

A CONSTRUCTIVE VIEW ON ERGODIC THEOREMS

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ABSTRACT. Let T be a positive L_1 - L_∞ contraction. We prove that the following statements are equivalent in constructive mathematics.

1. The projection in L_2 on the space $\mathcal{N} := \text{cl}\{x - Tx : x \in L_2\}$ exists;
2. The sequence $(T^n)_{n \in \mathbf{N}}$ Cesàro-converges in the L_2 norm;
3. The sequence $(T^n)_{n \in \mathbf{N}}$ Cesàro-converges almost everywhere.

Thus, we find necessary and sufficient conditions for the Mean Ergodic Theorem and the Dunford-Schwartz Pointwise Ergodic Theorem.

1. INTRODUCTION

Bishop [4] (p.233) put forward the following problem connected with the question of finding a constructive interpretation of ergodic theorems.

Think of X as the union of two equal tanks of fluid, and T as a motion of the fluid, which is supposed to keep the fluid confined to the tank in which it has been placed. Imagine that there may be a small leak, which would in fact allow the fluid in the two tanks to mix, but that we are not able to decide whether a leak actually exists. Since the leak if it exists, is small, there will be little mixing between the tanks after unit time (that is, under the transformation T), but after a long time (that is, under the transformation T^n for some large n) the mixing may be substantial.

He concluded that Birkhoff's Ergodic Theorem is non-constructive.

Bishop proved, using so-called upcrossings, a version of the Chacon-Ornstein Theorem. This theorem is a generalization of Dunford and Schwartz's version of the Pointwise Ergodic Theorem, a result which extends Birkhoff's Ergodic Theorem. In Bishop's ergodic theorem a limit is proved to exist in a constructively very weak sense. Bishop's result is a so-called *equal-hypothesis* substitute for the ergodic theorem. Bishop considered finding an *equal-conclusion* substitute to be 'an important open problem', see [2] (p55). His student Nuber [8][9] found such an *equal-conclusion substitute* for Birkhoff's Ergodic Theorem. His proof uses measure theoretic techniques and seems to work only for measure-preserving transformations. We use functional analytic techniques to give necessary and sufficient conditions for von Neumann's Mean Ergodic Theorem and the Dunford and Schwartz version of the Pointwise Ergodic Theorem to hold.

In the context of Bishop's constructive mathematics [3] we prove that for the Mean Ergodic Theorem to hold it is sufficient that the projection on the space of invariant functions exists. Conversely, from the convergence of the sequence in the conclusion of that theorem we obtain the projection. We also show that the Mean Ergodic Theorem is sufficient to prove the Dunford and Schwartz version of the Pointwise Ergodic Theorem, and again a converse is also true. The aim of this paper is to make these claims rigorous, see Theorem 16.

The paper is organized as follows. We first prove a Mean Ergodic Theorem. Then the Maximal Ergodic Theorem and Banach's Principle are proved and used to prove the Pointwise Ergodic Theorem. Our presentation loosely follows that of Krengel [7] (p.65,p.159) and Dunford and Schwartz [5]. We use [3] as a general reference for constructive mathematics.

2. THE MEAN ERGODIC THEOREM

The following definitions will be used throughout this chapter.

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Let T be an operator on a Banach space \mathcal{X} . Define the sum $S_n := \sum_{k=0}^{n-1} T^k$ and the average $A_n := \frac{1}{n}S_n$. Define the subspaces $\mathcal{M} := \{f \in \mathcal{X} : Tf = f\}$ and $\mathcal{N} := \text{cl}\{x - Tx : x \in \mathcal{X}\}$. An operator T is called a *contraction* if $\|Tx\| \leq \|x\|$ whenever $x \in \mathcal{X}$.

When X is a vector space and Y, Z are subspaces such that for every x in X there exist unique y in Y and z in Z such that $x = y + z$, we write $X = Y \oplus Z$.

Theorem 1. *Let T be a contraction on a Banach space \mathcal{X} . The sequence $(A_n)_{n \in \mathbf{N}}$ converges if and only if $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$, in which case $\lim_{n \rightarrow \infty} A_n = P_{\mathcal{M}}$, where $P_{\mathcal{M}}$ denotes the projection on \mathcal{M} parallel to \mathcal{N} .*

Proof. First suppose that $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$. Let $h = h_{\mathcal{M}} + h_{\mathcal{N}}$, where $h_{\mathcal{M}} \in \mathcal{M}$ and $h_{\mathcal{N}} \in \mathcal{N}$. We claim that $A_n h$ converges to $h_{\mathcal{M}}$. First consider $f = g - Tg$ for some $g \in \mathcal{X}$; then the sequence $A_n f = \frac{1}{n}(g - T^n g)$ converges to 0. When $f \in \mathcal{N}$, then there exists $g \in \mathcal{X}$ such that $\|f - (g - Tg)\| < \varepsilon$, so for all $n \in \mathbf{N}$, $\|A_n(f - (g - Tg))\| < \varepsilon$. Hence for large n , $\|A_n f\| < 2\varepsilon$. Consequently, the sequence $A_n f$ converges to 0 whenever $f \in \mathcal{N}$ and, using the notation above, $A_n h$ converges to $h_{\mathcal{M}}$.

Let $f \in \mathcal{X}$; then there exist $f_{\mathcal{M}}$ in \mathcal{M} and $f_{\mathcal{N}}$ in \mathcal{N} such that $f = f_{\mathcal{M}} + f_{\mathcal{N}}$. Consequently, $A_n f = f_{\mathcal{M}} + A_n f_{\mathcal{N}}$ which converges to $f_{\mathcal{M}}$ when n tends to ∞ .

Now suppose that the sequence $(A_n)_{n \in \mathbf{N}}$ converges to an operator P . The equalities $TP = P = PT$ follow easily from the definition of the sequence $(A_n)_{n \in \mathbf{N}}$. Consequently, $P = 0$ on \mathcal{N} and

$$P^2 = \lim_{n \rightarrow \infty} A_n P \stackrel{[TP=P]}{=} \lim_{n \rightarrow \infty} P = P.$$

If $z \in \mathcal{M} \cap \mathcal{N}$ and $\varepsilon > 0$, then there exists $u \in \mathcal{X}$ such that $\|z - (u - Tu)\| < \varepsilon$. Hence for all $n \in \mathbf{N}$, $\|A_n(z - (u - Tu))\| < \varepsilon$. Because $A_n(u - Tu)$ converges to 0 and for all $n \in \mathbf{N}$, $A_n z = z$ we see that $\|z\| \leq \varepsilon$. Consequently, $\mathcal{M} \cap \mathcal{N} = \{0\}$.

To see that $(I - P)x \in \mathcal{N}$ observe that

$$(I - T)\left(\frac{n-1}{n}I + \frac{n-2}{n}T + \dots + \frac{1}{n}T^{n-2}\right) = I - A_n$$

for all n in \mathbf{N} . Define

$$y_n := \left(\frac{n-1}{n}I + \frac{n-2}{n}T + \dots + \frac{1}{n}T^{n-2}\right)x;$$

then $(I - T)y_n \rightarrow (I - P)x$. Consequently, $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$. □

Let \mathcal{H} be a Hilbert space and let T be an operator on \mathcal{H} . Let $x \in \mathcal{H}$. There exists a vector x^* such that $\langle Ty, x \rangle = \langle x, x^* \rangle$ if and only if the functional $y \mapsto \langle Ty, x \rangle$ is normable. This follows from the Riesz representation theorem [3] (p.419). If such a vector exists we will denote it by T^*x even if the adjoint is not totally defined¹.

Theorem 2. [Mean Ergodic Theorem] *Let T be a contraction on a Hilbert space \mathcal{H} . Then the sequence $(A_n)_{n \in \mathbf{N}}$ converges if and only if \mathcal{N} is located; in this case the sequence $(A_n)_{n \in \mathbf{N}}$ converges to the orthogonal projection $P_{\mathcal{M}}$ on \mathcal{M} .*

Proof. We first prove that \mathcal{M} and \mathcal{N} are orthogonal. Suppose that $x \in \mathcal{M}$, i.e. $Tx = x$. We claim that the map $y \mapsto \langle Ty, x \rangle$ is normable. Since $|\langle Ty, x \rangle| \leq \|x\| \|y\|$ whenever $y \in \mathcal{H}$, $\|x\|$ is an upper bound on the norm. On the other hand this upper bound is attained at x . It follows that $x \in \text{Dom } T^*$ and $\|T^*x\| = \|x\|$. Now,

$$\begin{aligned} \|T^*x - x\|^2 &= \langle T^*x - x, T^*x - x \rangle \\ &= \|T^*x\|^2 + \|x\|^2 - \langle x, T^*x \rangle - \langle T^*x, x \rangle \\ &= \|T^*x\|^2 + \|x\|^2 - \langle Tx, x \rangle - \langle x, Tx \rangle \\ &\stackrel{[Tx=x]}{=} \|T^*x\|^2 + \|x\|^2 - 2\langle x, x \rangle = \|T^*x\|^2 - \|x\|^2 = 0. \end{aligned}$$

1. Classically, the adjoint of an operator is always totally defined. Constructively this is not the case.

Consequently, $T^*x = x$ and so

$$\langle x, (I - T)y \rangle = \langle (I - T)^*x, y \rangle = 0$$

for all y in \mathcal{H} . We see that \mathcal{M} and \mathcal{N} are orthogonal.

Suppose that the sequence $(A_n)_{n \in \mathbf{N}}$ converges. Theorem 1 shows that $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$. Because \mathcal{M} and \mathcal{N} are also orthogonal, \mathcal{M} and \mathcal{N} are located.

Conversely, suppose that \mathcal{N} is located. We know that $\mathcal{M} \subset \mathcal{N}^\perp$. We will prove that $\mathcal{N}^\perp \subset \mathcal{M}$. Let $x \in \mathcal{N}^\perp$. Then $\langle (I - T)y, x \rangle = 0$ whenever $y \in \mathcal{H}$. It follows that $\langle y, x \rangle = \langle Ty, x \rangle$, i.e. $x = T^*x$. By a similar argument as above we see that $Tx = x$. We conclude that $\mathcal{M} = \mathcal{N}^\perp$, so by Theorem 1 the sequence $(A_n)_{n \in \mathbf{N}}$ converges. \square

Let (X, μ) be a measure space. A *measure-preserving transformation* of X is a partial function from a full set to a full set such that for all integrable sets A , $\tau(A)$ is integrable and $\mu(\tau(A)) = \mu(A)$. If τ is a measure-preserving transformation, then $T_\tau f := f \circ \tau$ is a contraction on L_2 . This shows that our result generalizes the possibly more familiar formulation of the Theorem.

Bishop and Bridges [3] (problem 46, p.395) give the following version of the Mean Ergodic Theorem. Let T be a unitary operator on a Hilbert space H ; then for all $x \in H$ the sequence $(A_n x)_{n \in \mathbf{N}}$ converges if and only if the sequence $(\|A_n x\|)_{n \in \mathbf{N}}$ converges.

3. MAXIMAL ERGODIC THEOREMS

Let (X, μ) be a measure space. An operator T on $L_1(\mu)$ is an L_1 - L_∞ *contraction* if T is a contraction on L_1 that contracts the L_∞ -norm on $L_1 \cap L_\infty$ — that is, $\|f\|_1 \leq \|Tf\|_1$ and for all real numbers m , $|Tf| \leq m$ whenever $f \in L_1$ and $|f| \leq m$. An operator T on an ordered vector space is *positive*² if $Tf \geq 0$ whenever $f \geq 0$. When τ is a measure-preserving transformation, then T_τ is a positive L_1 - L_∞ contraction.

Let T be a positive L_1 - L_∞ contraction. Define the operator M_n by

$$M_n f := \sup_{k \leq n} A_k f$$

for all n in \mathbf{N} . Garcia's proof [7] (p.8) of the following Theorem 3 is constructive. Note however that we do not make any claims about M_∞ . This operator is defined classically as $\sup_{k \in \mathbf{N}} A_k$, but constructively $M_\infty f$ may not be a measurable function for all f in L_1 , i.e. we may not be able to find simple functions approximating $M_\infty f$.

In the constructive theory of measure spaces it is not always possible to compute the measure of the set $[f < \alpha] := \{x : f(x) < \alpha\}$. However, we can compute the measure for all but countably many α , such α are called *admissible*, see [3] for details.

Theorem 3. [Hopf's Maximal Ergodic Theorem] *Let T be a positive contraction on $L_1(\mu)$. Let n be a natural number. If $\alpha \geq 0$ is admissible for $M_n f$, then*

$$\int_{[M_n f \geq \alpha]} f \geq 0.$$

Corollary 4. [Wiener [7] (p.51)] *Let T be a positive L_1 - L_∞ contraction. Let $f \in L_1$, $n \in \mathbf{N}$ and $\alpha > 0$ be admissible for $M_n f$. Then for all n ,*

$$\mu[M_n f \geq \alpha] \leq \frac{1}{\alpha} \int_{[M_n f \geq \alpha]} f.$$

The following theorem is sometimes called the little Riesz theorem. The proof we give here is an adaptation of [7] (Lemma 1.7.4).

² When $p = 2$, this order-theoretic definition differs from the definition of a positive operator on a Hilbert space.

Proposition 5. *If T is a positive L_1 - L_∞ contraction, then T can be uniquely extended to an L_p contraction for all $p \geq 1$.*

Proof. Because for all $p \geq 1$, $L_1 \cap L_\infty$ is a dense subset of L_p , it is enough to prove that $\|Tf\|_p \leq \|f\|_p$ for all $f \in L_1 \cap L_\infty$. To achieve this goal we will prove that

$$Tf \leq T(f^p)^{\frac{1}{p}} \quad (1)$$

for all positive simple functions f and $p > 1$.

Assume for a moment that we have done so and let f be a positive simple function and $p > 1$. Then $(Tf)^p \leq T(f^p)$, so $\|(Tf)^p\|_1 \leq \|T(f^p)\|_1 \leq \|f^p\|_1$ and thus $\|Tf\|_p \leq \|f\|_p$. Observe that this inequality trivially holds for $p = 1$. Consequently, $\|Tf\|_p > \|f\|_p$ is impossible, i.e. $\|Tf\|_p \leq \|f\|_p$ holds for all $p \geq 1$, even if we are unable to decide $p = 1$ or $p > 1$. It follows that for all $p \geq 1$, T is a positive L_p -contraction on the positive simple functions, and consequently also on the simple functions. The simple functions are dense and T is a contraction, so the operator T restricted to the simple functions can be uniquely extended to L_p . This extension agrees with T on $L_1 \cap L_p$ for all $p \geq 1$.

We will now prove (1) for a positive simple function f and $p > 1$. We will assume that $Y := [f > 0]$ is integrable. Since the simple functions f with this property are also dense in L_p^+ , we do not lose generality. We note that $f = f\chi_Y$. We define $q := (1 - \frac{1}{p})^{-1}$ as usual. For all real numbers $a, b \geq 0$:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

It follows that for all real numbers $c, d > 0$:

$$\frac{f}{c} \cdot \frac{\chi_Y}{d} \leq \frac{f^p}{c^p p} + \frac{\chi_Y^q}{d^q q}.$$

Consequently,

$$T(f \cdot \chi_Y) \leq T(f^p) \frac{cd}{c^p p} + T(\chi_Y^q) \frac{dc}{d^q q} \quad a.e. \quad (2)$$

Let F be a full set on which (2) holds for rational c and d and thus by continuity for all $c, d > 0$. Compute $M, m \in \mathbf{R}^+$ such that $M \geq f \geq m\chi_Y > 0$. Then $f \leq m^{1-p}f^p$. Let $F' \subset F$ be a full set such that for all $x \in F'$

$$f(x) \leq M, \quad (Tf)(x) \leq m^{1-p}(Tf^p)(x), \quad (Tf)(x) \leq M(T\chi_Y)(x) \quad \text{and} \quad (T\chi_Y)^{\frac{1}{q}}(x) \leq 1.$$

Fix $x \in F'$.

If $T(f^p)(x) = 0$, then $T(f)(x) \leq m^{1-p}T(f^p)(x) = 0 = T(f^p)^{\frac{1}{p}}(x)$.

If $T(\chi_Y)(x) = 0$, then $T(f)(x) \leq MT(\chi_Y)(x) = 0 \leq T(f^p)^{\frac{1}{p}}(x)$.

If $T(f^p)(x) > 0$ and $T(\chi_Y)(x) > 0$, then we define $c := T(f^p)^{\frac{1}{p}}(x)$ and $d := T(\chi_Y)^{\frac{1}{q}}(x)$. The right hand side of (2) equals $cd(\frac{1}{p} + \frac{1}{q}) = cd$. Because $f = f\chi_Y$ we obtain:

$$T(f)(x) \leq T(f^p)^{\frac{1}{p}}T(\chi_Y)^{\frac{1}{q}}(x) \leq T(f^p)^{\frac{1}{p}}(x). \quad (3)$$

We conclude that in any case $T(f)(x) > T(f^p)^{\frac{1}{p}}(x)$ is impossible. It follows that (3) holds for all $x \in F'$. Consequently, (1) holds and we have thus completed the proof. \square

From this point onwards we will assume that a positive L_1 - L_∞ contraction is extended to L_p for all $p \geq 1$.

Theorem 6. [Dominated Ergodic Theorem] *Let T be a positive L_1 - L_∞ contraction. Then for all $n \in \mathbf{N}$, $p > 1$ and $f \in L_p$: $\|M_n f\|_p \leq \frac{p}{p-1} \|f\|_p$.*

Proof. Fix $n \in \mathbf{N}$, $p > 1$, and $f \in L_p^+$.

$$\begin{aligned}
\int (M_n f)^p d\mu &= \int \int_0^{M_n f(s)} p\alpha^{p-1} d\alpha d\mu(s) \\
&= p \int \int_0^\infty \alpha^{p-1} \chi_{[M_n f \geq \alpha]}(s) d\alpha d\mu(s) \\
&\stackrel{\text{Fubini}}{=} p \int_0^\infty \alpha^{p-1} \mu[M_n f \geq \alpha] d\alpha \\
&\stackrel{\text{Wiener}}{\leq} p \int_0^\infty \alpha^{p-2} \int_{[M_n f \geq \alpha]} f d\mu d\alpha \\
&= p \int_0^\infty \int \alpha^{p-2} \chi_{[M_n f \geq \alpha]}(s) f(s) d\mu(s) d\alpha \\
&\stackrel{\text{Fubini}}{=} p \int f(s) \int_0^\infty \alpha^{p-2} \chi_{[M_n f \geq \alpha]}(s) d\alpha d\mu(s) \\
&= p \int f(s) \int_0^{M_n f(s)} \alpha^{p-2} d\alpha d\mu(s) \\
&= \frac{p}{p-1} \int f (M_n f)^{p-1} d\mu \\
&\leq \frac{p}{p-1} \|f\|_p \|M_n f\|_p^{p-1}.
\end{aligned}$$

This last inequality follows from Hölder's inequality and the fact that $M_n f \leq \sum_{k=0}^n A_k f$, so that $M_n f \in L_p$. For all $f \in L_p$, $M_n f \leq M_n |f|$ and thus $\|M_n f\|_p \leq \|M_n |f|\|_p \leq \frac{p}{p-1} \|f\|_p$. \square

A function $f: \mathbf{R} \rightarrow \mathbf{R}$ is *convex* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

whenever $\lambda \in [0, 1]$ and $x, y \in \mathbf{R}$.

Theorem 7. *A (total) convex function from \mathbf{R} to \mathbf{R} has an non-decreasing derivative which is defined in all but countably many points.*

Proof. To prove this we will use Bishop's profile theorem, a constructive substitute for the classical lemma stating that every non-decreasing real function is continuous in all but countably many points. Like Bishop we define the set \mathcal{E} to be the set of piecewise linear functions

$$h_{xy}(z) := \frac{\min\{z, y\} - \min\{z, x\}}{y - x}$$

whenever $x < y$. These functions are 0 when $z \leq x$ and 1 when $x \geq y$. We define

$$\Lambda(h_{xy}) := \frac{f(y) - f(x)}{y - x}.$$

Then Λ is increasing³ on the set \mathcal{E} with the order inherited from the the functions from \mathbf{R} to \mathbf{R} — that is, (\mathcal{E}, Λ) is a profile. The profile theorem ensures that all but countably many points are smooth. In the present case this means that the function is differentiable at these points. \square

Lemma 8. [Jensen's inequality] *Let (X, μ) be a finite measure space and $\phi: \mathbf{R} \rightarrow \mathbf{R}$ be a convex function. If $f, \phi \circ f$ are in L_1 , then*

$$\phi\left(\frac{1}{\mu(X)} \int f\right) \mu(X) \leq \int \phi \circ f.$$

Proof. Let x be in the domain of ϕ' . Then for all real numbers y ,

$$\phi(y) \geq \phi'(x)(y - x) + \phi(x),$$

and hence

$$\phi(f(t)) \geq \phi'(x)(f(t) - x) + \phi(x)$$

³. This is most easily seen by a geometric argument considering the graph of a convex function.

for all $t \in \text{Dom} f$. By integrating we obtain

$$\int \phi \circ f \geq \phi'(x) \left(\left(\int f \right) - x\mu(X) \right) + \phi(x)\mu(X).$$

If we take a sequence $(x_n)_{n \in \mathbf{N}}$ in $\text{Dom} \phi'$ tending to $\frac{1}{\mu(X)} \int f$, we obtain the inequality above. \square

The L_p Ergodic Theorem can be proved classically for finite measure spaces using the point-wise ergodic theorem [14]. Another proof [7] (Thm. 2.1.2) uses a non-constructive compactness argument. Our proof works not only for finite measure spaces, but also for σ -finite measure spaces. We make some preparations.

Let μ be a finite measure. If $1 \leq p < q$ and $f \in L_q$, then $|f|^p$ is measurable and bounded by $|f|^q \vee 1$, hence $|f|^p$ is integrable and by taking $\phi(x) := x^{\frac{p}{q}}$ in Jensen's inequality we obtain

$$\|f\|_p \leq \|f\|_q \mu(X)^{\frac{1}{p} - \frac{1}{q}}. \quad (4)$$

We see that $L_q \subset L_p$.

Let μ be σ -finite. If $p \geq q \geq 1$ and $f \leq M$, then

$$\|f\|_p^p = \int |f|^p = \int |f|^q |f|^{p-q} \leq M^{p-q} \|f\|_q^q.$$

Consequently,

$$\|f\|_p \leq M^{1 - \frac{q}{p}} \|f\|_q^{\frac{q}{p}}. \quad (5)$$

Theorem 9. [L^p Ergodic Theorem] *Let $p, q \geq 1$ and (X, μ) be a finite measure space or let $p > 1, q \geq 1$ and (X, μ) a σ -finite measure space. Let T be a positive L_1 - L_∞ contraction. If the sequence $(A_n)_{n \in \mathbf{N}}$ converges in L_q , then it converges in L_p .*

Proof. Let $f \in L_p$ and choose a simple g such that $\|f - g\|_p < \frac{\varepsilon}{4}$. Let M be a bound for g . For all $n, m \in \mathbf{N}$:

$$\begin{aligned} \|A_n f - A_m f\|_p &\leq \|A_n f - A_n g\|_p + \|A_n g - A_m g\|_p + \|A_m g - A_m f\|_p \\ &\leq \frac{\varepsilon}{4} + \|A_n g - A_m g\|_p + \frac{\varepsilon}{4}. \end{aligned}$$

If we show that $\|A_n g - A_m g\|_p \rightarrow 0$, when $m, n \rightarrow \infty$, then $(A_n f)_{n \in \mathbf{N}}$ is a Cauchy sequence in L_p . That $\|A_n g - A_m g\|_p \rightarrow 0$ follows from the fact that the sequence $(A_n g)_{n \in \mathbf{N}}$ converges in L_q and the inequalities (4) and (5) for the case μ is finite or μ is σ -finite and $p \geq q$. The case μ is σ -finite and $1 < p \leq q$ is more difficult. We will now proceed to consider this case. If $p \geq 1, h \in L_p, n, m, l, r \in \mathbf{N}$ and $m = ln + r$, then

$$\begin{aligned} \|A_m h\|_p - \|A_n h\|_p &\leq \frac{1}{l} \|(I + T^n + \dots + T^{(l-1)n}) A_n h\|_p - \|A_n h\|_p \\ &\quad + \frac{1}{m} \|T^{ln}(I + \dots + T^{r-1}) h\|_p \\ &\leq 0 + \frac{r}{m} \|h\|_p \leq \frac{n}{m} \|h\|_p. \end{aligned}$$

It follows that the sequence $(\|A_n h\|_p)_{n \in \mathbf{N}}$ is *essentially decreasing*, that is for each $n \in \mathbf{N}$ and each $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $\|A_m h\|_p \leq \|A_n h\|_p + \varepsilon$ for all $m \geq N$.

Let $g \in L_q \cap L_p$. Let \tilde{g} be the limit of $A_n g$ in L_q . The function \tilde{g} is in L_p , because T contracts the L_p -norm. We will prove that $\lim_{n \rightarrow \infty} A_n g = \tilde{g}$ in L_p . By looking at $g - \tilde{g}$ we may assume that $\tilde{g} = 0$.

Because the sequence $(\|A_n g\|_p)_{n \in \mathbf{N}}$ is essentially decreasing, it is enough to find for each $\eta > 0$ and $k \in \mathbf{N}$, an $n > k$ such that $\|A_n g\|_p^p < \eta$. We will now proceed to find such n . Let $\beta = \frac{\eta}{4}$. Take an integrable set B_1 such that $\|g \chi_{X-B_1}\|_p^p < \beta$. We recall that $A_n g \rightarrow 0$ in L_q . We can thus compute n_1 such that $\|A_{n_1} g\|_q^p < \beta \mu(B_1)^{\frac{p}{q}-1}$. By (4)

$$\|(A_{n_1} g) \chi_{B_1}\|_p^p \leq \|(A_{n_1} g) \chi_{B_1}\|_q^p \mu(B_1)^{1 - \frac{p}{q}} < \beta.$$

Now

$$\|A_{n_1}g\|_p^p < \eta \quad \text{or} \quad \|A_{n_1}g\|_p^p > \eta - \beta.$$

In the former case we are done, so we may assume that $\|A_{n_1}g\|_p^p > \eta - \beta$ and thus $\|(A_{n_1}g)\chi_{X-B_1}\|_p^p > \eta - 2\beta$. We choose $B_2 \supset B_1$ such that $\|(A_{n_1}g)\chi_{B_2-B_1}\|_p^p \geq \eta - 3\beta$. Compute $n_2 > n_1$ such that $\|A_{n_2}g\|_q^p < \beta\mu(B_2)^{\frac{p}{q}-1}$. Then $\|A_{n_2}g\chi_{B_2-B_1}\|_p^p < \beta$.

We continue in this way until we find $N \in \mathbf{N}$ such that $\|A_{n_N}g\|_p^p < \eta$. That such an N exists we see as follows. Choose $K \in \mathbf{N}$ such that

$$\left(\left(\frac{p}{p-1}\right)^p - 1\right)\|g\|_p^p + \beta \leq K(\eta - 3\beta).$$

For all $N \leq K$, $\|A_{n_N}g\|_p^p > \eta - \beta$ or $\|A_{n_N}g\|_p^p < \eta$. Suppose that for all $N \leq K$, $\|A_{n_N}g\|_p^p > \eta - \beta$. Define $B_0 = \emptyset$ and $n_0 = 0$. Remark that for all $N \in \mathbf{N}$,

$$\begin{aligned} M_N|g| &\geq \sum_{i=0}^N M_N|g|\chi_{B_{i+1}-B_i} \\ &\geq \sum_{i=0}^N A_{n_i}|g|\chi_{B_{i+1}-B_i} \\ &\geq \sum_{i=0}^N A_{n_i}g\chi_{B_{i+1}-B_i}. \end{aligned}$$

It follows from the Dominated Ergodic Theorem that for all $N \leq K$,

$$\begin{aligned} \left(\frac{p}{p-1}\right)^p \|g\|_p^p &\geq \|M_N|g|\|_p^p \\ &\geq \left\| \sum_{i=0}^N (A_{n_i}g)\chi_{B_{i+1}-B_i} \right\|_p^p \\ &= \sum_{i=0}^N \|(A_{n_i}g)\chi_{B_{i+1}-B_i}\|_p^p \\ &> N(\eta - 3\beta) + \|g\|_p^p - \beta. \end{aligned}$$

It follows that there exists $N \leq K$ such that $\|A_{n_N}g\|_p^p < \eta$. Consequently, $A_n g \rightarrow \tilde{g}$ in L_p . Finally, we observe that even if we do not know whether $p \leq q$ or $p > q$ we can show that the sequence A_n converges in L_p . To see this we observe that either $p < q + \varepsilon$ or $p > q$, the latter case has been treated before. In the former case, the sequence also converges in $L_{q+\varepsilon}$ as we proved before, and we can thus proceed as above. \square

If μ is σ -finite, it is not true in general that if for some $p > 1$, the sequence A_n converges to 0 in L_p , then the sequence A_n converges in L_1 . To see this let $\tau(x) = x + 1$ on \mathbf{R} with Lebesgue measure, $T = T_\tau$ and $f = \chi_{[0,1]}$. For all $p > 1$, $A_n f$ converges to 0 in L_p , but the sequence does not converge in L_1 .

4. THE POINTWISE ERGODIC THEOREM

The following principle, called Banach's principle, will be used as follows: we prove that a sequence of operators converges almost everywhere (a.e.) on a dense set and then conclude that the sequence converges a.e. on the whole space.

The proof of the following theorem would be easier if we could prove constructively that M_∞ is measurable.

Theorem 10. [Banach's Principle] *Let (Y, μ) be a measure space. Let $(T_n)_{n \in \mathbf{N}}$ be a sequence of linear operators from a Banach space \mathcal{X} to the space μ -measurable real functions. Define for each $n \in \mathbf{N}$, an operator \tilde{M}_n by $\tilde{M}_n x := \sup_{k \leq n} |T_k x|$ for all $x \in \mathcal{X}$. Suppose that there exists a positive decreasing function C from \mathbf{R} to \mathbf{R} such that $\lim_{\alpha \rightarrow \infty} C(\alpha) = 0$ and for all x and n :*

$$\mu[\tilde{M}_n x \geq \alpha \|x\|] < C(\alpha),$$

whenever $\alpha\|x\|$ is admissible for $\tilde{M}_n x$. Then the set of elements $x \in \mathcal{X}$ for which the sequence $(T_n x)_{n \in \mathbf{N}}$ converges a.e. is closed.

Proof. Suppose that a sequence z_n converges to z in \mathcal{X} in norm and that there exists a sequence $(f_n)_{n \in \mathbf{N}}$ of measurable functions such that for all $m \in \mathbf{N}$, $T_m z_n \rightarrow f_n$ a.e. We will prove that the sequence $(T_m z)_{m \in \mathbf{N}}$ converges a.e.

For natural numbers a and b , $\omega \in Y$ and $x \in \mathcal{X}$ put

$$\Delta_{a,b}(\omega, x) := \sup_{a \leq n, m \leq b} |T_n x(\omega) - T_m x(\omega)|.$$

Then

$$|\Delta_{a,b}(\cdot, z) - \Delta_{a,b}(\cdot, z_n)| \leq |\Delta_{a,b}(\cdot, z - z_n)| \leq 2\tilde{M}_b(z - z_n).$$

Let $\varepsilon > 0$. Choose a natural numbers n such that $C(\frac{\varepsilon}{2}\|z - z_n\|) < \varepsilon$ and choose a natural number k and an integrable set $D \subset Y$ such that $\mu(D) < \varepsilon$ and for all $a \geq k$:

$$\{\omega: |T_a z_n(\omega) - f_n(\omega)| > \frac{\varepsilon}{2}\} \subset D.$$

For all $b \geq a \geq k$,

$$\begin{aligned} \mu[|\Delta_{a,b}(\cdot, z)| > 2\varepsilon] &\leq \mu[|\Delta_{a,b}(\cdot, z_n)| > \varepsilon] \\ &\quad + \mu[|\Delta_{a,b}(\cdot, z) - \Delta_{a,b}(\cdot, z_n)| > \varepsilon] \\ &\leq \mu(D) + \mu[2\tilde{M}_b(z - z_n) > \varepsilon] \\ &\leq \varepsilon + C(\frac{\varepsilon}{2}\|z - z_n\|) \leq 2\varepsilon. \end{aligned}$$

Construct ascending sequences $(n_m)_{m \in \mathbf{N}}$ and $(k_m)_{m \in \mathbf{N}}$ of natural numbers such that

$$C(\varepsilon 2^{-m-1}\|z - z_{n_m}\|) < \varepsilon 2^{-m}$$

and

$$\mu[|T_{n_m} z_{n_m} - f_{n_m}| > \varepsilon 2^{-m}] < \varepsilon 2^{-m}$$

for all $n \geq k_m$. Put

$$E := \bigcup_m \{\omega: |\Delta_{k_m, k_{m+1}}(\omega, z)| > \varepsilon 2^{-m}\}.$$

The set E is integrable and $\mu(E) \leq \sum_{m=1}^{\infty} 2\varepsilon 2^{-m} = 2\varepsilon$. For $\omega \in -E$ and $n, m \geq k_1$:

$$|T_n z(\omega) - T_m z(\omega)| \leq \varepsilon.$$

Consequently, the sequence $(T_n z)_{n \in \mathbf{N}}$ converges a.e. □

Bishop [4] (p.230) gave a constructive proof of Lebesgue's Differentiation Theorem. Using Banach's principle one can give another constructive proof, see [10] (p.101).

When $q \geq 1$ and T is a positive L_1 - L_∞ contraction, we define $\mathcal{M}_q = \{f \in L_q: T f = f\}$ and $\mathcal{N}_q = \text{cl}\{T f - f: f \in L_q\}$.

Theorem 11. [Pointwise Ergodic Theorem] *Let μ be a σ -finite measure. Let T be a positive L_1 - L_∞ contraction. If $L_q = \mathcal{M}_q \oplus \mathcal{N}_q$, for some $q \geq 1$, then for all $p \geq 1$, the sequence $(A_n f)_{n \in \mathbf{N}}$ converges a.e. for all $f \in L_p$.*

Proof. Suppose that $L_q = \mathcal{M}_q \oplus \mathcal{N}_q$, for some $q \geq 1$. First let $p > 1$. Theorem 9 and Theorem 1 show that $L_p = \mathcal{M}_p \oplus \mathcal{N}_p$. Suppose that $f = g + T h - h$, where $g \in L_p$, $T g = g$ and $h \in L_\infty \cap L_p$. The set of these f is dense in L_p . Because T contracts the L^∞ -norm, $\lim_{n \rightarrow \infty} A_n f = g$ a.e. For all admissible $\alpha > 0$,

$$\begin{aligned} \alpha \mu[M_n f \geq \alpha] &\stackrel{[\text{Wiener}]}{\leq} \int_{[M_n f \geq \alpha]} f \\ &\stackrel{[\text{Jensen}]}{\leq} \left(\int_{[M_n f \geq \alpha]} f^p \right)^{\frac{1}{p}} \mu[M_n f \geq \alpha]^{1 - \frac{1}{p}} \\ &\leq \|f\|_p \mu[M_n f \geq \alpha]^{1 - \frac{1}{p}}. \end{aligned}$$

Consequently, $\alpha\mu[M_n f \geq \alpha]^{\frac{1}{p}} \leq \|f\|_p$. Define $\tilde{M}_n f = \sup_{k \leq n} |A_k f|$, for all $f \in L_p$. Substituting $\alpha = \beta\|f\|_p$ and observing that $\tilde{M}_n f = M_n f$ when $f \geq 0$, we obtain

$$\mu[\tilde{M}_n f \geq \beta\|f\|_p] \leq \beta^{-p}. \quad (6)$$

For all $f \in L_p$, $\tilde{M}_n f \leq \tilde{M}_n |f|$, so inequality (6) holds for all $f \in L_p$. Banach's principle shows that the sequence $(A_n f)_{n \in \mathbb{N}}$ converges a.e. for all f in L_p .

Finally we remove the assumption that p is strictly greater than 1. Let $p \geq 1$. The set $L_{p+1} \cap L_p$ is dense in L_p , so one can apply Banach's principle since inequality (6) above holds for all $p \geq 1$. \square

The two-step argument above is used because in general we do not have convergence in the L_1 -norm. The space L_1 has an awkward geometrical structure: it is not uniformly convex.

Theorem 12. *Let $p \geq 1$. If the sequence $(A_n f)_{n \in \mathbb{N}}$ converges a.e. for all f in L_p , then $\mathcal{N} = \text{cl}\{x - Tx : x \in L_2\}$ is located in L_2 .*

Proof. Let f be an element of L_2 . Without loss of generality we may assume that $A_n f \rightarrow 0$ a.e. We claim that the sequence $(A_n f)_{n \in \mathbb{N}}$ converges weakly to 0. We may assume that f is a simple function. Since the sequence $(A_n f)_{n \in \mathbb{N}}$ is bounded, the sequence $\int (A_n f)g$ converges to 0 whenever g is a simple function. The inner product is continuous on L_2 , so $\langle A_n f, g \rangle \rightarrow 0$ for all $f, g \in L_2$ — that is, $(A_n f)_{n \in \mathbb{N}}$ converges weakly to 0.

Define $B_n := (\frac{n-1}{n}I + \frac{n-2}{n}T + \dots + \frac{1}{n}T^{n-2})$. By an argument similar to the proof of Theorem 1 we see that for each x , $(I - T)B_n x$ converges weakly to an element in the space $\text{wcl}\{y - Ty : y \in L_2\}$ and thus

$$L_2 = \mathcal{M} \oplus \text{wcl}\{(I - T)x : x \in L_2\},$$

where wcl denotes the weak closure. Since $(I - T)B_n x = (I - A_n)x$ is a bounded sequence and the weak closure of a bounded convex inhabited subset coincides with its strong closure, by Lemma 5.2.4 in [11], we see that $\mathcal{N} = \text{wcl}\{y - Ty : y \in L_2\}$. It follows that the display sum above is orthogonal and thus that \mathcal{N} is located. \square

Lemma 13. [6] *Let E be a uniformly convex and uniformly smooth Banach space and C a bounded convex subset of E . Then C is located if and only if $\sup\{f(z) : z \in C\}$ exists for each normable linear functional f on E .*

Lemma 14. [3][6] *For $p > 1$, the space L_p is uniformly convex and uniformly smooth and each normable functional may be represented by a functional $f \mapsto \int f g$, for some g in L_q , where $\frac{1}{p} + \frac{1}{q} = 1$.*

Lemma 15. *If \mathcal{N} is located in some L_p , then both \mathcal{M} and \mathcal{N} are located in all L_p , where $p > 1$.*

Proof. Locatedness is a property of closed sets, so we may restrict to a set which is dense in all the L_p -spaces, for instance the bounded L_1 functions or the simple functions. Moreover, an inhabited set A is located if we can compute the distance $\rho(x, A)$ for each x . Let a be an element of A . Then $\rho(x, A) = \rho(x, A \cap B(x, \rho(x, a) + 1))$, where $B(x, r)$ denotes the ball around x with radius r . Consequently, when considering located sets we may restrict to its bounded subsets.

We will now apply Lemma 14. Since the simple functions are dense in all the L_p -spaces, if $\sup\{(z, f) : z \in C\}$ exists for all f in L_p , then this supremum exists for all f in set of the simple functions, and thus for all f in any $L_{p'}$, where $p' > 1$. It follows that A is located in $L_{p'}$. In particular, if \mathcal{N} are located in L_p , then it is located in L_2 . Since these sets are orthogonal, they are thus both located in L_2 . Consequently, they are both located in any $L_{p'}$. \square

Theorem 16. *Let μ be a σ -finite measure and T a positive L_1 - L_∞ contraction. Denote by A_n the average $\frac{1}{n} \sum_{i=0}^{n-1} T^i$. The following statements are equivalent:*

1. *The set $\mathcal{N} = \text{cl}\{x - Tx : x \in L_2\}$ is located in L_2 ;*

2. The sequence $(A_n)_{n \in \mathbf{N}}$ converges in L_2 ;
3. For all $p > 1$, the sequence $(A_n)_{n \in \mathbf{N}}$ converges in L_p ;
4. There is $p > 1$ such that the sequence $(A_n)_{n \in \mathbf{N}}$ converges in L_p ;
5. For all $p \geq 1$ and $f \in L_p$, the sequence $(A_n f)_{n \in \mathbf{N}}$ converges a.e.;
6. There is $p \geq 1$ such that for all $f \in L_p$, the sequence $(A_n f)_{n \in \mathbf{N}}$ converges a.e.

For a finite measure we may replace $p > 1$ by $p \geq 1$.

Proof. (1) \Leftrightarrow (2) is Theorem 2. (4) \Rightarrow (3) is Theorem 9. (3) \Rightarrow (2) is trivial. (2) \Rightarrow (4) is trivial. (4) \Rightarrow (5) follows from Theorem 1 and Theorem 11. (5) \Rightarrow (6) is trivial. (6) \Rightarrow (1) follows from Theorem 12. \square

Some of these results have circulated in preprints for some time. They then appeared in my thesis [11]. The results have been used in [1] in the context of reverse mathematics.

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