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A New Class of Invertible Polynomial Maps

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In this paper we present a new large class of polynomial maps $F = X + H : A^n \to A^n$ (Definition 1.1) on every commutative ring A for which the Jacobian Conjecture is true. In particular H does not need to be homogeneous. We also show that for all H in this class satisfying $H(0) = 0$ the nth iterate $H \circ \cdots \circ H = 0$.

INTRODUCTION

In [1] it was shown that it suffices to prove the Jacobian Conjecture for cubic homogeneous polynomial maps, i.e., maps of the form

$$F = X + H : \mathbb{C}^n \to \mathbb{C}^n,$$

where $H = (H_1, \ldots, H_n)$ and each H_i is either zero or a homogeneous polynomial map of degree 3. In this case the Jacobian condition $\det(JF) \in \mathbb{C}^*$ is equivalent to JH is nilpotent. (JF and JH are the Jacobian matrices of F and H.) So understanding nilpotent Jacobian matrices is crucial in the study of the Jacobian Conjecture. In [14] Wright showed that if $n = 3$ all JH where H is cubic homogeneous are linearly triangularizable. In [10] the second author gave a complete description of all cubic homogeneous Jacobian matrices in case $n = 4$. They are no longer linearly triangularizable. However, it turns out that the rows of the Jacobian matrices are linearly dependent over $\mathbb{C}$ (or equivalently that H_1 H_2, H_3, and H_4 linearly dependent over $\mathbb{C}$). Already in [4] Drużkowski and Rusek conjectured that if $H_1 = l_1^2, \ldots, H_n = l_n^2$, where each l_i is a linear form, then the nilpotence of JH implies the linear dependence of $H_1, \ldots, H_n$. The same question of linear dependence of $H_1, \ldots, H_n$ was raised by Olech in [13] and Meisters in [12], in case $H_1, \ldots, H_n$ are cubic homogeneous.

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Then it was observed by the authors that the following more general dependence problem would imply the Jacobian Conjecture: does JH nilpotent (not necessarily homogeneous) and $H(0) = 0$ imply that $H_1, \dots, H_n$ are linearly dependent over $\mathbb{C}$? (Recently in [3] this dependence problem appeared as a conjecture, the Nilpotent Conjecture, where it was shown that an affirmative answer would imply the Jacobian Conjecture.) Our aim was to investigate what consequences could be deduced assuming that the dependence question had an affirmative answer.

The result is that for every commutative ring A we defined a large class, denoted $\overline{\mathscr{H}_n(A)}$, of polynomial maps $H \in A[X_1, \dots, X_n]^n$ such that the Jacobian matrix JH is nilpotent. It is shown that for all $H \in \overline{\mathscr{H}_n(A)}$ the map $F := X + H$ is invertible with $\det(JF) = 1$ and that the inverse is of the form $X + G$ with $G \in \overline{\mathscr{H}_n(A)}$. Furthermore we show that $H^n = H \circ \cdots \circ H = 0$ for all $H \in \overline{\mathscr{H}_n(A)}$, with $H(0) = 0$, a phenomenon first observed by Meisters in [11].

Then in Section 4 we consider the question if every H with JH nilpotent belongs to $\overline{\mathscr{H}_n(A)}$ (which, if true, would imply the Jacobian Conjecture). We show that the answer is yes if $n = 2$ and A is a $\mathbb{Q}$-algebra which is a $U.F.D.$ (this result was already obtained by the second author in [10]), and that the answer is no for all $n \geq 3$ and every domain A, which is a $\mathbb{Q}$-algebra. This last result is based on recent counterexamples to the dependence problem for all $n \geq 3$ obtained by the first author in [7].

Finally in Section 5 we show that all counterexamples found in [5, 8, 2] belong to $\mathscr{H}_n(\mathbb{C})$ (which is a subclass of $\overline{\mathscr{H}_n(\mathbb{C})}$).

In a subsequent paper [9] we undertake a detailed study of the class $\mathscr{H}_n(A)$ and show that all F of the form $X + H$ with $H \in \mathscr{H}_n(A)$ are stably tame. This implies that all cubic homogeneous maps in dimension 4, obtained in [10], are stably tame.

1. THE CLASS $\mathscr{H}_n(A)$

Throughout this section A denotes an arbitrary commutative ring and let us denote by $A[X] := A[X_1, \dots, X_n]$ the polynomial ring in n variables over A. Let $F = (F_1, \dots, F_n) \in A[X]^n$. Such an F is called *invertible* over A or $F_1, \dots, F_n$ is called a *coordinate system* of $A[X]$ if $A[F_1, \dots, F_n] = A[X_1, \dots, X_n]$. In other words, if there exist $G_1, \dots, G_n \in A[X]$ such that $X_i = G_i(F_1, \dots, F_n)$ for all i. It is an immediate consequence of the formal inverse function theorem that $G = (G_1, \dots, G_n)$ is uniquely determined and satisfies $F \circ G = X$.

Now we come to the main definition of this paper. First we put

$$\mathscr{N}_n(A) := \{H \in A[X]^n : JH \text{ is nilpotent}\}.$$

For each $n \in \mathbb{N}, n \geq 1$ and each commutative ring A we are going to define a set $\mathscr{H}_n(A) \subset A[X]^n$ which will turn out to be a subset of the set $\mathscr{N}_n(A)$. (cf. Theorem 2.3.)

DEFINITION 1.1. Put $\mathscr{H}_1(A) = A$, for each commutative ring A and inductively for $n \geq 2$ and $H \in A[X]^n$ we define that $H \in \mathscr{H}_n(A)$ if and only if there exist $T \in M_n(A)$, $c \in A^n$, and $\tilde{H} \in \mathscr{H}_{n-1}(A[X_n])$ such that

$$H = \operatorname{Adj}(T) \begin{pmatrix} \tilde{H} \\ 0 \end{pmatrix}_{|TX} + c, \tag{1}$$

where Adj(T) denotes the adjoint matrix of T and $|TX$ the "evaluation at the vector TX."

EXAMPLE 1.2. Let $H = \binom{H_1}{H_2} \in A[X_1, X_2]^2$. Then $H \in \mathscr{H}_2(A)$ if and only if there exist $T = \begin{pmatrix} t_1 & t_2 \\ a_1 & a_2 \end{pmatrix} \in M_2(A), c_1, c_2 \in A$, and $f(X_2) \in \mathscr{H}_1(A[X_2]) = A[X_2]$ such that

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} a_2 & -t_2 \\ -a_1 & t_1 \end{pmatrix} \begin{pmatrix} f(X_2) \\ 0 \end{pmatrix}\Bigg|_{\substack{t_1X_1 + t_2X_2 \\ a_1X_1 + a_2X_2}} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

In other words: if and only if H_1 and H_2 are of the form

$$H_1 = a_2 f(a_1X_1 + a_2X_2) + c_1$$
$$H_2 = -a_1 f(a_1X_1 + a_2X_2) + c_2$$

for some $a_1, a_2, c_1, c_2 \in A$ and $f \in A[X_2]$.

Remark 1.3. It was shown in [10, Theorem 3.1] that if A is a $\mathbb{Q}$-algebra and a unique factorization domain then $\mathscr{H}_2(A) = \mathscr{N}_2(A)$. We will give a short proof in Section 4 (Theorem 4.3). However, if A is a domain which is not a unique factorization domain it can happen that $\mathscr{H}_2(A) \subsetneqq \mathscr{N}_2(A)$ (see Sect. 3 below).

2. PROPERTIES OF $\mathscr{H}_n(A)$

LEMMA 2.1. Let $H \in \mathscr{H}_n(A)$, $r \in A$, $c \in A^n$. Then

(i) $rH + c \in \mathscr{H}_n(A)$.

(ii) If $S \in M_n(A)$ then Adj$(S)H_{|SX} \in \mathscr{H}_n(A)$.

(iii) If $\varphi : A \to S$ is a ring homomorphism then $\varphi(H) \in \mathscr{H}_n(S)$ where $\varphi(H)$ is obtained by applying φ to the coefficients of H.

Proof. (i) and (iii) follow readily by induction on n. It therefore remains to prove (ii). So let $S \in M_n(A)$ and $H \in \mathscr{H}_n(A)$. Then according to Definition 1.1 we get

$$\begin{aligned}\operatorname{Adj}(S)H_{|SX} &= \operatorname{Adj}(S)\left(\operatorname{Adj}(T)\begin{pmatrix}\tilde{H}\\0\end{pmatrix}_{|TX} + c\right)_{|SX}\\ &= \operatorname{Adj}(S)\operatorname{Adj}(T)\left(\begin{pmatrix}\tilde{H}\\0\end{pmatrix}_{|TX}\right)_{|SX} + c\\ &= \operatorname{Adj}(TS)\begin{pmatrix}\tilde{H}\\0\end{pmatrix}_{|(T\circ S)X} + c\end{aligned}$$

and this is of the desired form. ∎

COROLLARY 2.2. *Let $A[Y]$ be the polynomial ring in one variable over A. Let $a \in A$. If $H = (H_1, \ldots, H_n) \in \mathscr{H}_n(A[Y])$, then $H(Y = a) \in \mathscr{H}_n(A)$.*

Proof. Apply Lemma 2.1(iii) to the substitution homomorphism φ: $A[Y] \to A$ sending Y to a. ∎

To simplify notations we abbreviate (1) by $H = \tilde{H}[T, c]$ or by $H = \tilde{H}[T]$ in case $c = 0$. As before we denote the Jacobian matrix with respect to $X_1, \ldots, X_n$ of an element $H \in A[X]^n$ by JH or if confusion is possible by $J_n H$. One then easily verifies that

$$J_n\tilde{H}[T, c] = \operatorname{Adj}(T)\begin{pmatrix}(J_{n-1}\tilde{H})_{|TX} & * \\ 0 & 0\end{pmatrix}T. \tag{2}$$

THEOREM 2.3. *For all rings A and $n \in \mathbb{N}$, $n \geq 1$ we have $\mathscr{H}_n(A) \subset \mathscr{N}_n(A)$.*

Proof. Induction on n. The case $n = 1$ is obvious. So let $n \geq 2$ and let $H = \tilde{H}[T, c]$ for some $T \in M_n(A)$, $c \in A^n$, and $\tilde{H} \in \mathscr{H}_{n-1}(A[X_n])$. By the induction hypothesis we have that $J_{n-1}\tilde{H}$ is nilpotent. Hence so is $(J_{n-1}\tilde{H})_{|TX}$. From remark (2) it then follows that $J_n\tilde{H}[T, c] = J_n H$ is nilpotent. ∎

THEOREM 2.4. *Let $H \in \mathscr{H}_n(A)$ Put $F := X + H$. Then*

(i) $\det(JF) = 1$ *and*

(ii) *F is invertible over A. Furthermore $F^{-1} = X + G$ with $G \in \mathscr{H}_n(A)$.*

Before we can prove this theorem we need some preliminaries. Therefore consider the polynomial ring

$$A[T_{ij}] := A[T_{ij};\, 1 \leq i, j \leq n]$$

in n^2 indeterminates over A. Put $T_u = (T_{ij})$, $d := \det(T_u)$, and consider the ring $S := A[T_{ij}][d^{-1}]$. We claim that $A[T_{ij}] \subset S$. This follows immediately from

LEMMA 2.5. *d is not a zero-divisor in $A[T_{ij}]$.*

Proof. We use induction on n; the case $n = 1$ is obvious, so let $n \geq 2$. Write $d_{n-1} := \det(T_{ij})_{1 \leq i, j \leq n-1}$. Put $A_* := A[T_{ni}, T_{in}; 1 \leq i \leq n-1]$ and $B := A_*[T_{ij};\ 1 \leq i, j \leq n-1]$. So $B = A[T_{ij};\ (i,j) \neq (n,n)]$. If we now develop d with respect to the nth column we get

$$d = d_{n-1}T_{nn} + b$$

for some $b \in B$. In particular b does not contain any T_{nn}. Now suppose d is a zero-divisor in $A[T_{ij}]$. Then there exists an element $0 \neq g \in A[T_{ij}]$ with $dg = 0$. Now develop g after powers of T_{nn}, i.e.,

$$g = g_m T_{nn}^m + \cdots + g_0$$

with $m \geq 0$, $g_m \neq 0$, and $g_i \in B$ for all i. Looking at the coefficient of T_{nn}^{m+1} in the equation $dg = 0$ we get $d_{n-1}g_m = 0$. But if we apply the induction hypothesis to the ring A_* we get that d_{n-1} is no zero-divisor in $A_*[T_{ij};\ 1 \leq i, j \leq n-1]$. Consequently $g_m = 0$, a contradiction. Hence d is no zero-divisor in $A[T_{ij}]$. ∎

Proof of Theorem 2.4. By induction on n. Again the case $n = 1$ is clear. So let $n \geq 2$ and let $H = \tilde{H}[T, c]$ for some $T = (t_{ij}) \in M_n(A)$, $c \in A^n$, and $\tilde{H} \in \mathscr{H}_{n-1}(A[X_n])$. Since the transformation $T_c : X \mapsto x + c$ is bijective with inverse T_{-c}, we get that $T_{-c} \circ H = \tilde{H}[T]$ and hence we may assume that $c = 0$ without loss of generality.

(i) Let $S = A[T_{ij}][d^{-1}]$ as above and put $S_0 := A[T_{ij}]$. By Lemma 2.5 we have that S_0 is a subring of S. By Lemma 2.1 we can view $\tilde{H}$ as an element of $\mathscr{H}_{n-1}(S_0[X_n]) \subset \mathscr{H}_{n-1}(S[X_n])$. Now define the universal $H_u := \tilde{H}[T_u]$ and $F_u := X + H_u$. Note that

$$H_u = T_u^{-1}\begin{pmatrix} d\tilde{H} \\ 0 \end{pmatrix}_{|T_u X}.$$

So we get

$$\begin{aligned}
\det(J_n F_u) &= \det\big(J_n(X + H_u)\big) \\
&= \det\left(T_u^{-1} J_n\left(X + \begin{pmatrix} d\tilde{H} \\ 0 \end{pmatrix}\right)_{|T_u X} T_u\right) \\
&= \det\big(J_{n-1}(X' + d\tilde{H})\big)_{|T_u X},
\end{aligned}$$

where $X' = (X_1, \dots, X_{n-1})^t$. However, since $d\tilde{H} \in \mathscr{H}_{n-1}(S[X_n])$, the last determinant equals 1 by the induction hypothesis. So also $\det(JF_u) = 1$. Finally making the substitutions $T_{ij} \to t_{ij}$ we obtain $\det(JF) = 1$.

(ii) Since $\tilde{H} \in \mathscr{H}_{n-1}(S_0[X_n])$ and $d \in S_0$ we get $d\tilde{H} \in \mathscr{H}_{n-1}(S_0[X_n])$. So by the induction hypothesis we get that $X' + d\tilde{H}$ is invertible over $S_0[X_n]$ with inverse $X' + \tilde{G}$, where $\tilde{G} \in \mathscr{H}_{n-1}(S_0[X_n])$. The equation $(X' + d\tilde{H}) \circ (X' + \tilde{G}) = X'$ implies that $X' + \tilde{G} + d\tilde{H}(X' + \tilde{G}) = X'$ so $\tilde{G} = -d\tilde{H}(X' + \tilde{G})$. Now observe that

$$F_u = T_u^{-1}\left(X + \begin{pmatrix} d\tilde{H} \\ 0 \end{pmatrix}\right)_{|T_u X}$$

and that its inverse over S is given by

$$T_u^{-1}\left(X + \begin{pmatrix} \tilde{G} \\ 0 \end{pmatrix}\right)_{|T_u X} = X + \operatorname{Adj}(T_u)\begin{pmatrix} \dfrac{1}{d}\tilde{G} \\ 0 \end{pmatrix}_{|T_u X}.$$

Since $(1/d)\tilde{G} = -\tilde{H}(X' + \tilde{G})$ belongs to $S_0[X_n]^{n-1}$, it follows that F_u is in fact invertible over S_0. As in (i) we conclude the proof by making the substitutions $T_{ij} \to t_{ij}$ for all i, j. ∎

The next result shows a remarkable nilpotence property of the elements of $\mathscr{H}_n(A)$. For special examples this property was discovered by Meisters in [11].

THEOREM 2.6. *Let $H \in \mathscr{H}_n(A)$. Then $H^n = H \circ \cdots \circ H \in A^n$ for all $n \geq 1$. In particular if $H(0) = 0$, then $H^n = 0$.*

Before we can give the proof of this theorem we first present a lemma.

LEMMA 2.7. *Let $H = (H_1, \dots, H_n) \in A[X]^n$ and assume $H_n = c_n \in A$. Now write for each $1 \leq i \leq n-1$ $H_{i0} = H_i(X_n = c_n)$. Put*

$$H_0 := (H_{10}, \dots, H_{(n-1)0}) \in A[X_1, \dots, X_{n-1}]^{n-1}.$$

Then for all $p \geq 2$

$$H^p = \begin{pmatrix} H_0^{p-1}(H_1, \dots, H_{n-1}) \\ c_n \end{pmatrix}.$$

Proof. We use induction on p. First observe that for all $1 \leq i \leq n-1$

$$H_i(H_1, \dots, H_n) = H_{i0}(H_1, \dots, H_{n-1}) \tag{3}$$

which proves the case $p = 2$. Now let $p \geq 3$. Then by the induction hypothesis

$$H^p = H^{p-1} \circ H = \begin{pmatrix} H_0^{p-1}(H_1, \ldots, H_{n-1}) \\ c_n \end{pmatrix} \circ H.$$

So by (3) we get

$$\begin{aligned} H^p &= \begin{pmatrix} H_0^{p-1}\big(H_{10}(H_1, \ldots, H_n), \ldots, H_{(n-1)0}(H_1, \ldots, H_n)\big) \\ c_n \end{pmatrix} \\ &= \begin{pmatrix} H_0^p(H_1, \ldots, H_{n-1}) \\ c_n \end{pmatrix}. \end{aligned}$$

∎

Proof of Theorem 2.6. Let $H = \tilde{H}[T, c]$ for some $c \in A^n, T = (t_{ij}) \in M_n(A)$, and $\tilde{H} \in \mathscr{H}_{n-1}(A[X_n])$. As in the proof of Theorem 2.4 consider the ring S and $H_u \in S[X]^n$. It suffices to prove that $H_u^n \in S^n$, for then $H_u^n \in S^n \cap A[T_{ij}][X]^n = A[T_{ij}]^n$, so making the substitutions $T_{ij} \mapsto t_{ij}$ gives $H^n \in A^n$ as desired. Now observe that

$$H_u = T_u^{-1} \begin{pmatrix} d\tilde{H} \\ 0 \end{pmatrix}_{|T_u X} + T_u^{-1}(T_u c) = T_u^{-1} \begin{pmatrix} \tilde{\tilde{H}} \\ a \end{pmatrix}_{|T_u X},$$

where $\binom{\tilde{\tilde{H}}}{a} = \binom{d\tilde{H}}{0} + T_u c$. Observe that $a \in S$ and $\tilde{\tilde{H}} \in \mathscr{H}_{n-1}(S[X_n])$. We deduce that

$$H_u^n = T_u^{-1} \begin{pmatrix} \tilde{\tilde{H}} \\ a \end{pmatrix}^n_{|T_u X}.$$

So it suffices to show that $\binom{\tilde{\tilde{H}}}{a}^n \in S^n$. Therefore we may assume that

$$H = \begin{pmatrix} H_1 \\ \vdots \\ H_n \end{pmatrix} \in A[X]^n \qquad \text{with } \tilde{H} = \begin{pmatrix} H_1 \\ \vdots \\ H_{n-1} \end{pmatrix} \in \mathscr{H}_{n-1}(A[X_n])$$

and $H_n \in A$. Write c_n instead of H_n. So we need to show that $H^n \in A^n$. We use induction on n. First write $H_i = H_{i0} + (X_n - c_n)H_i^*$ as in Lemma 2.7 above and put $H_0 = (H_{10}, \ldots, H_{(n-1)0})$. Then Lemma 2.7 gives

$$H^n = \begin{pmatrix} H_0^{n-1}(H_1, \ldots, H_{n-1}) \\ c_n \end{pmatrix}. \tag{4}$$

Furthermore by Corollary 2.2 we have $H_0 \in \mathscr{H}_{n-1}(A)$. so if $n = 2$ then $H^2 \in A^2$. Finally if $n \geq 3$ then the induction hypothesis, applied to H_0, gives that $H_0^{n-1} \in A^{n-1}$, whence $H^n \in A^n$ by (4). ∎

3. A DOMAIN A WITH $\mathscr{H}_2(A) \subsetneq \mathscr{N}_2(A)$

Throughout this section A denotes the domain $\mathbb{Z}[X, Y, Z]/(X^2 + YZ)$.

THEOREM 3.1. *Let* $H_1 = c_1 X_1 + c_2 X_2$, $H_2 = d_1 X_1 + d_2 X_2$ *in* $A[X_1, X_2]$ *where* $c_1 = \bar{X}$, $c_2 = \bar{Y}$, $d_1 = \bar{Z}$, *and* $d_2 = -\bar{X}$. *Then*

(i) $H = (H_1, H_2) \in \mathscr{N}_2(A)$.

(ii) $H \notin \mathscr{H}_2(A)$.

(iii) $\bar{Y}H \in \mathscr{H}_2(A)$.

Proof. (i) $JH = \begin{pmatrix} \bar{X} & \bar{Y} \\ \bar{Z} & -\bar{X} \end{pmatrix}$. Since $\mathrm{Tr}(JH = 0$ and $\det(JH) = -(\bar{X}^2 + \overline{YZ}) = 0$ we deduce that $H \in \mathscr{N}_2(A)$.

(ii) Suppose $H \in \mathscr{H}_2(A)$. Then by Example 1.2 there exist $a_1, a_2 \in A$ and $f \in A[T]$ with $f(0) = 0$ such that

$$H_1 = a_2 f(a_1 X_1 + a_2 X_2)$$
$$H_2 = -a_1 f(a_1 X_1 + a_2 X_2).$$

Now since both $\deg(H_1) = \deg(H_2) = 1$ we deduce that $f(T) = bT$ for some $b \in A \backslash \{0\}$. Consequently $\bar{X} = ba_1 a_2$ and $\bar{Y} = ba_2^2$. Let $A_1, A_2, B \in \mathbb{Z}[X, Y, Z]$ such that $a_1 = \overline{A_1}$, $a_2 = \overline{A_2}$, and $b = \bar{B}$. Then multiplying $\bar{X}$ by a_2 and $\bar{Y}$ by a_1 we obtain $a_2 \bar{X} = a_1 \bar{Y}$, i.e., $A_2 X - A_1 Y = c(X^2 + YZ)$ for some $c \in \mathbb{Z}[X, Y, Z]$. Consequently $X(A_2 - cX) = Y(A_1 + cZ)$. So $A_2 - cX = dY$ for some $d \in \mathbb{Z}[X, Y, Z]$ and hence $A_1 + cZ = dX$. Summarizing

$$A_1 = dX - cZ \qquad \text{and} \qquad A_2 = cX + dY$$

with $c, d \in \mathbb{Z}[X, Y, Z]$. Consequently the equation $\bar{X} = ba_1 a_2$, i.e., $X - BA_1 A_2 \in (X^2 + YZ)$, implies $X \in (X, Y, Z)^2$, a contradiction. So $H \notin \mathscr{H}_2(A)$.

(iii) $\bar{Y}H = \begin{pmatrix} \bar{Y}\bar{X}X_1 + \bar{Y}^2 X_2 \\ \bar{Y}\bar{Z}X_1 - \bar{Y}\bar{X}X_2 \end{pmatrix}$. Since $\bar{Y}\bar{Z} = -\bar{X}^2$, we see that we can take $a_1 = \bar{X}$, $a_2 = \bar{Y}$, and $f(T) = T$ to get the desired form of Example 1.2. ∎

4. THE CLASS $\overline{\mathscr{H}_n(A)}$

In the previous section we saw that there exists a commutative domain A such that $H \in A[X]^2$, $H \notin \mathscr{H}_2(A)$ but $rH \in \mathscr{H}_2(A)$, for some $0 \neq r \in A$. This leads us to the following definition, where we take the closure of $\mathscr{H}_n(A)$ with respect to this property.

Throughout this section: A is a commutative domain.

DEFINITION 4.1. First define $\overline{\mathscr{H}_1(A)} = A$. Now let $n \geq 2$ and $H \in A[X]^n$. Then $H \in \overline{\mathscr{H}_n(A)}$ if and only if there exist $0 \neq r \in A$, $T \in M_n(A)$, $c \in A^n$, and $\tilde{H} \in \overline{\mathscr{H}_{n-1}(A[X_n])}$ such that

$$rH = \operatorname{Adj}(T) \begin{pmatrix} \tilde{H} \\ 0 \end{pmatrix}_{|TX} + c.$$

As in Section 2 we have the following result.

THEOREM 4.2. (i) $\overline{\mathscr{H}_n(A)} \subset \mathscr{N}_n(A)$, *for all* $n \geq 1$.

(ii) *Let* $H \in \overline{\mathscr{H}_n(A)}$ *and put* $F := X + H$. *Then* $\det(JF) = 1$ *and* F *is invertible with inverse* F^{-1} *equal to* $X + G$ *where* $G \in \overline{\mathscr{H}_n(A)}$.

(iii) *Let* $H \in \overline{\mathscr{H}_n(A)}$. *Then* $H^n \in A^n$, *for all* $n \geq 1$.

Proof. (Sketch) The proofs of these theorems are obtained from the proofs of Theorems 2.3, 2.4, and 2.6 given in Section 2 by replacing $\mathscr{H}_n(A)$ by $\overline{\mathscr{H}_n(A)}$ and using localizations. ∎

Finally we consider the question whether $\overline{\mathscr{H}_n(A)} = \mathscr{N}_n(A)$?

As already observed earlier, it was proved in [10] that in case A is a U.F.D., then $\mathscr{H}_2(A) = \mathscr{N}_2(A)$, hence $\overline{\mathscr{H}_2(A)} = \mathscr{N}_2(A)$. Since the paper [10] is not readily available we give a short proof of this result.

THEOREM 4.3 [10]. *Let* A *be a U.F.D. Then* $\mathscr{H}_2(A) = \mathscr{N}_2(A)$.

Proof. (i) First assume that $A = k$ is a field. Then the result is proved in [1].

(ii) Now let A be a U.F.D. and let $H = (H_1, H_2) \in \mathscr{N}_2(A)$. Then $H \in \mathscr{N}_2(K)$ where K is the quotient field of A. So by (i) there exist $g(T) \in K[T]$ with $g(0) = 0$ and $\nu_1, \nu_2, d_1, d_2 \in K$ such that

$$H_1 = \nu_2 g(\nu_1 X_1 + \nu_2 X_2) + d_1$$
$$H_2 = -\nu_1 g(\nu_1 X_1 + \nu_2 X_2) + d_2$$

(see example 1.2). So clearing denominators we get: there exist $a \in A$, $a \neq 0$, $f(T) \in A[T]$ with $f(0) = 0$ and $\mu_1, \mu_2, c_1, c_2 \in A$ such that

$$aH_1 = \mu_2 g(\mu_1 X_1 + \mu_2 X_2) + c_1$$
$$aH_2 = -\mu_1 g(\mu_1 X_1 + \mu_2 X_2) + c_2. \tag{5}$$

Substituting $X_1 = X_2 = 0$ in (5) we obtain that $c_1 = a\tilde{c}_1$ and $c_2 = a\tilde{c}_2$ for some $\tilde{c}_1, \tilde{c}_2 \in A$. So replacing H_i by $H_i - \tilde{c}_i$ we may assume that $c_1 = c_2 = 0$.

(iii) Now we show that we may assume that $\gcd(\mu_1, \mu_2) = 1$: therefore let $\mu_1 = \tilde{\mu}_1 d$, $\mu_2 = \tilde{\mu}_2 d$ where $d = \gcd(\mu_1, \mu_2)$. So $\gcd(\tilde{\mu}_1, \tilde{\mu}_2) = 1$ and $\mu_i f(\mu_1 X_1 + \mu_2 X_2) = \tilde{\mu}_i\, df(d(\tilde{\mu}_1 X_1 + \tilde{\mu}_2 X_2))$. Hence if we put $\tilde{f}(T) = df(dT)$ we get

$$\mu_i f(\mu_1 X_1 + \mu_2 X_2) = \tilde{f}(\tilde{\mu}_1 X_1 + \tilde{\mu}_2 X_2).$$

(iv) Consequently we may assume that $\gcd(\mu_1, \mu_2) = 1$. Write $f = \sum_{i=1}^{N} f_i T^i$, with $f_i \in A$. From (5) we see that we may assume that $\gcd(a, f_1, \dots, f_N) = 1$.

Claim. a is a unit in A (and hence we are done).

Suppose that p is a prime factor of a. Then (5) implies that p divides $f(\mu_1 X_1 + \mu_2 X_2)$ (since $\gcd(\mu_1, \mu_2) = 1$). So in particular p divides both $f(\mu_1 X_1)$ and $f(\mu_2 X_2)$, so p divides $f_i \mu_1^i$ and $f_i \mu_2^i$ for all $i \geq 1$ and hence p divides f_i for all $i \geq 1$ which contradicts $\gcd(a, f_1, \dots, f_N) = 1$. So a is a unit. ∎

In the remainder of this section we will show that such a result is no longer true if $n \geq 3$. More precisely we have:

THEOREM 4.4. *Let A be any $\mathbb{Q}$-algebra. Then $\overline{\mathscr{H}_n(A)} \subsetneq \mathscr{N}_n(A)$, for all $n \geq 3$.*

To prove this result we need the following lemma.

LEMMA 4.5. *Let A be a domain, $n \geq 1$, and $H \in \overline{\mathscr{H}_n(A)}$ with $H(0) = 0$. Then there exist $\lambda_1, \dots, \lambda_n \in A$, not all zero, such that $\lambda_1 H_1 + \cdots + \lambda_n H_n = 0$.*

Proof. If $H = 0$ we are done, so let $H \neq 0$. Then there exist $0 \neq r \in A$, $T \in M_n(A)$, $c \in A^n$ and $\tilde{H} \in \overline{\mathscr{H}_{n-1}(A[X_n])}$ such that

$$rH = \operatorname{Adj}(T) \begin{pmatrix} \tilde{H} \\ 0 \end{pmatrix}_{|TX} + c.$$

So, multiplying by T we get

$$rTH = \det(T) \begin{pmatrix} \tilde{H} \\ 0 \end{pmatrix}_{|TX} + Tc. \tag{6}$$

(i) If $\det(T) = 0$ it follows from (6) that $rTH = Tc$. Since $H(0) = 0$ and A is a domain we deduce that $TH = 0$. Since $T \neq 0$ (otherwise $H = 0$) there exists a non-zero row, say the ith, whence $t_{i1} H_1 + \cdots + t_{in} H_n = 0$, as desired.

(ii) If $\det(T) \neq 0$, then equating the nth components of the vectors in (6) we get $r(TH)_n = (Tc)_n$. Since $H(0) = 0$, we get $(Tc)_n = 0$, so $(TH)_n = 0$, i.e., $t_{n1}H_1 + \cdots + t_{nn}H_n = 0$. Obviously $t_{nj} \neq 0$ for some j (otherwise $\det(T) = 0$). ∎

Now let $n \geq 3$ and let A be a $\mathbb{Q}$-algebra. It was shown in [7] that the following $H = (H_1, \ldots, H_n) \in A[X]^n$ belongs to $\mathcal{N}_n(A)$: let

$$\alpha(X_1) := X_1^{n-1}$$

$$H_1 := X_2 - \alpha(X_1)$$

$$H_i := X_{i+1} + \frac{(-1)^i}{(i-1)!}\alpha^{(i-1)}(X_2 - \alpha(X_1))^{i-1} \qquad \text{for } 2 \leq i \leq n-1$$

$$H_n := \frac{(-1)^n}{(n-1)!}\alpha^{(n-1)}(X_2 - \alpha(X_1))^{n-1}.$$

Proof of Theorem 4.4. Let $n \geq 3$ and let H be as defined above. Then, as observed, $H \in \mathcal{N}_n(A)$. However, if $H \in \overline{\mathcal{H}_n(A)}$, then by Lemma 4.5 there exist $\lambda_1, \ldots, \lambda_n \in A$, but not all zero, such that $\lambda_1 H_1 + \cdots + \lambda_n H_n = 0$. It follows readily that $\lambda_2 = \cdots = \lambda_n = 0$ (look at the monomials $X_3, X_4, \ldots, X_n$, respectively). So $\lambda_1 H_1 + \lambda_n H_n = 0$, which easily implies that also $\lambda_1 = \lambda_n = 0$ ($n \geq 3$!), contradiction. So $H \notin \overline{\mathcal{H}_n(A)}$. ∎

In particular this proof shows that the dependence problem, or the Nilpotent Conjecture from [3], is false. (See [7] for more details.)

5. FINAL REMARKS

To conclude this paper we explain the counterexamples found in [5, 8, 2]. They all belong to $\mathcal{H}_n(\mathbb{C})$. To see this consider Example 1.2. First take $A = \mathbb{C}[X_3, X_4]$, $a_1 = X_3$, $a_2 = X_4$, $c_1 = c_2 = 0$, and $f(T) = T$. Then

$$(H_1, H_2) = (X_4(X_3X_1 + X_4X_2), -X_3(X_3X_1 + X_4X_2))$$

belongs to $\mathcal{H}_2(\mathbb{C}[X_4][X_3])$. Consequently $(H_1, H_2, 0)$ belongs to $\mathcal{H}_3(\mathbb{C}[X_4])$ and hence (H_1, H_2, X_4^3) belongs to $\mathcal{H}_3(\mathbb{C}[X_4])$. This implies that $(H_1, H_2, X_4^3, 0)$ belongs to $\mathcal{H}_4(\mathbb{C})$ and hence that $H := (H_1, H_2, X_4^3, 0, \ldots, 0)$ belongs to $\mathcal{H}_n(\mathbb{C})$ for all $n \geq 4$. Then $X + H$ is exactly the counterexample to Meisters' Linearization Conjecture given in [5].

Similarly, taking $f(T) = T^2$ we find the counterexamples to the Deng–Meisters–Zampieri Conjecture and the Discrete Markus–Yamabe problem given in [8].

Finally the counterexamples to the Markus–Yamabe Conjecture and the Discrete Markus–Yamabe problem in [2]: take $A = \mathbb{C}[X_3]$, $a_1 = 1$, $a_2 = X_3$, $c_1 = c_2 = 0$, and $f(T) = T^2$ in Example 1.2. Then

$$(H_1, H_2) = \left(X_3(X_1 + X_3X_2)^2, -(X_1 + X_3X_2)^2\right)$$

belongs to $\mathscr{H}_2(\mathbb{C}[X_3])$; hence $(H_1, H_2, 0)$ belongs to $\mathscr{H}_3(\mathbb{C})$ and consequently the n-dimensional map $(H_1, H_2, 0, \dots, 0)$ belongs to $\mathscr{H}_n(\mathbb{C})$ for all $n \geq 3$. Then $-X + H$ resp. $\frac{1}{2}X + H$ are exactly the counterexamples to the Markus–Yamabe Conjecture resp. the Discrete Markus–Yamabe problem given in [2].

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