative game was introduced by Davis and Maschler (1965). Two concepts which are important in the definition of the kernel are those of *excess* and *surplus*.

1.5.1 Definition *Let v be a cooperative game and let $x \in \mathbb{R}^n$ be a payoff vector. For any $S \subset N$ the excess $e(S, x)$ of S with respect to x is defined by*

$$e(S, x) := v(S) - \sum_{i \in S} x_i.$$

1.5.2 Definition *Let v be a cooperative game and let i and j be two different players. The maximum surplus $s_{ij}(x)$ of i against j with respect to a payoff vector x is defined by*

$$s_{ij}(x) := \max_{S \in \mathcal{M}_{ij}} e(S, x)$$

where $\mathcal{M}_{ij} := \{S \in 2^N \mid i \in S, j \notin S\}$.

The maximum surplus of player i against j with respect to x can be regarded as the highest payoff that player i can gain (or the minimal amount that i can lose if $s_{ij}(x)$ is negative) without the cooperation of player j . Player i can do this by forming a coalition without j but with other players

who are satisfied with their payoff according to x . Thus, s_{ij} can be regarded as the weight of a possible threat of i against j . If x is an imputation then player j can not be threatened by i or any other player when $x_j = v(j)$ because j can obtain $v(j)$ by working alone. It is said that i *outweights* j if

$$x_j > v(j) \text{ and } s_{ij}(x) > s_{ji}(x).$$

The kernel is the set of imputations with the property that no player outweights another player with respect to such an imputation.

1.5.3 Definition The kernel $\mathcal{K}(v)$ of a cooperative game v is defined by

$$\mathcal{K}(v) := \{x \in I(v) \mid s_{ij}(x) \leq s_{ji}(x) \text{ or } x_j = v(j) \text{ for all } i, j \in N, i \neq j\}.$$

Maschler, Peleg and Shapley (1972) defined the *pre-kernel* of a cooperative game.

1.5.4 Definition The *pre-kernel* $\mathcal{PK}(v)$ of a cooperative game v is defined by

$$\mathcal{PK}(v) := \{x \in PI(v) \mid s_{ij}(x) = s_{ji} \text{ for all } i, j \in N, i \neq j\}.$$

Davis and Maschler (1965) proved that the kernel of a cooperative game is always non-empty. Maschler, Peleg and Shapley (1972) proved that the pre-kernel of any cooperative game is non-empty and that for 0-monotonic games the kernel and the pre-kernel coincide. From the definition it follows that convex games are 0-monotonic and hence the kernel and the pre-kernel coincide for convex games.

For the definition of the *nucleolus* introduced by Schmeidler (1969) the following notions are required. Let $v \in G^N$. For any $x \in \mathbb{R}^n$ let $\theta(x)$ be the vector in $\mathbb{R}^n$ whose coordinates are the excesses of the 2^n subset of N arranged in decreasing order. That is,

$$\theta_i(x) \geq \theta_j(x) \text{ whenever } 1 \leq i \leq j \leq 2^n.$$

Consider the lexicographic order on the vectors $\theta(x)$. The lexicographic order is defined as follows. Given two vectors $y = (y_1, \dots, y_q)$ and $z =$

$(z_1, \dots, z_q)$, y is said to be lexicographically smaller than z if there exists an integer k with $1 \leq k \leq q$ such that

$$\begin{aligned} y_l &= z_l & \text{for } 1 \leq l < k \\ y_k &< z_k. \end{aligned}$$

This relation will be denoted by $y <_L z$ while $y \leq_L z$ is used to indicate that either $y <_L z$ or $y = z$.

1.5.5 Definition *The nucleolus $\nu(v)$ of a cooperative game v is defined by*

$$\nu(v) := \{x \in I(v) \mid \theta(x) \leq_L \theta(y) \text{ for all } y \in I(v)\}.$$

Thus, the nucleolus consists of those imputations which minimize the maximum excess. Schmeidler(1969) proved that the nucleolus of any game consists of exactly one point which is always an element of the kernel of the game and which lies in the core of the game whenever the core is not empty. Maschler, Peleg and Shapley (1972) proved that for a convex game the kernel consists of a single point and therefore this point is the nucleolus. Thus, for a convex game v , $\mathcal{PK}(v) = \mathcal{K}(v) = \nu(v)$.

1.5.6 Definition *The pre-nucleolus $p\nu(v)$ of a game v is defined by*

$$p\nu(v) := \{x \in PI(v) \mid \theta(x) \leq_L \theta(y) \text{ for all } y \in PI(v)\}.$$

The pre-nucleolus of any game consists of a unique point. For the characterization of the pre-nucleolus as given by Sobolev (1975) the *reduced game property* is needed.

1.5.7 Definition *Let v be a cooperative game and let $\emptyset \neq S \subset N$. Let $x \in \mathbb{R}^n$. The reduced game $\langle S, v_x^S \rangle$ of v with respect to S and x is defined by*

$$v_x^S(T) := \begin{cases} 0 & \text{if } T = \emptyset \\ v(N) - x(S) & \text{if } T = S \\ \max\{v(T \cup Q) - x(Q) \mid Q \subset N \setminus S\} & \text{if } T \subset S, T \neq \emptyset, S. \end{cases}$$

Reduced games were defined and investigated in Davis and Maschler (1965). The reduced game $\langle S, v_x^S \rangle$ describes a situation where all the members of N agree that each member j of $N \setminus S$ will receive x_j . Then the members of S can get $v(N) - x(N \setminus S)$. A coalition $T \subset S$, $T \neq S$ can cooperate with a coalition $Q \subset N \setminus S$ to obtain $v(T \cup Q)$. However, Q still has to receive $x(Q)$, thus by cooperating with Q , T will obtain $v(T \cup Q) - x(Q)$. The worth of T in the reduced game is equal to the maximum of all these expressions.

1.5.8 Definition *A solution concept f has the reduced game property if it satisfies the following condition.*

$$v \in G^N, \emptyset \neq S \subset N \text{ and } x \in f(v) \text{ imply } x_S \in f(S, v_x^S).$$

A systematic investigation of the reduced game property was first carried out by Aumann and Drèze (1974). They proved that the core has the reduced game property. The following theorem of Sobolev is stated here without a proof.

1.5.9 Theorem *The pre-nucleolus is the unique solution concept that satisfies the symmetry property, the invariance under strategic equivalence property and the reduced game property.*

Peleg (1985) noted that the Shapley-value, the kernel and the nucleolus do not satisfy the reduced game property. The pre-kernel does satisfy the reduced game property and Peleg gave a characterization of the pre-kernel with the aid of the reduced game property and five other axioms. One of them is the *converse reduced game property*.

1.5.10 Definition *A solution concept f satisfies the converse reduced game property if the following holds. If $v \in G^N$ and $x \in I(v)$ and for each $S \in 2^N$ with $|S| = 2$ the restriction of x to S belongs to $f(S, v_x^S)$ then x belongs to $f(v)$.*

Peleg (1985) also gave a characterization of the core using the reduced game property and the converse reduced game property.

1.6 The τ -value

The τ -value is a solution concept which was introduced by Tijs (1981). Originally it was defined only for the subclass of *quasi-balanced* games where it is a solution function.

1.6.1 Definition A game $v \in G^N$ is called *quasi-balanced* if

1. $\mu^v \leq M^v$
2. $\mu^v(N) \leq v(N) \leq M^v(N)$.

Note that every game with a non-empty core cover is quasi-balanced. With theorem 1.3.10 it follows that every balanced game is quasi-balanced. Let Q^N denote the set of quasi-balanced games in G^N . It is straightforward to see that Q^N is a convex cone. For a game $v \in Q^N$ the τ -value is the unique efficient payoff vector on the closed interval $[\mu^v, M^v]$.

1.6.2 Definition Let $v \in Q^N$. The τ -value $\tau(v)$ of v is defined by

$$\tau(v) := \lambda \mu^v + (1 - \lambda) M^v$$

where $\lambda \in [0, 1]$ is uniquely determined by $\sum_{i \in N} \tau_i(v) = v(N)$.

The τ -value of a game $v \in Q^N$ need not be an element of $C(v)$ even when $C(v)$ is not empty. In Driessen and Tijs (1985) necessary and sufficient conditions for $\tau(v)$ to be an element of $C(v)$ are given.

1.6.3 Definition Let $v \in G^N$. The gap function $g^v : 2^N \rightarrow \mathbb{R}$ of v is defined by

$$g^v(S) := M^v(S) - v(S) \text{ for all } S \in 2^N.$$

The gap function can be used to define a subclass of games for which a

simple expression for the τ -value exists.

1.6.4 Definition Let $v \in G^N$. Then v is called *semiconvex* if v is superadditive and $g^v(\{i\}) \leq g^v(S)$ for all $i \in N$ and $S \in 2^N$ with $i \in S$.

Driessen and Tijs (1985) proved that every convex game is semiconvex and that every semiconvex game is quasi-balanced. The following result is due to Driessen and Tijs (1985).

1.6.5 Theorem Let v be a semiconvex game. Then $\tau(v) = \lambda \underline{v} + (1 - \lambda)M^v$ where $\underline{v} = (v(1), v(2), \dots, v(n))$ and where λ is such that $\tau(v)$ is efficient.

Proof. Let $i \in N$ and suppose that $\mu_i^v = v(T) - \sum_{j \in T \setminus \{i\}} M_j^v$ where T is such that $i \in T$. Then $v(T) - \sum_{j \in T \setminus \{i\}} M_j^v \geq v(S) - \sum_{j \in S \setminus \{i\}} M_j^v$ for all $S \in 2^N$ with $i \in S$. This yields $v(T) - M^v(T) \geq v(S) - M^v(S)$ for all S with $i \in S$. From the definition of the gap function it follows that $g^v(T) \leq g^v(S)$ for all S with $i \in S$. Because v is semiconvex this yields $g^v(T) = g^v(\{i\})$ or equivalently, $v(T) - M^v(T) = v(i) - M_i^v$. Now $\mu_i^v = v(T) - M^v(T) + M_i^v = v(i)$ completes the proof. $\square$

1.6.6 Corollary Let v be a semiconvex zero-normalized game. Then $\tau(v) = M^v v(N) / M^v(N)$.

Proof. Immediate from theorem 1.6.5 $\square$

Driessen and Tijs (1985) introduced the subclass of *1-convex* games. They proved that these games have a non-empty core and that τ -value, nucleolus and center of gravity of the core all coincide for these games. Driessen (1985b) gave an characterization of the 1-convex games by means of the structure of their cores. In Driessen and Tijs (1984) an extension of the τ -value from Q^N to G^N was given. However, this extension does not possess the individual rationality and dummy player property which the τ -value does possess on Q^N . In Tijs and Driessen (1986a) another extension of the τ -value from Q^N to the subclass of games with a non-empty imputation set

was given that is individual rational and possesses the dummy player property. A thorough treatment of the τ -value and 1-convex games and their generalization, k -convex games can be found in Driessen (1985c).

1.6.7 Definition A solution concept f on Q^N is said to have the minimal right property if $f(v) = \mu^v + f(v - \mu^v)$ for all $v \in Q^N$. Here $v - \mu^v$ is the game defined by $(v - \mu^v)(S) := v(S) - \mu^v(S)$.

Note that if $v \in Q^N$ then $v - \mu^v \in Q^N$. The minimal right property says that the payoffs assigned by f in v are the same as in the case where the players get their minimal right first and then f is applied to the adapted game $v - \mu^v$. Note that any solution concept which is invariant under strategic equivalence satisfies the minimal right property.

1.6.8 Definition A solution function f on Q^N has the restricted proportionality property if for every $v \in Q^N$ with $\mu^v = 0$ $f(v)$ is proportional to M^v .

The following characterization of the τ -value is due to Tijs (1987).

1.6.9 Theorem The τ -value is the unique efficient solution function on Q^N with the minimal right property and the restricted proportionality property.

Proof. From the definition it follows that the τ -value is efficient. Tijs (1981) proved that the τ -value on Q^N is invariant under strategic equivalence and hence the τ -value satisfies the minimal right property. Again from the definition it is obvious that for a game $v \in Q^N$ with $\mu^v = 0$, $\tau(v)$ is proportional to M^v . Thus, the τ -value possesses the restricted proportionality property. Let $f : Q^N \rightarrow \mathbb{R}^n$ be a solution function with the three mentioned properties. Consider the game $w = v - \mu^v$. For this game the following holds.

$$M_i^w = v(N) - \mu^v(N) - v(N \setminus \{i\}) + \mu^v(N \setminus \{i\}) = M_i^v - \mu_i^v$$

and

$$\begin{aligned}
\mu_i^w &= \max_{S: S \ni i} (v(S) - \mu^v(S) - \sum_{j \in S \setminus \{i\}} M_j^v + \sum_{j \in S \setminus \{i\}} \mu_j^v) \\
&= \max_{S: S \ni i} (v(S) - M^v(S) + M_i^v - \mu_i^v) \\
&= \max_{S: S \ni i} (v(S) - M^v(S) + M_i^v) - \mu_i^v \\
&= \mu_i^v - \mu_i^v = 0.
\end{aligned}$$

From the restricted proportionality property and efficiency it follows that $f(v) = w(N)M^w/M^w(N) = \tau(w)$. Because of the minimal right property $f(v) = \mu^v + f(v - \mu^v) = \mu^v + f(v) = \mu^v + \tau(w) = \tau(v)$. Thus, f is equal to the τ -value and the proof is completed. $\square$.

1.7 Cooperative cost games

In the previous sections the definitions and concepts given were interpreted in terms of rewards. The purpose of this section is to show that it is possible to adjust these definitions and concepts in such a way that they are suitable for *cost situations* as well. The characteristic function of a cost game will be denoted in general with c instead of v . For a coalition S the amount $c(S)$ is the cost of S . A cost game can be used to describe a situation where a group of people can decrease their costs by working together. In chapter 4 four examples of cost games will be discussed. All the definitions in section 1.2 except that of balancedness stay the same for cost games. To obtain the definition of balancedness for cost games the inequality sign in 1.2.11 has to be reversed. If in a cost game the players decide to work together then the cost $c(N)$ has to be allocated among them. Just as in the case of reward games such an allocation can be represented by a vector $x \in \mathbb{R}^n$ where x_i denotes the cost allocated to player $i \in N$. While in a situation with rewards it is assumed that each player wants to have as much of the rewards as possible, in a cost situation it is logical to assume that each player wants to contribute as little as possible to the total cost. With this in mind the definitions of the solution concepts given in section 1.3 can be adjusted. The definitions which are not mentioned here stay the same. By reversing the inequality signs in the definitions of the imputation set and the core, the definitions of these two sets for cost games is obtained. With the new definitions theorem 1.3.6 stays valid. Proposition 1.3.7 has to be adjusted by changing superadditive into subadditive and reversing the

inequality sign. For a cost game c the vector M^c is defined in the same way as in 1.3.8. The definition of μ^c is obtained by changing “max” into “min”. The interpretation of these vectors changes. M_i^c can be seen as the minimum cost that i can expect to have to pay and μ_i^c as the highest cost that i will consent to pay. The core cover $CC(c)$ of c is defined by reversing the inequality signs in 1.3.9. With the new definitions theorem 1.3.10 still holds. Theorems 1.3.14 and 1.3.15 have to be adjusted by changing convex into concave. The definitions in 1.4 need no adjustment. The definition of the excess $e(S, x)$ of S with respect to an allocation x in a cost game c is given by $e(S, x) := \sum_{i \in S} x_i - c(S)$. With this new definition of excess the definitions of the (pre-)kernel and the (pre-)nucleolus can be given in the same way as in 1.5.3, 1.5.4, 1.5.5 and 1.5.6. The reduced game $\langle S, c_x^S \rangle$ of a cost game c with respect to a coalition S and an allocation x is obtained from 1.5.7 by changing “max” into “min” and leaving everything else as before. With the new definitions all the results in section 1.5 still hold. For cost games the definition of quasi-balancedness can be given by reversing the inequality signs in 1.6.1. The τ -value of a quasi-balanced cost game is defined in the same way as in 1.6.2 with the new definitions of M^c and μ^c . The gap function g^c of a cost game c is defined similarly as in 1.6.3. A cost game c is called *semiconcave* if c is subadditive and $g^c(\{i\}) \geq g^c(S)$ for all $i \in N$ and $S \in 2^N$ with $i \in S$. Theorem 1.6.5 and corollary 1.6.6 have to be adjusted by changing semiconvex into semiconcave. The characterization of the τ -value given in 1.6.9 can be given in the same way for the τ -value for cost games. Cooperative cost games are discussed e.g. in “Cost Allocation: Methods, Principles, Applications”, H.P. Young (1985b) (ed.) and in Tijs and Driessen (1986b).

CHAPTER 2

SIMPLE GAMES

2.1 Introduction

This chapter deals with the, so called, simple games. A cooperative game v is called a simple game if any coalition can either have worth 0 or 1. A coalition S is either all powerful, meaning that $v(S) = 1$ or completely powerless, meaning that $v(S) = 0$. Simple games were already introduced by von Neumann and Morgenstern (1944) and since then they have been used to model situations of committee control and of voting. Especially monotonic simple games, i.e. simple games in which the worth of any coalition is not less than the worth of any coalition it contains, have been used for these purposes. An outline of the descriptive theory of monotonic simple games is given in Shapley (1962). Lucas (1983) describes the use of a special type of these games, namely weighted majority games, to model weighted voting situations. A player in a simple game will want to know something about his power in the game. Several power indices have been proposed as a measure of the power of the players in monotonic simple games. Banzhaf (1953) and Coleman (1971) independently introduced power indices which are essentially the same. Shapley and Shubik (1954) restricted the Shapley-value to monotonic simple games to obtain the Shapley-Shubik power index. Deegan and Packel (1978) introduced a power index which depends only on the minimal winning coalitions of a monotonic simple game. In Packel and Deegan (1980) a generalization of this index is given. Bolger (1980) studies a class of power indices which includes the Deegan-Packel index. In this chapter which is based on Curiel (1987a) a class of non-normalized power indices is studied. The power indices in this class may not be efficient, i.e. summing up the power of all the players need not yield 1. They are regarded rather as functions describing the prospect of playing different roles in a simple game than as division rules for gains. Therefore, efficiency is not a necessary condition for them. In section 2 simple games and power indices are formally introduced. In section 3 situations where not all coalitions are equally likely to be formed are studied. With each coalition a weight is associated as a measure for the probability that the coalition will form. For each system of weights $\{r_S\}_{S \in 2^N \setminus \{\emptyset\}}$ a power index γ_r can be defined. When there can be no confusion about the system of weights involved, γ_r will be denoted by γ . Two axiomatic characterizations of γ_r are given for a general system of weights. The second one will be in terms of utility

functions for simple games. In section 4 a necessary and sufficient condition for the system of weights $\{r_S\}_{S \in 2^N \setminus \{\emptyset\}}$ is given to guarantee that γ_r has the symmetry property. It is proved that γ_r satisfies the dummy property and the behaviour of γ_r when two players quarrel is also examined.

2.2 Simple games

In this section formal definitions and notations needed throughout this chapter will be introduced.

2.2.1 Definition A cooperative game v is called a simple game if $v(N) = 1$ and $v(S) \in \{0, 1\}$ for all $S \in 2^N$.

2.2.2 Definition A simple game v is called monotonic if $v(S) \leq v(T)$ for all coalitions S and T with $S \subset T$.

Definition A coalition S in a simple game v is called losing if $v(S) = 0$, S is called winning if $v(S) = 1$.

2.2.4 Definition A coalition S in a monotonic simple game is called maximal losing if $v(S) = 0$ and $v(T) = 1$ for all $T \supset S$, $T \neq S$, S is called minimal winning if $v(S) = 1$ and $v(T) = 0$ for all $T \subset S$, $T \neq S$.

The collection of all winning coalitions in a simple game v is denoted by $W(v)$. The collection of all minimal winning coalitions in a monotonic simple game v is denoted by $W^m(v)$. For each $i \in N$ the collection of all minimal winning coalitions containing i is denoted by $W_i^m(v)$.

2.2.5 Definition A player $i \in N$ in a simple game v is said to be:

- a dummy player if $v(S) + v(i) = v(S \cup \{i\})$ for all $S \subset N \setminus \{i\}$
- a null player if $v(S) = v(S \cup \{i\})$ for all $S \subset N \setminus \{i\}$
- a dictator if $v(S) = 1$ iff $i \in S$
- a veto player if $v(S) = 1$ implies $i \in S$.

A dummy player i can be either a null player, in case $v(i) = 0$, or a dictator, in case $v(i) = 1$.

A well known result for simple games is that a simple game is balanced if and only if it has at least one veto player.

2.2.6 Theorem *A simple game v is balanced if and only if v has veto players.*

Proof. Let $i \in N$ be a veto player of v . Define $x \in \mathbb{R}^n$ by $x_i = 1$ and $x_j = 0$ for all $j \in N \setminus \{i\}$. Then $x(S) = 1 \geq v(S)$ for all S with $i \in S$ and $x(S) = 0 = v(S)$ for all $S \subset N \setminus \{i\}$. Thus, $x \in C(v)$ and the “if” part of the proof is completed. Suppose that v has no veto players. Then for each $i \in N$ there is a $S_i \subset N \setminus \{i\}$ with $v(S_i) = 1$. Hence if x were an element of the core of v then $x_i = 0$ for all $i \in N$, but this contradicts $x(N) = 1$ and thus the core of v is empty. This yields the “only if” part and thus the proof is completed. $\square$

Spinetto (1971) proved that every non-negative (0,1)-normalized balanced game can be written as a convex combination of balanced simple games. The following generalization of this result has been given by Derks (1987) and is stated here without a proof.

2.2.7 Theorem (Derks) *Every non-negative balanced game can be written as a positive linear combination of simple games.*

As already mentioned monotonic simple games can be used for modelling situations of committee control and of voting. A player in such a situation will want to know how much influence he has on the outcome of the game. He will need a measure of his power in the game. Such a measure is called a *power index*.

2.2.8 Definition *A power index is a function f which assigns to every monotonic simple game v a vector $f(v) \in \mathbb{R}^n$.*

The number $f_i(v)$ is considered to be a measure of the power of player i in the game v . Examples of power indices are the Shapley-Shubik index and the Banzhaf index. The Shapley-Shubik index is in fact the restriction of the Shapley-value to the monotonic simple games. The Banzhaf index counts the number of *swings* of a player i . A swing for a player i is a pair of sets of the form $(S \cup \{i\}, S \setminus \{i\})$ such that $S \cup \{i\}$ is winning and $S \setminus \{i\}$ is losing. The original characterization of the Shapley-value can not be used for the Shapley-Shubik index because the set of monotonic simple games is not a linear space. Dubey (1975) characterized the Shapley-Shubik index by replacing the additivity axiom by a so called *transfer* axiom. A power index f fulfils this axiom if $f(v \vee w) + f(v \wedge w) = f(v) + f(w)$ for all monotonic simple games v and w . The Banzhaf index can be characterized in a similar way as the Shapley-Shubik index by changing the efficiency property into an appropriate property. Bolger (1982) gives a characterization of the Banzhaf index and the Shapley-Shubik index without explicitly using the transfer axiom. Owen (1975a) extended the Banzhaf index to become a solution concept for all cooperative games and characterized this solution concept in terms of the multilinear extension of a game. Dubey and Shapley (1979) study various mathematical properties of the Banzhaf power index. In Owen (1977) and Owen (1980) modifications of respectively the Shapley-Shubik index and the Banzhaf index are given for games with a priori unions. Weber (1978) and Straffin (1983) describe several ways of interpreting the Shapley-Shubik and the Banzhaf indices.

Deegan and Packel (1978) introduced and characterized a power index which depends only on the minimal winning coalitions of a monotonic simple game. They assume that only minimal coalitions will form and that all minimal winning coalitions have the same probability of forming. In Packel and Deegan (1980) they relax this last assumption and give a generalization of their index to the case where only minimal winning coalitions with the same size are assumed to have equal probability of forming. The power index γ which will be introduced in section 2.3 will also depend only on the minimal winning coalitions of a monotonic simple game. However, in the definition of γ it is not assumed that minimal winning coalitions with the same size have the same probability to occur.

2.3 The power index γ

Consider a situation of committee control where not all the subsets S of the committee N are equally likely to occur. To each $\emptyset \neq S \subset N$ a weight $r_S > 0$

is attached which is interpreted as a measure for the probability that S will be formed. For all $S, T \in 2^N \setminus \{\emptyset\}$, $r_S > r_T$ means that the formation of S is more likely to occur than the formation of T . Coalitions with the same weight are equally likely to occur and vice versa. For every collection B of coalitions $\sum_{S \in B} r_S$ is denoted by $r(B)$. Let a specific structure of winning and losing coalitions in the committee N be described by a monotonic simple game v . In considering v it is assumed that only minimal winning coalitions in v will be formed. This assumption is justified by the fact that a minimal winning coalition can achieve as much as any coalition containing it. It is reasonable to assume that any profit resulting from the formation of a winning coalition has to be divided among its members so that there is no incentive to form larger coalitions which can not achieve more than smaller ones (cf. Riker (1962)). Let MSG^N denote the set of monotonic simple games with set of players equal to N . For each $v \in MSG^N$ and each $S \in 2^N$ define

$$p_S(v) := \begin{cases} 0 & \text{if } S \notin W^m(v) \\ r_S/r(W^m(v)) & \text{if } S \in W^m(v). \end{cases}$$

Then

$$p_S(v) \geq 0 \text{ for each } S \in 2^N \text{ and } p(W^m(v)) = 1. \quad (1)$$

The number $p_S(v)$ is interpreted as the probability that coalition S is formed in the game v . The power index γ assigns to each player $i \in N$ in a simple game v the probability that a minimal winning coalition containing him will be formed.

2.3.1 Definition *The power index γ is defined by*

$$\gamma_i(v) = p(W_i^m(v)) \text{ for each } i \in N.$$

Note that since the summation over the empty set is zero $\gamma_i(v) = 0$ if $W_i^m(v) = \emptyset$. By looking at $\gamma_i(v)$ player i can find out something about his importance in the game v . The larger $\gamma_i(v)$ the more important he is in the game. If i is a veto player nothing can be done without him and $\gamma_i(v)$ is equal to 1, the highest value it can take.

2.3.2 Definition *Let v_1 and v_2 be two monotonic simple games. Then*

v_1 and v_2 are called mergeable if for all $S \in W^m(v_1)$ and $T \in W^m(v_2)$, $S \not\subseteq T$ and $T \not\subseteq S$.

Note that for two mergeable games v_1 and v_2 :

$$W^m(v_1) \cap W^m(v_2) = \emptyset \text{ and } W^m(v_1) \cup W^m(v_2) = W^m(v_1 \vee v_2). \quad (2)$$

2.3.3 Definition A power index f is said to satisfy the merge property if

$$f(v_1 \vee v_2) = \frac{f(v_1)r(W^m(v_1)) + f(v_2)r(W^m(v_2))}{r(W^m(v_1)) + r(W^m(v_2))}$$

for all mergeable games v_1 and v_2 .

2.3.4 Definition A power index f is said to satisfy the null player property if $f_i(v) = 0$ whenever i is a null player in v .

2.3.5 Definition A power index f is said to satisfy the veto player property if $f_i(v) = 1$ whenever i is a veto player in v .

The following theorem provides a characterization of the power index γ .

2.3.6 Theorem The power index γ is the unique power index which satisfies (i) the merge property, (ii) the null player property and (iii) the veto player property.

Proof Let $i \in N$ be a null player in a monotonic simple game v . Then $W_i^m(v) = \emptyset$ and hence $\gamma_i(v) = 0$ by definition and it follows that γ satisfies (ii). Let $i \in N$ be a veto player in a monotonic simple game v . Then $W_i^m(v) = W^m(v)$ and $\gamma_i(v) = p(W_i^m(v)) = p(W^m(v)) = 1$ by (1). Hence γ satisfies (iii). Let v_1, v_2 be two mergeable simple games with $v_1 \vee v_2 = v$. Then it follows from (2) that

$$\gamma_i(v) = p(W_i^m(v)) = \sum_{S \in W_i^m(v_1)} p_S(v) + \sum_{S \in W_i^m(v_2)} p_S(v)$$

for each $i \in N$. For each $S \subset W^m(v_1)$ (2) yields

$$\begin{aligned} p_S(v) &= r_S / r(W^m(v)) = r_S / (r(W^m(v_1)) + r(W^m(v_2))) \\ &= p_S(v_1) r(W^m(v_1)) / (r(W^m(v_1)) + r(W^m(v_2))). \end{aligned} \quad (3)$$

Similarly, for each $S \in W^m(v_2)$

$$p_S(v) = p_S(v_2) r(W^m(v_2)) / (r(W^m(v_1)) + r(W^m(v_2))). \quad (4)$$

Together (3) and (4) lead to

$$\begin{aligned} \gamma_i(v) &= \frac{p(W_i^m(v_1) r(W^m(v_1)) + p(W_i^m(v_2)) r(W^m(v_2)))}{r(W^m(v_1)) + r(W^m(v_2))} \\ &= \frac{\gamma_i(v_1) r(W^m(v_1)) + \gamma_i(v_2) r(W^m(v_2))}{r(W^m(v_1)) + r(W^m(v_2))} \end{aligned}$$

and it follows that γ satisfies (i).

Let f be a power index satisfying (i), (ii) and (iii). Let v be a monotonic simple game with $W^m(v) = \{S_1, S_2, \dots, S_k\}$. For each $l \in \{1, \dots, k\}$ the game v_l is defined by

$$v_l(T) := \begin{cases} 1 & \text{if } T \supset S_l \\ 0 & \text{if } T \not\supset S_l. \end{cases}$$

The games $v_1, v_2, \dots, v_k$ are mergeable and $v = v_1 \vee v_2 \vee \dots \vee v_k$. From the null player and veto player property it follows that

$$f_i(v_l) = \begin{cases} 0 & \text{if } i \notin S_l \\ 1 & \text{if } i \in S_l \end{cases}$$

or shortly, $f_i(v_l) = 1_{S_l}(i)$. Using the inductive extension of the merge property to a merge of k mergeable games leads to

$$\begin{aligned} f_i(v) &= \frac{f_i(v_1) r(W^m(v_1)) + \dots + f_i(v_k) r(W^m(v_k))}{r(W^m(v_1)) + \dots + r(W^m(v_k))} \\ &= \frac{\sum_{l=1}^k 1_{S_l}(i) r_{S_l}}{\sum_{l=1}^k r_{S_l}} \\ &= r(W_i^m(v)) / r(W^m(v)) \\ &= p(W_i^m(v)) = \gamma_i(v). \end{aligned}$$

Thus, $f = \gamma$ and the uniqueness of γ is proved. $\square$

The ideas behind the null player and the veto player properties are clear. A null player has no power at all and this is reflected in the null player property. A veto player is very powerful and the veto player property assigns to him the highest measure of power possible. The merge property states that the power of a player in the merge of two mergeable games is the weighted mean of his power in the separate games. Another way of interpreting this property is as follows. Let v_1, v_2 be two mergeable games and suppose player $i \in N$ is playing v_1 with probability $r(W^m(v_1))/(r(W^m(v_1)) + r(W^m(v_2)))$ and v_2 with probability $r(W^m(v_2))/(r(W^m(v_1)) + r(W^m(v_2)))$. Then the merge property says that $\gamma_i(v_1 \vee v_2)$ is his expected power in this situation. For example, if he has absolute power in v_1 and is completely powerless in v_2 then his expected power is equal to the probability that v_1 occurs. The Shapley-Shubik index and the Banzhaf index were characterized by Roth (1977) as *utility functions* for simple games which differ in their postures toward risk. Here a characterization of γ as a utility function for simple games will be derived.

2.3.7 Definition A set M is said to be a mixture set if with any $a, b \in M$ and any $\lambda \in [0, 1]$ an element $\lambda a + (1 - \lambda)b$ can be associated which is again in M and where

- $1a + (1 - 1)b = a$
- $\lambda a + (1 - \lambda)b = (1 - \lambda)b + \lambda a$
- $\mu(\lambda a + (1 - \lambda)b) + (1 - \mu)b = (\mu\lambda a + (1 - \mu\lambda))b$

for all $a, b \in M$ and $\lambda, \mu \in [0, 1]$.

Let R be a complete, transitive, continuous preference relation on M . For any $a, b \in M$, aRb is interpreted as a is not less preferred than b . If aRb and bRa then there exists indifference between a and b and this is denoted by aIb . If aRb and not bRa then a is more preferred than b and this is denoted by aPb .

2.3.8 Definition Let M be a mixture set with a complete, transitive, con-

tinuous preference relation R defined on M . A function u from M to $\mathbb{R}$ is said to be a utility function which represents R if

(i) $u(a) > u(b)$ if and only if aPb

(ii) $u(\lambda a + (1 - \lambda)b) = \lambda u(a) + (1 - \lambda)u(b)$

for all $a, b \in M$ and all $\lambda \in [0, 1]$.

The following result of Herstein and Milnor (1953) is stated here without a proof.

2.3.9 Theorem *Let M be a mixture set with a complete, transitive, continuous preference relation R defined on M such that for all $a, b \in M$ if aIb then for any $c \in M$, $(\frac{1}{2}a + \frac{1}{2}c)I(\frac{1}{2}b + \frac{1}{2}c)$. Then there exists a utility function u which represents R . Furthermore, u is unique up to an affine transformation.*

Let M be the mixture space generated by $MSG^N \times N$. Let R be a complete, transitive, continuous preference relation on M which satisfies the condition of 2.3.9. For elements $(v, i), (w, j)$ of $MSG^N \times N$, $(v, i)R(w, j)$ is interpreted as: "It is as least as preferable to play position i in the game v as to play position j in the game w ." For elements $\lambda(v, i) + (1 - \lambda)(w, j)$ and $\mu(q, k) + (1 - \mu)(z, l)$ of M , $\lambda(v, i) + (1 - \lambda)(w, j)R\mu(q, k) + (1 - \mu)(z, l)$ is interpreted as: "It is as least as preferable to play with probability λ position i in the game v and with probability $1 - \lambda$ position j in the game w as it is to play with probability μ position k in the game q and with probability $1 - \mu$ position l in the game z . From theorem 2.3.9 it follows that there exists a utility function u which represents R . For each $S \in 2^N \setminus \{\emptyset\}$ let the game $v_S \in MSG^N$ be defined by

$$v_S(T) = \begin{cases} 1 & \text{if } T \supset S \\ 0 & \text{if } T \not\supset S. \end{cases}$$

Consider the following extra conditions which can be imposed on R .

C.1 Let $S \subset N$, $S \neq \emptyset$. Then

$$\begin{aligned} (v_{\{i\}}, i)I(v_S, i) & \quad \text{for all } i \in S \\ (v_{\{i\}}, j)I(v_S, j) & \quad \text{for all } j \notin S, i \in N \setminus \{j\} \\ (v_{\{i\}}, i)P(v_{\{j\}}, i) & \quad \text{for all } i, j \in N \text{ with } i \neq j \\ (v_{\{i\}}, i)R(v_{\{j\}}, i) & \quad \text{for all } v \in MSG^N, i, j \in N \text{ with } i \neq j. \end{aligned}$$

C.2 Let $v_1, v_2 \in MSG^N$ be mergeable. Then

$$(v_1 \vee v_2, i)I \frac{r(W^m(v_1))}{r(W^m(v_1)) + r(W^m(v_2))}(v_1, i) + \frac{r(W^m(v_2))}{r(W^m(v_1)) + r(W^m(v_2))}(v_2, i).$$

Condition C.2 states that indifference existst between playing the game $v_1 \vee v_2$ and playing with probability $r(W^m(v_1))/(r(W^m(v_1)) + r(W^m(v_2)))$ the game v_1 and with probability $r(W^m(v_2))/(r(W^m(v_1)) + r(W^m(v_2)))$ the game v_2 .

The utility function u representing R is unique up to an affine transformation. Therefore it can be normalized to satisfy

$$u(v_{\{i\}}, i) = 1 \text{ and } u(v_{\{i\}}, j) = 0 \text{ for all } i, j \in N \text{ with } i \neq j. \quad (5)$$

With the function u a *strategic utility* function $\bar{u}$ can be associated where $\bar{u}$ is a function from MSG^N to $\mathbb{R}^n$ defined by

$$\bar{u}_i(v) := u(v, i) \text{ for each } v \in MSG^N \text{ and each } i \in N. \quad (6)$$

2.3.10 Theorem *Let R be a complete, transitive, continuous preference relation on M satisfying the condition of theorem 2.3.9 and the conditions C.1 and C.2. Let u be the unique utility function representing R which satisfies (5). Then the strategic utility function $\bar{u}$ associated with u by (6) is equal to the power index γ .*

Proof. Let $v \in MSG^N$ with $W^m(v) = \{S_1, \dots, S_k\}$. For each $l \in \{1, \dots, k\}$ let v_{S_l} be denoted by v_l . The games $v_1, v_2, \dots, v_k$ are mergeable and $v = v_1 \vee v_2 \vee \dots \vee v_k$. From C.1, the definition of $\bar{u}$ and (5) it follows that

$$\bar{u}_i(v_l) = u(v_l, i) = 1_{S_l}(i) = \gamma_i(v_l) \text{ for all } l \in \{1, \dots, k\}, i \in N.$$

Inductive extension of condition C.2 yields

$$u_i(v) = u_i(v_1 \vee v_2 \vee \dots \vee v_k)$$

$$\begin{aligned}
&= (\tau_{S_1} 1_{S_1}(i) + \dots + \tau_{S_k} 1_{S_k}(i)) / (\tau_{S_1} + \dots + \tau_{S_k}) \\
&= (\tau(W^m(v_1)) \gamma_i(v) + \dots + \tau(W^m(v_k)) \gamma_i(v_k)) / \tau(W^m(v)) \\
&= \gamma_i(v_1 \vee v_2 \vee \dots \vee v_k) = \gamma_i(v).
\end{aligned}$$

Thus, $\bar{u}$ equals γ . □

2.4 Other properties of γ

In this section some other properties of γ will be studied. A property usually required for a power index is the *symmetry* property. A power index f possesses the symmetry property if $f_{\pi(i)}(\pi v) = f_i(v)$ for each $\pi \in \Pi_N$, $i \in N$ and $v \in MSG^N$. Recall that the game πv is defined by $\pi v(\pi(S)) = v(S)$ for each $S \in 2^N$. Because of the weights τ_S occurring in the definition of γ which need not be the same for coalitions with the same size, γ does not, in general, satisfy the symmetry property. This is in fact quite natural because if the probability that a coalition is formed really depends on the members of the coalition and is not the same for all coalitions with the same size, symmetry can not be expected in a power index evaluating such a situation. This is reflected in the following proposition.

2.4.1 Proposition *The power index γ satisfies the symmetry property if and only if $\tau_S = \tau_T$ for all $S, T \in 2^N$ with $|S| = |T|$.*

Proof. Suppose there exists S, T with $|S| = |T|$ and $\tau_S \neq \tau_T$. Let π be a permutation of N such that $\pi(S) = T$ and $\pi(T) = S$. Let $i \in S$, $i \notin T$. Consider the monotonic simple game v with $W^m(v) = \{S, T\}$. Then also $W^m(\pi v) = \{S, T\}$. Further,

$$\gamma_i(v) = \frac{\tau_S}{\tau_S + \tau_T} \neq \frac{\tau_T}{\tau_S + \tau_T} = \gamma_{\pi(i)}(\pi v)$$

and thus, γ does not satisfy the symmetry property. This proves the “only if” part.

Suppose that for all $S, T \in 2^N$ with $|S| = |T|$, $\tau_S = \tau_T$. Let $\pi \in \Pi_N$ and $v \in MSG^N$. For each $S \in 2^N$ the following holds

$$p_S(\pi v) = \begin{cases} 0 & \text{if } S \notin W^m(\pi v) \\ \frac{\tau_S}{\tau(W^m(v))} & \text{if } S \in W^m(\pi v) \end{cases}$$

yielding

$$p_S(\pi v) = \begin{cases} 0 & \text{if } \pi^{-1}(S) \notin W^m(v) \\ \frac{r_{\pi^{-1}(S)}}{r(W^m(v))} & \text{if } \pi^{-1}(S) \in W^m(v) \end{cases} = p_{\pi^{-1}(S)}(v).$$

For $i \in N$ this leads to

$$\begin{aligned} \gamma_{\pi(i)}(\pi v) &= \sum_{S \in W_{\pi(i)}^m(\pi v)} p_S(\pi v) \\ &= \sum_{\pi^{-1}(S) \in W_i^m(v)} p_S(\pi v) \\ &= \sum_{\pi^{-1}(S) \in W_i^m(v)} p_{\pi^{-1}(S)}(v) \\ &= \sum_{T \in W_i^m(v)} p_T(v) \\ &= \gamma_i(v). \end{aligned}$$

This proves the “if” part and the proof is completed. $\square$

A power index f satisfies the *dummy player* property if $f_i(v) = v(i)$ whenever i is a dummy player in $v \in MSG^N$.

2.4.2 Proposition *The power index γ satisfies the dummy player property.*

Proof. Let $v \in MSG^N$ and let i be a dummy player in v . Then i is either a null player or a dictator. If i is a null player then $\gamma_i(v) = 0 = v(i)$ by 2.3.6. If i is a dictator then $W_i^m(v) = W^m(v) = \{i\}$ and $\gamma_i(v) = v(i)$. $\square$

Next the behaviour of the power index γ when two players in a simple game *quarrel*, i.e. refuse to enter into a coalition together, will be investigated. Kilgour (1974) was the first to give a formal description of quarelling in the context of the Shapley-value. Brams (1975) gives an example in which the Shapley-Shubik index and the Banzhaf index of two quarrelling players increase after their quarrel. Several other things may happen with these indices when two players quarrel. Straffin (1983) gives the following list

of things that can happen with power as measured by the Shapley-Shubik index and the Banzhaf index when two players quarrel.

- (i) They may both gain power.
- (ii) They may both lose power.
- (iii) One may gain power while the other loses power.
- (iv) Their power may not be affected at all but the power of innocent bystanders may be changed.
- (v) If one of them is a null player the other may lose power.

When power is measured by the power index introduced by them, Deegan and Packel (1983) show that (i), (ii), (iii), (iv) can also occur but that quarrelling with a null player does not affect anyone's power. When the power index γ is taken as a measure neither (i), (iii), (iv) nor (v) will happen.

2.4.3 Definition Let $v \in MSG^N$, $i, j \in N$ and denote by v^Q the game arising from v when i and j quarrel. Then v^Q is defined by $W^m(v^Q) = \{S \mid S \in W^m(v), \{i, j\} \not\subset S\} = \{S \mid S \in W^m(v), S \not\subset W_i^m(v) \cap W_j^m(v)\}$.

The following theorem describes what can happen to the power index γ when two players quarrel.

2.4.4 Theorem Let $v \in MSG^N$, $i, j \in N$ and v^Q as defined in 2.4.3

- (a) If $W_i^m(v) \cap W_j^m(v) = \emptyset$ then $\gamma(v^Q) = \gamma(v)$.
- (b) If $W_i^m(v) \cap W_j^m(v) \neq \emptyset$ and $W_i^m \neq W^m(v)$, $W_j^m(v) \neq \emptyset$ then $\gamma_i(v^Q) < \gamma_i(v)$ and $\gamma_j(v^Q) < \gamma_j(v)$.
- (c) Let $z \in N$ with $W_z^m(v) \neq \emptyset$ and $W_i^m(v) \cap W_j^m(v) \cap W_z^m(v) = \emptyset$. If $W_i^m(v) \cap W_j^m(v) \neq \emptyset$ then $\gamma_z(v^Q) > \gamma_z(v)$.

Proof. (a) $W^m(v^Q) = \{S \mid S \in W^m(v), S \not\subset W_i^m(v) \cap W_j^m(v)\} = W^m(v)$ hence $p_S(v^Q) = p_S(v)$ for each $S \in 2^N$ and $\gamma(v^Q) = \gamma(v)$.

(b) Without loss of generality it is assumed that

$$\begin{aligned} W^m(v) &= \{S_1, S_2, \dots, S_k\} \\ W_i^m(v) &= \{S_1, S_2, \dots, S_l\} \quad l < k \\ W_j^m(v) &= \{S_{p+1}, \dots, S_l, S_{l+1}, \dots, S_q\} \quad q < k, p+1 \leq l. \end{aligned}$$

Then $W_i^m(v) \neq W^m(v)$, $W_j^m(v) \neq W^m(v)$ and $W_i^m(v) \cap W_j^m(v) = \{S_{p+1}, \dots, S_l\} \neq \emptyset$. Furthermore,

$$\begin{aligned} W^m(v^Q) &= \{S_1, \dots, S_p, S_{l+1}, \dots, S_q\} \\ W_i^m(v^Q) &= \{S_1, \dots, S_p\} \\ W_j^m(v^Q) &= \{S_{l+1}, \dots, S_q\}. \end{aligned}$$

Now

$$\begin{aligned} \gamma_i(v^Q) &< \gamma_i(v) \Leftrightarrow p(W_i^m(v^Q)) < p(W_i^m(v)) \\ \Leftrightarrow (r_{S_1} + \dots + r_{S_p}) / (r_{S_1} + \dots + r_{S_p} + r_{S_{l+1}} + \dots + r_{S_k}) \\ &< (r_{S_1} + \dots + r_{S_l}) / (r_{S_1} + \dots + r_{S_k}) \\ \Leftrightarrow (r_{S_1} + \dots + r_{S_p})(r_{S_1} + \dots + r_{S_k}) \\ &< (r_{S_1} + \dots + r_{S_l})(r_{S_1} + \dots + r_{S_p} + r_{S_{l+1}} + \dots + r_{S_k}) \\ \Leftrightarrow (r_{S_1} + \dots + r_{S_p})(r_{S_{l+1}} + \dots + r_{S_k}) \\ &< (r_{S_1} + \dots + r_{S_l})(r_{S_{l+1}} + \dots + r_{S_k}) \\ \Leftrightarrow 0 &< (r_{S_{p+1}} + \dots + r_{S_l})(r_{S_{l+1}} + \dots + r_{S_k}) \end{aligned}$$

and the last inequality holds because $r_S > 0$ for each $\emptyset \neq S \subset N$. In the same way $\gamma_j(v^Q) < \gamma_j(v)$ can be shown.

(c) Without loss of generality the same assumptions as in (b) are made with the difference that instead of $q < k$, $q \leq k$ is taken. Further $W_z^m(v)$ is assumed to be equal to

$$\{S_a, S_{a+1}, \dots, S_b, S_c, S_{c+1}, \dots, S_d\} \text{ with } a \leq b \leq p, c \geq l+1, d \leq k.$$

Then $W_z^m(v) \neq \emptyset$, $W_i^m(v) \cap W_j^m(v) \cap W_z^m(v) = \emptyset$ and $W_i^m(v) \cap W_j^m(v) = \{S_{p+1}, \dots, S_l\} \neq \emptyset$. Furthermore, $W_z^m(v^Q) = W_z^m(v)$ and $W^m(v) = \{S_1, \dots, S_p, S_{l+1}, \dots, S_k\}$ yielding

$$\begin{aligned} \gamma_z(v^Q) &= \frac{r_{S_a} + \dots + r_{S_b} + r_{S_c} + \dots + r_{S_d}}{r_{S_1} + \dots + r_{S_p} + r_{S_{l+1}} + \dots + r_{S_k}} \\ &> \frac{r_{S_a} + \dots + r_{S_b} + r_{S_c} + \dots + r_{S_d}}{r_{S_1} + \dots + r_{S_k}} \\ &= \gamma_z(v). \end{aligned}$$

Thus, the proof is completed. $\square$

From theorem 2.4.4 it follows that when two players who are not together in any minimal coalition quarrel, the γ -value of the game does not change. This implies that quarrelling with a null player can not hurt anybody. When two players who are not veto players and who are together in at least one minimal winning coalition quarrel, they will both lose power and a third one who is not in any minimal winning coalition with both of them will gain power. A quarreller will never benefit from the quareel when power is measured by γ . The power index γ does not lead to the quarrelling paradoxes induced by the other indices mentioned. However, a paradox which does occur for the Deegan-Packel index and also for γ but not for the Shapley-Shubik index and the Banzhaf index is the so called weighted voting paradox. Let v be the 5-person weighted majority game with $v(S) = 1$ if $\sum_{i \in S} w_i \geq 5$ and $v(S) = 0$ otherwise. Here $w_1 = 3$, $w_2 = 2$, $w_3 = w_4 = w_5 = 1$. Then

$$\gamma_2(v) = \frac{r_{\{1,2\}} + r_{\{2,3,4,5\}}}{r(W^m(v))} \text{ and}$$

$$\gamma_3(v) = \frac{r_{\{1,3,4\}} + r_{\{1,3,5\}} + r_{\{2,3,4,5\}}}{r(W^m(v))}.$$

For certain r_S it happens that $\gamma_2(v) < \gamma_3(v)$ while $w_2 < w_3$. This paradox may not be as serious as it seems at first sight; when a player with less weight is a member of more minimal coalitions then a player with more weight, the former can be regarded as being more crucial for the game and hence having more power.

In order to compute γ the weights r_S have to be known. The assumption that all minimal winning coalitions are equally likely to occur will lead to a special form of γ with $\gamma_i(v) = |W_i^m(v)|/|W^m(v)|$ for each $i \in N$. Another form of γ is obtained if the assumption $r_S = \frac{1}{|S|}$ is made, meaning that the probability of the formation of a coalition of a certain size is inversely proportional to the size of the coalition. Then $\gamma_i(v) = \sum_{S \in W_i^m(v)} \frac{1}{|S|} / \sum_{S \in W^m(v)} \frac{1}{|S|}$.

LINEAR PROGRAMMING GAMES

3.1 Introduction

In this chapter linear programming games will be considered. Linear programming games are games in which for each coalition S the worth $v(S)$ of S is given by a linear programming problem which depends on S . In section 2 linear programming games are introduced and used to classify totally balanced games. Two examples of linear programming games, namely, flow games and linear production games are studied. In section 3 linear programming games with committee control will be discussed. Since these games need not be balanced, sufficient conditions for them to be balanced are given. It is proved that every non-negative balanced game is a linear programming game with committee control. The non-balanced case is treated in section 4 where two ways to handle non-balanced linear programming games are discussed. Finally, in the last section of this chapter, linear programming games with claims are introduced. These games describe linear production situations with a fixed amount of resources which are claimed by the players. Claim games can be constructed to describe such a situation. These claim games turn out to be convex and therefore balanced and with the aid of core elements of the claim games a core element for the linear programming game with claims is constructed. Section 3 of this chapter is partly based on Curiel, Derks and Tijs (1987) and section 4 is based on work done in Curiel et al. (1988a).

3.2 Linear programming games

Consider the following linear programming problem.

$$\begin{aligned}
 & \text{maximize } c \cdot x \text{ subject to} \\
 & \quad xA \leq b \\
 & \quad xB = d \\
 & \quad x \geq 0
 \end{aligned} \tag{1}$$

Here $c \in \mathbb{R}^m$, $b \in \mathbb{R}^p$, $d \in \mathbb{R}^r$, A is an $m \times p$ -matrix, B is an $m \times r$ -matrix. Note that a linear programming problem like (1) but containing

only inequality constraints can be given the form of (1) by adding the void inequalities $xB = d$ where B is the $m \times 1$ -matrix containing only zeroes and d is equal to zero. In a similar way a problem containing only equalities can be handled. An $x \in \mathbb{R}^m$ which satisfies the above conditions is called a *solution* of this problem. If there exists a solution the problem is called *feasible*, otherwise it is called *infeasible*. A solution $\hat{x} \in \mathbb{R}^m$ of (1) is an *optimal solution* if $c \cdot \hat{x} \geq c \cdot x$ for all solutions x . In this case the value $v_p(A, B, b, d, c)$ of (1) is equal to $c \cdot \hat{x}$. If the problem is infeasible then $v_p(A, B, b, d, c) = -\infty$, if the problem is feasible but *unbounded* then $v_p(A, B, b, d, c) = \infty$.

The *dual* problem of (1) is

$$\begin{aligned} & \text{minimize } y \cdot b + z \cdot d \text{ subject to} \\ & Ay + Bz \geq c \\ & y \geq 0 \end{aligned} \tag{2}$$

In an obvious way the definitions of the terms (optimal) solution, (in)feasible and value can be extended to problem (2). The value of (2) is denoted by $v_d(A, B, b, d, c)$. From the duality theorem of linear programming theory it follows that (2) is feasible and bounded if and only if (1) is feasible and bounded and then $v_p(A, B, b, d, c) = v_d(A, B, b, d, c)$.

A cooperative game can be constructed from (1) by making all or some of the right hand sides in the constraints depend on the coalitions. This can be done in several ways. In this section the following case is studied. For every $k \in \{1, \dots, p\}$ and for all $S \in 2^N \setminus \{\emptyset\}$ let $b_k(S) = \sum_{i \in S} b_k(i)$ with $b_k(i) \in \mathbb{R}$ given for all $i \in N$. The vector in $\mathbb{R}^p$ with k -th coordinate equal to $b_k(S)$ for all $k \in \{1, \dots, p\}$ is denoted by $b(S)$.

A similar condition is assumed to hold for $d(S)$ as well. So for every $k \in \{1, \dots, r\}$ and for all $S \in 2^N \setminus \{\emptyset\}$ let $d_k(S) = \sum_{i \in S} d_k(i)$ with $d_k(i) \in \mathbb{R}$ given for all $i \in N$. The vector in $\mathbb{R}^r$ with k -th coordinate equal to $d_k(S)$ for all $k \in \{1, \dots, r\}$ is denoted by $d(S)$.

3.2.1 Definition A cooperative game v is a linear programming game if there exist an $m \times p$ -matrix A , an $m \times r$ -matrix B and vectors $b(S) \in \mathbb{R}^p$ and $d(S) \in \mathbb{R}^r$ for all $S \in 2^N \setminus \{\emptyset\}$ such that $v(S) = v_p(A, B, b(S), d(S), c)$.

3.2.2 Remark Note that in order to obtain a game $v_p(A, B, b(S), d(S), c)$ should be a real number for all $S \in 2^N \setminus \{\emptyset\}$. Therefore, for all $S \in 2^N \setminus \{\emptyset\}$ the linear programming problem with the right hand sides in the constraints equal to $b(S)$ and $d(S)$ should be feasible and bounded. Whenever the equality constraints are absent a sufficient but not necessary condition for this to hold is that all entries in the matrix A are non-negative with in each row at least one positive entry and that $b(S)$ is non-negative.

3.2.3 Theorem *A linear programming game is totally balanced.*

Proof. Let v be a linear programming game with for all $S \in 2^N \setminus \{\emptyset\}$, $v(S) = v_p(A, B, b(S), d(S), c)$. Then for every $T \in 2^N \setminus \{\emptyset\}$ the subgame v_T of v is also a linear programming game with $v_T(S) = v_p(A, B, b(S), d(S), c)$ for all $\emptyset \neq S \subset T$. Consequently it is sufficient to prove that v is balanced. Let $(\hat{y}, \hat{z}) \in \mathbb{R}^p \times \mathbb{R}^r$ be an optimal solution for the dual problem of the problem which determines $v(N)$. Define $u \in \mathbb{R}^n$ by

$$u_i := \hat{y} \cdot b(i) + \hat{z} \cdot d(i).$$

Then

$$\begin{aligned} \sum_{i \in N} u_i &= \hat{y} \cdot \sum_{i \in N} b(i) + \hat{z} \cdot \sum_{i \in N} d(i) \\ &= \hat{y} \cdot b(N) + \hat{z} \cdot d(N) = v(N). \end{aligned}$$

Further,

$$\sum_{i \in S} u_i = \hat{y} \cdot b(S) + \hat{z} \cdot d(S) \geq v(S) \text{ for all } S \in 2^N \setminus \{\emptyset\}.$$

Here the inequality follows from the fact that $(\hat{y}, \hat{z})$ is a solution of the dual problem of the problem which determines $v(S)$ as well. Thus it follows that $u \in C(v)$ and v is totally balanced. $\square$

3.2.4 Theorem *Every totally balanced game is a linear programming game.*

Proof. Let v be a totally balanced game. For each $T \in 2^N \setminus \{\emptyset\}$ let z^T

be a vector in $\mathbb{R}^n$ with $z_i^T = M + 1$ for all $i \notin T$ and such that its projection on $\mathbb{R}^T$ is an element of $C(v_T)$. Here $M \geq v(S)$ for all $S \in 2^N$. Let $T_1, T_2, \dots, T_{2^n-1}$ be an ordering of the non-empty coalitions. Let A be the $n \times 2^n - 1$ -matrix with all entries equal to one. For each $i \in N$ define the vector $b(i) \in \mathbb{R}^{2^n-1}$ by $b_k(i) = z_i^{T_k}$. Then, for every $S \in 2^N \setminus \{\emptyset\}$, $v(S)$ is equal to the value of the linear programming problem

$$\begin{aligned} & \text{maximize } x \cdot c \text{ subject to} \\ & xA \leq b(S). \end{aligned} \tag{3}$$

Here c is the vector with all coordinates equal to one and $b(S) = \sum_{i \in S} b(i)$. Problem (3) is equivalent to the following problem.

$$\begin{aligned} & \text{maximize } x^1 \cdot c - x^2 \cdot c \text{ subject to} \\ & x^1 A - x^2 A \leq b(S) \\ & x^1 \geq 0 \\ & x^2 \geq 0 \end{aligned} \tag{4}$$

Let A' be the $2n \times 2^n - 1$ -matrix with j -th row equal to the j -th row of A for $1 \leq j \leq n$ and equal to minus the $(j-n)$ -th row of A for $n+1 \leq j \leq 2n$. Let $c' \in \mathbb{R}^{2^n-1}$ be the vector with the first n coordinates equal to 1 and the last n coordinates equal to -1. Then (4) is equivalent to

$$\begin{aligned} & \text{maximize } x \cdot c' \text{ subject to} \\ & xA' \leq b(S) \\ & x \leq 0. \end{aligned}$$

Thus it follows that v is a linear programming game with $v(S) = v_p(A', B, b(S), d(S), c)$ for all $S \in 2^N \setminus \{\emptyset\}$ where B is the $m \times 1$ -matrix with all entries equal to zero and $d(S) = 0$ for all $S \in 2^N \setminus \{\emptyset\}$. $\square$

3.2.5 Flow games are examples of linear programming games which were introduced by Kalai and Zemel (1982a). Let G be a directed graph with finite set of vertices P and finite set of arcs L . For every $p \in P$ let $B(p)$ denote the set of arcs which start in p and $B(p)$ the set of arcs which end in p .

Two functions c and o are given on L . The function c is called the *capacity function* and assigns to every $l \in L$ a non-negative number, the capacity $c(l)$ of l . The function o is called the *ownership function* and assigns to every $l \in L$ a player, the owner $o(l)$ of l . Two different vertices are distinguished from the others, a *source* s and a *sink* t . A *flow* from source to sink in such a graph is a function f from L to $\mathbb{R}$ with $0 \leq f(l) \leq c(l)$ for each $l \in L$ and such that for every $p \in P \setminus \{s, t\}$ the amount of flow which enters p is equal to the amount of flow which leaves p , formally, $\sum_{l \in B(p)} f(l) = \sum_{l \in E(p)} f(l)$. The *value* of such a flow is the net amount which enters the sink and is equal to the net amount which leaves the source. A flow with maximum value is called a maximum flow and this value is denoted by $w(G)$. So,

$$w(G) := \sum_{l \in B(s)} f(l) - \sum_{l \in E(s)} f(l) = \sum_{l \in E(t)} f(l) - \sum_{l \in B(t)} f(l)$$

where f is a maximum flow in G .

For each $S \in 2^N \setminus \{\emptyset\}$ a graph G_S can be obtained from G by keeping all vertices and removing all arcs which are not owned by a member of S . This new set of arcs is denoted by L_S . Note that $G_N = G$ and $L_N = L$.

3.2.6 Definition A game v is said to be a flow game if there exists a graph G with capacity function c and ownership function o such that $v(S) = w(G_S)$ for each $S \in 2^N \setminus \{\emptyset\}$.

3.2.7 Theorem Flow games are linear programming games.

Proof. Let v be a flow game with $v(S) = w(G_S)$ for all $S \in 2^N \setminus \{\emptyset\}$, where $G_N = G$ is a directed graph with capacity function c and ownership function o . Let $l_1, l_2, \dots, l_r$ be an ordering of L such that $\{l_1, l_2, \dots, l_g\} = B(s)$ and $\{l_{g+1}, \dots, l_h\} = E(s)$. For every $i \in N$ define the vector $b(i) \in \mathbb{R}^r$ by $b_k(i) = c(l_k)$ if $o(l_k) = i$, $b_k(i) = 0$ otherwise. Let $s = p_1, p_2, \dots, p_q = t$ be an ordering of the vertices. Define the $r \times q$ -matrix $H = [h_{km}]_{k=1}^{r} \substack{m=q \\ m=1}$ by $h_{k1} = h_{kq} = 0$ for each $k \in \{1, \dots, r\}$; for $m \neq 1$ and $m \neq q$, $h_{km} = 1$ if $l_k \in B(p_m)$, $h_{km} = -1$ if $l_k \in E(p_m)$, $h_{km} = 0$ otherwise. Let $d(i)$ be the zero vector in $\mathbb{R}^q$ for every $i \in N$ and let $c \in \mathbb{R}^r$ be the vector with the first g coordinates equal to 1, the following $h - g$ coordinates equal to -1 and the remainder of the coordinates equal to 0. Then $v(S) = v_p(I, B, b(S), d(S), c)$ for each $S \in 2^N \setminus \{\emptyset\}$. Thus, v is a linear programming game. $\square$

From the theorems 3.2.3 and 3.2.7 it follows that flow games are totally balanced. Kalai and Zemel (1982a) proved this by using the Ford-Fulkerson theorem which is a special case of the duality theorem of linear programming. In fact, Kalai and Zemel proved that every non-negative totally balanced game is a flow game.

3.2.8 A slightly more general model is that of *flow games with losses*. Especially when considering the transportation of perishable goods it seems plausible to incorporate losses into the model. Here the case is considered where the losses depend on the amount which is put through an arc and not on the arc. In other words, there exists a λ with $0 \leq \lambda \leq 1$ such that if a flow $f(l)$ is put through an arc l going from p_1 to p_2 only $\lambda f(l)$ will reach p_2 . A *graph with losses* is a graph as in 3.2.5 for which a $0 \leq \lambda \leq 1$ is given to define the losses in the way described above. Here a flow from source to sink is a function f from L to $\mathbb{R}$ with $0 \leq f(l) \leq c(l)$ for each $l \in L$ and such that for each $p \in P \setminus \{s, t\}$ the following holds

$$\sum_{l \in B(p)} f(l) = \lambda \sum_{l \in E(p)} f(l).$$

The value of such a flow is the net amount which reaches the sink. Note that, contrary to the case without losses this will not be equal to the net amount that leaves the source. The value of a maximum flow in this graph is denoted by $w(G, \lambda)$. So,

$$w(G, \lambda) = \lambda \sum_{l \in E(t)} f(l) - \sum_{l \in B(t)} f(l).$$

In a similar way as for ordinary graphs a game can be constructed from a graph with losses. Such a game is called a *flow game with losses*.

3.2.9 Definition A game v is said to be a *flow game with losses* if there exists a directed graph G with losses described by a $0 \leq \lambda \leq 1$ such that $v(S) = w(G_S, \lambda)$ for each $S \in 2^N \setminus \{\emptyset\}$.

3.2.10 Theorem *Flow games with losses are linear programming games.*

Proof. Let v be a flow game with losses arising from a graph G with losses described by λ , capacity function c and ownership function o . Let $l_1, \dots, l_r$ be an ordering of L such that $\{l_{h+1}, \dots, l_r\} = E(t)$ and $\{l_{g+1}, \dots, l_h\} = B(t)$. For every $i \in N$ define the vector $b(i) \in \mathbb{R}^r$ by $b_k(i) = c(l_k)$ if $o(l_k) = i$, $b_k(i) = 0$ otherwise. Let $s = p_1, p_2, \dots, p_q = t$ be an ordering of the vertices. Define the $r \times q$ -matrix $H = [h_{km}]_{k=1}^{r} \substack{m=1 \\ m=q}$ by $h_{k1} = h_{kq} = 0$ for each $k \in \{1, \dots, r\}$; for $m \neq 1$ and $m \neq q$, $h_{km} = 1$ if $l_k \in B(p_m)$, $h_{km} = -\lambda$ if $l_k \in E(p_m)$, $h_{km} = 0$ otherwise. Let $d(i)$ be the zero vector in $\mathbb{R}^q$ for every $i \in N$ and let $c \in \mathbb{R}^r$ be the vector with the last $r - h$ coordinates equal to 1, the $h - g$ coordinates before those equal to -1 and the remainder of the coordinates equal to 0. Then $v(S) = v_p(I, H, b(S), d(S), c)$ for each $S \in 2^N \setminus \{\emptyset\}$. Thus, v is a linear programming game. $\square$

From the theorems 3.2.10 and 3.2.3 it follows that flow games with losses are totally balanced games.

3.2.11 Owen (1975b) introduced *linear production games*. In these games each player has a certain amount of m different resources described by a vector $b(i)$ where $b_k(i)$ is the amount of the k -th resource that player i possesses. It is possible to produce r different products with the resources. Each of these r products can be sold at a given market price. The quantities needed of the resources to produce one unit of a certain product are given in a production matrix $A = [a_{jk}]_{j=1}^r \substack{k=1 \\ k=r}$. Here a_{jk} is the amount of the k -th resource needed for the production of one unit of the j -th product. Every player wants to maximize his profit by producing a bundle of products that will yield most when sold. The players may consider to work together and pool their resources in doing this. This will lead to a linear production game.

3.2.12 Definition A game v is a linear production game if there exists a production matrix A , a price vector c and a resource vector $b(i)$ for each $i \in N$ such that $v(S)$ is the value of the problem

$$\begin{aligned} & \text{maximize } c \cdot x \text{ subject to} \\ & xA \leq b(S) \\ & x \geq 0. \end{aligned} \tag{5}$$

It is obvious that a linear production game is a linear programming game. In the context of production and resources the coordinates of an optimal vector $\hat{y}$ for the dual problem of (5) for the coalition N can be interpreted as shadow prices of the corresponding resources. The core element constructed in the proof of theorem 3.2.3 can then be viewed as a way of paying to every player the amount that he would obtain, if instead of producing anything he would sell his resources by charging the shadow prices.

Owen (1975b) showed that not all core elements can be obtained from the solutions of the dual problem. However, if the players are replicated, then in the limit the core will converge to the set of payoffs generated by the optimal solutions of the dual problem. In a special case it is shown by Owen that this will happen after finitely many replications.

Samet and Zemel (1984) gave a necessary and sufficient condition for finite convergence of the core to the set of payoffs corresponding to optimal solutions of the dual problem. They also gave a necessary and sufficient condition for the equality of these sets even without replication.

Dubey and Shapley (1984) considered totally balanced games arising from, not necessarily linear, programming problems with constraints controlled by the players.

3.3 Linear programming games with committee control

The concept of committee control in a linear programming game can be illustrated best with two examples.

Suppose that the resources in the production situation are available in portions. For $k \in \{1, \dots, m\}$ there are g_k portions of resource k . Such a portion is denoted by B_k^q . The total amount of resource k which is available is $B_k = \sum_{q=1}^{g_k} B_k^q$. Each portion is controlled by committees of players. These controls are modeled with the aid of simple games. For every B_k^q a simple game w_k^q describes the control. Coalition S can use B_k^q only if $w_k^q(S) = 1$. Let $B_k^q(S) := B_k^q w_k^q(S)$. The total amount of the k -th resource available to coalition S is $B_k(S) := \sum_{q=1}^{g_k} B_k^q(S)$. The resource vector $B(S)$ of coalition S has coordinates $B_1(S), B_2(S), \dots, B_m(S)$. Then $v(S)$ is the value of problem (5) with $b(S)$ replaced by $B(S)$.

Another example of a *linear programming game with committee control* is a flow game where the arcs are not owned by the players, but are controlled by

committees of players. So instead of an ownership function there is a control function which assigns to every arc l a simple game w_l which describes the control of l . Coalition S may use l only if $w_l(S) = 1$. From a graph $G = G_N$ with the control of an arc l described by a simple game w_l a network G_S can be obtained for all $S \in 2^N \setminus \{\emptyset\}$, by removing from G all arcs l with $w_l(S) = 0$. The worth of S is then again equal to $w(G_S)$.

In general it is possible to consider linear programming games with committee control.

3.3.1 Definition A game v is a linear programming game with committee control if there exist an $m \times p$ -matrix A , an $m \times r$ -matrix H , a vector $c \in \mathbb{R}^m$ and for each $k \in \{1, \dots, p\}$ and each $j \in \{1, \dots, r\}$ there exist numbers $g_k \in \mathbb{N}$ and $h_j \in \mathbb{N}$ such that there are g_k real non-negative numbers $b_k^1, b_k^2, \dots, b_k^{g_k}$ and g_k simple games $w_k^1, \dots, w_k^{g_k}$ and h_j real non-negative numbers $d_j^1, d_j^2, \dots, d_j^{h_j}$ and h_j simple games $u_j^1, \dots, u_j^{h_j}$ with $b_k(S) = \sum_{q=1}^{g_k} b_k^q w_k^q(S)$ and $d_j(S) = \sum_{q=1}^{h_j} d_j^q u_j^q(S)$ such that $v(S) = v_p(A, H, b(S), d(S), c)$ for each $S \in 2^N \setminus \{\emptyset\}$.

Linear programming games are in fact special cases of linear programming games with committee control. A linear programming game v with $v(S) = v_p(A, H, b(S), d(S), c)$ for each $S \in 2^N \setminus \{\emptyset\}$ where A is an $m \times p$ -matrix, H is an $m \times r$ -matrix, $b(S) = \sum_{i \in S} b(i)$ with $b(i) \in \mathbb{R}^p$ given for each $i \in N$ and $d(S) = \sum_{i \in S} d(i)$ with $d(i) \in \mathbb{R}^r$ given for each $i \in N$ can be seen as a linear programming game with committee control with $g_k = n$ for each $k \in \{1, \dots, p\}$, $h_j = n$ for each $j \in \{1, \dots, r\}$, $b_k^i = b_k(i)$ and w_k^i is the simple game with player i as dictator for each $k \in \{1, \dots, p\}$ and each $i \in \{1, \dots, g_k = n\}$; $d_j^i = d_j(i)$ and u_j^i is the simple game with player i as dictator for each $j \in \{1, \dots, r\}$ and $i \in \{1, \dots, h_j = n\}$.

A linear programming game with committee control need not be balanced as the following example from Curiel, Derks and Tijs (1987) shows.

3.3.2 Example Let $N = \{1, 2, 3\}$ and let G be the graph with only a source s and a sink t as vertices and a single arc l with capacity 10. the committee control is described by the simple game w_l with $w_l(S) = 1$ for all S with $|S| \geq 2$ and $w_l(S) = 0$ otherwise. Then in the corresponding flow

game with committee control v , $v(S) = 10$ for all S with $|S| \geq 2$, $v(S) = 0$ otherwise and it follows that $C(v) = \emptyset$.

However, if all the simple games which describe the controls in a linear programming game are balanced then the linear programming game is also balanced.

3.3.3 Theorem *Linear programming games with committee control for which all the simple games that describe the controls are balanced, have a non-empty core.*

Proof. Let v be a linear programming game with committee control described by balanced simple games. Suppose that $v(S) = v_p(A, H, b(S), d(S), c)$ for each $S \in 2^N \setminus \{\emptyset\}$ with the k -th coordinate of $b(S)$ equal to $\sum_{q=1}^{g_k} b_k^q w_k^q(S)$ and the j -th coordinate of $d(S)$ equal to $\sum_{q=1}^{h_j} d_j^q u_j^q(S)$. Let $z_k^q \in C(w_k^q)$ and $y_j^q \in C(u_j^q)$. Let $(\hat{y}, \hat{z})$ be an optimal solution of the dual problem of the problem that determines $v(N)$. Define the vector $u \in \mathbb{R}^n$ by

$$u_i := \sum_{k=1}^p \hat{y}_k \sum_{q=1}^{g_k} b_k^q (z_k^q)_i + \sum_{j=1}^r \hat{z}_j \sum_{q=1}^{h_j} d_j^q (y_j^q)_i.$$

Then

$$\begin{aligned} \sum_{i \in N} u_i &= \sum_{k=1}^p \hat{y}_k \sum_{q=1}^{g_k} b_k^q + \sum_{j=1}^r \hat{z}_j \sum_{q=1}^{h_j} d_j^q \\ &= \sum_{k=1}^p \hat{y}_k b_k^q(N) + \sum_{j=1}^r \hat{z}_j d_j^q(N) \\ &= \hat{y}b(N) + \hat{z}d(N) = v(N). \end{aligned}$$

Further,

$$\begin{aligned} \sum_{i \in S} u_i &= \sum_{k=1}^p \hat{y}_k \sum_{q=1}^{g_k} b_k^q z_k^q(S) + \sum_{j=1}^r \hat{z}_j \sum_{q=1}^{h_j} d_j^q y_j^q(S) \\ &\geq \sum_{k=1}^p \hat{y}_k \sum_{q=1}^{g_k} b_k^q w_k^q(S) + \sum_{j=1}^r \hat{z}_j \sum_{q=1}^{h_j} d_j^q u_j^q(S) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^p \hat{y}_k \sum_{q=1}^{g_k} b_k^q(S) + \sum_{j=1}^r \hat{z}_j \sum_{q=1}^{h_j} d_j^q u_j^q(S) \\
&= \hat{y}b(S) + \hat{z}d(S) \geq v(S).
\end{aligned}$$

Here the last inequality follows from the fact that $(\hat{y}, \hat{z})$ is a solution of the dual problem of the problem that determines $v(S)$ as well. Thus, it follows that $u \in C(v)$. $\square$

3.3.4 Theorem *Linear programming games with committee control for which all the simple games that describe the controls are monotonic and non-balanced have an empty core.*

Proof. Let v be a linear programming game with committee control described by simple games w_k^q and u_k^q . Because $w_k^q(u_k^q)$ is not balanced it follows that $w_k^q(u_k^q)$ has no veto players, cf. theorem 2.2.6. From this and the monotonicity of $w_k^q(u_k^q)$ it follows that $w_k^q(N \setminus \{i\}) = w_k^q(N)$ ($u_k^q(N \setminus \{i\}) = u_k^q(N)$) for each $i \in N$. Therefore, $b(N \setminus \{i\}) = b(N)$ and $d(N \setminus \{i\}) = d(N)$ for each $i \in N$ which yields $v(N) = v(N \setminus \{i\})$ for each $i \in N$. Consider the balanced collection $\{N \setminus \{i\}\}_{i \in N}$ with all weights equal to $1/(n-1)$. Then

$$\frac{1}{n-1} \sum_{i \in N} v(N \setminus \{i\}) = \frac{n}{n-1} v(N) \geq v(N).$$

So this balancedness condition is not satisfied by v and v is not balanced. $\square$

3.3.5 Theorem *Every non-negative balanced game is a linear programming game with committee control.*

Proof. Let v be a balanced game. From theorem 2.2.7 it follows that there exist $\alpha_1, \dots, \alpha_p \geq 0$ and balanced simple games $w_1, \dots, w_p$ such that $v = \sum_{j=1}^p \alpha_j w_j$. Thus $v(S) = v_p(A, H, b(S), d(S), c)$ for each $S \in 2^N \setminus \{\emptyset\}$ where A is the 1×1 -zero matrix, H is the 1×1 -identity matrix, $b(S) = 0$, $d(S) = \sum_{j=1}^p \alpha_j w_j(S)$ and $c = 1$. Thus, v is a linear programming game with committee control. $\square$

Curiel, Derks and Tijs (1987) considered flow games and linear production games with committee control and proved that every non-negative balanced game is a flow game with committee control. Dubey and Shapley (1984) also studied committee control in their model of games arising from controlled programming problems.

3.4 Non-balanced linear programming games

As was already noted in section 3.3 a linear programming game with committee control need not be balanced. Still, a way to divide $v(N)$ must be found. In this section two escapes which are treated in Curiel et al. (1988a) are considered. First the *least tax core* introduced by Tijs and Driessen (1986a) will be discussed. Let v be a game such that $v(N) \geq \sum_{i \in N} v(i)$; then each player can be guaranteed at least an individual rational payoff. If this is not the case the players will be better off by not forming the grand coalition. The relative profit derived from the formation of a coalition S is $v(S) - \sum_{i \in S} v(i)$, that is, the difference between the worth of S and the sum of the worths of the members of S when they work alone. Now a tax relative to this profit, is imposed on any coalition $S \neq N$ which forms. So there is an ε with $0 \leq \varepsilon \leq 1$ such that for each $S \neq N$ the tax is equal to $\varepsilon(v(S) - \sum_{i \in S} v(i))$. Thus a new game, the *multiplicative ε -tax game* arises.

3.4.1 Definition *The multiplicative ε -tax game v^ε of a game v is defined by*

$$v^\varepsilon := \begin{cases} v(S) & \text{if } S = N, \emptyset \\ v(S) - \varepsilon(v(S) - \sum_{i \in S} v(i)) & \text{if } S \neq N, \emptyset. \end{cases}$$

Now the smallest ε for which $C(v^\varepsilon) \neq \emptyset$ is considered. If $C(v) \neq \emptyset$ then this ε is equal to 0. For $\varepsilon = 1$ the multiplicative ε -tax game is given by $v^1(S) = \sum_{i \in S} v(i)$ for $S \neq N, \emptyset$ and $C(v) \neq \emptyset$. For every game v with $\sum_{i \in N} v(i) \leq v(N)$ define

$$\varepsilon(v) := \min\{\varepsilon \mid 0 \leq \varepsilon \leq 1, C(v^\varepsilon) \neq \emptyset\}.$$

Then $\varepsilon(v)$ is well defined since the balanced games form a convex polyhedral cone and $C(v^1) \neq \emptyset$.

3.4.2 Definition The least tax core $LTC(v)$ of a game v is the core of the game $v^{\epsilon(v)}$.

If $C(v) \neq \emptyset$ then $\epsilon(v) = 0$, $v^{\epsilon(v)} = v$ and $LTC(v) = C(v)$. So the least tax core can be seen as a generalization of the core.

Let w be a simple game with an empty core and with $w(N) \geq \sum_{i \in N} w(i)$. Two cases can be distinguished.

1. There is an $i^* \in N$ such that $w(i^*) = 1$.
2. $w(i) = 0$ for all $i \in N$.

In the first case there is an $S \subset N \setminus \{i^*\}$ with $w(S) = 1$ because i^* is not a veto player. For $\epsilon < 1$ $w^{\epsilon}(S) = 1 - \epsilon > 0$ and $w^{\epsilon}(i^*) = 1$ and it is evident that $C(v^{\epsilon}) = \emptyset$. Hence $\epsilon(w) = 1$ and

$$w^{\epsilon(w)}(S) = \begin{cases} 1 & \text{if } i^* \in S \\ 0 & \text{if } i^* \notin S. \end{cases}$$

In the second case

$$w^{\epsilon(w)}(S) = \begin{cases} 0 & \text{if } w(S) = 0 \\ 1 - \epsilon(w) & \text{if } w(S) = 1, S \neq N. \end{cases}$$

In case 1 $w^{\epsilon(w)}$ is still a simple game and i^* has become a unique veto player. In case 2 $w^{\epsilon(w)}$ is no longer a simple game and a new interpretation of committee control as described by $w^{\epsilon(w)}$ has to be found. Let w be the simple game which describes the control of a constraint b . Then $w^{\epsilon(w)}$ can be seen as describing the situation where only part of b is controlled by a coalition $S \neq N$ with $w(S) = 1$, namely the part given by $(1 - \epsilon)b$.

Let v be a linear programming game with committee control described by simple games w_k^q and u_k^q . Then the *least tax programming game* corresponding to v is the linear programming game with control described by the least tax game constructed from the simple games w_k^q and u_k^q .

3.4.3 Theorem Let v be a linear programming game with committee control. Then the corresponding least tax linear programming game is balanced.

Proof. Similar to the proof of theorem 3.3.3 by taking elements of the least

tax core of the simple games which describe the controls. $\square$

Another way to divide $v(N)$ in the case of non-balancedness of the game can be constructed by means of normalized power indices. A normalized power index ψ can be used to compute the power of the players in the controlling committees. If $\psi_i(w_k^q)$ is a measure of the power of player i in the controlling game w_k^q which describes the the control of b_k^q , then this can be regarded as if player i controls $\psi_i(w_k^q)b_k^q$. Let $(\hat{y}, \hat{z})$ be an optimal solution for the dual of the problem which determines $v(N)$. Then $v(N)$ can be divided by giving to player i the amount $\sum_{k=1}^p \hat{y}_k \sum_{q=1}^{g_k} \psi_i(w_k^q) + \sum_{j=1}^r \hat{z}_j \sum_{q=1}^{h_j} \psi_i(u_k^q)$. An advantage of these two methods is that just as in the balanced case it is not necessary to compute all the $v(S)$, which would involve solving $2^n - 1$ linear programming problems, but it is sufficient to solve only one linear programming problem and to analyze the simple games.

3.5 A linear production game with claims

Consider the following variation on linear production games. Instead of the resources being controlled by players or coalitions of players there is a fixed amount of each resource. Each player *claims* a certain amount of the resources. The amount claimed by a player depends on the production matrix and on his *production capacity*. Let A be the non-negative $r \times m$ -production matrix and $c \in \mathbb{R}^r$ the non-negative price vector. For each player $i \in N$ the production capacity of i is equal to $k(i) \in \mathbb{R}_+^m$. This means that i can produce at most $k_j(i)$ of the j -th product. Let $B = (B_1, B_2, \dots, B_m)$ be the total available amount of each resource. Then player i can be considered to claim a part of the resources which he will need to maximize his profit while keeping his limited capacity in mind. For each player i the following linear programming problem determines his claim.

$$\begin{aligned} & \text{maximize } c \cdot x \text{ subject to} \\ & \quad xA \leq B \\ & \quad 0 \leq x \leq k(i) \end{aligned} \tag{6}$$

Let x^i be an optimal solution of this problem, then player i claims $b(i) = x^i A$ of the resources. If there are more optimal solutions of (6) it is assumed that player i claims of each resource the maximal amount given by such a

solution.

3.5.1 Definiton *The claim vector $b(i)$ of player i is the vector with k -th coordinate equal to $\max\{(x^i A)_k | x^i \in O^i\}$ where O^i is the set of optimal solutions of (6).*

The production capacity of a coalition S is taken to be the sum of the production capacities of its members. Hence, $k(S) = \sum_{i \in S} k(i)$. The claim vector $b(S)$ of coalition S is the sum of the claim vectors of its members. In determining the worth of a coalition S in a linear production game a somewhat pessimistic view is taken. Any coalition considers the maximal profile it can achieve when its complement has taken all the resources that the complement claims.

3.5.2 Definition *In a linear production game v with claims $v(S)$ is for each $S \in 2^N \setminus \{\emptyset\}$ equal to the value of the linear programming problem*

$$\begin{aligned} & \text{maximize } c \cdot x \text{ subject to} \\ & xA \leq (B - b(N \setminus S))_+ \\ & 0 \leq x \leq k(S). \end{aligned} \tag{7}$$

3.5.3 Definititon *For each resource j let w_j be the claim game of j defined by*

$$w_j(S) := (B_j - b_j(N \setminus S))_+ \text{ for each } S \in 2^N \setminus \{\emptyset\}.$$

3.5.4 Theorem *For each $j \in \{1, \dots, m\}$ the claim game w_j is convex.*

Proof. Similar to the proof of theorem 5.3.2

□

3.5.5 Theorem *Linear production games with claims are balanced.*

Proof. Let v be a linear production game with claims. Then all the claim

games are convex and therefore balanced. Let $q \in C(w_j)$ for each claim game w_j . Consider the following problem.

$$\begin{aligned} & \text{minimize } By + k(N) \cdot z \text{ subject to} \\ & Ay + z \geq c \\ & y \geq 0 \\ & z \geq 0 \end{aligned}$$

This is the dual problem of (7) for $S = N$. Let $(\hat{y}, \hat{z})$ be an optimal solution of this problem. Define $u \in \mathbb{R}^n$ by

$$u_i := \sum_{j=1}^m q_i^j \hat{y}_j + \sum_{l=1}^r k_l(i) \hat{z}_l.$$

Then $u \in C(v)$ and v is balanced. □

In chapter 5 the claim games will be studied more extensively.

COMBINATORIAL GAMES

4.1 Introduction

While the characteristic functions of the games in chapter 3 were determined by linear programming problems, the characteristic functions of the games in this chapter will mostly be determined by combinatorial optimization problems. A cooperative game v with $v(S)$ equal to the value of a combinatorial optimization problem for all $S \in 2^N$ is called a combinatorial game. The three following sections are dedicated to two kinds of combinatorial situations, namely, matching and permutation situations. In section 4.2 they are introduced and some examples are discussed. Section 4.3 deals with combinatorial games arising from matching and permutation situations. In section 4.4 economies with indivisibilities arising from permutation situations are studied and section 4.5 handles ordinal matching situations. In section 4.6 combinatorial games arising from sequencing situations will be studied. This section is based on Curiel, Pederzoli and Tijs (1988b). These, so called, sequencing games are convex and hence balanced. A division rule for sequencing situations is defined without game theoretic considerations. Yet, it is proved that this rule always yields a division which is in the core of the corresponding sequencing game. Simple expressions for the Shapley-value and the τ -value of a sequencing game are derived. In section 4.7 which is based on Potters, Curiel and Tijs (1987) traveling salesman games are treated. These are combinatorial games arising from the well known traveling salesman problem. Traveling salesman games are not balanced in general and some subclasses containing only balanced games are discussed. A variation on traveling salesman games, namely routing games are defined. In fact, these games are not combinatorial games since only $v(N)$ is determined by a combinatorial optimization problem. It is proved that routing games are balanced. Section 4.8 discusses minimum cost spanning tree games. Both these games as well as the location games dealt with in section 4.9 are balanced. All the games in section 4.7, 4.8, 4.9 are cost games and the definitions given in section 1.7 apply to them.

4.2 Matching and permutation situations

Consider the following situation. In a group consisting of n women and n men each man has to be married to a woman. Each individual has a certain

preference over the members of the other sex. What criteria should one keep in mind when making an assignment of the men to the women and is it always possible to satisfy these criteria? The above *marriage problem* is a very well known example of a *matching situation*. Matching situations have been studied extensively in the literature. Cf. Gale and Shapley (1962), Dubins and Freedman (1981), Crawford and Knoer (1981), Roth (1982) and (1984) and Gale and Sotomayor (1985). In a matching situation there is a group of people or institutions which have to be matched one to one or one to many, where many can be two or more. If the group can be partitioned into two sets such that any element of one set can only be matched with a members or members of the other set, the matching is called bipartite. In the following mostly bipartite matchings will be considered. Other examples of bipartite matching situations are the *house market* where each seller owns a house and each buyer wants to purchase a house and where the sets of sellers and buyers are disjoint, and the *college admission problem* where students have to be assigned to colleges. The preferences of the individuals can be given just by ranking lists, saying that they prefer this partner to that partner, that one to another one, and so forth or by real numbers assigned to each possible partner to denote his or her or its worth. In the former case the preferences are called *ordinal*, in the latter they are called *cardinal*. Cooperative games arising from matching situations are studied in Shapley and Shubik (1972).

An example of a quite different situation is the following. A group $N = \{1, 2, \dots, n\}$ of customers is standing in a queue waiting to be served. They all have the same service time. Each one of them has some cost which depends on when he is finished. Since all the service times are the same, the cost for any customer can be seen as depending only on his position in the queue. The position of all the customers can be described with the aid of a permutation of N . So the cost of a customer can be seen as depending on a permutation. This situation is an example of a *permutation situation*. In a permutation situation each member of a group has a preference on the set of permutations of the group. Just as in matching situations these preferences can be either ordinal or cardinal. Another example of a permutation situation is the *machine-job permutation situation* where each producer owns one machine and has one job to be processed. Any machine can process any job, but no machine can process more than one job. Each producer has a preference over the machines. A permutation of the group of producers corresponds to a redistribution of the machines among the producers. Cooperative games arising from permutation situations are studied in Shapley

and Scarf (1974), Tijs et al. (1984) and Curiel and Tijs (1986).

In the two following sections matching and permutation situations with cardinal preferences will be discussed. In section 4.3 which is partly based on Curiel and Tijs (1986) cooperative games arising from matching and permutation situations will be discussed and in section 4.4 economies with indivisibilities arising from permutation situations will be considered. Section 4.5 will conclude the sections on matching and permutation situations by treating ordinal matching situations.

4.3 Assignment games and permutation games

Consider the following bipartite matching situations. Let B and M be two disjoint sets. For each $i \in B$ the value of being matched to $j \in M$ is g_{ij} and for each $j \in M$ the value of being matched to $i \in B$ is h_{ij} . The value of a matched pair (i, j) is equal to $a_{ij} := g_{ij} + h_{ij}$. From this situation the following cooperative game can be constructed. For each coalition $S \subset B \cup M$ the worth $v(S)$ of S is defined to be the maximum that S can achieve by making suitable pairs from its members. If $S \subset B$ or $S \subset M$ then no suitable pairs can be made and therefore $v(S) = 0$. If $S \not\subset B$ and $S \not\subset M$ but for all $i \in S \cap B$ and all $j \in S \cap M$ the pair (i, j) has value $a_{ij} < 0$ then the maximum for S is achieved by making no pairs and is equal to 0. Formally, $v(S)$ is equal to the value of the following integer programming problem.

$$\begin{array}{ll} \max \sum_{i \in B} \sum_{j \in M} a_{ij} x_{ij} & \text{subject to} \\ \sum_{j \in M} x_{ij} \leq 1_S(i) & \text{for all } i \in B \\ \sum_{i \in B} x_{ij} \leq 1_S(j) & \text{for all } j \in M \\ x_{ij} \in \{0, 1\} & \text{for all } i \in B, j \in M. \end{array} \quad (1)$$

4.3.1 Definition *A cooperative game v is an assignment game if the set of players can be partitioned into two sets B and M such that for each $i \in B$ and each $j \in M$ there exist real numbers g_{ij} and h_{ij} with $a_{ij} = g_{ij} + h_{ij}$ such that for each coalition S the worth $v(S)$ of S is equal to the value of problem (1).*

Some examples of matching situations which lead to assignment games are the following. Suppose M is a set of merchants who possesses one indivisible good, e.g. a house, each and B is a set of buyers who wants to buy exactly

one house each. Then g_{ij} can be taken to be the worth of the house of merchant $j \in M$ to buyer $i \in B$. Let c_j be the worth to merchant j of his own house. If he is matched with an $i \in B$ then this can be considered as losing his house to i and the value of this situation to him is $-c_j$. Hence h_{ij} can be taken to be $-c_j$ for each $j \in M$ and each $i \in B$ and then $a_{ij} := g_{ij} - c_j$ is the value of the pair (i, j) . Since $a_{ij} > 0$ iff buyer i values the house of merchant j more than j himself, it is clear that it is only profitable to match a merchant and a buyer if the former possesses something which is valued more by the latter than by himself. This model is a slight variation of the model of Shapley and Shubik (1972).

Another well known example is obtained by taking M to be a set of men, B a set of women and by letting the numbers g_{ij} and h_{ij} express the satisfaction of woman i in being married to man j and the satisfaction of man j in being married to woman i respectively. In this way a cardinal variation of the marriage problem discussed by Gale and Shapley (1962) is obtained. Let M be a set of firms and B a set of workers. Each firm wants to hire exactly one worker and each worker wants to work for exactly one firm. Let $h_{ij} \in \mathbb{R}$ be the profit of firm j if it hires worker i and let $w_{ij} \in \mathbb{R}$ be the measure of worker i of how hard he has to work if he is hired by firm j . Then the value of a match with firm j is $g_{ij} = -w_{ij}$ to worker i and the value of a pair $a_{ij} = h_{ij} + g_{ij} = h_{ij} - w_{ij}$. It is again obvious that it is only worthwhile to match a firm with a worker if the value of his work for the firm is higher than the cost of his work to himself. A more general example of such a *job matching* situation can be found in Crawford and Knoer (1981).

Consider the following permutation situation. A set $N = \{1, 2, \dots, n\}$ is given. Each $i \in N$ places the value $k_{i\pi(i)}$ on a permutation π of N . Each subgroup $S \subset N$ has the power to originate a permutation π such that only the members of S are permuted, that is, $\pi(i) = i$ for all $i \in N \setminus S$. In the cooperative game corresponding to this situation the worth $v(S)$ of a coalition S is defined to be the maximum of the sum of the values of all the members of S taken over all the permutations that S can achieve. Let Π_S be the set of all permutations π with $\pi(i) = i$ for all $i \in N \setminus S$. Then

$$v(S) = \max_{\pi \in \Pi_S} \sum_{i \in S} k_{i\pi(i)}. \quad (2)$$

Another way of denoting (2) is with the aid of permutation matrices. For each permutation $\pi \in \Pi_S$ a corresponding permutation matrix $P =$

$[p_{ij}]_{i=1}^{i=n} j=1^n$ can be defined as follows.

$$p_{ij} := \begin{cases} 1 & \text{if } i \in S, j \in S \text{ and } \pi(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

Let $P(S)$ be the set of S -admissible permutation matrices, i.e. permutation matrices which correspond to a $\pi \in \Pi_S$. Let K be the $n \times n$ -matrix $[k_{ij}]_{i=1}^{i=n} j=1^n$. Then (2) is equivalent to

$$v(S) = \max_{P \in P(S)} K * P \quad (3)$$

where $K * P := \sum_{i=1}^n \sum_{j=1}^n k_{ij} p_{ij}$.

4.3.2 Definition A cooperative game v is a permutation game if there exists a matrix K such that $v(S)$ is given by (3) for all $S \in 2^N \setminus \{\emptyset\}$.

4.3.3 Remark A third way to describe a permutation game v is by means of an integer programming problem. Let k_{ij} be given for all $i, j \in N$. Then $v(S)$ is equal to the value of the following integer programming problem.

$$\begin{aligned} \max \quad & \sum_{i \in N} \sum_{j \in N} k_{ij} x_{ij} \quad \text{subject to} \\ & \sum_{j \in N} x_{ij} = 1_{S(i)} \quad \text{for all } i \in N \\ & \sum_{i \in N} x_{ij} = 1_{S(j)} \quad \text{for all } j \in N \\ & x_{ij} \in \{0, 1\} \quad \text{for all } i, j \in N. \end{aligned} \quad (4)$$

An example of a permutation game is obtained by looking at the permutation situation where customers are waiting in a queue. For $i \in N$ let c_{ij} denote the cost of i if he is standing in the j -th position. Originally, i is standing in the i -th position. Define the matrix $K = [k_{ij}]_{i=1}^{i=n} j=1^n$ by $k_{ij} = c_{ii} - c_{ij}$. Then the permutation game corresponding to this situation is given by (3). Another example is obtained by looking at the machine-job permutation situation. Suppose that the cost for $i \in N$ is equal to c_{ij} if his job is processed on machine j owned by $j \in N$. The value k_{ij} of machine j for $i \in N$ is defined by $k_{ij} = c_{ii} - c_{ij}$. The characteristic function of the permutation game corresponding to this situation is given by (3).

4.3.4 Theorem Every assignment game is a permutation game.

Proof. Let v be an assignment game with a partition of the set of players into two sets B and M and with a_{ij} given for all $i \in B, j \in M$ such that $v(S)$ is defined by (1) for each $S \in 2^N$. Define for every $i, j \in B \cup M$

$$k_{ij} := \begin{cases} a_{ij} & \text{if } i \in B, j \in M \\ 0 & \text{otherwise.} \end{cases}$$

Let w be the permutation game defined by (4) with k_{ij} as given above and $N = B \cup M$. Note that the number of variables in the integer programming problem which defines $v(S)$ is $|B| \times |M|$ while the number of variables in the integer programming problem which defines $w(S)$ is $(|B| \times |M|)^2$. For $S \subset B$ or $S \subset M$, $w(S) = 0 = v(S)$. Let $S \subset M \cup B$ with $S \not\subset B$ and $S \not\subset M$. Let $x \in \{0, 1\}^{|B| \times |M|}$ be an optimal solution of the problem which determines $v(S)$. Define $\bar{x} \in \{0, 1\}^{(|B| \times |M|)^2}$ by

$$\begin{aligned} \bar{x}_{ij} &:= x_{ij} && \text{if } i \in B, j \in M \\ \bar{x}_{ji} &:= x_{ji} && \text{if } i \in M, j \in B \\ \bar{x}_{ii} &:= 1_S(i) - \sum_{j \in M} x_{ij} && \text{if } i \in B \\ \bar{x}_{jj} &:= 1_S(j) - \sum_{i \in B} x_{ij} && \text{if } j \in M \\ \bar{x}_{ij} &:= 0 && \text{in all other cases.} \end{aligned}$$

Then $\bar{x}$ is a, not necessarily optimal, solution of the problem which determines $w(S)$. Hence,

$$w(S) \geq \sum_{i \in B \cup M} \sum_{j \in B \cup M} k_{ij} \bar{x}_{ij} = \sum_{i \in B} \sum_{j \in M} a_{ij} x_{ij} = v(S).$$

On the other hand let $z \in \{0, 1\}^{(|B| \times |M|)^2}$ be an optimal solution of the problem which determines $w(S)$. Let $\bar{z} \in \{0, 1\}^{|B| \times |M|}$ be defined by

$$\bar{z}_{ij} := z_{ij} \text{ for } i \in B, j \in M.$$

Then $\bar{z}$ is a solution of the problem which determines $v(S)$. Hence,

$$v(S) \geq \sum_{i \in B} \sum_{j \in M} a_{ij} \bar{z}_{ij} = \sum_{i \in B} \sum_{j \in M} k_{ij} z_{ij} = w(S).$$

It follows that $v = w$ and therefore v is a permutation game. $\square$

4.3.5 Theorem Assignment games and permutation games are totally balanced.

Proof. Because of theorem 4.3.4 it is sufficient to prove that permutation games are totally balanced. Since any subgame of a permutation game is again a permutation game it is sufficient to prove that permutation games are balanced. Let v be a permutation game with matrix $K = [k_{ij}]_{i=1}^n_{j=1}^n$. Then $v(S)$ is given by (4). From the theorem of Birkhoff-von Neumann which says that the extreme points of the set of doubly stochastic matrices are exactly the permutation matrices it follows that (4) is equivalent to

$$\begin{aligned} \max \quad & \sum_{i \in N} \sum_{j \in N} k_{ij} x_{ij} \quad \text{subject to} \\ & \sum_{j \in N} x_{ij} = 1_{S(i)} \quad \text{for all } i \in N \\ & \sum_{i \in N} x_{ij} = 1_{S(j)} \quad \text{for all } j \in N \\ & x_{ij} \geq 0 \quad \text{for all } i, j \in N. \end{aligned} \quad (5)$$

The dual problem of (5) is

$$\begin{aligned} \min \quad & \sum_{i \in N} 1_{S(i)} u_i + \sum_{j \in N} 1_{S(j)} v_j \quad \text{subject to} \\ & u_i + v_j \geq k_{ij} \quad \text{for all } i, j \in N. \end{aligned} \quad (6)$$

Let $(\hat{u}, \hat{v})$ be an optimal solution of (6) for $S = N$. Then

$$\sum_{i \in N} (\hat{u}_i + \hat{v}_i) = v(N)$$

and for all $S \in 2^N$

$$\sum_{i \in S} (\hat{u}_i + \hat{v}_i) = \sum_{i \in N} 1_{S(i)} \hat{u}_i + \sum_{i \in N} 1_{S(i)} \hat{v}_i \geq v(S)$$

where the inequality follows from the fact that $(\hat{u}, \hat{v})$ is a solution of problem (6) for all $S \in 2^N$. Thus the vector $z \in \mathbb{R}^n$ defined by

$$z_i := \hat{u}_i + \hat{v}_i$$

is an element of the core of v and v is balanced. $\square$

Generalizations of the assignment game and the permutation game are the *multi-assignment game* and the *multi-permutation game*, respectively. An example of a *multi-assignment situation* which yields a multi-assignment

game is the sellers-buyers market where each seller possesses one item of different types of indivisible goods, e.g. one house, one car, one television, one compact disc player, which he wants to sell and where each buyer wants to buy exactly one item of each type of good. A buyer need not buy all the goods from the same seller. Suppose there are p different type of goods. Let B be the set of buyers and M the set of sellers. Let $(j_1, j_2, \dots, j_p) \in M^p$ denote a bundle consisting of the good of type 1 of seller j_1 , the good of type 2 of seller j_2 and so forth. For each $i \in B$ let $g_i(j_1, j_2, \dots, j_p)$ denote the worth of a bundle $(j_1, j_2, \dots, j_p)$ to buyer i . For each $j \in M$ and each $q \in \{1, \dots, p\}$ let $h_j(q, i)$ denote the worth for seller j of selling his good of type q to buyer i . This situation, where for each of the p types of goods an assignment of buyers to sellers must be made is called a p -assignment situation. A p -assignment game v corresponding to a p -assignment situation can be defined by taking for each $S \in 2^N$ the worth $v(S)$ of S equal to the maximum that S can achieve by making suitable matchings among the sellers and buyers belonging to S . Let $a(i, j_1, \dots, j_p) := g_i(j_1, \dots, j_p) + h_{j_1}(1, i) + h_{j_2}(2, i) + \dots + h_{j_p}(p, i)$ denote the value of assigning the bundle $(j_1, \dots, j_p)$ to buyer i . Then $v(S)$ is the value of the problem

$$\begin{array}{ll}
 \max \sum_{(i, j_1, \dots, j_p) \in B \times M^p} a(i, j_1, \dots, j_p) x(i, j_1, \dots, j_p) & \text{subject to} \\
 \sum_{(i, j_2, \dots, j_p) \in B \times M^{p-1}} x(i, j, j_2, \dots, j_p) \leq 1_S(j) & \text{for all } j \in M \\
 \sum_{(i, j_1, j_3, \dots, j_p) \in B \times M^{p-1}} x(i, j_1, j, j_3, \dots, j_p) \leq 1_S(j) & \text{for all } j \in M \\
 \sum_{(i, j_1, j_2, j_4, \dots, j_p) \in B \times M^{p-1}} x(i, j_1, j_2, j, j_4, \dots, j_p) \leq 1_S(j) & \text{for all } j \in M \\
 \vdots & \vdots \\
 \sum_{(i, j_1, \dots, j_{p-1}) \in B \times M^{p-1}} x(i, j_1, \dots, j_{p-1}, j) \leq 1_S(j) & \text{for all } j \in M \\
 \sum_{(j_1, \dots, j_p) \in M^p} x(i, j_1, \dots, j_p) \leq 1_S(i) & \text{for all } i \in B \\
 x(i, j_1, \dots, j_p) \in \{0, 1\} & \text{for all } i \in B, \\
 & (j_1, \dots, j_p) \in M^p.
 \end{array} \tag{7}$$

4.3.6 Definition A cooperative game v is a p -assignment game for a $p \in \mathbb{N}$ if the set of players can be partitioned into two sets B and M such that there exist real numbers $g_i(j_1, \dots, j_p)$ for each $i \in B$ and each $(j_1, \dots, j_p) \in M^p$ and real numbers $h_j(q, i)$ for each $j \in M$, $q \in \{1, \dots, p\}$ and $i \in B$ with $a(i, j_1, \dots, j_p) = g_i(j_1, \dots, j_p) + h_{j_1}(1, i) + h_{j_2}(2, i) + \dots + h_{j_p}(p, i)$ such that for each coalition S the worth $v(S)$ of S is given by (7).

Note that whenever $|B| < |M|$ it is possible to add extra elements to B without really changing the game by assuming that for such an extra element i^* , $g_{i^*}(j_1, \dots, j_p) = 0$ for all $(j_1, \dots, j_p) \in M^p$ and $h_j(q, i^*) = 0$ for all $j \in M$ and $q \in \{1, \dots, p\}$. Therefore it will be assumed in the following that $|B| \geq |M|$.

An example of a multi-permutation situation is given by a variation of the machine-job situation where each job has to be processed on different types of machines and each producer owns one machine of each type. Suppose there are p different types of machines. For each producer i and each $q \in \{1, \dots, p\}$ let $i(q)$ denote the machine of type q that i owns. Let $k_i(\pi_1(i), \pi_2(i), \dots, \pi_p(i))$ be the value for producer i of using the machines $\pi_1(i)$ of type 1, $\pi_2(i)$ of type 2 and so forth. Here $\pi_1, \pi_2, \dots, \pi_p$ are permutations of N , the set of producers. Such a p -permutation situation yields a p -permutation game v where the worth $v(S)$ of a coalition S is the maximum of the sum of the values of all the members of S taken over all the permutations for which nothing changes for the players not in S . Formally, $v(S)$ is the value of the problem

$$\begin{array}{ll}
\max \sum_{(i, j_1, \dots, j_p) \in N^{p+1}} k_i(j_1, j_2, \dots, j_p) x(i, j_1, j_2, \dots, j_p) & \text{subject to} \\
\sum_{(i, j_2, \dots, j_p) \in N^p} x(i, j, j_2, \dots, j_p) = 1_S(j) & \text{for all } j \in N \\
\sum_{(i, j_1, j_3, \dots, j_p) \in N^p} x(i, j_1, j, j_3, \dots, j_p) = 1_S(j) & \text{for all } j \in N \\
\vdots & \vdots \\
\sum_{(i, j_1, \dots, j_{p-1}) \in N^p} x(i, j_1, j_2, \dots, j_{p-1}, j) = 1_S(j) & \text{for all } j \in N \\
\sum_{(j_1, \dots, j_p) \in N^p} x(i, j_1, j_2, \dots, j_p) = 1_S(i) & \text{for all } i \in N \\
x(i, j_1, j_2, \dots, j_p) \in \{0, 1\} & \text{for all } i \in N, \\
& (j_1, \dots, j_p) \in N^p.
\end{array} \tag{8}$$

4.3.7 Definition A cooperative game v is a p -permutation game for a $p \in \mathbb{N}$ if there exist real numbers $k_i(j_1, \dots, j_p)$ for each $(i, j_1, \dots, j_p) \in N^{p+1}$ such that $v(S)$ is given by (8) for each $S \in 2^N$.

4.3.8 Theorem For $p \geq 2$ both p -assignment games as well as p -permutation games need not be balanced.

Proof. Let $p = 2$. Let the p -assignment game $\langle B \cup M, v \rangle$ be given by $B = \{1, 2\}$, $M = \{3, 4\}$, $g_1(3, 3) = 5$, $g_1(3, 4) = 2$, $g_1(4, 3) = 4$, $g_1(4, 4) = 5$,

$g_2(3,3) = 3$, $g_2(3,4) = 3$, $g_2(4,3) = 7$, $g_2(4,4) = 3$, $h_3(1,1) = -1$, $h_3(1,2) = -1$, $h_3(2,1) = -2$, $h_3(2,2) = -2$, $h_4(1,1) = -2$, $h_4(1,2) = -2$, $h_4(2,1) = -1$, $h_4(2,2) = -1$. Then $a(1,3,3) = 2$, $a(1,3,4) = 0$, $a(1,4,3) = 0$, $a(1,4,4) = 2$, $a(2,3,3) = 0$, $a(2,3,4) = 1$, $a(2,4,3) = 3$, $a(2,4,4) = 0$. The characteristic function v is given by $v(S) = 0$ for $S \subset B$ or $S \subset M$ and $v(1,3) = v(1,4) = v(1,2,3) = v(1,2,4) = v(1,3,4) = 2$, $v(2,3) = v(2,4) = 0$, $v(2,3,4) = v(1,2,3,4) = 3$. Consider the balanced collection $\mathcal{B} = \{\{2,3,4\}, \{1,3\}, \{1,4\}, \{2\}\}$ with weights $\lambda_S = \frac{1}{2}$ for each $S \in \mathcal{B}$. Then

$$\sum_{S \in \mathcal{B}} \lambda_S v(S) = \frac{1}{2}3 + \frac{1}{2}2 + \frac{1}{2}2 + \frac{1}{2}0 = 3\frac{1}{2} > 3 = v(N).$$

Thus, v is not balanced. From this example a p -assignment game which is not balanced can be constructed for any $p > 2$ by taking $g_i(j_1, j_2, \dots, j_p) = g_i(j_1, j_2)$ for each $i \in B$ and each $(j_1, \dots, j_p) \in M^p$ and $h_j(q, i)$ as given above for $q \in \{1, 2\}$, $h_j(q, i) = 0$ for $q > 2$ for each $j \in M$, $i \in B$.

Again for $p = 2$, let $\langle N, v \rangle$ be the p -permutation game given by $N = \{1, 2, 3\}$, $k_1(1,1) = 0$, $k_1(1,2) = 0$, $k_1(1,3) = 0$, $k_1(2,1) = 1$, $k_1(2,2) = 0$, $k_1(2,3) = 0$, $k_1(3,1) = 0$, $k_1(3,2) = 0$, $k_1(3,3) = 2$, $k_2(1,1) = 0$, $k_2(1,2) = 1$, $k_2(1,3) = 0$, $k_2(2,1) = 0$, $k_2(2,2) = 0$, $k_2(2,3) = 0$, $k_2(3,1) = 0$, $k_2(3,2) = 0$, $k_2(3,3) = 2$, $k_3(1,1) = 1$, $k_3(1,2) = 0$, $k_3(1,3) = 0$, $k_3(2,1) = 0$, $k_3(2,2) = 2$, $k_3(2,3) = 0$, $k_3(3,1) = 0$, $k_3(3,2) = 0$, $k_3(3,3) = 0$. Then $v(1) = v(2) = v(3) = 0$, $v(1,2) = 2$, $v(1,3) = 3$, $v(2,3) = 4$, $v(1,2,3) = 4$. Hence,

$$\frac{1}{2}v(1,2) + \frac{1}{2}v(1,3) + \frac{1}{2}v(2,3) = 4\frac{1}{2} > 4 = v(1,2,3)$$

and it follows that v is not balanced. For $p > 2$ a p -permutation game which is not balanced can be constructed by taking $k_i(j_1, \dots, j_p) = k_i(j_1, j_2)$ for each $(i, j_1, \dots, j_p) \in N^{p+1}$. $\square$

4.3.9 Remark For a p -assignment game v with $|B| = 1$ or $|M| = 1$ a core element is obtained by assigning $v(B \cup M)$ to the unique element of B or M . For a p -permutation game v with $|N| \leq 2$ it follows that the core is not empty from the superadditivity of the game.

Although it was shown in theorem 4.3.8 that in general p -assignment games and p -permutation games need not be balanced, it is possible to define a

class of p -assignment games and p -permutation games which contains only balanced games.

4.3.10 Definition A p -assignment game $\langle B \cup M, v \rangle$ is said to have additive revenues if there exist real numbers $g_i^1(j_1), g_i^2(j_2), \dots, g_i^p(j_p)$ such that $g_i(j_1, \dots, j_p) = g_i^1(j_1) + \dots + g_i^p(j_p)$ for each $i \in B$ and each $(j_1, \dots, j_p) \in M^p$. A p -permutation game $\langle N, v \rangle$ is said to have additive revenues if there exist real numbers $k_i^1(j_1), k_i^2(j_2), \dots, k_i^p(j_p)$ such that $k_i(j_1, \dots, j_p) = k_i^1(j_1) + \dots + k_i^p(j_p)$ for each $(i, j_1, \dots, j_p) \in N^{p+1}$.

A concrete situation which may give rise to a p -assignment game with additive revenues is the sellers-buyers market with p indivisible commodities where the value of any commodity is independent of the other commodities. This may be the case when the commodities are not related to each other, such as, for example, a house, a car, a television. The value of a bundle consisting of different commodities is then simply the sum of the values of the separate commodities. It is not always reasonable to assume that the revenues are additive. The commodities may be parts of a big machinery which can only be operated with all parts together and such that not all parts of different types can be fitted together. Then the value of a bundle really depends on the combination of the commodities.

A situation which may lead to a p -permutation game with additive revenues is the following. Suppose each player needs to process a job for which he needs p different types of machines. Each player possesses one machine of each type. The value of using a certain combination of machines is equal to the sum of the values of using each machine.

4.3.11 Theorem If v is a p -assignment game or a p -permutation game with additive revenues then v is totally balanced.

Proof. Let $\langle B \cup M, v \rangle$ be a p -assignment game with additive revenues $g_i(j_1, \dots, j_p) = g_i^1(j_1) + \dots + g_i^p(j_p)$ for each $i \in B$ and each $(j_1, \dots, j_p) \in M^p$. Define p assignment games $\langle B \cup M, v^1 \rangle, \langle B \cup M, v^2 \rangle, \dots, \langle B \cup M, v^p \rangle$ by taking $g_{ij}^q = g_i^q(j)$ and $h_{ij}^q = h_j(q, i)$ for each $q \in \{1, \dots, p\}$ and each $i \in B, j \in M$. Then $v(S) = v^1(S) + v^2(S) + \dots + v^p(S)$ for each $S \in 2^N$. Now theorem 4.3.5 implies that all the games $v^1, v^2, \dots, v^p$ are totally balanced,

therefore v is totally balanced as well. The proof of the total balancedness of a p -permutation game with additive revenues can be given in the same way as this proof. $\square$

Consider the machine-job permutation situation with p types of machines. Suppose that the value of processing on a certain combination $(j_1, \dots, j_p)$ of machines is the same for each job, namely, $k(j_1, \dots, j_p)$. Further, there is an additional value which depends on the job. This situation gives rise to the following class of games.

4.3.12 Definition A p -permutation game $\langle N, v \rangle$ is said to have separable revenues if there exist $k(j_1, \dots, j_p)$ and l_i such that $k_i(j_1, \dots, j_p) = l_i + k(j_1, \dots, j_p)$ for each $(i, j_1, \dots, j_p) \in N^{p+1}$. A p -assignment game $\langle B \cup M, v \rangle$ is said to have separable revenues if there exist $f_i \geq 0$ and $a(j_1, \dots, j_p) \geq 0$ such that $a_i(j_1, \dots, j_p) = f_i + a(j_1, \dots, j_p)$ for each $i \in B$ and $(j_1, \dots, j_p) \in M^p$.

A p -assignment game with separable revenues may arise from a sellers-buyers situation with p indivisible commodities where each bundle $(j_1, \dots, j_p)$ has a certain intrinsic value $g(j_1, \dots, j_p)$ and where f_i denotes the value for buyer i of possessing any bundle of goods and where a seller is indifferent to whom he sells his good, i.e. $h_j(q, i) = h_j(q)$ for all $j \in M$, $i \in B$, $q \in \{1, \dots, p\}$.

4.3.13 Theorem 2-permutation games and 2-assignment games with separable revenues are totally balanced.

Proof. Let v be a 2-permutation game with separable revenues $k_i(j_1, j_2) = k(j_1, j_2) + l_i$ for each $(i, j_1, j_2) \in N^3$. Let $\langle N, l \rangle$ be the additive game defined by $l(S) := \sum_{i \in S} l_i$ for each $S \in 2^N$. Let $\langle N, w \rangle$ be the permutation game with $k_{ij} = k(i, j)$ for each $(i, j) \in N^2$. Then $v(S) = l(S) + w(S)$ for each $S \in 2^N$. From the total balancedness of l and w the total balancedness of v follows.

Let $\langle B \cup M, v \rangle$ be a 2-assignment game with separable revenues $a(i, j_1, j_2) = f_i + a(j_1, j_2)$. Consider the permutation game $\langle M, w \rangle$ with $k_{j_1 j_2} = a(j_1, j_2)$ for all $j_1, j_2 \in M$. This game is balanced. Let $y \in C(w)$. Recall that $|B| \geq |M|$. Let B' consists of $|M|$ elements of B with high-

est f -values and let $B'' = B \setminus B'$. Denote $\max_{i \in B''} f_i$ by f . Define $x \in \mathbb{R}^{|B| \times |M|}$ by $x_j = y_j + f$ for all $j \in M$, $x_i = f_i - f$ for all $i \in B'$ and $x_i = 0$ for all $i \in B''$. Then $x(N) = v(N)$ and for all $S \in 2^N$, $x(S) = y(S \cap M) + f(S \cap B') + |S \cap M|f - |S \cap B'|f \geq v(S)$. Thus, $x \in C(v)$ and it follows that v is balanced. The totally balancedness of v follows from the fact that each subgames of v is again a 2-assignment game with separable revenues. $\square$

4.3.14 Theorem For $p > 2$ p -permutation games and p -assignment games with separable revenues need not be balanced.

Proof. Let $p = 3$ and let $\langle N, v \rangle$ be the p -permutation game with separable revenues given by $N = \{1, 2, 3\}$, $k_i(j_1, j_2, j_3) = l_i + k(j_1, j_2, j_3)$ where $l_i = 0$ for each $i \in N$ and $k(j_1, j_2, j_3)$ is equal to $k_{j_1}(j_2, j_3)$ as given in the proof of theorem 4.3.8 for each $(j_1, j_2, j_3) \in N^3$. Then v is equal to the 2-permutation game in the proof of theorem 4.3.8 and it follows that v is not balanced. In the same way a p -permutation game with separable revenues which is not balanced can be constructed from a $p - 1$ -permutation game that is not balanced for any $p > 2$.

For $p = 3$ let $\langle M \cup B, w \rangle$ be the p -assignment game with separable revenues given by $B = \{5, 6, 7, 8\}$, $M = \{1, 2, 3, 4\}$ and $a(i, j_1, j_2, j_3) = f_i + a(i, j_1, j_2, j_3)$ where $f_i = 0$ for each $i \in B$ and $a(1, 3, 3)$, $a(1, 3, 4)$, $a(1, 4, 3)$, $a(1, 4, 4)$, $a(2, 3, 3)$, $a(2, 3, 4)$, $a(2, 4, 3)$, $a(2, 4, 4)$ are the same as in the proof of theorem 4.3.8 and $a(j_1, j_2, j_3) = 0$ for all other combinations of $(j_1, j_2, j_3) \in M^3$. Then $v(B \cup M) = 3$. Consider the balanced collection $\mathcal{B} = \{\{2, 3, 4, 5, 6, 7\}, \{1, 3, 5, 6\}, \{1, 4, 7, 8\}, \{2, 8\}\}$ with weight $\lambda_S = \frac{1}{2}$ for all $S \in \mathcal{B}$. Then

$$\sum_{S \in \mathcal{B}} \lambda_S v(S) = \frac{1}{2}3 + \frac{1}{2}2 + \frac{1}{2}2 + \frac{1}{2}0 = 3\frac{1}{2} > 3 = v(B \cup M).$$

Hence $\langle B \cup M, v \rangle$ is not balanced and the proof is completed. $\square$

4.4 Economies with indivisibilities

Economies with indivisibilities arising from permutation situations have been studied by many authors. Cf. Shapley and Scarf (1974), Kaneko (1982), Kaneko (1983), Kaneko and Yamamoto (1986), Gale (1984), Quinzii (1984), Wako (1984) and Wako (1986). However, all these papers consider only

the 1-permutation case. Here economies with indivisibilities arising from multi-permutation situations will be considered. An economy with indivisibilities E is an ordered tuple $\langle N, p, ((e_i^q)_{q \in \{1, \dots, p\}})_{i \in N}, (\omega_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where the finite set N is the set of agents and $p \in \mathbb{N}$ is the number of different types of indivisible commodities in the economy. For every type of indivisible commodity q , the commodity of this type possessed by agent i at the beginning is denoted by e_i^q . The amount of money that agent i possesses at the beginning is denoted by $\omega_i \in \mathbb{R}_+$. For each agent $i \in N$, u_i is a utility function which describes the value for agent i of a bundle consisting of one indivisible commodity of each type and a certain amount of money. Let $T^q = \bigcup_{i \in N} e_i^q$ for each $q \in \{1, \dots, p\}$ then u_i is a function defined on $T^1 \times T^2 \times \dots \times T^p \times \mathbb{R}_+$ with values in $\mathbb{R}$. An allocation in such an economy is a distribution of the commodities and the money among the agents whereby every agent obtains exactly one commodity of each type. Such an allocation can be denoted with the aid of p permutations, each one describing how the commodities of one type are redistributed among the agents and a vector m describing how the money is redistributed among the agents. Thus, $(\pi^1, \pi^2, \dots, \pi^p, m)$ is an allocation which assigns to agent i the indivisible commodities $e_{\pi^1(i)}^1, e_{\pi^2(i)}^2, \dots, e_{\pi^p(i)}^p$ and an amount of money equal to m_i , the i -th coordinate of m .

Every subgroup $S \subset N$ of agents can redistribute the indivisible commodities and the money available in S among its members. Let $A(S)$ be the set of *feasible allocations* for S , i.e.

$$A(S) := \{ (\pi^1, \pi^2, \dots, \pi^p, m) \in \Pi_N^p \times \mathbb{R}^n \mid \pi^q(S) = S \text{ for each } q \in \{1, \dots, p\} \text{ and } m(S) \leq \omega(S) \}.$$

A coalition S can *improve upon* an allocation $(\pi^1, \pi^2, \dots, \pi^p, m)$ if there exists an allocation $(\sigma^1, \dots, \sigma^p, m') \in A(S)$ such that

$$u_i(e_{\sigma^1(i)}^1, e_{\sigma^2(i)}^2, \dots, e_{\sigma^p(i)}^p, m'_i) > u_i(e_{\pi^1(i)}^1, \dots, e_{\pi^p(i)}^p, m_i) \text{ for each } i \in S.$$

4.4.1 Definition The core $C(E)$ of an economy E with indivisibilities is the set of allocations that no coalition can improve upon.

A coalition $S \subset N$ can *weakly improve upon* an allocation $(\pi^1, \dots, \pi^p, m)$

if there exists an allocation $(\sigma^1, \dots, \sigma^p, m') \in A(S)$ such that

$$\begin{aligned} u_i(e_{\sigma^1(i)}^1, \dots, e_{\sigma^p(i)}^p, m'_i) &\geq u_i(e_{\pi^1(i)}^1, \dots, e_{\pi^p(i)}^p, m_i) \quad \text{for all } i \in S, \\ u_j(e_{\sigma^1(j)}^1, \dots, e_{\sigma^p(j)}^p, m'_j) &> u_j(e_{\pi^1(j)}^1, \dots, e_{\pi^p(j)}^p, m_j) \quad \text{for some } j \in S. \end{aligned}$$

4.4.2 Definition The strong core $SC(E)$ of an economy E with indivisibilities is the set of allocations that no coalition can weakly improve upon.

It is evident that the strong core of an economy is contained in the core of the economy.

4.4.3 Definition A price equilibrium for an economy with indivisibilities E is a tuple $\langle p^1, \dots, p^p, (\pi^1, \dots, \pi^p, m) \rangle$ where $p^q \in \mathbb{R}_+^n$ is a price vector for the q -th commodity such that

- (a) $p_{\pi^1(i)}^1 + \dots + p_{\pi^p(i)}^p + m_i \leq p_i^1 + \dots + p_i^p + \omega_i$ for all $i \in N$.
- (b) For all $i \in N$ and $(e_{j_1}^1, \dots, e_{j_p}^p, m'_i) \in T^1 \times \dots \times T^p \times \mathbb{R}_+$ if $u_i(e_{j_1}^1, \dots, e_{j_p}^p, m'_i) > u_i(e_{\pi^1(i)}^1, \dots, e_{\pi^p(i)}^p, m_i)$ then $p_{j_1}^1 + \dots + p_{j_p}^p + m'_i > p_i^1 + \dots + p_i^p + \omega_i$.

If $\langle p^1, \dots, p^p, (\pi^1, \dots, \pi^p, m) \rangle$ is a price equilibrium then the vectors $p^1, \dots, p^p$ are called *equilibrium price vectors* and the allocation $(\pi^1, \dots, \pi^p, m)$ is called an *equilibrium allocation*.

4.4.4 Theorem Let $\langle p^1, \dots, p^p, (\pi^1, \dots, \pi^p, m) \rangle$ be a price equilibrium. Then $(\pi^1, \dots, \pi^p, m) \in C(E)$.

Proof. Let $S \subset N$. Suppose there exists a $(\sigma^1, \dots, \sigma^p, m') \in A(S)$ such that $u_i(e_{\sigma^1(i)}^1, \dots, e_{\sigma^p(i)}^p, m'_i) > u_i(e_{\pi^1(i)}^1, \dots, e_{\pi^p(i)}^p, m_i)$ for all $i \in S$. Then $p_{\sigma^1(i)}^1 + \dots + p_{\sigma^p(i)}^p + m'_i > p_i^1 + \dots + p_i^p + \omega_i$ for all $i \in S$. Since $\sigma^q(S) = S$ for all $q \in \{1, \dots, p\}$ it follows that $\sum_{i \in S} m'_i > \sum_{i \in S} \omega_i$ contradicting $(\sigma^1, \dots, \sigma^p, m') \in A(S)$. Therefore no S can improve upon $(\pi^1, \dots, \pi^p, m)$ and $(\pi^1, \dots, \pi^p, m) \in C(E)$. $\square$

4.4.5 Theorem *An economy with indivisibilities E can have an empty core.*

Proof. Let $N = \{1, 2, 3\}$, $p = 2$, $\omega_1 = \omega_2 = \omega_3 = 3$, $u_1(e_2^1, e_1^2, m_1) = m_1 + 1$, $u_1(e_3^1, e_2^2, m_1) = m_1 + 2$, $u_2(e_1^1, e_2^2, m_2) = m_2 + 1$, $u_2(e_3^1, e_2^2, m_2) = m_2 + 2$, $u_3(e_1^1, e_2^2, m_3) = m_3 + 1$, $u_3(e_2^1, e_2^2, m_3) = m_3 + 2$ and $u_i(e_j^1, e_k^2, m_i) = m_i$ for all other combinations of i, j, k . For any allocation (π^1, π^2, m) the following holds.

$$u_1(e_{\pi^1(1)}^1, e_{\pi^2(1)}^2, m_1) + u_2(e_{\pi^1(2)}^1, e_{\pi^2(2)}^2, m_2) + u_3(e_{\pi^1(3)}^1, e_{\pi^2(3)}^2, m_3) \leq 13 \quad (9)$$

For an allocation $(\hat{\pi}, \hat{\sigma}, \hat{m})$ to be in the core of E the following must hold.

$$u_1(e_{\hat{\pi}(1)}^1, e_{\hat{\sigma}(1)}^2, m_1) + u_2(e_{\hat{\pi}(2)}^1, e_{\hat{\sigma}(2)}^2, m_2) \geq 8 \quad (10)$$

$$u_1(e_{\hat{\pi}(1)}^1, e_{\hat{\sigma}(1)}^2, m_1) + u_3(e_{\hat{\pi}(3)}^1, e_{\hat{\sigma}(3)}^2, m_3) \geq 9$$

$$u_2(e_{\hat{\pi}(2)}^1, e_{\hat{\sigma}(2)}^2, m_2) + u_3(e_{\hat{\pi}(3)}^1, e_{\hat{\sigma}(3)}^2, m_3) \geq 10$$

Straightforward verification shows that it is impossible for both (9) and (10) to hold simultaneously. Therefore, $C(E) = \emptyset$. $\square$

Under the assumption that u_i is continuous and non-decreasing with respect to the amount of money for all $i \in N$, Quinzii (1984) proved that the core of the economy E is non-empty for $p = 1$. Under certain additional assumptions Quinzii proved that in this case the core of E and the set of equilibrium allocations of E coincide.

4.4.6 Theorem *Let $E = \langle N, 2, ((e_i^q))_{q \in \{1,2\}}, (\omega_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be an economy with $u_i(e_j^1, e_k^2, m_i) = u(e_j^1, e_k^2, m_i) + f_i$ for all $i, j, k \in N$, where u is a function from $T^1 \times T^2 \times \mathbb{R}_+$ to $\mathbb{R}$ which is continuous and non-decreasing with respect to the amount of money and where $f_i \in \mathbb{R}$. Then $C(E) \neq \emptyset$.*

Proof. Define the economy $\hat{E} = \langle N, 1, (\hat{e}_i)_{i \in N}, (\hat{\omega}_i)_{i \in N}, (\hat{u}_i)_{i \in N} \rangle$ by $\hat{e}_i = e_i^2$, $\hat{\omega}_i = \omega_i$ and $\hat{u}_i(\hat{e}_j, m_i) = u(e_j^1, e_j^2, m_i)$ for all $i, j \in N$. Then Quinzii's result yields $C(\hat{E}) \neq \emptyset$. Let $(\hat{\pi}, m) \in C(\hat{E})$. Then $(\sigma, \hat{\pi}, m)$, where σ is the identity permutation, is an allocation that can not be improved upon by any coalition. Namely, suppose that a coalition S could improve upon the allocation $(\sigma, \hat{\pi}, m)$ by means of an allocation (π^1, π^2, m') . Consider the allocation $(\pi^2(\pi^1)^{-1}, m')$ in $\hat{E}$. Let $i \in S$, then there is a j such that $i = \pi^1(j)$

and

$$\begin{aligned} \hat{u}_i(\hat{e}_{\pi^2(\pi^1)^{-1}(i)}, m_i^t) &= \hat{u}_{\pi^1(j)}(\hat{e}_{\pi^2(j)}, m_i^t) = \\ u(e_{\pi^1(j)}^1, e_{\pi^2(j)}^2, m_i^t) &> u(e_i^1, e_{\pi(i)}^2, m_i^t) = \\ \hat{u}_i(\hat{e}_{\pi(i)}, m_i^t). \end{aligned}$$

This contradicts $(\hat{\pi}, m) \in C(\hat{E})$ and therefore it is not possible for an S to improve upon $(\sigma, \hat{\pi}, m)$. Hence $(\sigma, \hat{\pi}, m) \in C(E)$ and $C(E) \neq \emptyset$. $\square$

Note that theorem 4.4.6 is the analogue of theorem 4.3.14. in terms of economies with indivisible commodities.

4.5 Ordinal matching situations

In this section matching situations with ordinal rather than cardinal preferences are considered. The matching situations studied here are many to many matching situations. These are bipartite matching situations in which any member of one of the two groups is matched to more than one member of the other group. Let H and G be two disjoint, finite sets. As an example to fix the ideas H could represent the set of hospitals in a certain area while G represents the set of medical students in that area. Each hospital offers p types of jobs, e.g. a job in surgery, one in psychiatry, and so forth. Each hospital offers exactly one job of each type. Every medical student has to fulfil each type of job once for the practical part of his study. Students have to be matched to jobs. A student need not fulfil all the jobs at the same hospital. The students have preferences over combinations of jobs. For instance, a student may prefer to fulfil all the jobs at the same hospital while another one prefers to work at as many hospitals as possible. The preferences of the students are defined on the set H^p . It is assumed that these preferences are complete, transitive and strict and hence they can be represented by ordered lists. Typically the preference $P(g)$ of student $g \in G$ can be described as follows.

$P(g)$: most preferred combination least preferred combination

With respect to the preferences of the hospitals two different models will be studied. In the first model each hospital $h \in H$ has separate preference relations $P^1(h), P^2(h), \dots, P^p(h)$ defined on the set G of students. The preference relation $P^j(h)$ describes the preference of h over the students for the j -th job. These preferences are taken to be complete, transitive and strict

and can be described by ordered lists in the same way as the preferences of the students. In the second model each hospital $h \in H$ has a joint preference $P(h)$ defined on G^p . In the first case the matching situation is called a *p-matching situation with separate preferences* while in the second case it is called a *p-matching situation with joint preferences*. Note that in both models the preferences of the students are joint preferences and that it is the difference in the type of preferences of the hospitals that accounts for the difference in the names of the matching situations.

4.5.1 Definition A *p-matching situation with separate preferences* consists of two disjoint sets H and G which have the same finite number of elements and where for each $g \in G$ a complete, transitive and strict preference relation $P(g)$ on H^p is given and for each $h \in H$ p complete, transitive and strict preference relations $P^1(h), P^2(h), \dots, P^p(h)$ on G are given.

4.5.2 Definition A *p-matching situation with joint preferences* consists of two disjoint sets H and G which have the same finite number of elements and where for each $g \in G$ a complete, transitive and strict preference relation $P(g)$ on H^p is given and for each $h \in H$ a complete, transitive and strict preference relation $P(h)$ on G^p is given.

Let μ be a function from G to H^p . For all $g \in G$ let $\mu(g)$ be denoted by $(\mu_1(g), \mu_2(g), \dots, \mu_p(g))$.

4.5.3 Definition A function μ from G to H^p is called a *matching* if and only if

$$\mu_q(g) \neq \mu_q(g') \text{ for all } q \in \{1, \dots, p\} \text{ and all } g, g' \in G \text{ with } g \neq g'.$$

In the example considered above a matching assigns to every student a combination of the P types of jobs in such a way that no job is assigned twice. Let μ be a matching. The element g of G for which $\mu_j(g) = h$ is denoted by $\mu_j^{-1}(h)$.

4.5.4 Definition A matching μ in a p -matching situation with separate preferences is called *stable* if there is no pair $(g, h) \in G \times H$ such that g prefers $(\mu_1(g), \dots, \mu_{j-1}(g), h, \mu_{j+1}(g), \dots, \mu_p(g))$ to $(\mu_1(g), \dots, \mu_{j-1}(g), \mu_j(g), \mu_{j+1}(g), \dots, \mu_p(g))$ and according to $P^j(h)$, h prefers g to $\mu_j^{-1}(h)$.

For the case with $p = 1$ Gale and Shapley (1962) proved that there exist stable matchings and gave an algorithm to arrive at a stable matching. Note that in this case the two types of matching situations defined above are the same.

4.5.5 Theorem For $p \geq 2$ there need not exist a stable matching for each p -matching situation with separate preferences.

Proof. Consider the following 2-matching situation with separate preferences. $G = \{g_1, g_2\}$, $H = \{h_1, h_2\}$ and the preferences are given by

$$\begin{aligned} P(g_1) &: h_1 h_1 & h_2 h_2 & h_1 h_2 & h_2 h_1 \\ P(g_2) &: h_2 h_1 & h_1 h_2 & h_1 h_1 & h_2 h_2 \\ \\ P^1(h_k) &: g_1 & g_2 & \text{for } k \in \{1, 2\} \\ P^2(h_k) &: g_2 & g_1 & \text{for } k \in \{1, 2\}. \end{aligned}$$

There are four possible matchings, none of which is stable.

The matching μ with $\mu(g_1) = (h_1, h_1)$, $\mu(g_2) = (h_2, h_2)$ is not stable since g_2 prefers (h_2, h_1) to (h_2, h_2) and according to $P^2(h_1)$ h_1 prefers g_2 to g_1 .

The matching μ with $\mu(g_1) = (h_2, h_2)$, $\mu(g_2) = (h_1, h_1)$ is not stable since g_2 prefers (h_1, h_2) to (h_1, h_1) and according to $P^2(h_2)$ h_2 prefers g_2 to g_1 .

The matching μ with $\mu(g_1) = (h_1, h_2)$, $\mu(g_2) = (h_2, h_1)$ is not stable since g_1 prefers (h_2, h_2) to (h_1, h_2) and according to $P^1(h_2)$ h_2 prefers g_1 to g_2 .

The matching μ with $\mu(g_1) = (h_2, h_1)$, $\mu(g_2) = (h_1, h_2)$ is not stable since g_1 prefers (h_1, h_1) to (h_2, h_1) and according to $P^1(h_1)$ h_1 prefers g_1 to g_2 .

This matching situation can be extended in a straightforward way to a p -matching situation with separate preferences for which no stable matching exists for any $p > 2$. $\square$

An extra condition on the preferences of the elements of G is sufficient

to ensure the existence of a stable matching. This condition was introduced by Roth (1985) in a slightly different context.

4.5.6 Definition A preference relation P defined on a product set $X_1 \times X_2 \times \cdots \times X_k$ is said to be responsive if there exist separate, complete, transitive and strict preference relations $P^1, P^2, \dots, P^k$ on $X_1, X_2, \dots, X_k$ such that if two elements of $X_1 \times X_2 \times \cdots \times X_k$ differ only in the j -th coordinate, the element which contains the most preferred job according to P^j is preferred by P .

Note that if a preference relation is responsive then the separate preference relations are uniquely determined.

4.5.7 Theorem If in a p -matching situation with separate preferences the preferences of the elements of G are responsive, then there exists a stable matching.

Proof. Let $P^1(g), P^2(g), \dots, P^p(g)$ be the separate preferences of $g \in G$. Consider p 1-matching situations given by the sets G and H and the preference relations $P^1(g), P^2(g), \dots, P^p(g)$ for all $g \in G$ and the preference relations $P^1(h), P^2(h), \dots, P^p(h)$ for all $h \in H$. With the Gale-Shapley algorithm it is possible to find stable matchings $\hat{\mu}_1, \hat{\mu}_2, \dots, \hat{\mu}_p$ for all these situations. Let μ be the matching in the p -matching situation defined by $\mu_q(g) = \hat{\mu}_q(g)$ for each $g \in G$ and each $q \in \{1, \dots, p\}$. Then μ is stable, since if there existed a pair $(g, h) \in G \times H$ such that g prefers $(\mu_1(g), \dots, \mu_{j-1}(g), h, \mu_{j+1}(g), \dots, \mu_p(g))$ to $(\mu_1(g), \dots, \mu_{j-1}(g), \mu_j(g), \mu_{j+1}(g), \dots, \mu_p(g))$ and g is preferred by $P^j(h)$ to $\mu_j^{-1}(h)$ then this would imply that in the 1-matching situation with preferences $P^j(g)$ and $P^j(h)$ g prefers h to $\mu_j(g) = \hat{\mu}_j(g)$ and h prefers g to $\mu_j^{-1}(h) = \hat{\mu}_j^{-1}(h)$ contradicting the stability of $\hat{\mu}_j$ in this situation. Thus, it follows that μ is stable. $\square$

For the hospitals-students example it follows that when the jobs are assigned to the students according to a stable matching μ , it is not possible that a hospital h prefers another student g for the q -th job to the one who

obtains the job and that this student g prefers the combination where $\mu_q(g)$ is replaced by h to the original combination assigned to him. However, it is still possible, even under a stable matching μ , for a hospital to prefer a student g for several jobs $q_1, q_2, \dots, q_k$ to the ones who obtained the jobs while the student g prefers the combination where $\mu_{q_1}(g), \mu_{q_2}(g), \dots, \mu_{q_k}(g)$ are replaced by h to the original combination assigned to him. Hence even an assignment according to a stable matching can be destabilized if the hospitals and the students are allowed to change more than one of the students, respectively, jobs which are assigned to them. Therefore, *strongly stable matchings* are introduced.

4.5.8 Definition A matching μ in a p -matching situation with separate preferences is said to be *strongly stable* if there is no pair $(g, h) \in G \times H$ such that for some $\{q_1, q_2, \dots, q_k\} \subset \{1, \dots, p\}$ g prefers the element of H^p arising from $\mu(g)$ by replacing $\mu_{q_1}(g), \mu_{q_2}(g), \dots, \mu_{q_k}(g)$ with h to $\mu(g)$ and for all $q \in \{q_1, \dots, q_k\}$ $P^q(h)$ prefers g to $\mu_q^{-1}(h)$.

It is obvious that every strongly stable matching is stable. Hence it follows from theorem 4.5.5 that a strongly stable matching need not exist.

4.5.9 Theorem If in a p -matching situation with separate preferences the preferences of the elements of G are responsive, then every stable matching is strongly stable.

Proof. Let μ be a stable matching. Then for each $g \in G$ and each $h \in H$ if g is preferred by $P^j(h)$ to $\mu_j^{-1}(h)$ then g prefers $(\mu_1(g), \dots, \mu_{j-1}(g), \mu_j(g), \mu_{j+1}(g), \dots, \mu_p(g))$ to $(\mu_1(g), \dots, \mu_{j-1}(g), h, \mu_{j+1}(g), \dots, \mu_p(g))$. Suppose there exists an $h \in H$ and a set $\{q_1, \dots, q_k\} \subset \{1, \dots, p\}$ such that g is preferred by $P^q(h)$ to $\mu_q^{-1}(h)$ for all $q \in \{q_1, \dots, q_k\}$. From the stability of μ it follows that g prefers $(\mu_1(g), \dots, \mu_{q-1}(g), \mu_q(g), \mu_{q+1}(g), \dots, \mu_p(g))$ to $(\mu_1(g), \dots, \mu_{q-1}(g), h, \mu_{q+1}(g), \dots, \mu_p(g))$ for all $q \in \{q_1, \dots, q_k\}$. This and the responsiveness and the transitivity of $P(g)$ lead to the conclusion that g prefers $\mu(g)$ to the element of H^p arising from $\mu(g)$ by replacing $\mu_{q_1}(g), \dots, \mu_{q_k}(g)$ with h . Since this argument holds for every $(g, h) \in G \times H$ the strong stability of μ is proved. $\square$

Combining theorem 4.5.7 and theorem 4.5.9 yields the following corollary.

4.5.10 Corollary *If in a p -matching situation with separate preferences the preferences of the elements of G are responsive then there exists a strongly stable matching.*

4.5.11 Definition *A matching μ in a p -matching situation with joint preferences is called stable if there does not exist a pair $(g, h) \in G \times H$ such that g prefers $(\mu_1(g), \dots, \mu_{j-1}(g), h, \mu_{j+1}(g), \dots, \mu_p(g))$ to $(\mu_1(g), \dots, \mu_{j-1}(g), \mu_j(g), \mu_{j+1}(g), \dots, \mu_p(g))$ and h prefers $(\mu_1^{-1}(h), \dots, \mu_{j-1}^{-1}(h), g, \mu_{j+1}^{-1}(h), \dots, \mu_p^{-1}(h))$ to $(\mu_1^{-1}(h), \dots, \mu_{j-1}^{-1}(h), \mu_j^{-1}(h), \mu_{j+1}^{-1}(h), \dots, \mu_p^{-1}(h))$.*

4.5.12 Theorem *For $p \geq 2$ there need not exist a stable matching in a p -matching situation with joint preferences.*

Proof. Consider the following 2-matching situation with joint preferences. $G = \{g_1, g_2\}$, $H = \{h_1, h_2\}$ and the preferences are given by

$$\begin{array}{llll} g_1 : & h_1 h_1 & h_2 h_2 & h_1 h_2 & h_2 h_1 \\ g_2 : & h_2 h_1 & h_1 h_2 & h_1 h_1 & h_2 h_2 \\ h_k : & g_1 g_2 & g_1 g_1 & g_2 g_2 & g_2 g_1 \text{ for } k \in \{1, 2\}. \end{array}$$

There are four possible matchings, none of which is stable.

The matching μ with $\mu(g_1) = (h_1, h_1)$, $\mu(g_2) = (h_2, h_2)$ is not stable since g_2 prefers (h_2, h_1) to (h_2, h_2) and h_1 prefers (g_1, g_2) to (g_1, g_1) .

The matching μ with $\mu(g_1) = (h_2, h_2)$, $\mu(g_2) = (h_1, h_1)$ is not stable since g_2 prefers (h_1, h_2) to (h_1, h_1) and h_2 prefers (g_1, g_2) to (g_1, g_1) .

The matching μ with $\mu(g_1) = (h_1, h_2)$, $\mu(g_2) = (h_2, h_1)$ is not stable since g_1 prefers (h_2, h_2) to (h_1, h_2) and h_2 prefers (g_1, g_1) to (g_2, g_1) .

The matching μ with $\mu(g_1) = (h_2, h_1)$, $\mu(g_2) = (h_1, h_2)$ is not stable since g_1 prefers (h_1, h_1) to (h_2, h_1) and h_1 prefers (g_1, g_1) to (g_2, g_1) .

This matching situation can be extended in a straightforward way to a p -matching situation with joint preferences for which there is no stable matching for any $p > 2$. $\square$

4.5.13 Theorem *If in a p -matching situation with joint preferences the preferences of the elements of both G as well as H are responsive, then there exists a stable matching.*

Proof. Consider p 1-matching situations given by the sets G and H where the preferences of a $g \in G$ and an $h \in H$ are taken to be the separate preferences $P^1(g), \dots, P^p(g)$ and $P^1(h), \dots, P^p(h)$ defined by $P(g)$ and $P(h)$ respectively. With the Gale-Shapley algorithm it is possible to find stable matchings $\hat{\mu}_1, \hat{\mu}_2, \dots, \hat{\mu}_p$ for all these situations. Let μ be the matching in the p -matching situation defined by $\mu_j(g) = \hat{\mu}_j(g)$ for each $g \in G$ and each $j \in \{1, \dots, p\}$. Suppose that there exists a pair $(g, h) \in G \times H$ such that g prefers $(\mu_1(g), \dots, \mu_{j-1}(g), h, \mu_{j+1}(g), \dots, \mu_p(g))$ to $(\mu_1(g), \dots, \mu_{j-1}(g), \mu_j(g), \mu_{j+1}(g), \dots, \mu_p(g))$ and h prefers $(\mu_1^{-1}(h), \dots, \mu_{j-1}^{-1}(h), g, \mu_{j+1}^{-1}(h), \dots, \mu_p^{-1}(h))$ to $(\mu_1^{-1}(h), \dots, \mu_{j-1}^{-1}(h), \mu_j^{-1}(h), \mu_{j+1}^{-1}(h), \dots, \mu_p^{-1}(h))$. Then in the 1-matching situation with preferences $P^j(g)$ and $P^j(h)$ g prefers h to $\mu_j(g) = \hat{\mu}_j(g)$ and h prefers g to $\mu_j^{-1}(h) = \hat{\mu}_j^{-1}(h)$ contradicting the stability of $\hat{\mu}_j$. Therefore such a pair (g, h) can not exist and μ is stable. $\square$

4.5.14 Definition *A matching μ in a p -matching situation is called strongly stable if there is no pair $(g, h) \in G \times H$ such that for some $\{q_1, q_2, \dots, q_k\} \subset \{1, \dots, p\}$ g prefers the element of H^p arising from $\mu(g)$ by replacing $\mu_q(g)$ with h for all $q \in \{q_1, \dots, q_k\}$ to $\mu(g)$ and h prefers the element of G^p arising from $\mu^{-1}(h)$ by replacing $\mu_q^{-1}(h)$ with g for all $q \in \{q_1, \dots, q_k\}$ to $\mu^{-1}(h)$.*

Again a strongly stable matching is necessarily stable. But, contrary to the case of p -matching situations with separate preferences in this case the responsiveness of the preferences does not guarantee that a stable matching is strongly stable as well. Indeed, even the existence of a strongly stable matching is not guaranteed by the responsiveness of the preferences.

4.5.15 Theorem *For $p \geq 2$ there need not exist a strongly stable matching in a p -matching situation with joint preferences where all the preferences are responsive.*

Proof. Consider the 2-matching situation with $G = \{g_1, g_2\}$, $H = \{h_1, h_2\}$ and where the preferences are given by

$$\begin{array}{llll} P(g_i) : & h_1 h_2 & h_1 h_1 & h_2 h_2 & h_2 h_1 \text{ for } i \in \{1, 2\} \\ P(h_1) : & g_1 g_2 & g_2 g_2 & g_1 g_1 & g_2 g_1 \\ P(h_2) : & g_1 g_2 & & & g_2 g_1 \end{array}$$

where the open places in $P(h_2)$ are not important. These preferences are responsive. The separate preferences are

$$\begin{array}{llll} P^1(g_i) : & h_1 & h_2 & \text{for } i \in \{1, 2\} \\ P^2(g_i) : & h_2 & h_1 & \text{for } i \in \{1, 2\} \\ P^1(h_k) : & g_1 & g_2 & \text{for } k \in \{1, 2\} \\ P^2(h_k) : & g_2 & g_1 & \text{for } k \in \{1, 2\}. \end{array}$$

The only stable matching here is the matching μ with $\mu(g_1) = (h_1, h_1)$, $\mu(g_2) = (h_2, h_2)$ but μ is not strongly stable since g_2 prefers (h_1, h_1) to (h_2, h_2) and h_1 prefers (g_2, g_2) to (g_1, g_1) . Since every strongly stable matching is stable it follows that there does not exist a strongly stable matching in this situation. $\square$

A corollary to theorem 4.5.15 is given by 4.5.16.

4.5.16 Corollary *For $p \geq 2$ a stable matching in a p -matching situation with joint responsive preferences need not be strongly stable.*

A stronger condition on the preferences in a p -matching situation with joint preferences will give the existence of a strongly stable matching.

4.5.17 Definition *A preference relation P defined on a set X^k is said to be strongly responsive if there exists a complete, transitive and strict preference relation $\hat{P}$ on X such that if two elements of X^k differ only in the j -th coordinate then P prefers that element with most preferred j -th coordinate according to $\hat{P}$.*

Note that a strongly responsive preference relation on X^k defines a unique preference relation on X .

In the hospitals-students example if a student g has a strongly responsive preference relation this means that he can rank the hospitals in such a way that if he prefers hospital h to hospital h' then he always prefers to fulfil a certain job at h rather than at h' regardless of where he fulfils the other jobs. In a similar way strongly responsiveness of the preference relation of a hospital can be interpreted.

It is clear that a strongly responsive preference relation is responsive.

4.5.18 Theorem *If in a p -matching situation with joint preferences all the preferences are strongly responsive then there exists a strongly stable matching.*

Proof. Consider the 1-matching situation given by the sets G and H and with preference relation $\hat{P}(g)$ of $g \in G$ defined on H by the strongly responsive preference relation $P(g)$ of g on H^p and the preference relation $\hat{P}(h)$ of $h \in H$ defined on G by the strongly responsive preference relation $P(h)$ of h on G^p . Let $\hat{\mu}$ be a stable matching for this 1-matching situation. Define a matching μ in the p -matching situation by $\mu_j(g) = \hat{\mu}(g)$ for all $j \in \{1, \dots, p\}$ and all $g \in G$. Let $(g, h) \in G \times H$ and $\{q_1, q_2, \dots, q_k\} \subset \{1, \dots, p\}$ be such that g prefers the element of H^p which arises from $\mu(g)$ by replacing $\mu_{q_i}(g)$ by h for all $q_i \in \{q_1, \dots, q_k\}$ to $\mu(g)$. This implies that in the 1-matching situation g prefers h to $\hat{\mu}(g)$. From the stability of $\hat{\mu}$ it follows that h prefers $\hat{\mu}^{-1}(h)$ to g in the 1-matching situation. The strong responsiveness and the transitivity of $P(h)$ yield that in the p -matching situation h prefers $\mu^{-1}(h)$ to the element of G^p arising from $\mu^{-1}(h)$ by replacing $\mu_{q_i}^{-1}(h)$ with g for all $q_i \in \{q_1, \dots, q_k\}$. Since this argument holds for any $(g, h) \in G \times H$ it follows that μ is strongly stable. $\square$

4.6 Sequencing games

Sequencing situations and *sequencing games* are closely related to permutation situations and permutation games in the sense that in a sequencing situation just as in a permutation situation the people involved place some value on each permutation of N and each subgroup $S \subset N$ can achieve some permutations. However, while in a permutation situation S can achieve all

permutations π with $\pi(i) = i$ for all $i \in N \setminus S$, in a sequencing situation an original permutation which need not be the identity is given and S has in general less possibilities to change this permutation. Another difference is that while in a permutation situation the value for $i \in N$ of a permutation π only depends on $\pi(i)$, in a sequencing situation this value depends on $P(\pi, i)$, the set of predecessors of i according to π . An example will help to clarify the ideas involved in sequencing situations. Consider a situation where a set of customers N is standing in a queue before a counter waiting to be served. The original position of each customer is given by a permutation σ of N . For $i \in N$, $\sigma(i) = j$ means that i has the j -th position in the queue. Each customer i has a certain *service time* of s_i units of time where $s_i > 0$ and a *cost function* $c_i : \mathbb{R}_+ \rightarrow \mathbb{R}$. The cost of a customer i depends on his waiting time plus his service time. If the sum of these two equals t then his cost is $c_i(t)$. In the following it is assumed that c_i is linear for all $i \in N$ and therefore there exist $\alpha_i, \beta_i \in \mathbb{R}$ such that $c_i(t) = \alpha_i t + \beta_i$. The *total cost* of the group N if served according to σ is given by

$$C_\sigma := c_1\left(\sum_{i \in P(\sigma, 1)} s_i + s_1\right) + c_2\left(\sum_{i \in P(\sigma, 2)} s_i + s_2\right) + \cdots + c_n\left(\sum_{i \in P(\sigma, n)} s_i + s_n\right).$$

By rearranging the customers, N can decrease this cost. Each rearrangement corresponds to a permutation $\pi \in \Pi_N$. What is gained by this rearrangement is equal to $C_\sigma - C_\pi$. To compute this difference the β_i 's are not important. That is why they are not included in the following definition of a sequencing situation.

4.6.1 Definition A sequencing situation consists of a finite set $N = \{1, \dots, n\}$ and an ordered triplet $(\sigma; \alpha; s)$ where $\sigma \in \Pi_N$, $\alpha \in \mathbb{R}^n$ and $s \in \mathbb{R}_+^n$.

Whenever no confusion is possible about the identity of the set N it will not be mentioned explicitly.

To find an optimal permutation, i.e. a permutation where the total cost of N is minimal, the *urgency indices* of the members of N can be used.

4.6.2 Definition In a sequencing situation $(\sigma; \alpha; s)$ the urgency index u_i of $i \in N$ is given by $u_i := \alpha_i / s_i$.

For a certain permutation τ of N let $i, j \in N$ be neighbours, i.e. $|\tau(i) - \tau(j)| = 1$. Assume that $\tau(j) = \tau(i) + 1$, then if i and j switch positions the total cost changes by an amount $\alpha_j s_i - \alpha_i s_j$ regardless of their position in the queue. If $u_i < u_j$ then this amount will be positive and the total cost will decrease, if $u_i > u_j$ then it will be negative and the total cost will increase, if $u_i = u_j$ then it will be zero and the total cost stays the same. This leads to the following proposition.

4.6.3 Proposition *Let $(\sigma; \alpha; s)$ be a sequencing situation. Then $C_\pi = \min_{\tau \in \Pi_N} C_\tau$ iff $u_{\pi^{-1}(1)} \geq u_{\pi^{-1}(2)} \geq \dots \geq u_{\pi^{-1}(n)}$.*

Proof. Let τ be a permutation which does not satisfy the condition on the right hand side. Then there exist $i, j \in N$ with $\tau(j) = \tau(i) + 1$ and $u_i < u_j$. By switching i and j the cost will decrease and it follows that C_τ is not minimal. Let ρ be a permutation which does satisfy the right hand side condition. Then it is possible to go from any permutation τ of N to ρ by switching only neighbours. In fact only neighbours i, j with $\tau(j) = \tau(i) + 1$ and $u_j \geq u_i$ have to be switched in this process. Such switchings will either decrease the cost of N or leave it unchanged. It follows that $C_\tau \geq C_\rho$ and therefore that $C_\rho = \min_{\pi \in \Pi_N} C_\pi$. $\square$

Not only the group N but every subgroup S of N can decrease its cost by rearranging its members. However, in this process of rearrangement it is not allowed for any member of S to jump over non-members of S . Thus, if $i, j \in S$ and there is a $k \in N \setminus S$ standing between them then i and j are not allowed to change position. This situation can be modeled with the aid of a cooperative game, a *sequencing game*. The set of players will be, of course, equal to N . Before defining the characteristic function v some concepts which will be helpful for the definition of v will be introduced. First the *gain* g_{ij} that two players can achieve when standing next to one another with i in front of j in a sequencing situation $(\sigma; \alpha; s)$ will be defined. This gain is equal to the difference of the sums of the costs of i and j before and after they change position. If $u_i \geq u_j$ then i and j can gain nothing by switching positions, if $u_j > u_i$ then i and j can gain $\alpha_j s_i - \alpha_i s_j$. Therefore,

$$g_{ij} := (\alpha_j s_i - \alpha_i s_j)_+ .$$

4.6.4 Definition Let $(\sigma; \alpha; s)$ be a sequencing situation then $T \subset N$ is called a *connected coalition* if $i, j \in T$ and $k \in N$ with $\sigma(i) < \sigma(k) < \sigma(j)$ imply $k \in T$.

In a connected coalition there are no non-members of the coalition standing between the members and therefore all rearrangements of its members are allowed. From proposition 4.6.3 it follows that an optimal way to rearrange the members of a connected coalition is in order of decreasing urgency index. This can be done by switching neighbours who are not standing in the right order. Every time this is done in a sequencing situation $(\sigma; \alpha; s)$ with two neighbours i and j such that $\sigma(i) < \sigma(j)$ and $u_i < u_j$ the gain will be equal to g_{ij} . This leads to the following definition of the worth $v(T)$ of a connected coalition.

$$v(T) := \sum_{i \in T} \sum_{k \in P(\sigma, i) \cap T} g_{ki}. \quad (11)$$

4.6.5 Definition Let $S \subset N$ be a non-connected coalition in a sequencing situation $(\sigma; \alpha; s)$. Then $T \subset N$ is said to be a *component* of S if (i) $T \subset S$, (ii) T is connected and (iii) $T \cup \{i\}$ is not connected for each $i \in S \setminus T$.

The components of a non-connected coalition S form a partition of S which is denoted by S/σ . Maximal cost savings are achieved by S when the members in all its components are rearranged in decreasing urgency index order. The total cost savings of S is the sum of the cost savings of all its components. This leads to the following definition of the worth $v(S)$ for a non-connected coalition S .

$$v(S) := \sum_{T \in S/\sigma} v(T) \quad (12)$$

4.6.6 Definition A cooperative game v is a *sequencing game* if there exist a sequencing situation $(\sigma; \alpha; s)$ such that for a connected coalition T the definition of $v(T)$ is given by (11) and for a non-connected coalition S the definition of $v(S)$ is given by (12).

4.6.7 Theorem Sequencing games are convex games.

Proof. Let v be a sequencing game arising from the sequencing situation $(\sigma; \alpha; s)$. Let $S_1 \subset S_2 \subset N \setminus \{i\}$. Then there exist $T_1, U_1 \in S_1/\sigma \cup \{\emptyset\}$ and $T_2, U_2 \in S_2/\sigma \cup \{\emptyset\}$ with $T_1 \subset T_2$ and $U_1 \subset U_2$ such that

$$v(S_p \cup \{i\}) - v(S_p) = \sum_{k \in T_p} g_{ki} + \sum_{j \in U_p} g_{kj} \text{ for } p \in \{1, 2\}.$$

From the non-negativeness of the g_{ij} 's it follows that $v(S_1 \cup \{i\}) - v(S_1) \leq v(S_2 \cup \{i\}) - v(S_2)$ and therefore v is convex. $\square$

Not all convex games are sequencing games since sequencing games are zero-normalized and convex games need not be zero-normalized. Actually, even not all zero-normalized convex games are sequencing games. In the following proposition a characterization of the 3-person zero-normalized convex games which are sequencing games is given.

4.6.8 Proposition *A 3-person zero-normalized convex game v is a sequencing game iff (i) at least one 2-person coalition has worth zero, (ii) $v(N) = v(S) + v(T)$ with $|S| = |T| = 2$ implies $v(S) = 0$ or $v(T) = 0$ and (iii) $v(S) = 0$ for all coalitions S with $|S| = 2$ implies $v(N) = 0$.*

Proof. Let $N = \{1, 2, 3\}$ and suppose that the game v arises from the sequencing situation $(\sigma; \alpha; s)$. Without loss of generality it is assumed that $\sigma(1) = 1, \sigma(2) = 2, \sigma(3) = 3$. Then $v(1, 3) = 0$ by definition. Suppose $v(N) = v(S) + v(T)$ with $|S| = |T| = 2$ and $v(S) > 0, v(T) > 0$, say $S = \{1, 2\}$ and $T = \{2, 3\}$. Then $u_2 > u_1$ and $u_3 > u_2$ and it follows that $v(N) = g_{12} + g_{23} + g_{13} > g_{12} + g_{23} = v(S) + v(T)$ which contradicts $v(N) = v(S) + v(T)$. Therefore, either $v(S) = 0$ or $v(T) = 0$. Suppose $v(S) = 0$ for all S with $|S| = 2$, then $u_1 \geq u_2 \geq u_3$ and hence $v(N) = 0$ as well. This completes the proof of the "only if" part. For the "if" part, note that there are three possible cases: either only one 2-person coalition or two 2-person coalitions or all three 2-person coalitions have worth zero. In the last case v is the game with $v(S) = 0$ for all $S \in 2^N$. This a sequencing game arising from a sequencing situation $(\sigma; \alpha; s)$ with $u_{\sigma^{-1}(1)} \geq u_{\sigma^{-1}(2)} \geq u_{\sigma^{-1}(3)}$. Without loss of general-

ity it is assumed that in the first case $v(1, 3) = 0$ and in the second case $v(1, 3) = v(1, 2) = 0$. Let $(\rho; \gamma; t)$ and $(\rho; \delta; w)$ be sequencing situations with $\rho(1) = 1, \rho(2) = 2, \rho(3) = 3, \gamma = (0, v(1, 2), v(1, 2, 3) - v(1, 2) - v(2, 3)), t = (1, (v(2, 3) + v(1, 2)) / (v(1, 2, 3) - v(1, 2) - v(2, 3)), 1), \delta = (v(2, 3), 0, v(2, 3))$ and $w = (v(1, 2, 3) / v(2, 3), 1, 1)$. Straightforward verification shows that in the first case v is the sequencing game arising from $(\rho; \gamma; t)$ and in the second case v is the sequencing game arising from $(\rho; \delta; w)$. This completes the proof. $\square$

Because sequencing games are convex games they are also balanced and the Shapley-value is the barycenter of the core. There exists a simple expression for the Shapley-value of a sequencing game which does not involve computing the characteristic function for all the coalitions.

4.6.9 Theorem *Let v be a sequencing game corresponding to the sequencing situation $(\sigma; \alpha; s)$. Then the Shapley-value $\varphi(v)$ of v is given by*

$$\varphi_i(v) = \sum_{\sigma(k) \leq \sigma(i) \leq \sigma(j)} \frac{g_{kj}}{\sigma(j) - \sigma(k) + 1} \text{ for each } i \in N.$$

Proof. For each $i, j \in N$ with $\sigma(j) > \sigma(i)$ define a game v_{ij} by

$$v_{ij}(S) := \begin{cases} g_{ij} & \text{if } \{l \mid \sigma(i) \leq \sigma(l) \leq \sigma(j)\} \subset S \\ 0 & \text{otherwise.} \end{cases}$$

Let T be a connected coalition; then $v_{ij}(T) = g_{ij}$ iff $\{i, j\} \subset T$. For a non-connected coalition S , $v_{ij}(S) = \sum_{T \in S/\sigma} v_{ij}(T)$. It follows that for a connected coalition T

$$\sum_{\sigma(k) < \sigma(j)} v_{kj}(T) = \sum_{j \in T} \sum_{k \in P(\sigma, j) \cap T} g_{kj} = v(T)$$

and for a non-connected coalition S

$$\sum_{\sigma(k) < \sigma(j)} v_{kj}(S) = \sum_{\sigma(k) < \sigma(j)} \sum_{T \in S/\sigma} v_{kj}(T) = \sum_{T \in S/\sigma} v(T) = v(S).$$

Thus, $v = \sum_{\sigma(k) < \sigma(j)} v_{kj}$. Because of the efficiency, symmetry and dummy player property of the Shapley-value (cf. 1.4.6) it follows that

$$\varphi_i(v_{kj}) = \begin{cases} g_{kj} / (\sigma(j) - \sigma(k) + 1) & \text{if } i \in \{l \mid \sigma(k) \leq \sigma(l) \leq \sigma(j)\} \\ 0 & \text{otherwise.} \end{cases}$$

From the linearity of the Shapley-value it follows that

$$\varphi_i(v) = \sum_{\sigma(k) < \sigma(j)} \varphi_i(v_{kj}) = \sum_{\sigma(k) \leq \sigma(i) \leq \sigma(j)} \frac{g_{kj}}{\sigma(j) - \sigma(k) + 1}$$

and this completes the proof. $\square$

Since sequencing games are balanced and balanced games are quasi-balanced the τ -value of a sequencing game v is given by 1.6.2.

4.6.10 Theorem *Let v be a sequencing game corresponding to the sequencing situation $(\sigma; \alpha; s)$. Then the τ -value $\tau(v)$ of v is given by*

$$\tau_i(v) = \sum_{\sigma(k) \leq \sigma(i) \leq \sigma(j)} g_{kj} \lambda \text{ for all } i \in N$$

where $\lambda = \sum_{j \in N} \sum_{k \in P(\sigma, j)} g_{kj} / \sum_{j \in N} \sum_{\sigma(k) \leq \sigma(j) \leq \sigma(l)} g_{kl}$.

Proof. From 1.6.5 it follows that for a convex game w , $\tau_i(w) = \lambda M_i^w + (1 - \lambda)w(i)$ where $\lambda \in [0, 1]$. Since v is convex and $v(i) = 0$ this yields $\tau_i(v) = \lambda M_i^v$. Further, $M_i^v = v(N) - v(N \setminus \{i\}) = \sum_{\sigma(k) \leq \sigma(i) \leq \sigma(j)} g_{kj}$. Efficiency of the τ -value leads to

$$\lambda = \frac{v(N)}{\sum_{j \in N} M_j(v)} = \frac{\sum_{j \in N} \sum_{k \in P(\sigma, j)} g_{kj}}{\sum_{j \in N} \sum_{\sigma(k) \leq \sigma(j) \leq \sigma(l)} g_{kl}}$$

and the proof is completed. $\square$

4.6.11 Remark As was already noted at the beginning of this section there are some similarities and some differences between permutation games and sequencing games. The situation where customers are waiting in a queue to be served which was used as an example of a sequencing situation is an example of a permutation situation whenever all the service times are the same. However, in general it is not possible to define a matrix K as used in the definition of permutation games for sequencing games. The reason for this is that although for both permutation games as well as sequencing games the worth of the grand coalition can be found by maximizing a function on the set of permutation matrices which correspond to permutations of N , in the case of permutation games it is possible to give a linear extension

of this function to the set of doubly stochastic matrices while in the case of sequencing games this need not be possible as the following example from Curiel (1987) illustrates.

4.6.12 Example Let $N = \{1, 2, 3\}$, $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 3$, $s_1 = 7$, $s_2 = 3$, $s_3 = 5$, $\alpha_1 = 10$, $\alpha_2 = 20$, $\alpha_3 = 30$. Consider the six permutation matrices corresponding to the permutations of N .

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} P_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Then the function k to be maximized on $\{P_1, P_2, P_3, P_4, P_5, P_6\}$ in order to find $v(N)$ has the following values: $k(P_1) = 0$, $k(P_2) = -10$, $k(P_3) = 110$, $k(P_4) = 150$, $k(P_5) = 270$, $k(P_6) = 260$. Let D be the doubly stochastic matrix defined by $D := \frac{1}{3}P_1 + \frac{1}{3}P_4 + \frac{1}{3}P_5$ then $D = \frac{1}{3}P_2 + \frac{1}{3}P_3 + \frac{1}{3}P_6$. For k to have a linear extension on the set of doubly stochastic matrices $\frac{1}{3}P_1 + \frac{1}{3}P_4 + \frac{1}{3}P_5$ has to equal $\frac{1}{3}P_2 + \frac{1}{3}P_3 + \frac{1}{3}P_6$. But the first expression equals 140 and the last equals 120. Therefore it is impossible to extend k to a linear function on the set of doubly stochastic matrices.

A way of dividing the gains in a sequencing situation without using game-theoretic concepts is given by the *Equal Gain Splitting* or *EGS*-rule.

4.6.13 Definition A division rule for sequencing situations is a function f which assigns to every sequencing situation consisting of a set N and an ordered triplet $(\sigma; \alpha; s)$ a vector $f(\sigma; \alpha; s) = (f_1(\sigma; \alpha; s), \dots, f_n(\sigma; \alpha; s))$ such that

- (i) $f_i(\sigma; \alpha; s) \geq 0$ for each $i \in N$. (individual rationality)
- (ii) $\sum_{i \in N} f_i(\sigma; \alpha; s) = C_\sigma - \min_{\tau \in \Pi_N} C_\tau$. (efficiency)

An example of a division rule is the *equal division* rule which divides the cost savings equally among the members of N . A drawback of this method

is that it does not distinguish between customers who actually contribute to the savings and those who do not. The *EGS*-rule avoids this disadvantage.

4.6.14 Definition *The Equal Gain Splitting rule (EGS-rule) is the division rule for sequencing situations which assigns to every sequencing situation $(\sigma; \alpha; s)$ the vector $(EGS_1(\sigma; \alpha; s), \dots, EGS_n(\sigma; \alpha; s))$ defined by*

$$EGS_i(\sigma; \alpha; s) := \frac{1}{2} \sum_{k \in P(\sigma, i)} g_{ki} + \frac{1}{2} \sum_{j: i \in P(\sigma, j)} g_{ij} \text{ for each } i \in N.$$

In a sequencing situation $(\sigma; \alpha; s)$ an optimal arrangement can be reached by switching neighbours who are not standing in decreasing order of urgency index. Every time such a switching takes place between i and j with $\sigma(j) = \sigma(i) + 1$ and $u_i < u_j$ the gain equals g_{ij} . The *EGS*-rule divides the gain that arises whenever two neighbours are switched equally between them. That the *EGS*-rule is indeed a division rule can easily be checked by the reader. The *EGS*-rule can be characterized by three properties for division rules.

4.6.15 Definition *Let $(\sigma; \alpha; s)$ be a sequencing situation. Then $i \in N$ is said to be a dummy in $(\sigma; \alpha; s)$ if $\sigma(j) > \sigma(i)$ implies $u_j \leq u_i$ and $\sigma(k) < \sigma(i)$ implies $u_k \geq u_i$ for all $j, k \in N$.*

4.6.16 Definition *Two sequencing situations $(\sigma; \alpha; s)$ and $(\tau; \alpha; s)$ are called i -equivalent if $P(\sigma, i) = P(\tau, i)$.*

4.6.17 Definition *Let $(\sigma; \alpha; s)$ be a sequencing situation and let $i, j \in N$ with $|\sigma(i) - \sigma(j)| = 1$. Then $(\tau; \alpha; s)$ with $\tau(i) = \sigma(j)$, $\tau(j) = \sigma(i)$ and $\tau(k) = \sigma(k)$ for all $k \in N \setminus \{i, j\}$ is called the ij -inverse of $(\sigma; \alpha; s)$.*

4.6.18 Definition *A division rule f for sequencing situation is said to satisfy the dummy property if $f_i(\sigma; \alpha; s) = 0$ whenever i is a dummy in $(\sigma; \alpha; s)$; f is said to possess the equivalence property if $f_i(\sigma; \alpha; s) = f_i(\tau; \alpha; s)$ for*

each $i \in N$ and each pair of i -equivalent sequencing situations $(\sigma; \alpha; s)$ and $(\tau; \alpha; s)$; f is said to possess the switch property if $f_i(\tau; \alpha; s) - f_i(\sigma; \alpha; s) = f_j(\tau; \alpha; s) - f_j(\sigma; \alpha; s)$ for all $i, j \in N$ and all pairs of sequencing situations $(\sigma; \alpha; s)$ and $(\tau; \alpha; s)$ where $(\tau; \alpha; s)$ is the ij -inverse of $(\sigma; \alpha; s)$.

4.6.19 Theorem *The EGS-rule is the unique division rule for sequencing situations that possesses the dummy property, the equivalence property and the switch property.*

Proof. Let $(\sigma; \alpha; s)$ be a sequencing situation such that i is a dummy. Then $g_{ki} = 0$ for all $k \in P(\sigma, i)$ and $g_{ij} = 0$ for all j with $i \in P(\sigma, j)$. From the definition of the EGS-rule it follows that $EGS_i(\sigma; \alpha; s) = 0$ and hence the EGS-rule possesses the dummy property.

Let $(\sigma; \alpha; s)$ and $(\tau; \alpha; s)$ be two i -equivalent sequencing situations. Then $P(\sigma, i) = P(\tau, i)$ and hence $\{j \in N \mid i \in P(\sigma, j)\} = \{j \in N \mid i \in P(\tau, j)\}$. Now $EGS_i(\sigma; \alpha; s) = EGS_i(\tau; \alpha; s)$ from the definition of the EGS-rule and so the EGS-rule possesses the equivalence property.

Let $(\sigma; \alpha; s)$ be a sequencing situation and $(\tau; \alpha; s)$ its ij -inverse. Without loss of generality it is assumed that $\sigma(i) < \sigma(j)$. Then $EGS_i(\tau; \alpha; s) - EGS_i(\sigma; \alpha; s) = \frac{1}{2}g_{ji} - \frac{1}{2}g_{ij} = EGS_j(\tau; \alpha; s) - EGS_j(\sigma; \alpha; s)$. Thus, the EGS-rule possesses the switch property.

Suppose f is a division rule for sequencing situations which possesses these three properties. Let the set of misplaced pairs of neighbours in a sequencing situation $(\sigma; \alpha; s)$ be defined by $\mathcal{M}_\sigma := \{(i, j) \mid \sigma(i) = \sigma(j) + 1, u_i > u_j\}$. By induction on the cardinality of $\mathcal{M}_\sigma$ it will be proved that f is equal to the EGS-rule. Let $(\sigma; \alpha; s)$ be a sequencing situation with $\mathcal{M}_\sigma = \emptyset$, then everybody is a dummy and it follows that $f(\sigma; \alpha; s) = (0, 0, \dots, 0) = EGS(\sigma; \alpha; s)$. Suppose $f(\sigma; \alpha; s) = EGS(\sigma; \alpha; s)$ for all sequencing situations $(\sigma; \alpha; s)$ with $|\mathcal{M}_\sigma| \leq m$, where $m \geq 0$. Let $(\tau; \alpha; s)$ be such that $|\mathcal{M}_\tau| = m + 1$. Then there exist a sequencing situation $(\sigma; \alpha; s)$ and a pair $(k, l) \in \mathcal{M}_\tau$ such that $\sigma(i) = \tau(i)$ for all $i \notin \{k, l\}$ and $\sigma(k) = \tau(l)$, $\sigma(l) = \tau(k)$. Then $\mathcal{M}_\sigma = \mathcal{M}_\tau \setminus \{(k, l)\}$ and the equivalence property and the induction assumption yield $f_i(\tau; \alpha; s) = f_i(\sigma; \alpha; s) = EGS_i(\sigma; \alpha; s) = EGS_i(\tau; \alpha; s)$ for all $i \notin \{k, l\}$. Further, $C_\tau - C_\sigma = g_{kl}$ and the efficiency, the switch property and the induction assumption yield $f_k(\tau; \alpha; s) = f_k(\sigma; \alpha; s) + \frac{1}{2}g_{kl} = EGS_k(\sigma; \alpha; s) + \frac{1}{2}g_{kl} = EGS_k(\tau; \alpha; s)$. In the same way $f_l(\tau; \alpha; s) = EGS_l(\tau; \alpha; s)$ is obtained. Hence $f(\tau; \alpha; s) = EGS(\tau; \alpha; s)$ and the proof is completed. $\square$

4.6.20 Theorem Let v be the sequencing game corresponding to the sequencing situation $(\sigma; \alpha; s)$. Then $EGS(\sigma; \alpha; s) \in C(v)$.

Proof. Because of the efficiency of the EGS -rule the equality $\sum_{i \in N} EGS_i(\sigma; \alpha; s) = v(N)$ holds. Let $S \in 2^N$, then

$$\begin{aligned} \sum_{i \in S} EGS_i(\sigma; \alpha; s) &= \sum_{i \in S} \frac{1}{2} \left(\sum_{k \in P(\sigma, i)} g_{ki} + \sum_{j \in P(\sigma, j)} g_{ij} \right) \geq \\ \sum_{i \in S} \frac{1}{2} \left(\sum_{k \in P(\sigma, i) \cap S} g_{ki} + \sum_{j \in S, i \in P(\sigma, j)} g_{ij} \right) &= \sum_{i \in S} \sum_{k \in P(\sigma, i) \cap S} g_{ki} \geq v(S). \end{aligned}$$

Thus, $EGS(\sigma; \alpha; s) \in C(v)$.

The meaning of theorem 4.6.20 is that even though the EGS -rule has been introduced without considering the sequencing game arising from a sequencing situation it still yields a division which is in the cores of the game.

4.6.21 Remark A kind of abstract sequencing situation can be defined where only the original permutation σ and the non-negative g_{ij} 's are given, with the property that if $g_{ij} > 0$ then $g_{ji} = 0$, without the interpretation that they result from costs and service times. From these abstract sequencing situations, abstract sequencing games can be defined in the same way as in (11) and (12) because only the g_{ij} 's and σ are needed in these definitions. In fact, everything that has been defined and proved for sequencing situations and sequencing games can also be defined and proved for abstract sequencing situations and abstract sequencing games. The following example shows that there exists indeed abstract sequencing situations which are not sequencing situations.

4.6.22 Example Let $N = \{1, 2, 3\}$, $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 3$ and $g_{12} = g_{23} = g_{31} = 1$, $g_{13} = g_{32} = g_{21} = 0$. Suppose that there existed $\alpha \in \mathbb{R}^3$ and $s \in \mathbb{R}_+^3$ such that $g_{ij} = (\alpha_j s_i - \alpha_i s_j)_+$ for all $i, j \in \{1, 2, 3\}$. Then $\alpha_2/s_2 = \alpha_1/s_1 + 1/s_1 s_2$, $\alpha_3/s_3 = \alpha_2/s_2 + 1/s_2 s_3$ and $\alpha_1/s_1 = \alpha_3/s_3 + 1/s_1 s_3$ which lead to the contradiction $\alpha_2/s_2 > \alpha_1/s_1 > \alpha_3/s_3$ and $\alpha_3/s_3 > \alpha_2/s_2$. Therefore, such an α and s do not exist and this situation is an abstract

sequencing situation but not a sequencing situation.

The idea of connectedness can also be found in a somewhat different framework in Myerson (1977) and Owen (1984) Myerson considers games with *cooperation structures* described by graphs. The nodes of such a graph are the players and the edges are unordered pairs of distinct players. Given a set of players N and a graph which describes the cooperation structure in N , for each $S \in 2^N$ two players in S are *connected* iff there is a path going from one to the other and staying in S . For a game v Myerson introduces an *allocation rule* for v as a function defined on the set of all possible graphs describing cooperation structures in N , with values in $\mathbb{R}^n$ such that for each component S of a given graph the sum of the quantities allocated to the members of S is equal to $v(S)$. Myerson calls such an allocation rule *fair* if for any two players $i, j \in N$ the difference between the quantities the rule assigns to i in a situation where $\{i, j\}$ is an edge and a situation where this is not the case, is equal to the difference between the quantities the rule assigns to j in these two situations. He calls an allocation rule *stable* if for any two player $i, j \in N$ the quantity the rule assigns to $i(j)$ in a situation where $\{i, j\}$ is an edge is at least equal to the quantity assigned to $i(j)$ when this is not the case. Myerson proves that there exists a unique fair allocation rule and that this rule is stable whenever v is superadditive. This rule assigns to a certain graph g , describing a cooperation structure in N , the Shapley-value of the *graph restricted game* v/g defined by

$$v/g(S) := \sum_{T \in S/g} v(T) \text{ for each } S \in 2^N,$$

where S/g is the partition of S into components in the graph g .

For a given graph g , Owen(1984) studies the mapping which assigns to each game v the game v/g . This mapping is linear and Owen proves that its image is the set spanned by the unanimity games u_T , where T is connected in g . For several games v , Owen computes the Shapley-value of v/g in the case that g is a tree. The difference between the games with cooperation structure considered by Myerson and Owen and sequencing games is that for sequencing games the cooperation structure is described by a directed graph rather than an undirected one. For a coalition containing $i, j \in N$ it really matters whether (i, j) or (j, i) is an arc of the graph.

4.7 Traveling salesman games

The *traveling salesman* problem is a very well known combinatorial optimization problem. In one of the formulations of the problem a *directed graph* is given with *weights* on the arcs and a cycle which visits each node exactly once and has *minimal weight* has to be found. Here the weight of a cycle is the sum of the weights of the arcs belonging to it. In a arbitrarily given graph there need not exist a cycle which visits each node exactly once. Even when such a cycle exists there may be a cycle with lesser weight which visits some nodes more than once. However if the graph is complete and the weights satisfy the *triangle inequality* (i.e. if for any two nodes i, j of the graph w_{ij} denotes the weight of the arc beginning in i and ending in j then $w_{ij} + w_{jk} \geq w_{ik}$ for all nodes i, j, k) then there is a cycle in the graph which visits each node exactly once and such that no cycle which visits some nodes more than once has lesser weight. Every traveling salesman problem on a graph can be transformed into a problem on a complete graph with weights satisfying the triangle inequality by defining the complete graph on the same set of nodes and taking the weight w_{ij} in the new graph to be the weight of a minimal path from i to j in the original graph. In the following it is assumed that the traveling salesman problems under consideration are on a complete graph with weights satisfying the triangle inequality unless stated otherwise.

Traveling salesman games arise from situations like the following. Each of n universities has invited a speaker to deliver a lecture. The speaker has to travel from his home town to all the universities and back. The problem of finding a cheapest way for him to visit all the universities and then return home is a traveling salesman problem. The problem of how to divide his travel cost among all the universities which invited him is a *cost allocation problem* which can be modeled with the aid of a cooperative game, namely a traveling salesman game. In the definition of the characteristic function of a traveling salesman game it is assumed that every coalition (in the example above every subset of universities) looks at the traveling cost if the speaker visits only members of the coalition and then returns home. Such a round trip of the speaker can be described with the aid of a bijection $e : \{1, \dots, |S|\} \rightarrow S$ where $e(1)$ is the university visited first, $e(2)$ the one visited second and so forth. Let the set of all such functions be denoted by $E(S)$. Let 0 denote the home of the speaker and let for each $S \in 2^N$ the set $S \cup 0$ be denoted by S_0 .

Definition 4.7.1 A cooperative game c is called a traveling salesman game whenever for all $i, j \in N_0$ there exist $w_{ij} > 0$ such that $c(S)$ is given for all $S \in 2^N$ by

$$c(S) := \min\{w_{0e(1)} + w_{e(1)e(2)} + \cdots + w_{e(|S|)0} \mid e \in E(S)\}.$$

Another way to express $c(S)$ is with the following integer programming problem. (Cf. Tucker 1960)

$$\begin{aligned} c(S) = \min \quad & \sum_{i \in S_0} \sum_{j \in S_0} w_{ij} x_{ij} \quad \text{subject to} \\ & \sum_{j \in S_0 \setminus \{i\}} x_{ij} = 1_{S(i)} \quad \text{for all } i \in N \\ & \sum_{i \in S_0 \setminus \{j\}} x_{ij} = 1_{S(j)} \quad \text{for all } j \in N \\ & u_i - u_j + nx_{ij} \leq n - 1 \quad \text{for all } i, j \in S \text{ with } i \neq j \\ & x_{ij} \in \{0, 1\}, u_i \in \mathbb{R} \quad \text{for all } i, j \in N. \end{aligned} \tag{13}$$

Here $x_{ij} = 1$ means that the speaker travels directly from university i to university j and $x_{ij} = 0$ means that he does not visit university j directly after university i . The first $2n$ conditions ensure that each university is visited exactly once, while the next $|S|^2$ conditions guarantee that the travel plan described by the x_{ij} 's does not contain loops without 0.

4.7.2 Remark Note that while in definition 4.7.1 no town including 0 can be visited more than once, in the definition given by (13) it is possible for 0 to be visited more than once. However, because the weights satisfy the triangle inequality the two minima are the same.

4.7.3 Theorem Traveling salesman games need not be balanced.

Proof. Let $N = \{1, 2, 3, 4\}$ and let the w_{ij} 's for $i, j \in N_0$ be given by:

$$\begin{array}{cccc} w_{01} = 1 & w_{02} = 2 & w_{03} = 2 & w_{04} = 1 \\ w_{10} = 1 & w_{12} = 1 & w_{13} = 2 & w_{14} = 2 \\ w_{20} = 2 & w_{21} = 1 & w_{23} = 1 & w_{24} = 2 \\ w_{30} = 1 & w_{31} = 2 & w_{32} = 2 & w_{34} = 2 \\ w_{40} = 1 & w_{41} = 2 & w_{42} = 1 & w_{43} = 1. \end{array}$$

Then a minimal cycle for N is given by $e(1) = 1, e(2) = 2, e(3) = 3, e(4) = 4$ which yields $c(N) = 6$. A minimal cycle for $\{1, 2, 3\}$ is given by $e(1) = 1, e(2) = 2, e(3) = 3$ yielding $c(1, 2, 3) = 4$, a minimal cycle for $\{1, 2, 4\}$ is given by $e(1) = 4, e(2) = 2, e(3) = 1$ yielding $c(1, 2, 4) = 4$, a minimal cycle for $\{3, 4\}$ is given by $e(1) = 4, e(3) = 2$ yielding $c(3, 4) = 3$. Then

$$\frac{1}{2}c(1, 2, 3) + \frac{1}{2}c(1, 2, 4) + \frac{1}{2}c(3, 4) = 5\frac{1}{2} < 6 = c(1, 2, 3, 4)$$

violating a balancedness condition. Thus, c is not balanced. $\square$

For a traveling salesman game to have an empty core it must have at least 4 players as the following theorem shows.

4.7.4 Theorem *Let c be a traveling salesman game with three or less players. Then c is balanced.*

Proof. If c is a game with only one player then c is trivially balanced. If c is a traveling salesman game with two players then the balancedness of c follows from the subadditivity of c . Let c be a traveling salesman game with set of players $N = \{1, 2, 3\}$. To prove that c is balanced only the inequality

$$c(1, 2) + c(1, 3) + c(2, 3) \geq 2c(1, 2, 3) \quad (14)$$

has to be checked. (Cf.1.3.7) Two cases will be distinguished.

(i) In the expression $c(1, 2) + c(1, 3) + c(2, 3)$ there is a w_{0i} occurring twice. Without loss of generality it is assumed that w_{01} is occurring twice. Then

$$c(1, 2) + c(1, 3) + c(2, 3) = w_{01} + w_{12} + w_{20} + w_{01} + w_{13} + w_{30} + \min\{w_{02} + w_{23} + w_{30}, w_{03} + w_{32} + w_{20}\}.$$

Let the minimum be equal to $w_{02} + w_{23} + w_{30}$. The other case can be treated similarly. Then

$$\begin{aligned} c(1, 2) + c(1, 3) + c(2, 3) &= w_{01} + w_{12} + w_{20} + w_{01} + w_{13} + w_{30} + w_{02} + w_{23} + w_{30} \\ &= w_{01} + w_{12} + w_{23} + w_{30} + w_{02} + w_{20} + w_{01} + w_{13} + w_{30} \\ &\geq c(1, 2, 3) + w_{02} + w_{20} + w_{01} + w_{13} + w_{30} \\ &\geq c(1, 2, 3) + w_{02} + w_{21} + w_{13} + w_{30} \\ &\geq 2c(1, 2, 3) \end{aligned}$$

(ii) In the expression $c(1, 2) + c(1, 3) + c(2, 3)$ there is no w_{0i} occurring twice. Then for each $i \in N$ w_{0i} occurs exactly once. Then

$$c(1, 2) + c(1, 3) + c(2, 3) = w_{01} + w_{12} + w_{20} + w_{02} + w_{23} + w_{30} + w_{03} + w_{31} + w_{10}$$

or

$$c(1, 2) + c(1, 3) + c(2, 3) = w_{01} + w_{13} + w_{30} + w_{03} + w_{32} + w_{20} + w_{02} + w_{21} + w_{10}.$$

Both cases can be treated similarly. Consider the first case. Then

$$\begin{aligned} c(1, 2) + c(1, 3) + c(2, 3) &= w_{01} + w_{12} + w_{20} + w_{02} + w_{23} + \\ &\quad w_{30} + w_{03} + w_{31} + w_{10} \\ &= w_{01} + w_{12} + w_{23} + w_{30} + w_{20} + \\ &\quad w_{02} + w_{03} + w_{31} + w_{10} \\ &\geq c(1, 2, 3) + w_{20} + w_{02} + w_{03} + w_{31} + w_{10} \\ &\geq c(1, 2, 3) + w_{02} + w_{23} + w_{31} + w_{10} \\ &\geq 2c(1, 2, 3) \end{aligned}$$

It follows that inequality (14) holds and hence c is balanced. $\square$

4.7.5 Proposition *Let c be a traveling salesman game such that for each $i \in N_0$ there exist u_i, z_i with $w_{ij} = u_i + z_i$ for all $i, j \in N_0$. Then*

$$C(c) = \{ x \in \mathbb{R}^n \mid x_i = u_i + z_i + \lambda_i(u_0 + z_0) \text{ with } 0 \leq \lambda \leq 1 \text{ for all } i \in N \text{ and } \sum_{i \in N} \lambda_i = 1 \}.$$

Proof. For such a traveling salesman game all travel plans for a certain coalition S have the same cost, namely $\sum_{i \in S_0} (u_i + z_i)$. Let $y \in C(c)$. Then $y_i \leq u_i + z_i + u_0 + z_0$ and $y_i \geq c(N) - c(N \setminus \{i\}) = u_i + z_i$ for all $i \in N$. This and the efficiency of y imply that y is an element of the set on the right hand side. Let y be an element of this set. Then $y(N) = c(N)$ and for all $S \in 2^N$, $y(S) \leq \sum_{i \in S_0} (u_i + z_i) = c(S)$. Hence $y \in C(c)$ and the proof is completed. $\square$

4.7.6 Remark The numbers u_i, z_i could be seen as taxes which have to be paid when entering and leaving a town respectively. If there are no other travel costs that matter, the cost of traveling from i to j is simply equal to $u_i + z_j$.

4.7.7 Proposition Consider a traveling salesman problem with set of nodes N_0 where $N = \{1, 2, \dots, n\}$ and weights w_{ij} for $i, j \in N_0$. Then there exist u_i, z_i for all $i \in N_0$ such that $w_{ij} = u_i + z_i$ for all $i, j \in N_0$ iff for all $j \in N$, $w_{ij} - w_{i0}$ is the same for all $i \in N_0$ where w_{00} is taken to be equal to 0.

Proof. Let w_{ij} be such that there exist u_i, z_i for all $i \in N_0$ with $w_{ij} = u_i + z_j$ for all $i, j \in N_0$. Then $w_{ij} - w_{i0} = u_i + z_j - u_i - z_0 = z_j - z_0$ for all $i \in N_0$. This proves the “only if” part. Suppose $w_{ij} - w_{i0}$ is the same for all $i \in N_0$, say $w_{ij} - w_{i0} = b_j$. Define $a_i = w_{i0}$ for all $i \in N_0$ and $b_0 = 0$. Then $w_{ij} = a_i + b_j$ for all $i, j \in N_0$. Thus, the “if” part is proved. $\square$

It seems intuitively clear that when the home town is relatively far away from the universities while these are more or less clustered together, causing the travel cost of a trip around all the universities to be rather cheap compared with the cost of traveling from the home town to any university and back to the home town (not necessarily from the same university), it is profitable for the universities to cooperate and it should be possible to distribute the travel costs in such a way that no coalition will prefer to work on its own. That this is indeed the case is shown in the following theorem.

4.7.8 Theorem Let c be a traveling salesman game such that for all $i, j \in N$

$$(w_{0i} + w_{j0})/n \geq \max\{w_{e(1)e(2)} + w_{e(2)e(3)} + \dots + w_{e(n-1)e(n)} \mid e \in E(N)\}.$$

Then c is balanced.

Proof. Without loss of generality it is assumed that $c(N) = w_{01} + w_{12} + \dots + w_{n0}$. Let $x \in \mathbb{R}^n$ be defined by

$$\begin{aligned} x_i &= w_{ii+1} + (w_{01} + w_{n0})/n \quad \text{for } i \in n \setminus \{n\} \\ x_n &= (w_{01} + w_{n0})/n. \end{aligned}$$

Then $x(N) = \sum_{i \in N \setminus \{n\}} w_{ii+1} + w_{01} + w_{n0} = c(N)$.

Let $S \subset N$, $S \neq N$ with $e \in E(S)$ such that $c(S) = w_{0e(1)} + w_{e(1)e(2)} + \dots + w_{e(|S|)0}$. In the following $w_{e(1)e(2)} + w_{e(2)e(3)} + \dots + w_{e(|S|-1)e(|S|)}$ is denoted by M_S and $\max\{w_{e(1)e(2)} + \dots + w_{e(n-1)e(n)} \mid e \in E(N)\}$ is denoted by M .

Then

$$\begin{aligned}
c(S) &= w_{0e(1)} + w_{e(|S|)0} + M_S \\
&= \frac{|S|}{n}(w_{0e(1)} + w_{e(|S|)0}) + \frac{n - |S|}{n}(w_{0e(1)} + w_{e(S)0}) + M_S \\
&\geq \frac{|S|}{n}(w_{0e(1)} + w_{e(|S|)0}) + (n - |S|)M + M_S \\
&= \frac{|S|}{n}(w_{0e(1)} + w_{e(|S|)0}) + \frac{|S|}{n}M + (n - |S| - \frac{|S|}{n})M + M_S \\
&\geq \frac{|S|}{n}(w_{01} + w_{n0} + \sum_{i \in S \setminus \{n\}} w_{ii+1}) + (n - |S| - \frac{|S|}{n})M + M_S \\
&\geq \frac{|S|}{n}(w_{01} + w_{n0} + \sum_{i \in S \setminus \{n\}} w_{ii+1}) + \frac{n - |S|}{n}M + M_S \\
&\geq \frac{|S|}{n}(w_{01} + w_{n0}) + \sum_{i \in S \setminus \{n\}} w_{ii+1} \\
&= x(S).
\end{aligned}$$

Here the second inequality follows from the fact that $w_{0e(1)} + w_{e(|S|)0} + M < w_{01} + w_{n0} + \sum_{i \in S \setminus \{n\}} w_{ii+1} < w_{01} + w_{n0} + \sum_{i \in N \setminus \{n\}} w_{ii+1}$ would contradict $c(N) = w_{01} + w_{12} + \dots + w_{n0}$ because then any travel plan starting with going from 0 to $e(1)$ and ending with going from $e(|S|)$ to 0 would be cheaper. Hence $x \in C(c)$ and c is balanced. $\square$

4.7.9 Remark A situation as described in theorem 4.7.7 could arise when a speaker from the United States or Canada is invited to deliver a lecture at some universities in the Netherlands. The cost of traveling around the universities in the Netherlands would be negligible compared with the cost of the transatlantic flights.

Consider the following variation on traveling salesman games. In the example of the speaker and the universities the cost $c(N)$ of the grand coalition N is again taken to be the total cost of a cheapest travel plan of the speaker. The cost $c(S)$ of a coalition $S \subset N$, $S \neq N$ however, is no longer the cost of a cheapest travel plan along the universities in S but it is the cost arising from skipping the universities not in S and visiting those in S in the same order as in the travel plan for N .

4.7.10 Definition A game c is called a routing game if for all $i, j \in N_0$ there exist $w_{ij} > 0$ such that there is a $e_N \in E(N)$ with

$$\begin{aligned} c(N) &= w_{0e_N(1)} + w_{e_N(1)e_N(2)} + \cdots + w_{e_N(n)0} \\ &= \min\{w_{0e(1)} + w_{e(1)e(2)} + \cdots + w_{e(n)0} \mid e \in E(N)\} \end{aligned}$$

and

$$c(S) = w_{0e_S(1)} + w_{e_S(1)e_S(2)} + \cdots + w_{e_S(|S|)0} \text{ for all } S \in 2^N$$

where e_S is defined by

$$e_S^{-1}(i) > e_S^{-1}(j) \Leftrightarrow e_N(i) > e_N(j) \text{ for all } i, j \in S.$$

4.7.11 Remark A justification for using a routing game rather than a traveling salesman game to model a situation like the one described above could be that each coalition $S \subset N$, $S \neq N$ prefers to construct a travel plan from an optimal travel plan for the grand coalition rather than computing an optimal plan for itself since the finding of an optimal plan tends to take a lot of computation time which could induce more costs than benefits. In fact routing games are not combinatorial games since only the computation of $c(N)$ involves solving a combinatorial optimization problem. Note that beside the grand coalition the one-person coalitions also have the same cost in a routing game as in the corresponding traveling salesman game so that the imputation set is the same for the two games.

4.7.12 Definition Let c be a routing game and let $e \in E(N)$. A coalition T is said to be an e -connected coalition if T is of the form $\{e^{-1}(i), e^{-1}(i+1), \dots, e^{-1}(i+p)\}$ with $\{i, i+1, \dots, i+p\} \subset N$.

4.7.13 Definition Let c be a routing game and let $e \in E(N)$. Let $S \subset N$. A coalition T is called an e -component of S if (i) $T \subset S$, (ii) T is e -connected and (iii) $T \cup \{i\}$ is not e -connected for all $i \in S \setminus T$.

4.7.14 Theorem Let $\mathcal{B}$ be a balanced collection with weights $\{\lambda_T\}_{T \in \mathcal{B}}$.

If there exists an $e \in E(N)$ such that T is e -connected for each $T \in \mathcal{B}$ then there are partitions $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_q$ of N with $\bigcup_{k=1}^q \mathcal{B}_k = \mathcal{B}$ and $\alpha_1, \alpha_2, \dots, \alpha_q \geq 0$ with $\sum_{k=1}^q \alpha_k = 1$ such that $\lambda_T = \sum_{k=1}^q \alpha_k 1_{\mathcal{B}_k}(T)$ for all $T \in \mathcal{B}$.

Proof. Let $\mathcal{B}$ be a balanced collection with weights $\{\lambda_T\}_{T \in \mathcal{B}}$ such that each $T \in \mathcal{B}$ is e -connected for an $e \in E(N)$. Without loss of generality it is assumed that $e(i) = i$ for each $i \in N$. Hence each $T \in \mathcal{B}$ is of the form $\{i, i+1, \dots, i+p\} \subset N$. The proof will proceed by induction on the number of elements of $\mathcal{B}$. If $\mathcal{B}$ has only one element then $\mathcal{B} = N$ and the property stated in the theorem is trivially satisfied. Suppose that all balanced collections with m or less elements satisfy the property. Here $m \geq 1$. Let $\mathcal{B}$ be a balanced collection with $m+1$ elements. There is a $T_1 \in \mathcal{B}$ with $T_1 = \{1, 2, \dots, j_1\}$ where $j_1 < n$. Because $\mathcal{B}$ is a balanced collection there is a $T_2 \in \mathcal{B}$ with $T_2 = \{j_1 + 1, \dots, j_2\}$. If $j_2 < n$ then there is a $T_3 \in \mathcal{B}$ with $T_3 = \{j_2 + 1, \dots, j_3\}$. Proceeding in this way a row $T_1, T_2, T_3, T_4, \dots, T_r$ of elements of $\mathcal{B}$ is obtained with $T_1 = \{1, \dots, j_1\}$, $T_i = \{j_{i-1} + 1, \dots, j_i\}$ for $i \in \{2, \dots, r\}$ with $j_r = n$. Let $\mathcal{B}_1 = \{T_1, T_2, \dots, T_r\}$ then $\mathcal{B}_1$ is a partition of N . Define

$$\alpha_1 := \min_{T \in \mathcal{B}_1} \lambda_T.$$

Let $\mathcal{B}'$ be the collection containing all the elements of $\mathcal{B}$ except those T in $\mathcal{B}_1$ with $\lambda_T = \alpha_1$. If $\mathcal{B}'$ is empty then the induction step is completed with $\mathcal{B}_1$ and α_1 as defined above. Otherwise, define

$$\begin{aligned} \lambda'_S &:= \lambda_S / (1 - \alpha_1) & \text{if } S \in \mathcal{B}' \setminus \mathcal{B}_1 \\ \lambda'_S &:= (\lambda_S - \alpha_1) / (1 - \alpha_1) & \text{if } S \in \mathcal{B}' \cap \mathcal{B}_1. \end{aligned}$$

Then $\lambda'_S > 0$ for all $S \in \mathcal{B}'$ and

$$\begin{aligned} \sum_{S \in \mathcal{B}'} \lambda'_S 1_S &= \sum_{S \in \mathcal{B} \setminus \mathcal{B}_1} 1_S \lambda_S / (1 - \alpha_1) + \sum_{T \in \mathcal{B}_1} 1_T (\lambda_T - \alpha_1) / (1 - \alpha_1) \\ &= 1 / (1 - \alpha_1) \sum_{S \in \mathcal{B}} \lambda_S 1_S - \alpha_1 / (1 - \alpha_1) \sum_{T \in \mathcal{B}_1} 1_T \\ &= 1_N / (1 - \alpha_1) - 1_N \alpha_1 / (1 - \alpha_1) = 1_N. \end{aligned}$$

Thus, $\mathcal{B}'$ is a balanced collection with weights $\{\lambda'_S\}_{S \in \mathcal{B}'}$ and m or less elements. From the induction assumption it follows that there exist partitions $\mathcal{B}_2, \mathcal{B}_3, \dots, \mathcal{B}_q$ of N with $\bigcup_{k=2}^q \mathcal{B}_k = \mathcal{B}'$ and $\gamma_2, \gamma_3, \dots, \gamma_q \geq 0$ with $\sum_{k=2}^q \gamma_k = 1$ such that $\lambda'_S = \sum_{k=2}^q \gamma_k 1_{\mathcal{B}_k}(S)$ for all $S \in \mathcal{B}'$. Define $\alpha_2, \alpha_3, \dots, \alpha_q$ by

$\alpha_k = \gamma_k(1 - \alpha_1)$ for all $k \in \{2, \dots, q\}$. Then $\alpha_k \geq 0$ for all $k \in \{2, \dots, q\}$ and $\sum_{k=1}^q \alpha_k = \alpha_1 + (1 - \alpha_1) = 1$. Furthermore,

$$\sum_{k=1}^q \alpha_k 1_{\mathcal{B}_k}(S) = \sum_{k=2}^q \alpha_k 1_{\mathcal{B}_k}(S) = (1 - \alpha_1) \lambda'_S = \lambda_S \text{ for } S \in \mathcal{B} \setminus \mathcal{B}_1$$

and

$$\sum_{k=1}^q \alpha_k 1_{\mathcal{B}_k}(T) = \sum_{k=2}^q \alpha_k 1_{\mathcal{B}_k}(T) + \alpha_1 = (1 - \alpha_1) \lambda'_T + \alpha_1 = \lambda_T \text{ for } T \in \mathcal{B}_1.$$

Because $\bigcup_{k=1}^q \mathcal{B}_k = \mathcal{B}_1 \cup \mathcal{B}' = \mathcal{B}$ the induction step and the proof are completed. $\square$

Theorem 4.7.14 is needed for the proof of the following theorem.

4.7.15 Theorem *Routing games are balanced.*

Proof. Let c be a routing game. Without loss of generality it is assumed that

$$\begin{aligned} c(N) &= w_{01} + w_{12} + \dots + w_{n0} \quad \text{and} \\ c(S) &= w_{0e_S(1)} + \dots + w_{e_S(|S|)0} \quad \text{for all } S \in 2^N \end{aligned}$$

where e_S is defined by

$$e_S(i) < e_S(j) \Leftrightarrow i < j.$$

In the following e_N is denoted by e . Let $\mathcal{B}$ be a balanced collection with weights $\{\lambda_S\}_{S \in \mathcal{B}}$. For each $S \in \mathcal{B}$ let $S/e = \{T_1^S, T_2^S, \dots, T_{p(S)}^S\}$ be the partition of S into e -components. For each $S \in \mathcal{B}$ and each $k \in \{1, \dots, p(S)\}$ let the e -component T_k^S of S be of the form $\{i_k(S), i_k(S) + 1, \dots, j_k(S)\}$ with $i_{k+1}(S) > j_k(S)$. Then for all $S \in \mathcal{B}$

$$c(S) = \sum_{k=1}^{p(S)} c(T_k) + \sum_{k=1}^{p(S)-1} w_{j_k(S)i_{k+1}(S)} - \sum_{k=1}^{p(S)-1} w_{j_k(S)0} - \sum_{k=2}^{p(S)} w_{0i_k(S)}.$$

This yields

$$\sum_{S \in \mathcal{B}} \lambda_S c(S) = \sum_{S \in \mathcal{B}} \lambda_S \left(\sum_{k=1}^{p(S)} c(T_k) + \sum_{k=1}^{p(S)-1} w_{j_k(S)i_{k+1}(S)} - \right) \quad (15)$$

$$\sum_{k=1}^{p(S)-1} w_{j_k(S)0} - \sum_{k=2}^{p(S)} w_{0i_k(S)}.$$

Let the collection $\mathcal{B}'$ of elements of 2^N be defined by

$$T \in \mathcal{B}' \Leftrightarrow T \in S/e \text{ for an } S \in \mathcal{B}.$$

Then each element of $\mathcal{B}'$ is e -connected. For each $T \in \mathcal{B}'$ let λ'_T be defined by

$$\lambda'_T := \sum_{S \in \mathcal{B}: T \in S/e} \lambda_S.$$

Then

$$\begin{aligned} \sum_{T \in \mathcal{B}'} \lambda'_T 1_T &= \sum_{T \in \mathcal{B}'} \sum_{S \in \mathcal{B}: T \in S/e} \lambda_S 1_T \\ &= \sum_{S \in \mathcal{B}} \lambda_S \sum_{T \in S/e} 1_T \\ &= \sum_{S \in \mathcal{B}} \lambda_S 1_S = 1_N. \end{aligned}$$

Thus $\mathcal{B}'$ is a balanced collection with weights $\{\lambda'_T\}_{T \in \mathcal{B}'}$ containing only e -connected sets. From theorem 4.7.14 it follows that there exist partitions $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_q$ of N with $\bigcup_{k=1}^q \mathcal{B}_k = \mathcal{B}'$ and $\alpha_1, \alpha_2, \dots, \alpha_q \geq 0$ with $\sum_{k=1}^q \alpha_k = 1$ such that $\lambda'_T = \sum_{k=1}^q \alpha_k 1_{\mathcal{B}_k}(T)$ for all $T \in \mathcal{B}'$. Then

$$\begin{aligned} \sum_{S \in \mathcal{B}} \lambda_S \sum_{k=1}^{p(S)} c(T_k^S) &= \sum_{T \in \mathcal{B}'} \lambda'_T c(T) \\ &= \sum_{T \in \mathcal{B}'} \sum_{k=1}^q \alpha_k 1_{\mathcal{B}_k}(T) c(T) \\ &= \sum_{k=1}^q \alpha_k \sum_{T \in \mathcal{B}_k} c(T) \end{aligned} \tag{16}$$

Let $\mathcal{B}^*$ be a partition of N containing only e -connected sets. Say $\mathcal{B}^* = \{T_1, T_2, \dots, T_m\}$ with $T_k = \{i_k, i_k + 1, \dots, j_k\}$ where $i_1 = 1$ and $j_m = n$ and $i_{k+1} = j_k + 1$. Then

$$\sum_{T \in \mathcal{B}^*} c(T) = c(N) + \sum_{k=1}^{m-1} w_{j_k 0} + \sum_{k=2}^m w_{0 i_k} - \sum_{k=1}^{m-1} w_{j_k i_{k+1}}.$$

Because $\bigcup_{k=1}^q B_k = B'$ and $T \in B'$ iff $T \in S/e$ for an $S \in B$ and

$$\sum_{k=1}^q \alpha_k 1_{B_k}(T) = \lambda'_T = \sum_{S \in B: T \in S/e} \lambda_S$$

it follows that

$$\begin{aligned} \sum_{k=1}^q \alpha_k \sum_{T \in B_k} c(T) &= \sum_{k=1}^q \alpha_k c(N) + \\ \sum_{S \in B} \lambda_S &\left(\sum_{k=1, j_k(S) \neq n}^{p(S)} w_{j_k(S)0} + \sum_{k=1, i_k(S) \neq 1} w_{0i_k(S)} - \sum_{k=1}^{p(S)-1} w_{j_k(S)j_k(S)+1} \right). \end{aligned} \quad (17)$$

Combining (15),(16) and (17) yields

$$\begin{aligned} \sum_{S \in B} \lambda_S c(S) &= c(N) + \sum_{S \in B, j_p(S) \neq n} \lambda_S w_{j_p(S)0} + \sum_{S \in B, i_1(S) \neq 1} \lambda_S w_{0i_1(S)} + \\ &\sum_{S \in B} \lambda_S \sum_{k=1}^{p(S)-1} w_{j_k(S)i_{k+1}(S)} - \sum_{S \in B} \lambda_S \sum_{k=1}^{p(S)-1} w_{j_k(S)j_k(S)+1} \\ &= c(N) + \sum_{S \in B, j_p(S) \neq n} \lambda_S w_{j_p(S)0} + \sum_{S \in B, i_1(S) \neq 1} \lambda_S w_{0i_1(S)} + \\ &\sum_{S \in B} \lambda_S \sum_{k=1}^{p(S)-1} w_{j_k(S)i_{k+1}(S)} + \sum_{S \in B} \lambda_S \sum_{k=1}^{p(S)} \sum_{i \in T_k^S \setminus \{j_k(S)\}} w_{ii+1} + \\ &w_{01} + w_{n0} - \sum_{S \in B} \lambda_S \sum_{k=1}^{p(S)-1} w_{j_k(S)j_k(S)+1} - w_{01} - w_{n0} - \\ &\sum_{S \in B} \lambda_S \sum_{k=1}^{p(S)} \sum_{i \in T_k^S \setminus \{j_k(S)\}} w_{ii+1}. \end{aligned}$$

Because

$$\begin{aligned} \sum_{S \in B} \lambda_S \sum_{k=1}^{p(S)-1} w_{j_k(S)j_k(S)+1} + \sum_{S \in B} \lambda_S \sum_{k=1}^{p(S)} \sum_{i \in T_k^S \setminus \{j_k(S)\}} w_{ii+1} + w_{01} + w_{n0} &= \\ \sum_{S \in B} \lambda_S \sum_{i \in S, i \neq n} w_{ii+1} + w_{01} + w_{n0} &= \sum_{i \in N, i \neq n} w_{ii+1} + w_{01} + w_{n0} = c(N) \end{aligned}$$

the following holds.

$$\begin{aligned} \sum_{S \in \mathcal{B}} \lambda_S c(S) &= \sum_{S \in \mathcal{B}, j_{p(S)}(S) \neq n} \lambda_S w_{j_{p(S)}(S)0} + \sum_{S \in \mathcal{B}, i_1(S) \neq 1} \lambda_S w_{0i_1(S)} + \\ &\sum_{S \in \mathcal{B}} \lambda_S \sum_{k=1}^{p(S)-1} w_{j_k(S)i_{k+1}(S)} + \sum_{S \in \mathcal{B}} \lambda_S \sum_{k=1}^{p(S)} \sum_{i \in T_k^S \setminus \{j_k(S)\}} w_{ii+1} + w_{01} + w_{n0}. \end{aligned} \quad (18)$$

Because $\mathcal{B}$ is a balanced collection the sum of the weights of the terms w_{ji} with $j \in N_0 \setminus \{i\}$ equals 1 for all $i \in N$ in the expression (18). The same holds for the sum of the weights of the terms w_{ij} with $j \in N_0 \setminus \{i\}$ for all $i \in n$. Therefore, for each $i \in N$ there exist $\alpha_k^i \geq 0$ for each $k \in N_0 \setminus \{i\}$ with $\sum_{k \in N_0 \setminus \{i\}} \alpha_k^i = 1$ such that

$$\sum_{S \in \mathcal{B}} \lambda_S c(S) = \sum_{i \in N} \sum_{k \in N_0 \setminus \{i\}} \alpha_k^i w_{ki} + \sum_{k \in N} \alpha_k^0 w_{k0} \quad (19)$$

where $\alpha_k^0 \geq 0$ for all $k \in N$ and $\sum_{k \in N} \alpha_k^0 \geq 0$. The expression in (19) equals

$$\gamma_1 C_1 + \gamma_2 C_2 + \dots + \gamma_l C_l$$

with $\gamma_q \geq 0$ for all $q \in \{1, \dots, l\}$ and $\sum_{q=1}^l \lambda_i = 1$ and C_q equal to the cost of a semi-travel plan for each $q \in \{1, \dots, k\}$, i.e. a travel plan in which the home town 0 may be visited more than once and each other town is visited exactly once. Because of the triangle inequality each C_q is higher than $c(N)$. This yields

$$\begin{aligned} \sum_{S \in \mathcal{B}} \lambda_S c(S) &= \sum_{i \in N_0} \sum_{k \in N_0 \setminus \{i\}} \alpha_k^i w_{ki} \\ &= \sum_{q=1}^l \gamma_q C_q \geq \sum_{q=1}^l \gamma_q c(N) = c(N) \end{aligned}$$

and the proof is completed. $\square$

An example will clarify the last part of the proof of theorem 4.7.15.

4.7.16 Example Let $N = \{1, 2, 3, 4, 5\}$, $\mathcal{B} = \{\{1, 3, 5\}, \{2, 3, 4\}, \{1, 2, 5\}, \{1, 4, 5\}, \{1, 3, 4, 5\}, \{1, 2, 3, 4\}\}$ with weights $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}$ respectively.

Then $\{1, 3, 5\}/e = \{\{1\}, \{3\}, \{5\}\}$, $\{2, 3, 4\}/e = \{\{2, 3, 4\}\}$, $\{1, 2, 5\}/e = \{\{1, 2\}, \{5\}\}$, $\{2, 4, 5\}/e = \{\{2\}, \{4, 5\}\}$, $\{1, 3, 4, 5\}/e = \{\{1\}, \{3, 4, 5\}\}$ and $\{1, 2, 3, 4\}/e = \{\{1, 2, 3, 4\}\}$. Further,

$$\begin{aligned}
\sum_{S \in \mathcal{B}} \lambda_{sc}(S) &= \frac{1}{6}(w_{01} + w_{13} + w_{35} + w_{50}) + \frac{1}{3}(w_{02} + w_{23} + w_{34} + w_{40}) + \\
&\quad \frac{1}{3}(w_{01} + w_{12} + w_{25} + w_{50}) + \frac{1}{6}(w_{02} + w_{24} + w_{45} + w_{50}) + \\
&\quad \frac{1}{3}(w_{01} + w_{13} + w_{34} + w_{45} + w_{50}) + \\
&\quad \frac{1}{6}(w_{01} + w_{12} + w_{23} + w_{34} + w_{40}) \\
&= \left(\frac{1}{6} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6}\right)w_{01} + \frac{1}{3}w_{02} + \frac{1}{3}w_{12} + \frac{1}{6}w_{02} + \frac{1}{6}w_{12} + \\
&\quad \frac{1}{6}w_{13} + \frac{1}{3}w_{23} + \frac{1}{3}w_{13} + \frac{1}{6}w_{23} + \frac{1}{3}w_{34} + \frac{1}{6}w_{24} + \frac{1}{3}w_{34} + \\
&\quad \frac{1}{6}w_{34} + \frac{1}{6}w_{35} + \frac{1}{3}w_{25} + \frac{1}{6}w_{45} + \frac{1}{3}w_{45} + \\
&\quad \left(\frac{1}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{3}\right)w_{50} + \frac{1}{3}w_{40} + \frac{1}{6}w_{40}
\end{aligned}$$

This last expression is the analogue of (19) and equals

$$\begin{aligned}
&\frac{1}{2}(w_{01} + w_{12} + w_{23} + w_{34} + w_{45} + w_{50}) + \\
&\frac{1}{3}(w_{01} + w_{13} + w_{34} + w_{40} + w_{02} + w_{25} + w_{50}) + \\
&\frac{1}{6}(w_{01} + w_{13} + w_{35} + w_{50} + w_{02} + w_{24} + w_{40}) + \\
&\frac{1}{6}(w_{01} + w_{13} + w_{35} + w_{50} + w_{02} + w_{24} + w_{40}) \\
&\geq \frac{1}{2}c(N) + \frac{1}{3}(w_{01} + w_{13} + w_{34} + w_{42} + w_{25} + w_{50}) + \\
&\quad \frac{1}{6}(w_{01} + w_{13} + w_{35} + w_{52} + w_{24} + w_{40}) \\
&\geq \frac{1}{2}c(N) + \frac{1}{3}c(N) + \frac{1}{6}c(N) = c(N).
\end{aligned}$$

4.8 Minimum cost spanning tree games

Consider the following situation studied by Claus and Kleitman (1973). Several customers who are geographically separated have to be linked to a

certain supplier. The supplier could be for instance an electricity plant, and the customers could be various towns. A customer can be linked directly to the supplier or through other customers. Each link induces a non-negative cost. Claus and Kleitman raised the question of how to allocate the total cost, incurred by connecting all the customers to the supplier, among the customers. They considered several methods to allocate this cost and discussed their pro's and contra's. They also gave a list of desirable criteria for a *cost allocation* in this situation.

Formally, this situation can be described as follows. Let $N = \{1, 2, \dots, n\}$ be the set of customers and let the supplier be denoted by 0. Let G_N be the complete graph with set of nodes $N \cup \{0\}$. Let E_N denote the *set of edges* of G_N . A typical edge of G_N is denoted by $l_{ij} = l_{ji}$ where $i, j \in N_0$ are the endpoints of the edge. To each edge $l_{ij} \in E_N$ a cost $k(l_{ij})$ is attached where $k(l_{ij})$ is the cost induced when i and j are linked. The problem of finding a cheapest way to connect all the customers to the supplier is equivalent to the problem of finding a *minimum cost spanning tree* of the graph G_N . Bird (1976) was the first to apply cooperative game theory to the cost allocation problem described above. The cost of a coalition $S \in 2^N$ is defined to be the cost of linking all members of S to the supplier in a cheapest possible way without using links involving customers not in S . Let G_S be the complete graph with set of nodes equal to $S_0 = S \cup \{0\}$. Then the cost $c(S)$ of a coalition S is the cost of a minimum cost spanning tree of the graph G_S . Let E_S be the set of edges of G_S .

4.8.1 Definition A cooperative game c is a minimum cost spanning tree game if there exists a graph G_N with set of nodes equal to N_0 and set of edges E_N , with costs $k(l_{ij}) \geq 0$ attached to each edge l_{ij} where $i, j \in N_0$, such that for each $S \in 2^N$, $c(S)$ is the cost of a minimum cost spanning tree of the graph G_S obtained from G_N by removing all nodes in $N \setminus S$ and all edges which at least one endpoint in $N \setminus S$.

4.8.2 Theorem (Bird) Minimum cost spanning tree games are balanced.

Proof. Let c be a minimum cost spanning tree game with corresponding graph G_N . Let T_N be a minimum cost spanning tree of G_N . For each $i \in N$ let e_i be the edge on the unique path from 0 to i in T_N with one endpoint equal to i . Let $x \in \mathbb{R}^n$ be defined by $x_i := k(e_i)$ for each $i \in N$. Then

$x(N) = \sum_{i \in N} k(e_i) = c(N)$. For $S \in 2^N \setminus \{\emptyset\}$ let T_S be a minimum cost spanning tree of G_S . For each $i \in S$ let e_i^S be the edge on the unique path from 0 to i in T_S with one endpoint equal to i . Then $c(S) = \sum_{i \in S} k(e_i^S)$. Suppose $x(S) > c(S)$, then there exists an $i \in S$ with $k(e_i^S) < k(e_i)$ and $k(e_j^S) \geq k(e_j)$ for all j on the path from 0 to i in T_S . Replace in T_N the edge e_i by e_i^S . This results in a spanning tree of G_N with lesser cost than T_N which contradicts the fact that T_N is a minimum cost spanning tree of G_N . Therefore, $x(S) \leq c(S)$ for all $S \in 2^N$ and $x \in C(c)$. $\square$

The Shapley-value of a minimum cost spanning tree game need not be in the core of the game. However, for every minimum cost spanning tree game it is possible to define at least one *weighted* Shapley-value which will lie in the core of the game. The following result is due to Bird (1976). Let c be a minimum cost spanning tree game with corresponding graph G_N . A permutation $\pi \in \Pi_N$ is called *feasible* if there exists a minimum cost spanning tree T_N of G_N such that T_N contains minimum cost spanning trees of G_S for all coalitions S of the form $\{\pi^{-1}(1), \pi^{-1}(2), \dots, \pi^{-1}(q)\}$ where $q \in \{1, \dots, n\}$. Let T_N be a minimum cost spanning tree of G_N . Then any permutation $\pi \in \Pi_N$ for which $\pi(i) > \pi(j)$ whenever j is on the unique path from 0 to i is feasible. Consider the weighted Shapley-value with weights $\{p_\pi\}_{\pi \in \Pi_N}$ where $p_\pi \geq 0$ for all $\pi \in \Pi_N$, $\sum_{\pi \in \Pi_N} p_\pi = 1$ and $p_\pi = 0$ whenever π is not feasible. Bird proves that the weighted Shapley-values defined in this way will always be in the core of c .

Granot and Huberman (1982) introduced *permutationally convex (concave) games*.

4.8.3 Definition A game c is called *permutationally concave* if there exists a permutation π such that for all $i \in N$ and all $S \subset T \subset N \setminus \{i\}$ with S and T of the form $\{\pi^{-1}(1), \pi^{-1}(2), \dots, \pi^{-1}(q)\}$ where $q \in \{1, \dots, n\}$

$$c(S \cup U) - c(S) \geq c(T \cup U) - c(T).$$

By reversing the inequality sign in 4.8.3 the definition of a *permutationally convex* game is obtained.

Granot and Huberman proved that the core of permutationally concave game is not empty and that minimum cost spanning tree games are permu-

tationally concave. The following relation exists between the result of Bird and that of Granot and Huberman. If c is a minimum cost spanning tree game then the inequality in 4.8.3 holds for each feasible permutation. Further, for every permutationally concave game c , if π is a permutation such that this inequality holds, then the marginal vector $m^\pi(c)$ is in the core of c . Hence, every convex combination of such marginal vectors is in the core of c . This implies that for minimum cost spanning tree games every convex combination of marginal vectors corresponding to feasible permutations is an element of the core of the game and therefore the weighted Shapley-values as defined above are in the core of the game. In another paper Granot and Huberman (1981) proved that if a minimum cost spanning tree game $\langle N, c \rangle$ can be decomposed into q minimum cost spanning tree games $\langle N_k, c_k \rangle$ where the N_k 's form a partition of N , then the core and the nucleolus of $\langle N, c \rangle$ is the cartesian product of the cores and the nucleoli of the games $\langle N_k, c_k \rangle$.

Another way of denoting the cost $c(S)$ of a coalition S in a minimum cost spanning tree game is by means of an integer programming problem. For each $S \in 2^N$ the cost $c(S)$ is equal to the value of the following integer programming problem.

$$\begin{aligned} \min \quad & \sum_{i \in N_0} \sum_{j \in N_0} k(l_{ij}) x_{ij} && \text{subject to} \\ \sum_{i \in S_0} \sum_{j \in S_0} x_{ij} 1_{T(i)}(1 - 1_{T(j)}) \geq 1 && \text{for all } \emptyset \neq T \subset S \\ x_{ij} \in \{0, 1\} && \text{for all } i, j \in N_0. \end{aligned} \quad (20)$$

Here $x_{ij} = 1$ means that l_{ij} is an element of the edge set of the minimum cost spanning tree T_S for S and $x_{ij} = 0$ means that l_{ij} is not an element of the edge set of T_S . Edmonds (1967) proved that (20) is equivalent to

$$\begin{aligned} \min \quad & \sum_{i \in N_0} \sum_{j \in N_0} k(l_{ij}) x_{ij} && \text{subject to} \\ \sum_{i \in S_0} \sum_{j \in S_0} x_{ij} 1_{T(i)}(1 - 1_{T(j)}) \geq 1 && \text{for all } \emptyset \neq T \subset S \\ x_{ij} \geq 0 && \text{for all } i, j \in N_0. \end{aligned} \quad (21)$$

This leads to a new proof of the non-emptiness of the core of a minimum cost spanning tree game. Consider the dual problem of (21), namely

$$\begin{aligned} \max \quad & \sum_{\emptyset \neq T \subset S} y_T && \text{subject to} \\ \sum_{T \subset N} y_T 1_{T(i)}(1 - 1_{T(j)}) \leq k(l_{ij}) && \text{for all } i, j \in N_0 \\ y_T \geq 0 && \text{for all } \emptyset \neq T \subset N. \end{aligned} \quad (22)$$

Let $\hat{y}$ be an optimal solution of (22) with $S = N$. Define $z \in \mathbb{R}^n$ by

$$z_i := \sum_{\emptyset \neq T \subset N} 1_{T(i)} \hat{y}_T |T|^{-1}.$$

From the duality theorem of linear programming it follows that

$$\sum_{i \in N} z_i = \sum_{\emptyset \neq T \subset N} \hat{y}_T = c(N).$$

Let $S \in 2^N \setminus \{\emptyset\}$ be a coalition with minimum cost spanning tree T_S . For each $i \in S$ let $p(i)$ be the node which immediately precedes i on the unique path from 0 to i in T_S . Then

$$c(S) = \sum_{i \in S} k(l_{ip(i)})$$

and

$$\begin{aligned} \sum_{i \in S} z_i &= \sum_{T \cap S \neq \emptyset} \hat{y}_T |T \cap S| |T|^{-1} \leq \sum_{T \cap S \neq \emptyset} \hat{y}_T \\ &\leq \sum_{i \in S} \sum_{T \subset N} \hat{y}_T 1_T(i) (1 - 1_T(p(i))) \\ &\leq \sum_{i \in S} k(l_{ip(i)}) = c(S). \end{aligned}$$

Thus, $z \in C(c)$. The second inequality follows from the fact that a $T \subset N$ with $T \cap S \neq \emptyset$ can not contain both i and $p(i)$ for all $i \in T \cap S$ because then T_S would not be a spanning tree of G_S .

Megiddo (1978a) studies a situation where the customers may use other links besides those connecting two customers or a customer with the supplier. The cooperative game arising from such a situation can have an empty core.

Another type of game related to minimum cost spanning tree games was studied in Megiddo (1978b). In these games a spanning tree T_N of G_N is given and the cost of a coalition S is defined to be the total cost of the edges that belong to some path from 0 to any node of S . In a straightforward way it can be proved that these games are concave and therefore balanced. Megiddo gives algorithms to compute the nucleolus of such a game within $O(n^3)$ operations and the Shapley-value within $O(n^3)$ operations.

4.9 Location games

Location game as considered here were introduced by Tamir (1980a). Let G be a given graph with set of nodes V and set of edges E . Each edge has a certain positive length. The length of a path in G is the sum of the lengths of the edges that belong to the path. For two nodes $v_1, v_2 \in V$ the distance $d(v_1, v_2)$ between v_1 and v_2 is defined to be the length of a shortest path from v_1 to v_2 . Two, not necessarily disjoint, subsets N and Q of V are given. $N = \{1, 2, \dots, n\}$ is the set of players. Each player is considered to be located in the corresponding node. *Service centers* have to be located in G in order to provide services to the players. The set $Q = \{q_1, \dots, q_t\}$ denotes

the possible locations for service centers. The cost of establishing a center at q_j is $k_j \geq 0$. Player i demands that at least one center shall be located at a distance of at most $r_i \geq 0$ from him. The problem is to locate the centers in such a way that all the demands are fulfilled and the cost is minimized. It is assumed that all the demands can be met. The cooperative game c constructed from this situation is called a *location game*. In this game $c(S)$ is defined to be the minimum cost needed to locate service centers in such a way that the demands of all the members of S are fulfilled. Let $A = [a_{ij}]_{i=1}^m, j=1}^t$ be the $m \times t$ -matrix defined by

$$a_{ij} := \begin{cases} 1 & \text{if } d(i, q_j) \leq r_i \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

4.9.1 Defintion The cost $c(S)$ of a coalition S in a location game c is given by

$$\begin{aligned} c(S) := \min & k \cdot x \quad \text{subject to} \\ & Ax \geq e^S \\ & x_j \in \{0, 1\} \quad \text{for all } j \in \{1, \dots, t\} \end{aligned}$$

where $e^S \in \mathbb{R}^n$ is defined by $e_i^S = 1_S(i)$ and where the matrix A is as defined in (23) and the vector $k \in \mathbb{R}^t$ is as given above.

4.9.2 Theorem (Tamir) let c be a location game for which the given graph G is a tree. Then $C(c) \neq \emptyset$.

Proof. From Tamir (1980b) it follows that

$$\begin{aligned} c(S) = \min & k \cdot x \quad \text{subject to} \\ & Ax \geq e^S \\ & x \geq 0. \end{aligned} \quad (24)$$

The duality theorem of linear programming yields

$$\begin{aligned} c(S) = \max & y \cdot e^S \quad \text{subject to} \\ & yA \leq c \\ & y \geq 0. \end{aligned} \quad (25)$$

Let $\hat{y}$ be an optimal solution of (25) with e^N . Then $\hat{y}(N) = c(N)$ and $\hat{y}$ is feasible for all problems (25) with $S \in 2^N \setminus \{\emptyset\}$. Therefore, $\hat{y}(S) = \hat{y} \cdot e^S \leq c(S)$ for all $S \in 2^N \setminus \{\emptyset\}$ and it follows that $\hat{y} \in C(c)$. $\square$

CHAPTER 5

BANKRUPTCY GAMES

5.1 Introduction

In this chapter the claim games introduced in section 5.3 will be studied in more detail. However, here they will be called bankruptcy games as this is how they are usually called in the current literature. They are considered here as arising from so called bankruptcy situations. The essential feature of such situations is that a certain amount of money has to be divided among claimants who have valid claims on it and who together demand more than the available amount. An event which could lead to such a situation is the bankruptcy of a firm which has borrowed money from various money lenders. Now they all claim back the amount they have lent but there is not enough to satisfy all the claims. O'Neill (1982) discusses an example of a bankruptcy situation where a man dies leaving behind four sons and four wills assigning to one his entire estate, to the second half of his estate, to the third one third and to the fourth one quarter. It is evident that there is not enough to satisfy the claims of the four brothers. This example has been given around the year 1140 A.D. by Rabbi Abraham Ibn Ezra. Other examples of bankruptcy situations can be found in the Babylonian Talmud, a great collection of Jewish religious and legal decisions set down during the first five centuries A.D.

This chapter is based on the papers by O'Neill (1982), Aumann and Maschler (1985) and Curiel, Maschler and Tijs (1987). Three division rules for bankruptcy problems which play a prominent role in these papers will be discussed here. In section 5.2 bankruptcy problems and division rules for bankruptcy problems will be defined formally. Several division rules will be given and three of them, the recursive completion rule introduced by O'Neill, the contested garment consistent rule introduced by Aumann and Maschler and the adjusted proportional rule introduced by Curiel et al. will be examined in more detail. Various properties of these rules are treated. In section 5.3 bankruptcy games are defined, these are cooperative games arising from bankruptcy problems. It is proved that bankruptcy games are convex and that all 3-person zero-normalized convex games are bankruptcy games. For cooperative games which are in some sense majorized by bankruptcy games it is proved that the core and the core cover coincide. Game theoretic division rules are defined and a necessary and sufficient condition for a division rule to be a game theoretic division rule is established. It is proved that the

three division rules mentioned above are game theoretic division rules and that they correspond to three well known solution concepts from cooperative game theory, namely, the Shapley-value, the nucleolus and the τ -value. A necessary and sufficient condition for a division rule to generate an outcome for a bankruptcy problem which is in the core of the corresponding bankruptcy game is given. In section 5.4 an axiomatic characterization of the adjusted proportional rule is given. Other papers which treat situations closely related to bankruptcy situations are Moulin (1985) and Young (1987).

5.2 Bankruptcy problems

Consider a situation where an *estate* E has to be divided among n *claimants*. The *set of claimants* is denoted by $N = \{1, 2, \dots, n\}$. Claimant i advances a *claim* d_i on E . The problem is how to divide E among the claimants. Unless stated otherwise it is assumed in this chapter that $d_1 \leq d_2 \leq \dots \leq d_n$. This leads to the following definitions.

5.2.1 Definition A *bankruptcy problem* is an ordered pair $(E; d) \in \mathbb{R} \times \mathbb{R}^n$, where $0 \leq d_1 \leq d_2 \leq \dots \leq d_n$ and $0 \leq E \leq d_1 + \dots + d_n =: D$.

5.2.2 Definition A *division rule* is a function f that assigns to every bankruptcy problem $(E; d)$ for every positive number of claimants a solution $f(E; d) = (f_1(E; d), \dots, f_n(E; d))$ such that

- (i) $f_i(E; d) \geq 0$ for each $i \in N$ (individual rationality)
- (ii) $\sum_{i \in N} f_i(E; d) = E$ (efficiency)

If $E = 0$ then (i) and (ii) imply that $f_i(E; d) = 0$ for all $i \in N$. In the following it is assumed that $E > 0$.

Examples of division rules are the *proportional division* rule which assigns to claimant i the amount $\frac{E d_i}{D}$, the *constraint equal award* or *CEA*-rule which assigns $\alpha \wedge d_i$ to claimant i , where $\alpha \in [0, d_n]$ is uniquely determined by the efficiency property. The *constraint equal loss* or *CEL*-rule assigns to claimant i the maximum of $d_i - \beta$ and 0 where $\beta \in [0, d_n]$ is uniquely determined by the efficiency property. O'Neill (1982) discusses several ways

to divide E . One of them is a generalization of a division method given by Rabbi Abraham Ibn Ezra around the year A.D. for a specific problem. In this problem $E = 120$, $d_1 = 30$, $d_2 = 40$, $d_3 = 60$, and $d_4 = 120$. Ibn Ezra divides E by giving $7\frac{1}{2}$ to the first claimant, $10\frac{5}{6}$ to the second, $20\frac{5}{6}$ to the third and $80\frac{5}{6}$ to the fourth. The reasoning behind this division is as follows. The first claimant claims 30. This amount is considered to be a specific part of the estate which is also claimed by the others. Therefore it is divided equally among the four claimants, each one receiving $7\frac{1}{2}$. This settles the claim of the first claimant. The second claimant claims 40 but he has already received his share of 30, so his actual claim is 10, which is also claimed by 3 and 4, therefore he receives $\frac{1}{3}$ of the 10. Together with the $7\frac{1}{2}$ he already had this gives him $7\frac{1}{2} + 3\frac{1}{3} = 10\frac{5}{6}$. The third claimant claims 60 but he has already received his share of 40 so his claim becomes 20 which is also claimed by 4. Therefore he receives 10 of this 20 and his total share is $\frac{1}{2} \times 20 + \frac{1}{3} \times 10 + \frac{1}{4} \times 30 = 20\frac{5}{6}$. The fourth claimant claims 120, but he has already received his share of 60 and since the remaining 60 is claimed by him alone, he obtains it all and his total share is $60 + \frac{1}{2} \times 20 + \frac{1}{3} \times 10 + \frac{1}{4} \times 30 = 80\frac{5}{6}$. A special feature of this example is that there is one claimant who claims all of the estate. When this is not the case it is not clear from this example what Ibn Ezra would have done. O'Neill gives a division rule for general bankruptcy problems which yields the solution arising from Ibn Ezra's way of reasoning whenever there is one claimant who claims all of the estate. Another method to divide E introduced by O'Neill is the *method of recursive completion*. This method was designed to fulfil the following consistency property which is different from the consistency property considered in the papers of Aumann and Maschler (1985), Young (1987) and other authors. Let $(E; d)$ be a bankruptcy problem with set of claimants N . From this bankruptcy problem n different bankruptcy problems with one claimant less can be derived by giving to an $i \in N$ the minimum of his claim d_i and E and considering the bankruptcy problem $(E^i; d^i)$ with total estate $E^i = (E - d_i)_+$, set of claimants $N \setminus \{i\}$ and claims $d_j^i = d_j$ for each $j \in N \setminus \{i\}$.

5.2.3 Definition A division rule f is said to be consistent according to O'Neill if for all bankruptcy problems $(E; d)$

$$f_i(E; d) = \frac{1}{n} (E \wedge d_i + \sum_{j \in N \setminus \{i\}} f_i(E^j; d^j)). \quad (1)$$

The method of recursive completion solves a problem $(E; d)$ by recursion as the name already indicates. The problems $(E^j; d^j)$ are constructed from $(E; d)$ for each $j \in N$. From each problem $(E^j; d^j)$ $n - 1$ problems with $n - 2$ claimants are derived in the same way as the problems $(E^j; d^j)$ were derived from $(E; d)$. This process continues until the stage where each bankruptcy problem has only one claimant who then receives everything which is left. Note that this amount will always be less than or equal to his claim because $D \geq E$. With the aid of formula (1) and the outcomes of the one-claimant problems the outcome of the two-claimant problems can be computed and with these outcomes and (1) the outcomes for the three-claimant problems can be computed etc. until $f(E; d)$ has been computed.

Another way to look at the recursive completion method is given by the following process. Suppose that the claimants arrive one after the other to stake their claims. The one who arrives first receives the minimum of his claim and E . The second one receives the minimum of his claim and what is left. Continuing in this way each claimant who arrives receives his claim or what is left of the estate after the ones before him have obtained their shares. The expected outcome of this procedure if all orders of arrival are considered to be equally likely is equal to the outcome of the recursive completion method. This way of looking at the recursive completion method suggests the following formula for this method. An order of arrival corresponds to a permutation $\pi \in \Pi_N$ of the set of claimants N . Let $RC_i(E; d)$ denote the amount assigned by the recursive completion method to claimant i . Then

$$RC_i(E; d) = \frac{1}{n!} \sum_{\pi \in \Pi_N} x_i^\pi \text{ where} \quad (2)$$

$$x_i^\pi = \begin{cases} d_i & \text{if } E \geq \sum_{j \in P(\pi, i)} d_j + d_i \\ E - \sum_{j \in P(\pi, i)} d_j & \text{if } 0 \leq E - \sum_{j \in P(\pi, i)} d_j \leq d_i \\ 0 & \text{if } E \leq \sum_{j \in P(\pi, i)} d_j. \end{cases}$$

Aumann and Maschler (1985) consider the following three examples given in the Talmud.

E	d	$f(E; d)$
100	(100, 200, 300)	$(33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3})$
200	(100, 200, 300)	(50, 75, 75)
300	(100, 200, 300)	(50, 100, 150)

In all these examples there are three claimants with claims 100, 200 and 300. The estate is respectively equal to 100, 200 and 300 and the numbers in the

table show the division of the estate prescribed by the Talmud in this three cases. At a first glance it is not clear what rule f lies behind these divisions. Aumann and Maschler give a division rule for bankruptcy problems which for the three examples of the Talmud has the same outcome as given by the Talmud. This rule is called the *Contested Garment consistent* or *CG-consistent* rule because it is consistent with the division prescribed by the Talmud in a case where two people have claims on a garment, one claiming all of it and the other half of it. The Talmud then assigns $\frac{3}{4}$ to the first claimant and $\frac{1}{4}$ to the other. The principle behind this division is that only half of the garment is considered to be at stake. The lesser claimant has no claim on the other half, so that goes directly to the greater claimant and the half which is at stake is divided equally between both claimants. The *CG-consistent* rule applied to a bankruptcy problem $(E; d)$ gives a solution $(x_1, \dots, x_n)$ such that for any two claimants i and j the division of $x_i + x_j$ prescribed by the contested garment principle is (x_i, x_j) . The *CG-consistent* rule distinguishes between bankruptcy problems with $E \leq \frac{1}{2}D$ and bankruptcy problems with $E \geq \frac{1}{2}D$. Let $CG(E; d)$ denote the solution assigned by the *CG-consistent* rule to a bankruptcy problem $(E; d)$. Then

$$CG(E; d) = \begin{cases} CEA(E; \frac{d}{2}) & \text{if } E \leq \frac{1}{2}D \\ d - CEA(D - E; \frac{d}{2}) & \text{if } E \geq \frac{1}{2}D. \end{cases} \quad (3)$$

The *CG-consistent* rule satisfies the following *self-consistency* property.

5.2.4 Definition A division rule f is said to be *self-consistent* if

$$f(E; d) = x \text{ implies } f(x(S); d_S) = x_S$$

where $d_S(x_S)$ means the restriction of $d(x)$ to S and $x(S) = \sum_{i \in S} x_i$.

Let f be a self-consistent rule; then for any set of claimants S in a bankruptcy problem $(E; d)$, if the members of S create a new bankruptcy problem with estate equal to the sum of the amounts assigned to them by $f(E; d)$ and their old claims and apply f to find a division for this new problem, each member of S gets his original payoff back.

Curiel, Maschler and Tijs (1987) propose a division rule which is an adjustment of the proportional rule. The reason given for this is that the proportional rule is not invariant under strategic equivalence. Consider the

following two bankruptcy problems with the same two claimants 1 and 2. In the first problem the estate equals 60 and 1 claims 40, 2 claims 60. In the second problem the estate equals 80 and 1 claims 40, 2 claims 80. The proportional rule divides the estate in parts 24, 36 and $26\frac{2}{3}$, $53\frac{1}{3}$ respectively and does not reflect the fact that the second problem seems to arise from the first one through an extra loan from claimant 2 to the holder of the estate which should be given back to him without any dispute. Therefore the proportional rule is not invariant under strategic equivalence. The *adjusted proportional* or *AP*-rule introduced by Curiel et al. is invariant under strategic equivalence and proceeds as follows. First each claimant i receives his *minimal right* m_i , that is the amount that is not claimed by the others. Formally,

$$m_i := (E - d(N \setminus \{i\}))_+.$$

Here and in the following $d(S) := \sum_{i \in S} d_i$ for all $S \subset N$. In section 5.3 it will be shown that it is possible to give each claimant his minimal right because $m(N) := \sum_{i \in N} m_i \leq E$. The amount which is left, that is $E' = E - m(N)$, has to be divided yet. Since claimant i has already received m_i his claim is lowered and becomes $d'_i = d_i - m_i$. Note that $d'_i = d_i - (E - d(N \setminus \{i\}))_+ \geq d_i - (D - d(N \setminus \{i\}))_+ = d_i - d_i = 0$. Further, it is considered irrational to claim more than what is available. So each claim greater than E' is truncated to E' . This defines new claims d''_i with $d''_i = E' \wedge d'_i \geq 0$ for each $i \in N$. Now E' is divided proportionally to the claims d''_i . Let $AP(E; d)$ denote the outcome of the *AP*-rule for a bankruptcy problem $(E; d)$ and let $m \in \mathbb{R}^n$ ($d'' \in \mathbb{R}^n$) be the vector with coordinates m_i (d''_i) for each $i \in N$. Then

$$AP(E; d) = \begin{cases} m & \text{if } E' = 0 \\ m + \frac{d'' E'}{\sum_{i \in N} d''_i} & \text{otherwise.} \end{cases} \quad (4)$$

Note that $\sum_{i \in N} d'_i = D - m(N) \geq E - m(N) = E'$ and hence from $\sum_{i \in N} d''_i = 0$ it follows that $E' = 0$ because $\sum_{i \in N} d''_i = 0$ implies $\sum_{i \in N} d'_i = 0$ or $E' = 0$.

A bankruptcy problem for which all the minimal rights of all the claimants are equal to zero is called *zero-normalized*. A bankruptcy problem $(E; d)$ with $d_i \leq E$ for each $i \in N$ is called a *simple claims* bankruptcy problem. For a zero-normalized simple claims bankruptcy problem the *AP*-rule reduces to the proportional division rule.

A property for division rules introduced by Aumann and Maschler is the *self-duality* property.

5.2.5 Definition A division rule f is said to be self-dual if

$$f(E; d) = d - f(D - E; d). \quad (5)$$

If a self-dual rule is used to compute the losses of each claimant the outcome will be the same as when the rule is applied directly to divide the estate. From (3) it follows in a straightforward way that the CG -consistent rule is self-dual. The self-duality of the AP -rule will be proved in section 5.3 and the self-duality of the RC -rule is proved in the following theorem.

5.2.6 Theorem The RC -rule is a self-dual rule

Proof. Let $(E; d)$ be a bankruptcy problem. From (2) it follows that $RC_i(D - E; d) = \frac{1}{n!} \sum_{\pi \in \Pi_N} y_i^\pi$ where

$$y_i^\pi = \begin{cases} d_i & \text{if } D - E \geq \sum_{j \in P(\pi, i)} d_j + d_i \\ D - E - \sum_{j \in P(\pi, i)} d_j & \text{if } 0 \leq D - E - \sum_{j \in P(\pi, i)} d_j \leq d_i \\ 0 & \text{if } D - E \leq \sum_{j \in P(\pi, i)} d_j. \end{cases}$$

Let $F(\pi, i) = \{j \mid \pi(j) > \pi(i)\}$ for all $\pi \in \Pi_N$. $F(\pi, i)$ is the set of claimants who arrive after i in the order according to π . Then

$$y_i^\pi = \begin{cases} d_i & \text{if } E \leq \sum_{j \in F(\pi, i)} d_j \\ \sum_{j \in F(\pi, i)} d_j + d_i - E & \text{if } 0 \leq E - \sum_{j \in F(\pi, i)} d_j \leq d_i \\ 0 & \text{if } E \geq \sum_{j \in F(\pi, i)} d_j + d_i \end{cases}$$

and

$$d_i - y_i^\pi = \begin{cases} 0 & \text{if } E \leq \sum_{j \in F(\pi, i)} d_j \\ E - \sum_{j \in F(\pi, i)} d_j & \text{if } 0 \leq E - \sum_{j \in F(\pi, i)} d_j \leq d_i \\ d_i & \text{if } E \geq \sum_{j \in F(\pi, i)} d_j + d_i. \end{cases}$$

Consider the following bijection on Π_N . To each $\pi \in \Pi_N$ a $\pi' \in \Pi_N$ is assigned with π' defined by

$$\pi'(j) = n - \pi(j) + 1.$$

Then $F(\pi, i) = P(\pi', i)$ and it follows that

$$d_i - y_i^\pi = \begin{cases} 0 & \text{if } E \leq \sum_{j \in P(\pi', i)} d_j \\ E - \sum_{j \in P(\pi', i)} d_j & \text{if } 0 \leq E - \sum_{j \in P(\pi', i)} d_j \leq d_i \\ d_i & \text{if } E \geq \sum_{j \in P(\pi', i)} d_j + d_i. \end{cases} \quad (6)$$

Further,

$$d_i - RC_i(E; d) = d_i - \frac{1}{n!} \sum_{\pi \in \Pi_N} y_i^\pi = \frac{1}{n!} \sum_{\pi \in \Pi_N} (d_i - y_i^\pi) = RC_i(E; d).$$

Here the last inequality follows from (2) and (6). $\square$

For a self-dual rule it is sufficient to know the outcome in all cases with $E \leq \frac{1}{2}D$. The self-duality yields the outcome in the other cases. With this in mind explicit formulas for $AP(E; d)$ can be given. Let $E \leq \frac{1}{2}D$. Two cases are distinguished: (1) $d_n \leq \frac{1}{2}D$ and (2) $d_n \geq \frac{1}{2}D$. Let $I_0 := [0, d_n]$, $I_1 := [d_1, d_2], \dots, I_{n-1} := [d_{n-1}, d_n \wedge d(N \setminus \{n\})]$, $I_n = [d_n \wedge d(N \setminus \{n\}), \frac{1}{2}D]$. In case (1) $I_{n-1} = [d_{n-1}, d_n]$, $I_n = [d_n, \frac{1}{2}D]$. In case (2) $I_{n-1} = [d_{n-1}, d(N \setminus \{n\})]$, $I_n = [d(N \setminus \{n\}), \frac{1}{2}D]$. Straightforward calculations yield:

Case 1

$$AP_i(E; d) = \begin{cases} Ed_i / (\sum_{k \leq j} d_k + (n-j)E) & \text{for } E \in I_j \text{ with } i \leq j \\ E^2 / (\sum_{k \leq j} d_k + (n-j)E) & \text{for } E \in I_j \text{ with } i > j \end{cases} \quad (7)$$

Case 2

$$AP_i(E; d) = \begin{cases} \text{the same as in case 1} & \text{for } E \in I_j \text{ with } j \leq n-1 \\ \frac{1}{2}d_i & \text{for } E \in I_n, i \neq n \\ \frac{1}{2}d_n + E - \frac{1}{2}D & \text{for } E \in I_n, i = n \end{cases} \quad (8)$$

From the formulas (7) and (8) it follows that $AP_i(E; d)$ is a continuous function of E on $[0, \frac{1}{2}D]$ and hence the self-duality property implies that it is a continuous function of E on the whole interval $[0, D]$.

5.2.7 Definition A division rule f satisfies the monotonicity property if

$$E' \geq E \text{ implies } f_i(E'; d) \geq f_i(E; d) \text{ for all } i \in N. \quad (9)$$

The monotonicity property is a reasonable property to ask for in a division rule as it seems unfair that a claimant should suffer when the estate becomes higher. From (2) and (3) it follows that the RC -rule and the CG -consistent rule are monotonic. Because of the self-duality property of

the AP -rule it suffices to show that (9) holds for $\frac{1}{2}D \geq E' \geq E$ to prove that the AP -rule is monotonic. This will be done by calculating the partial derivatives $\frac{\partial AP_i(E; d)}{\partial E}$ in the open intervals $I_0^o, I_1^o, \dots, I_n^o$ with $I_0, I_1, \dots, I_n$ as defined above. From (7) and (8) the following is derived.

Case 1

$$\frac{\partial AP_i(E; d)}{\partial E} = \begin{cases} \frac{d_i}{(\sum_{k \leq i} d_k + (n-i)E)^2} & \text{for } E \in I_j \text{ with } i \leq j \\ \frac{2E \sum_{k \leq i} d_k + (n-i)E^2}{(\sum_{k \leq i} d_k + (n-i)E)^2} & \text{for } E \in I_j \text{ with } i > j \end{cases}$$

Case 2

$$\frac{\partial AP_i(E; d)}{\partial E} = \begin{cases} \text{the same as in case 1} & \text{for } E \in I_j, j \leq n-1 \\ 0 & \text{for } E \in I_n, i \neq n \\ 1 & \text{for } E \in I_n, i = n \end{cases}$$

$AP_i(E; d)$ is not differentiable at the points $d_1, d_2, \dots, d_{n-1}, d_n \wedge d(N \setminus \{n\})$. From the fact that $\frac{\partial AP_i(E; d)}{\partial E} \geq 0$ on the open intervals it follows that $AP_i(E; d)$ is monotonic on these intervals. Together with the continuity of $AP_i(E; d)$ with respect to E this yields the monotonicity of $AP_i(E; d)$ on $[0, \frac{1}{2}D]$.

The following property was introduced by Aumann and Maschler.

5.2.8 Definition A division rule f is called order preserving if

$$\begin{aligned} 0 \leq f_1(E; d) \leq f_2(E; d) \leq \dots \leq f_n(E; d) \quad \text{and} \\ 0 \leq d_1 - f_1(E; d) \leq \dots \leq d_n - f_n(E; d) \end{aligned}$$

for every bankruptcy problem $(E; d)$.

An order preserving division rule assigns to a claimant with a higher claim an award and a loss that is not less than the award and the loss assigned to a claimant with a lower claim. For a self-dual rule f only the inequalities $0 \leq f_1(E; d) \leq \dots \leq f_n(E; d)$ have to be checked to see whether f is order preserving. Because the RC -rule, the CG -consistent rule and the AP -rule all three satisfy these inequalities it follows that they are all order preserving.

5.2.9 Definition A division rule f satisfies the homogeneity property if for every bankruptcy problem $(E; d)$

$$f(\lambda E; \lambda d) = \lambda f(E; d) \text{ for all } \lambda > 0.$$

It is easy to verify that all division rules mentioned in this chapter satisfy the homogeneity property.

5.3 Bankruptcy games

In this section a game theoretic analysis of bankruptcy problems will be given. For every bankruptcy problem a corresponding cooperative game can be defined. The set of players in this game will be the same as the set of claimants in the problem. The worth of a coalition S in the game is defined to be that amount of the estate which is not claimed by the complement of S .

5.3.1 Definition A cooperative game v is a bankruptcy game if there exists a bankruptcy problem $(E; d)$ such that

$$v(S) = (E - d(N \setminus S))_+ \text{ for all } S \subset N.$$

In fact, $v(S)$ in a bankruptcy game v is a generalization of the minimal right concept. Note that $v(i) = m_i$.

5.3.2 Theorem Bankruptcy games are convex.

Proof. Let v be a bankruptcy game arising from a bankruptcy situation $(E; d)$. Let $T \subset S \subset N \setminus \{i\}$ and denote $E - D$ by A . Then

$$\begin{aligned} v(S \cup \{i\}) + v(T) &= \max\{A + d(S) + d_i, 0\} + \max\{A + d(T), 0\} \\ &= \max\{2A + d(S) + d(T) + d_i, A + d(S) + d_i, \\ &\quad A + d(T), 0\} \end{aligned}$$

and

$$\begin{aligned} v(T \cup \{i\}) + v(S) &= \max\{A + d(T) + d_i, 0\} + \max\{A + d(S), 0\} \\ &= \max\{2A + d(S) + d(T) + d_i, A + d(T) + d_i, \\ &\quad A + d(S), 0\}. \end{aligned}$$

From $d(S) \geq d(T)$ it follows that $A + d(T) + d_i \leq A + d(S) + d_i$. Furthermore, $A + d(S) \leq A + d(S) + d_i$. Therefore, $v(S \cup \{i\}) + v(T) \geq v(T \cup \{i\}) + v(S)$ yielding $v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$ and thus, v is convex. $\square$

Now it can be proved that $m(N) \leq E$ as stated in section 5.2. From the definition of v and the superadditivity of v it follows that $m(N) = \sum_{i \in N} v(i) \leq v(N) = E$.

Not all non-negative convex games are bankruptcy games as the following example shows.

5.3.3 Example Let v be a cooperative game with set of players $N = \{1, \dots, n\}$ and $v(N) = 1$, $v(N \setminus \{i\}) = \frac{1}{2}$ for all $i \in N$, $v(N \setminus \{i, j\}) = \frac{1}{2}$ for all $i, j \in N$ and $v(S) = 0$ for all other $S \in 2^N$. Then v is a convex game. Suppose v were a bankruptcy game corresponding to the bankruptcy problem (E, d) . Then $E = 1$ and $d_i = E - v(N \setminus \{i\}) = \frac{1}{2}$ for all $i \in N$. But then $v(N \setminus \{i, j\}) = (E - d_i - d_j)_+ = 0$, contradicting $v(N \setminus \{i, j\}) = \frac{1}{2}$. Therefore, there does not exist a bankruptcy problem $(E; d)$ with $v(S) = (E - d(N \setminus S))_+$ for all $S \in 2^N$ and v is not a bankruptcy game.

5.3.4 Proposition *All 3-person zero-normalized convex games are bankruptcy games.*

Proof. Let v be a 3-person zero-normalized convex game with set of players $N = \{1, 2, 3\}$. Define $E = v(N)$ and $d_i = v(N) - v(j, k)$ for each $i \in N$, where $\{j, k\} = N \setminus \{i\}$. Then $d_i = v(N) - v(j, k) \geq v(i) - v(\emptyset) = 0$ and $D = 3v(N) - v(i, j) - v(i, k) - v(j, k) \geq v(N) = E$ where the inequality follows from the balancedness of v . Hence, $(E; d)$ is a bankruptcy problem with $(E - d(N \setminus \{j, k\}))_+ = (E - d_i)_+ = (E - E + v(j, k))_+ = v(j, k)$ for all $\{j, k\} \subset N$ and $(E - d(N \setminus \{i\}))_+ = (E - E + v(i, k) - E + v(i, j))_+ = (v(i, k) + v(i, j) - v(N))_+ = 0$, where the last inequality follows because v is convex and zero-normalized. $\square$

The following theorem states that the core cover $CC(w)$ (defined in 1.3.9) of a game w which is in a certain sense majorized by a bankruptcy game, coincides with the core of w .

5.3.5 Theorem *Let w be a game such that there exists a bankruptcy game v with $0 \leq w \leq v$ and $w(S) = v(S)$ if $|S| \in \{0, n-1, n\}$. Then $CC(w) = C(w)$.*

Proof. Let $(E; d)$ be a bankruptcy problem corresponding to the game v . Note that $x \in C(v)$ implies $x \in C(w)$, so $C(w) \neq \emptyset$. Let $x \in CC(w)$, then

$$\begin{aligned} x(S) &= x(N) - x(N \setminus S) \geq w(N) - M^w(N \setminus S) \\ &= w(N) - \sum_{i \in N \setminus S} (w(N) - w(N \setminus \{i\})) \\ &= v(N) - \sum_{i \in N \setminus S} (v(N) - v(N \setminus \{i\})) \\ &= E - \sum_{i \in N \setminus S} (E - (E - d_i)_+) \\ &\geq E - d(N \setminus S). \end{aligned}$$

Furthermore, $x(S) \geq \mu^v(S) \geq \sum_{i \in S} v(i) \geq 0$. Thus, it follows that $x(S) \geq (E - d(N \setminus S))_+ = v(S)$ and $x \in C(v)$ implying $x \in C(w)$. Since the inclusion $C(w) \subset CC(w)$ was already proved in 1.3.10 it follows that $CC(w) = C(w)$ and the proof is completed. $\square$

Let v be a bankruptcy game corresponding to a bankruptcy problem $(E; d)$. Then $M_i^v = v(N) - v(N \setminus \{i\}) = (E - (E - d_i)_+) = E \wedge d_i$ for all $i \in N$. Because v is a convex game, $\mu_i^v = v(i) = m_i$ for all $i \in N$. (cf. 1.6.5) and $C(v) = CC(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N), m_i \leq x_i \leq E \wedge d_i \text{ for all } i \in N\}$. It follows that the extreme points of $C(v)$ are of the form $((m_i)_{i \in S}, x_k, (E \wedge d_j)_{j \in N \setminus (S \cup \{k\})})$ with $m_k \leq x_k \leq E \wedge d_k$ and with coordinates summing up to $v(N)$.

5.3.6 Definiton *A division rule f is a game theoretic division rule if there exists a solution concept F for cooperative games such that $f(E; d) = F(v_{E;d})$ for all bankruptcy problems $(E; d)$ where $v_{E;d}$ is the bankruptcy game corresponding to $(E; d)$.*

Because the bankruptcy games corresponding to the problems $(E; d)$ and $(E; \hat{E} \wedge d)$, where $\hat{E} \wedge d \in \mathbb{R}^n$ is the vector with i -th coordinate equal to $E \wedge d_i$ are the same, a necessary condition for a division rule f to be a game theoretic division rule is that $f(E; d) = f(E; \hat{E} \wedge d)$. The following theorem states that this is also a sufficient condition.

5.3.7 Theorem *A division rule f is a game theoretic division rule if and only if $f(E; d) = f(E; \hat{E} \wedge d)$.*

Proof. The “only if” part was shown above. Suppose f is a division rule with $f(E; d) = f(E; \hat{E} \wedge d)$. Define a solution concept F for cooperative games as follows. For every cooperative game v let $F(v) := f(v(N); K^v)$ where K^v is the vector with i -th coordinate $K_i^v = (M_i^v + (v(N) - M^v(N))_+)/n$. For a bankruptcy game $v_{E;d}$ this yields $F(v_{E;d}) = f(E; M^v) = f(E; \hat{E} \wedge d) = f(E; d)$ and the proof is completed. $\square$

It is immediate that the proportional rule is not a game theoretic rule because in general it assigns different outcomes to $(E; d)$ and $(E; \hat{E} \wedge d)$. For the same reason the *CEL*-rule is not a game theoretic rule. Since this is less clear than for the proportional rule it will be illustrated by the following example. Let $(E; d)$ be the 3-claimant bankruptcy problem with $E = 100$, $d_1 = 40$, $d_2 = 50$, $d_3 = 110$, then $CEL(E; d) = (6\frac{2}{3}, 16\frac{2}{3}, 76\frac{2}{3})$ while $CEL(E; \hat{E} \wedge d) = (10, 20, 70)$.

Recall that $CEA_i(E; d) = d_i \wedge \alpha$ where $\alpha \in [0, d_n]$ is determined by efficiency. Therefore $\alpha \leq E$ and $CEA_i(E; \hat{E} \wedge d) = E \wedge d_i \wedge \alpha = d_i \wedge \alpha = CEA_i(E; d)$ for each $i \in N$ and the *CEA*-rule is a game theoretic rule. The *RC*-rule, the *CG*-consistent rule and the *AP*-rule are all three game theoretic rules and it will be proved that they correspond to three well-known solution concepts from game theory.

5.3.8 Theorem *Let $(E; d)$ be a bankruptcy problem and let v be the corresponding bankruptcy game. Then $RC(E; d) = \varphi(v)$ where $\varphi(v)$ is the Shapley-value of the game v .*

Proof. Recall that the marginal vector $m^\pi(v)$ of a game v with regard to a permutation π is defined by

$$m_i^\pi(v) = v(P(\pi, i) \cup \{i\}) - v(P(\pi, i)).$$

The Shapley-value $\varphi(v)$ assigns to each player $i \in N$ the amount

$$\varphi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi_N} m_i^\pi(v)$$

where

$$\begin{aligned} m_i^\pi(v) &= v(P(\pi, i) \cup \{i\}) - v(P(\pi, i)) \\ &= (E - \sum_{j \in F(\pi, i)} d_j)_+ - (E - \sum_{j \in F(\pi, i)} d_j - d_i)_+ . \end{aligned}$$

It follows that

$$m_i^\pi(v) = \begin{cases} d_i & \text{if } E \geq \sum_{j \in F(\pi, i)} d_j + d_i \\ E - \sum_{j \in F(\pi, i)} d_j & \text{if } 0 \leq E - \sum_{j \in F(\pi, i)} d_j + d_i \\ 0 & \text{if } E \leq \sum_{j \in F(\pi, i)} d_j . \end{cases}$$

Consider the bijection defined in the proof of theorem 5.2.6 which assigns to each $\pi \in \Pi_N$ a $\pi' \in \Pi_N$ defined by

$$\pi'(j) = n - \pi(j) + 1.$$

Then

$$m_i^\pi(v) = \begin{cases} d_i & \text{if } E \geq \sum_{j \in P(\pi', i)} d_j + d_i \\ E - \sum_{j \in P(\pi', i)} d_j & \text{if } 0 \leq E - \sum_{j \in P(\pi', i)} d_j \leq d_i \\ 0 & \text{if } E \leq \sum_{j \in P(\pi', i)} d_j . \end{cases}$$

With (6) it follows that $\varphi_i(v) = RC_i(E; d)$ and the proof is completed. $\square$

The *CG*-consistent rule assigns to a bankruptcy problem $(E; d)$ the nucleolus of the corresponding bankruptcy game. However, this can not be shown as directly as in the proof of theorem 5.3.8 and some preliminary lemma's are needed.

5.3.9 Definition *A division rule is called consistent if for any bankruptcy problem $(E; d)$ the division of $f_i(E; d) + f_j(E; d)$ prescribed by the contested garment principle for claims d_i, d_j is $f_i(E; d), f_j(E; d)$ respectively.*

5.3.10 Lemma *The CG-consistent rule is the unique consistent division rule.*

Proof. That the CG-consistent rule is consistent has already been noted. Suppose there existed a division rule $f \neq$ CG-consistent rule such that f is consistent. Then there exists a bankruptcy problem $(E; d)$ for which it is possible to find claimants i and j with $f_i(E; d) > CG_i(E; d)$ and $f_j(E; d) < CG_j(E; d)$. If the total estate is $f_i(E; d) + f_j(E; d)$ and just i and j are involved consistency of f implies that the amount awarded to i is $f_i(E; d)$ and the amount awarded to j is $f_j(E; d)$. If the total estate is $CG_i(E; d) + CG_j(E; d)$ and just i and j are involved consistency of the CG-rule implies that the amount awarded to i is $CG_i(E; d)$ and the amount awarded to j is $CG_j(E; d)$. If $CG_i(E; d) + CG_j(E; d) \geq f_i(E; d) + f_j(E; d)$ then $CG_i(E; d) \geq f_i(E; d)$ because the CG-consistent rule is monotonic. However, this contradicts $f_i(E; d) > CG_i(E; d)$. If $CG_i(E; d) + CG_j(E; d) \leq f_i(E; d) + f_j(E; d)$ then $f_j(E; d) \geq CG_j(E; d)$ because any consistent rule is monotonic for the two-claimant case. This contradicts $f_j(E; d) < CG_j(E; d)$. Thus, there cannot exist another consistent division rule f and the CG-consistent rule is the unique consistent division rule. $\square$

5.3.11 Lemma *Let x be a solution of the bankruptcy problem $(E; d)$ such that $0 \leq x_i \leq d_i$. Let v be the bankruptcy game corresponding to $(E; d)$. Then for any coalition S the reduced game of v with respect to S and x is the bankruptcy game corresponding to $(x(S); d_S)$.*

Proof. Let v^S be the reduced game of v with respect to S and x . Recall that in 1.5.7 v^S was defined as follows.

$$v^S(T) = \begin{cases} x(T) & \text{if } T = S \text{ or } T = \emptyset \\ \max\{v(Q \cup T) - x(Q) \mid Q \subset N \setminus S\} & \text{if } T \subset S, T \neq \emptyset, S \end{cases}$$

Let w be the bankruptcy game corresponding to $(x(S); d_S)$. Then $v^S(S) = w(S)$ and $v^S(\emptyset) = w(S)$. Let $T \subset S, T \neq S, \emptyset$ and let the maximum in the definition of $v^S(T)$ be attained at Q . Then

$$\begin{aligned} v^S(T) &= v(T \cup Q) - x(Q) = (E - d(N \setminus (T \cup Q)))_+ - x(Q) \quad (10) \\ &\leq (x(N) - d(N \setminus (T \cup Q)) - x(Q))_+ \\ &= (x(S) - d(S \setminus T) - (d - x)(N \setminus (S \cup Q)))_+ \\ &\leq (x(S) - d(S \setminus T))_+. \end{aligned}$$

Further,

$$\begin{aligned}
v^S(T) &\geq v(T \cup N \setminus S) - x(N \setminus S) \\
&= (E - d(N \setminus (T \cup N \setminus S)))_+ - (x(N) - x(S)) \\
&\geq (E - d(S \setminus T)) - (E - x(S)) \\
&= x(S) - d(S \setminus T)
\end{aligned} \tag{11}$$

and

$$v^S(T) \geq v(T \cup \emptyset) - x(\emptyset) = v(T) = (E - d(N \setminus T))_+ \geq 0 \tag{12}$$

Formulas (11) and (12) yield

$$v^S(T) \geq (x(S) - d(S \setminus T))_+$$

and together with (10) this yields

$$v^S(T) = (x(S) - d(S \setminus T))_+ = w(T).$$

It follows that $v^S = w$ and the proof is completed. $\square$

5.3.12 Lemma *Let v be the bankruptcy game corresponding to the bankruptcy problem $(E; d)$. Let x be in the pre-kernel of the game v and let $S = \{i, j\}$ be a coalition with two players. Then $x_S = CG(x_i + x_j; d_S)$.*

Proof. From the definition it follows that

$$\begin{aligned}
CG_i(x_i + x_j; d_S) &= x_i + x_j - d_j + \frac{1}{2}(d_i + d_j - x_i - x_j) \\
&= \frac{1}{2}x_i + \frac{1}{2}x_j + \frac{1}{2}d_i - \frac{1}{2}d_j
\end{aligned}$$

and

$$\begin{aligned}
CG_j(x_i + x_j; d_S) &= x_i + x_j - d_i + \frac{1}{2}(d_i + d_j - x_i - x_j) \\
&= \frac{1}{2}x_j + \frac{1}{2}x_i + \frac{1}{2}d_j - \frac{1}{2}d_i.
\end{aligned}$$

Let v^S be the reduced game of v with respect to S and x . From theorem 5.3.11 it follows that

$$v^S(S) = x_i + x_j, \quad v^S(i) = x_i + x_j - d_j \text{ and } v^S(j) = x_i + x_j - d_i.$$

Because the pre-kernel satisfies the reduced game property it follows that x_S is an element of the pre-kernel of v^S . Therefore,

$$v^S(i) - x_i = v^S(j) - x_j$$

yielding

$$x_j - d_j = x_i - d_i$$

which leads to

$$\begin{aligned} x_i &= x_j - d_j + d_i \text{ and} \\ x_j &= x_i - d_i + d_j. \end{aligned}$$

Adding x_i to the first expression and x_j to the second gives

$$\begin{aligned} 2x_i &= x_j + x_i - d_j + d_i \\ 2x_j &= x_i + x_j - d_i + d_j. \end{aligned}$$

It follows that

$$\begin{aligned} x_i &= \frac{1}{2}x_i + \frac{1}{2}x_j + \frac{1}{2}d_i - \frac{1}{2}d_j = CG_i(x_i + x_j; d_S) \\ x_j &= \frac{1}{2}x_j + \frac{1}{2}x_i + \frac{1}{2}d_j - \frac{1}{2}d_i = CG_j(x_i + x_j; d_S) \end{aligned}$$

and the proof is completed. $\square$

5.3.13 Theorem *Let $(E; d)$ be a bankruptcy problem and let v be the corresponding bankruptcy game. Then $CG(E; d) = \nu(v)$ where $\nu(v)$ is the nucleolus of the game v .*

Proof. Since v is a convex game (theorem 5.3.2) it follows that $\nu(v)$ is the unique element of the pre-kernel of v . Cf. section 1.5. Because of theorem 5.3.12 the restriction of $\nu(v)$ to any two-person coalition $S = \{i, j\}$ is equal to $CG(\nu_i(v) + \nu_j(v); d_S)$. Lemma 5.3.10 now implies $\nu(v) = CG(E; d)$. $\square$

5.3.14 Theorem *Let $(E; d)$ be a bankruptcy problem and let v be the corresponding bankruptcy game. Then $AP(E; d) = \tau(v)$ where $\tau(v)$ is the τ -value of the game v .*

Proof. Because v is a convex game $\tau(v) = \underline{v} + \lambda(M^v - \underline{v}) = \underline{v} + \lambda((\hat{E} \wedge d) - \underline{v})$ where $\underline{v} = (v(1), v(2), \dots, v(n))$. (Cf. 1.6.5) Because $m_i = v(i)$ for all $i \in N$ it follows that $AP(E; d) = \underline{v} + \gamma((d - \underline{v}) \wedge \hat{E}')$ where $E' = E - \sum_{j \in N} v(j)$ and $(d - \underline{v}) \wedge \hat{E}'$ is the vector whose i -th coordinate is equal to $(d_i - v(i)) \wedge E'$. It will be shown that $(\hat{E} \wedge d) - \underline{v} = (d - \underline{v}) \wedge \hat{E}'$.

Suppose $d_i \leq E$. Then $E - d_i = v(N \setminus \{i\}) \geq \sum_{j \in N \setminus \{i\}} v(j)$ where the inequality follows from the superadditivity of v . This yields $E - \sum_{j \in N} v(j) \geq d_i - v(i)$ and hence $(E \wedge d_i) - v(i) = (d_i - v(i)) \wedge E'$.

Suppose $d_i > E$. Then $E - d_i < 0 = \sum_{j \in N \setminus \{i\}} v(j)$ leading to $E - \sum_{j \in N} v(j) < d_i - v(i)$ and hence $(E \wedge d_i) - v(i) = (d_i - v(i)) \wedge E'$.

Thus, $(\hat{E} \wedge d) - \underline{v} = (d - \underline{v}) \wedge \hat{E}'$ and the efficiency of both the τ -value and the AP -rule yields $\lambda = \gamma$ and hence $AP(E; d) = \tau(v)$. $\square$

Now the self-duality of the AP -rule will be proved as was announced in section 5.2.

5.3.15 Theorem *The AP -rule is self-dual.*

Proof. Let $(E; d)$ be a bankruptcy problem and v the corresponding bankruptcy game. It has to be shown that $AP(E; d) = d - AP(D - E; d)$. This will be done by proving that $\tau(v) = d - \tau(w)$ where w is the bankruptcy game corresponding to $(D - E; d)$. By definition of the τ -value and convexity of the games v and w

$$\tau(v) = \underline{v} + \lambda(M^v - \underline{v}) \text{ and } \tau(w) = \underline{w} + \gamma(M^w - \underline{w}).$$

It is easy to verify that $w(S) = d(S) - E + v(N \setminus S)$ for all $S \in 2^N$. Therefore, $w(i) = d_i - M_i^v$ and $M_i^w = d_i - v(i)$ for each $i \in N$ and $\tau(w) = d - M^v + \gamma(M^v - \underline{v})$. From the definition it follows that $M_i^v \geq v(i)$ for all $i \in N$. Hence $M^v(N) = \sum_{i \in N} v(i)$ implies $v(i) = M_i^v$ for all $i \in N$ and then it follows immediately that $\tau(v) = d - \tau(w)$. In the following it is assumed that $M^v(N) \neq \sum_{i \in N} v(i)$. Then $\sum_{i \in N} \tau_i(w) = D - \sum_{i \in N} \tau_i(v)$ yields $\gamma = 1 - \lambda$ leading to $d - \tau(w) = M^v - (1 - \lambda)(M^v - \underline{v}) = \underline{v} + \lambda(M^v - \underline{v}) = \tau(v)$ and the proof is completed. $\square$

With the aid of the self-duality property an alternative way of computing the AP -rule can be provided. Let $(E; d)$ be a bankruptcy problem with $E \leq \frac{1}{2}D$, then $AP(E; d) = \lambda d^*$ where λ is determined by efficiency and

$$d_i^* = \begin{cases} d_i & \text{if } d_i \leq E \\ E & \text{if } E \leq d_i \leq D - E \\ d_i - D + 2E & \text{if } d_i \geq D - E \end{cases}$$

The self-duality property now makes it possible to compute $AP(E; d)$ for problems with $E \geq \frac{1}{2}D$ as well.

Because bankruptcy games are convex games they have a non-empty core. The question arises under what conditions a division rule gives an outcome to a bankruptcy problem which will always lie in the core of the corresponding bankruptcy game.

5.3.16 Definition A division rule f is said to be reasonable if $f_i(E; d) \leq d_i$ for every bankruptcy problem $(E; d)$ and each claimant i in $(E; d)$.

Assigning more than his claim to claimant i seems highly unreasonable as there is not even enough to satisfy all the claims.

5.3.17 Theorem Let f be a division rule for bankruptcy problems. Then $f(E; d) \in C(v_{E;d})$ for all bankruptcy problems $(E; d)$ and the corresponding bankruptcy games $v_{E;d}$ if and only if f is reasonable.

Proof. From theorem 5.3.5 it follows that $C(v_{E;d}) = CC(v_{E;d}) = \{x \in \mathbb{R}^n \mid x(N) = v(N), m \leq x \leq \hat{E} \wedge d\}$ yielding the “only if” part. Further, efficiency and $f_i(E; d) \leq d_i$ for all $i \in N$ imply that $f_i(E; d) = E - \sum_{j \in N \setminus \{i\}} f_j(E; d) \geq E - d(N \setminus \{i\})$ for each $i \in N$. Individually rationality implies that $f_i(E; d) \geq 0$ for all $i \in N$, hence $f_i(E; d) \geq m_i$ for all $i \in N$ and $f(E; d) \in C(v_{E;d})$. $\square$

It follows that all the division rules mentioned in this chapter assign an outcomes to bankruptcy problem which is in the core of the corresponding game.

5.4 An axiomatic characterization of the *AP*-rule

In this section where for sake of convenience the property $d_1 \leq d_2 \leq \dots \leq d_n$ is deleted, an axiomatic characterization of the *AP*-rule will be given. In order to do that three new properties for division rules will be introduced. The first one is the *minimal right* property which states that it does not matter whether a division rule is applied directly to a bankruptcy problem or the minimal rights are handed out first and the division rule is applied then.

5.4.1 Definition *A division rule f satisfies the minimal right property if $f(E; d) = m + f(E - m(N); d - m)$.*

From the way the *AP*-rule has been introduced it follows that it satisfies the minimal right property. In fact, this is the case for all game theoretic division rules that are invariant under strategic equivalence.

5.4.2 Definition *A division rule f is called symmetric if $d_i = d_j$ implies that $f_i(E; d) = f_j(E; d)$ for all bankruptcy problems $(E; d)$.*

Symmetry is a weak form of the anonymity property which requires that f is a symmetric function of $d_1, \dots, d_n$. (Cf. Moulin (1985)) From the definition it follows that the *AP*-rule satisfies the anonymity property and hence the symmetry property.

The third property deals with a situation where one of the claimants dies leaving behind parts of his claim to different heirs. These heirs become new claimants, each one claiming the part of the original claim he received. Together their claims sum up to the original claim. The third property states that if the bankruptcy problem is a zero-normalized simple claims problem then nothing changes for the other claimants.

5.4.3 Definition *A division rule f is said to satisfy the additivity of claims property if for every zero-normalized simple claims bankruptcy problem $(E; d) = (E; (d_1, d_2, \dots, d_n))$ which changes by splitting up the claim d_i in*

smaller claims $d_{i,1}, d_{i,2}, \dots, d_{i,k}$ into a bankruptcy problem

$(E; d') = (E; (d_1, \dots, d_{i-1}, d_{i,1}, d_{i,2}, \dots, d_{i,k}, d_{i+1}, \dots, d_n))$ the following holds:
 $f_j(E; d) = f_j(E; d')$ for each $j \in N \setminus \{i\}$.

This property can also be found in O'Neill (1982) where it is defined for all simple claims problems and where it is called strategy-proofness. Together with four other axioms it is used there to characterize the proportional division rule. In the context of cost allocation it has also been defined and used by Banker (1981) to characterize the proportional cost allocation rule. One of the axioms O'Neill uses requires the continuity of the division rule in at least one point for all coordinates while Banker characterizes the rule in the case that it allocates cost in rational numbers only and remarks that an axiom requiring continuity is needed in order to relax this assumption. In the proof of the following theorem the proportional rule is characterized axiomatically by symmetry, additivity of claims, efficiency and individual rationality without the use of a continuity axiom.

5.4.4 Theorem *The AP-rule is the unique game theoretic division rule for bankruptcy problems that satisfies (i) the minimal right property. (ii) the symmetry property and (iii) the additivity of claims property.*

Proof. It was already noted that the AP-rule is a game theoretic rule and that it satisfies (i) and (ii). That it also satisfies (iii) follows from the fact that for zero-normalized simple claims bankruptcy problems the AP-rule reduces to the proportional division rule. Let f be a game theoretic division rule which satisfies these three properties. Let $(E; d)$ be a zero-normalized simple claims bankruptcy problem. In the following $f_j(E; d)$ is denoted by f_j for all $j \in N$. For an $i \in N$ with $d_i > 0$ let $k \in \mathbb{N}$ be such that $\frac{d_i}{k} \leq d_j$ whenever $d_j > 0$. Consider the bankruptcy problem arising from $(E; d)$ by replacing claimant i by k claimants, each one with claim equal to $\frac{d_i}{k}$. By efficiency and properties (ii) and (iii) f assigns to each one of these claimants the amount $\frac{f_i}{k}$. Let $j \in N$, consider the bankruptcy problem arising from the last problem by replacing claimant j by $q_k := [\frac{d_j}{\frac{d_i}{k}}]$ claimants with claim $\frac{d_i}{k}$ and one claimant with claim $r_{j,k} := d_j - q_k \frac{d_i}{k}$. Here $[x]$ stands for the largest integer smaller than or equal to x . So $\frac{d_j}{\frac{d_i}{k}} - 1 < q_k \leq \frac{d_j}{\frac{d_i}{k}}$. Because of symmetry all the claimants with claim $\frac{d_i}{k}$ receive $\frac{f_i}{k}$. Let $f_{j,k}$

be the amount received by the claimant with claim $r_{j,k}$. By definition $r_{j,k} < \frac{d_j}{k}$. This inequality, property (iii) and individual rationality imply that $0 \leq f_{j,k} \leq \frac{f_j}{k}$. Property (iii) also yields $f_j = q_k \frac{f_j}{k} + f_{j,k}$ and hence $\frac{d_j}{d_i} - \frac{1}{k} \leq \frac{f_j}{f_i} \leq \frac{d_j}{d_i} + \frac{1}{k}$. Because it is possible to go through the above procedure for every k which is big enough it follows that $\frac{f_j}{f_i} = \frac{d_j}{d_i}$. Therefore the division rule f is the same as the proportional division rule for zero-normalized simple claims bankruptcy problems and hence f is equal to the AP -rule for such problems. Let $(E; d)$ be a general bankruptcy problem. Then $f(E; d) = m + f(E - m(N); d - m) = m + f(E - m(N); \hat{E}' \wedge (d - m))$ where $\hat{E}'$ is the vector in $\mathbb{R}^n$ with all coordinates equal to $E - m(N)$ and where the first inequality follows from (i) and the second from the fact that f is a game theoretic division rule. Now $(E - m(N); \hat{E}' \wedge (d - m))$ is a zero-normalized simple claims bankruptcy problem and hence $m + f(E - m(N); \hat{E}' \wedge (d - m)) = m + AP(E - m(N); \hat{E}' \wedge (d - m)) = m + AP(E - m(N); d - m) = AP(E; d)$. Here the second equality follows because the AP -rule is a game theoretic division rule and the third equality follows from the minimal right property. Thus, $f(E; d) = AP(E; d)$ for all bankruptcy problems $(E; d)$ and hence the AP -rule is the unique game theoretic division rule satisfying (i), (ii) and (iii). $\square$

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COÖPERATIEVE SPELTHEORIE EN TOEPASSINGEN

Coöperatieve spelen werden voor het eerst systematisch bestudeerd in het boek "Theory of games and economic behaviour" door von Neumann en Morgenstern (1944). Coöperatieve spelen zijn bijzonder geschikt om situaties waarbij een groep mensen samenwerkt te modelleren. In dit proefschrift worden coöperatieve spelen in karakteristieke funktievorm bestudeerd. Een coöperatief spel in karakteristieke funktievorm is een geordend paar bestaande uit een eindige verzameling, de spelersverzameling en een funktie, de karakteristieke funktie die aan iedere deelverzameling van spelers een reëel getal toekent welke de waarde van zo'n deelverzameling in het spel voorstelt. In een situatie gemodelleerd door zo'n spel kunnen de spelers besluiten om alleen te werken of in deelgroepen genoemd coalities, of allemaal samen in de grote coalitie. De karakteristieke funktie beschrijft hun winst in al deze gevallen. Een vraag die veel aandacht heeft gekregen in de coöperatieve speltheorie is de volgende. Stel dat alle spelers besluiten samen te werken. Hoe moet de winst die hierdoor verkregen wordt onder hen verdeeld worden? In hoofdstuk 1 worden verscheidene bekende methoden uit de coöperatieve speltheorie om deze winst te verdelen behandeld. Als er een verdeelmethode bestaat die de stabiliteit van de grote coalitie garandeert dan heet het spel gebalanceerd. Verder worden er in dit inleidend hoofdstuk coöperatieve spelen gedefinieerd en worden er verscheidene eigenschappen van deze spelen behandeld.

Hoofdstuk 2 is gewijd aan simpele spelen. Simpele spelen zijn coöperatieve spelen met de eigenschap dat de karakteristieke funktie alleen de waarden 0 en 1 kan aannemen. Deze spelen zijn uitermate geschikt om comité controle. Bij de bestudering van simpele spelen vervullen machtsindices een belangrijke rol. In hoofdstuk 2 worden verscheidene machtsindices uit de literatuur behandeld en een nieuwe machtsindex word geïntroduceerd en op twee manieren gekarakteriseerd.

Hoofdstuk 3 is gewijd aan coöperatieve spelen met de waarde van een coalitie gelijk aan de waarde van een lineair programmeringsprobleem dat van de coalitie afhangt. Zulke spelen worden lineaire programmeringsspele genoemd. Deze spelen blijken gebalanceerd te zijn. Twee voorbeelden van lineaire programmeringsspele, namelijk transportspele en lineaire productiespele worden behandeld. Comité controle in lineaire programmeringsspele wordt beschreven. De simpele spelen uit hoofdstuk 2 worden hiervoor gebruikt. Als alle simpele spelen die de comité controle beschrijven in een

lineair programmeringspel gebalanceerd zijn, dan is het spel zelf ook gebalanceerd. Er wordt bewezen dat elk niet negatief gebalanceerd spel een lineair programmeringspel met comité controle is. Voor niet gebalanceerde lineaire programmeringspelen worden twee methoden om de winst te verdelen beschreven. Een variatie op lineaire programmeringspelen waarbij claims een rol spelen wordt bestudeerd. De hiertoe ingevoerde claim spelen worden in hoofdstuk 5 grondig bestudeerd.

Terwijl in hoofdstuk 3 een lineair programmeringsprobleem de waarde van een coalitie bepaalt gebeurt dit in hoofdstuk 4 door een combinatorisch programmeringsprobleem. Zo wordt een combinatorisch spel verkregen. In de paragrafen 4.2., 4.3, 4.4 en 4.5 worden koppel situaties en permutatie situaties bestudeerd. In paragraaf 4.3 worden combinatorische spelen die uit deze situaties ontstaan behandeld. In het algemeen hoeven deze spelen niet gebalanceerd te zijn maar verscheidene deelklassen bestaande uit gebalanceerde spelen worden gegeven. Paragraaf 4.4 behandelt economieën met ondeelbare goederen die uit permutatie situaties ontstaan. In paragraaf 4.5 worden koppel situaties met ordinale preferenties behandeld. Paragraaf 4.6 is gewijd aan een ander combinatorisch spel, het wachtrijspel. Dit spel blijkt mooie eigenschappen te bezitten. Eén daarvan is gebalanceerdheid. Voor twee verdeelregels uit de coöperatieve speltheorie, de Shapley-waarde en de τ -waarde, kunnen simpele uitdrukkingen gevonden worden in het geval van wachtrijspelen. Een verdeelregel voor wachtrij situaties die gedefinieerd wordt zonder rekening te houden met speltheoretische overwegingen blijkt mooie eigenschappen te bezitten. Deze regel wordt axiomatisch gekarakteriseerd. In de paragrafen 4.7, 4.8, 4.9 worden combinatorische spelen behandeld die te maken hebben met kosten in plaats van winsten. Paragraaf 4.7 behandelt handelsreizigersspelen. In het algemeen hoeven deze spelen niet gebalanceerd te zijn. Deelklassen bestaande uit gebalanceerde spelen worden gegeven. Een klasse van spelen die gerelateerd zijn aan handelsreizigersspelen, namelijk routingsspelen worden gedefinieerd. In tegenstelling tot handelreizigersspelen zijn routingsspelen altijd gebalanceerd. Paragraaf 4.8 behandelt minimum kosten opspannende boomspelen. Zowel deze spelen als de lokatiespelen die in paragraaf 4.9 behandeld worden zijn gebalanceerd.

Hoofdstuk 5 behandelt bankroetspelen. Deze spelen ontstaan uit bankroet situaties. Drie verdeelregels voor bankroetproblemen worden behandeld. Er wordt bewezen dat ze corresponderen met drie bekende verdeelregels uit de coöperatieve speltheorie.

CURRICULUM VITAE

De schrijfster van dit proefschrift werd op 6 november 1960 op Curaçao geboren. Daar behaalde zij in 1979 haar VWO-diploma aan het Maria Immaculata Lyceum. Hierna volgde de studie wiskunde aan de Katholieke Universiteit Nijmegen. In 1982 werd het candidaatsexamen wis- en natuurkunde afgelegd (met genoegen) en in 1984 het doctoraalexamen wiskunde (cum laude).

Van 16 september 1984 tot 1 september 1988 was zij werkzaam op een universitaire onderzoekpoolplaats op het Mathematisch Instituut van de Katholieke Universiteit Nijmegen. Onder leiding van Prof. Dr. S.H. Tijs verrichtte zij aldaar onderzoek op het gebied van de coöperatieve speltheorie. De resultaten van dit onderzoek zijn grotendeels beschreven in dit proefschrift.

Per 1 september 1988 is zij werkzaam als assistent professor aan de Department of Mathematics and Statistics van de University of Maryland te Baltimore, Maryland, U.S.A.

STELLINGEN

behorende bij het proefschrift
“Cooperative game theory and applications”

van

I.J. Curiel

1. Laat de operator M op G^N gedefinieerd zijn door:

$$M(v)(S) := \min_{\sigma \in \Pi_N} \sum_{i \in S} m_i^\sigma(v) \text{ voor alle } v \in G^N \text{ en alle } S \in 2^N.$$

Laat v een symmetrisch spel zijn. Dan is $M(v)$ een convex spel.

2. De vaste punten van de operator M uit stelling 1 zijn precies de convexe spelen in G^N .
3. Laat $\Gamma^T = \langle N, X_1, X_2, \dots, X_n, K_1^T, K_2^T, \dots, K_n^T \rangle$ een n -persoonsspel in normale vorm zijn met $K_i^T(x) = -m_i(x_i) - T_i(C(x))$ voor alle $i \in N$ en alle $x \in X = X_1 \times X_2 \times \dots \times X_n$. Hier is m_i een functie van X_i naar $\mathbb{R}$ voor alle $i \in N$ en C is een functie van X naar $\mathbb{R}$ met $L := \{z \in X \mid C(z) = \inf_{x \in X} C(x)\} \neq \emptyset$. Laat $R(C)$ het bereik van C zijn. Een functie T van $R(C)$ naar $\mathbb{R}^n$ heet efficiënt als $\sum_{i \in N} T_i(y) = y$ voor alle $y \in R(C)$. Een functie T van $R(C)$ naar $\mathbb{R}^n$ heet geschikt als er een $x \in L$ is die een Nash evenwicht is in Γ^T . Het spel Γ^T heet oplosbaar als T efficiënt en geschikt is.

Als C injectief is dan bestaat er een T zo dat Γ^T oplosbaar is.

Curiel, I.J. (1987). Cost allocation in a decentralized firm.
Methods of Operations Research 57 465-475.

4. Als de functie C in de definitie van Γ^T uit stelling 3 niet injectief is dan bestaat er een T zo dat Γ^T oplosbaar is dan en slechts dan als er een $x^* \in L$ bestaat met $y = C(x_{-1}^*, x_1) = C(x_{-2}^*, x_2) = \dots = C(x_{-n}^*, x_n)$ voor zekere $x_1 \in X_1, x_2 \in X_2, \dots, x_n \in X_n$ impliceert $y - C(x^*) \geq \sum_{i \in N} (m_i(x_i^*) - m_i(x_i))$. Hier is (x_{-i}^*, x_i) de vector met j -de coördinaat gelijk aan x_j^* voor alle $j \neq i$ en met i -de coördinaat gelijk aan x_i .
5. Laat E een eindige verzameling zijn. Voor alle $e \in E$ is een gewicht $w(e) > 0$ gegeven. Een coöperatief matroïdespel $\langle N, v \rangle$ wordt als volgt gedefinieerd. Voor iedere $S \in 2^N \setminus \{\emptyset\}$ geldt

$$v(S) := \max_{A \in \mathcal{M}_S} \sum_{e \in A} w(e)$$

waarin $(E, \mathcal{M}_S)$ een matroïde is. Stel dat

(i) $\mathcal{M}_S \cap \mathcal{M}_T = \mathcal{M}_{S \cap T}$ voor alle $S, T \in 2^N \setminus \{\emptyset\}$ en

(ii) $\mathcal{M}_{S \cup T} \supset \{C \mid C = A \cup B, A \in \mathcal{M}_S, B \in \mathcal{M}_T\}$ voor alle $S, T \in 2^N \setminus \{\emptyset\}$.

Dan is v gebalanceerd.

6. Het Latijns-Amerikaanse ballroom dansen dat in Europa beoefend wordt heeft weinig te maken met het dansen in Latijns-Amerika.
7. Pijnlijkler dan de discriminatie zelf is het goedpraten ervan door mensen die men tot zijn vrienden rekent.
8. Veel wrijving bevordert het gevoel van zelfvertrouwen.