Chaotic Polynomial Automorphisms; counterexamples to several conjectures

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Abstract

We give a polynomial counterexample to a discrete version of the Markus-Yamabe Conjecture and a conjecture of Deng, Meisters and Zampieri, asserting that if \( F : \mathbb{C}^n \to \mathbb{C}^n \) is a polynomial map with \( \det(JF) \in \mathbb{C}^* \), then for all \( \lambda \in \mathbb{R} \) large enough \( \lambda F \) is global analytic linearizable. These counterexamples hold in any dimension \( \geq 4 \).

Introduction

In [4] a new approach to the Jacobian Conjecture is introduced. The authors conjecture that if \( F : \mathbb{C}^n \to \mathbb{C}^n \) is a polynomial map with \( F(0) = 0 \) and \( JF(0) = I \), then for all \( \lambda > 1 \), \( \lambda \) large enough there exists an analytic automorphism \( \varphi_\lambda : \mathbb{C}^n \to \mathbb{C}^n \) such that \( \varphi_\lambda^{-1} \circ \lambda F \circ \varphi_\lambda = \lambda I \) i.e. \( \varphi_\lambda \) conjugates \( \lambda F \) to its linear part. We also say that \( \lambda F \) is analytic linearisable to its linear part. We call this conjecture the DMZ-conjecture (after Deng, Meisters and Zampieri). Of course this conjecture, if true, would imply the Jacobian Conjecture since it follows readily that \( \lambda F \) and hence \( F \) is injective. The local existence of \( \varphi_\lambda \) is guaranteed by the Poincaré-Siegel theorem (cf. [1, section 25, p. 193]) since if \( \lambda > 1 \) the eigenvalues of \( \lambda I \) are non-resonant. Furthermore \( \varphi_\lambda(0) = 0 \) and \( \varphi_\lambda \) is unique if we assume that \( J\varphi_\lambda(0) = I \), which we can do without loss of generality. It was shown in [4] that \( \varphi_\lambda^{-1} \) is entire, however the convergence of \( \varphi_\lambda \) could only be proved in some neighbourhood of 0. Meisters in [8] restricted the problem to polynomial maps of the form \( F = X + H \) with \( H \) cubic homogeneous and \( \det(JF) = 1 \) (or equivalently \( JH \) nilpotent) and conjectured that for such maps \( \lambda F \) can be conjugated to its linear part \( \lambda I \) by means of polynomial automorphisms \( \varphi_\lambda \), for almost all \( \lambda \in \mathbb{C} \), except a finite number of roots of unity. In [5] the first author gave a
counterexample to this conjecture for any dimension $\geq 4$. On the other hand it was recently shown by Gorni and Zampieri in [7] that this example can be conjugated to its linear part for all $\lambda$ with $|\lambda| \neq 1$ by means of an analytic automorphism $\varphi_\lambda$. So the DMZ-conjecture remained open.

Another proof of the fact that the counterexample of [5] satisfies the DMZ-conjecture was even more recently given by Deng in [3]. In his very elegant and short paper he proves that an analytic map $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with $F(0) = 0$ can be analytically conjugated to its linear part if and only if $F$ is an analytic automorphism of $\mathbb{C}^n$ and 0 is a global attractor of $F$ (i.e. for every $x \in \mathbb{C}$ the sequence $x, F(x), F^2(x), \ldots$ tends to 0). In the same paper he conjectured that if $F = X + H$ with $H$ cubic homogeneous and $JH$ nilpotent then 0 is a global attractor of $F \circ \lambda$ for all $\lambda$ with $|\lambda| < 1$. (In fact in the argument he gave to motivate this conjecture he does not use that $H$ is of degree 3.)

A similar kind of question was brought up independently by Cima, Gasull and Mañosas in [2]. They studied the problem that if $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a polynomial map with $F(0) = 0$ and such that the eigenvalues of $JF(x)$ are smaller then 1 in absolute value for all $x \in \mathbb{R}^n$, then 0 is a global attractor of $F$. They call it the discrete Markus-Yamabe Question and show that this problem implies the Jacobian Conjecture and that it is true for triangular maps.

In this paper we give a counterexample to the DMZ-conjecture of the form $F = X + H$, where $H$ is homogeneous of degree 5 in any dimension $n \geq 4$. Furthermore we show that if $0 < \lambda < 1$ $\lambda F$ is a counterexample to the discrete Markus-Yamabe Question.

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1 A counterexample to the discrete Markus-Yamabe Question

Let $n \geq 4$ and consider the polynomial ring $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$. In $\mathbb{R}[X]$ define the element

$$d(X) := X_3X_1 + X_4X_2$$

**Theorem 1.1** Let $n \geq 4$ and $m \in \mathbb{N}, m \geq 1$. Define the polynomial automorphism

$$F = (X_1 + X_4d(X)^2, X_2 - X_3d(X)^2, X_3 + X_4^m, X_4, \ldots, X_n).$$

Then for each $0 < \lambda < 1$, $\lambda F$ is a counterexample to the discrete Markus-Yamabe Question. More precisely, if $0 < \lambda < 1$ and $a \in \mathbb{R}$ is such that $a\lambda > 1$ then the first component of $(\lambda F)^k(a, a, \ldots, a)$ tends to infinity if $k$ tends to infinity.

**Definition 1.2** For each $\lambda > 0$ and $a > 0$ we put $(\lambda F)^k(a) := (\lambda F)^k(a, a, \ldots, a)$ and denote the first component of this vector by $f_k(\lambda, a)$. So

$$f_k(\lambda, a) := ((\lambda F)^k(a))_1,$$

for all $k \geq 1$. Furthermore we put

$$d_k(\lambda, a) := d((\lambda F)^k(a)),$$

for all $k \geq 1$.

**Lemma 1.3** i). $d(\lambda F(X)) = \lambda^2[X_4^m+1d(X)^2 + d(X) + X_4^mX_1]$

ii). $d_{k+1}(\lambda, a) \geq \lambda^2(\lambda^k a)^{m+1}(d_k(\lambda, a))^2$, for all $k \geq 1$.

iii). $f_{k+1}(\lambda, a) \geq \lambda^k a(d_k(\lambda, a))^2$, for all $k \geq 1$.

**Proof.** i) is easy to verify. Consequently, since all monomials in $d(\lambda F(X))$ have positive coefficients, we get

$$d_{k+1}(\lambda, a) = d((\lambda F)(\lambda F)^k(a))$$

$$\geq \lambda^2((\lambda F)^k(a))^{m+1}d((\lambda F)^k(a))^2$$

$$= \lambda^2(\lambda^k a)^{m+1}(d_k(\lambda, a))^2$$

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since the fourth component of $(\lambda F)^k(a)$ equals $\lambda^ka$. This proves ii). Finally
\[ f_{k+1}(\lambda, a) = (\lambda F)_1((\lambda F)^k(a)) \geq \lambda((\lambda F)^k(a))d((\lambda F)^k(a)) \]
(\text{using that } (\lambda F)_1 = \lambda X_1d(X)^2 + \lambda X_1). \text{ So } f_{k+1}(\lambda, a) \geq \lambda^{k+1}a(d_k(\lambda, a))^2, \text{ which proves iii).}

\textbf{Proposition 1.4} We have:
\[ f_k(\lambda, a) \geq \lambda^{p_k}a^{p_k + (2m+1)(k-1)+4} \]
\[ d_k(\lambda, a) \geq \lambda^{p_k + m(k-1)+1}a^{p_k + (2m+1)(k-1)+m+4} \]
for all $k \geq 1$, where $p_1 = 1$ and $p_{k+1} = 2p_k + (2m+1)(k-1) + 4$ for all $k \geq 1$.

\textbf{Proof.} Use induction on $k$. Details are left to the reader. \hfill \Box

\textbf{Proof of theorem 1.1.} It follows immediately from the estimation of $f_k(\lambda, a)$ in proposition 1.4 that \(\lim_{k \to \infty} f_k(\lambda, a) = \infty\) if $\lambda a > 1$. Furthermore one easily verifies that $\lambda F = \lambda X + H$ with $JH$ nilpotent. So for all $x \in \mathbb{R}^n$ the eigenvalues of $JF(x)$ are equal to $\lambda$. \hfill \Box

\textbf{Corollary 1.5} Let $m = 5$ and $0 < \lambda < 1$. Put $\bar{F} := \lambda F\lambda^{-1}$. Then $\bar{F} = X + H$ with $H$ homogeneous of degree 5 and $JH$ is nilpotent. However 0 is not a global attractor of $\bar{F} \circ \lambda (= \lambda F)$.

2 \textbf{A counterexample to the DMZ-conjecture}

Let $n \geq 4$ and consider the polynomial ring $\mathbb{C}[X] := \mathbb{C}[X_1, \ldots, X_n]$. In $\mathbb{C}[X]$ define the element $d(X) := X_3X_1 + X_4X_2$.

\textbf{Theorem 2.1} Let $n \geq 4$ and $m \geq 3$, $m$ odd. Define the polynomial automorphism
\[ F = (X_1 + X_4d(X))^2, X_2 - X_3d(X)^2, X_3 + X_4^m, X_4, \ldots, X_n). \]
Then $F$ is a counterexample to the DMZ-conjecture. More precisely, for every $\lambda > 0$, $\lambda \neq 1$, $\lambda F$ is not global analytic linearisable to $\lambda X$. 

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The proof of this theorem is based on the following observation which is due to Bo Deng (cf [3]).

**Lemma 2.2** Let \( F : \mathbb{C}^n \rightarrow \mathbb{C}^n \) be an analytic map with \( F(0) = 0 \). Put \( A := JF(0) \) and suppose that the eigenvalues of \( A \) are smaller than 1 in absolute value. If \( F \) is global analytic linearisable to its linear part \( A \) then 0 is a global attractor of \( F \).

**Proof.** Let \( x \in \mathbb{C}^n \) and let \( \varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n \) be the analytic automorphism of \( \mathbb{C}^n \) such that \( \varphi^{-1}F\varphi = A \). Then \( F = \varphi A\varphi^{-1} \) and hence \( F^k(x) = \varphi A^k\varphi^{-1}(x) \), for all \( k \geq 1 \). By the hypothesis on the eigenvalues of \( A \) it follows that \( A^k\varphi^{-1}(x) \rightarrow 0 \) if \( k \rightarrow \infty \). Consequently \( F^k(x) = \varphi(A^k\varphi^{-1}(x)) \rightarrow 0 \) if \( k \rightarrow \infty \). \( \square \)

**Proof of theorem 2.1.**

i). From lemma 2.2 and theorem 1.1 it follows that \( \lambda F \) is not analytic linearisable if \( 0 < \lambda < 1 \).

ii). Now let \( \lambda > 1 \). Suppose that \( \lambda F \) is analytic linearisable. We derive a contradiction. Then \( (\lambda F)^{-1} = F^{-1} \circ \lambda^{-1} \) is also analytic linearisable. Put \( \mu := \lambda^{-1} \) and \( G := F^{-1} \). So \( G \circ \mu \) is analytic linearisable. One easily verifies that

\[
G = (X_1 - X_4 \tilde{d}(X)^2, X_2 + (X_3 - X_4^m)\tilde{d}(X)^2, X_3 - X_4^m, X_4, \ldots, X_n) \quad (1)
\]

where

\[
\tilde{d}(X) := d(X) - X_4^m X_1. \quad (2)
\]

Since \( 0 < \mu < 1 \) it follows from lemma 2.2 that 0 is a global attractor of \( G \circ \mu \). However we will show below (corollary 2.6) that for every \( 0 < \mu < 1 \) 0 is not a global attractor of \( G \circ \mu \). Hence we have derived a contradiction. \( \square \)

So it remains to show that 0 is not a global attractor of \( G \circ \mu \). First we show that 0 is not a global attractor of \( \mu G \) if \( 0 < \mu < 1 \). To prove this we need some lemmas. So let \( G \) and \( \tilde{d}(X) \) be as in (1) resp. (2).

For each \( a > 0 \) let \( a^* := (a, -a, a, -a, \ldots, a) \in \mathbb{R}^n \). Then we define for each \( a > 0 \) and \( \mu > 0 \):

\[
g_k(\mu, a) := (\mu G)^k(a^*)_1
\]

\[
d_k(\mu, a) := \tilde{d}((\mu G)^k(a^*))
\]

for all \( k \geq 1 \).
Lemma 2.3  

i). \( d(G(X)) = \tilde{d}(X) \).

ii). \( \tilde{d}((\mu G)(X)) = \mu^2 \tilde{d}(X) - \mu^{m+1} X_4^m X_1 + \mu^{m+1} X_4^{m+1} \tilde{d}(X)^2 \).

iii). \( \tilde{d}_{k+1}(\mu, a) = (\mu^{k+1} a)^m + (\mu^{k+1} a)^m g_k(\mu, a) \) for all \( k \geq 1 \).

iv). \( g_{k+1}(\mu, a) = \mu^{k+1} a (\tilde{d}_k(\mu, a))^2 + g_k(\mu, a) \) for all \( k \geq 1 \).

Proof. The proofs of i) and ii) are straightforward and left to the reader. From ii) we deduce that

\[
\tilde{d}_{k+1}(\mu, a) = \tilde{d}((\mu G)^{k+1}(a^*)) \\
= \tilde{d}((\mu G)((\mu G)^k(a^*))) \\
= \mu^2 \tilde{d}((\mu G)^k(a^*)) - \mu^{m+1}((\mu G)^k(a^*))^m((\mu G)^k(a^*))_1 + \mu^{m+1}((\mu G)^k(a^*))^m \tilde{d}((\mu G)^k(a^*))^2 
\]

Now observe that \( ((\mu G)^k(a^*))_1 = \mu^k (-a) \), hence since \( m \) is odd \( ((\mu G)^k(a^*))^m = -(\mu^k a)^m \). So we get

\[
\tilde{d}_{k+1}(\mu, a) = \mu^2 \tilde{d}_k(\mu, a) + \mu^{m+1}(\mu^k a)^m g_k(\mu, a) + \mu^{m+1}(\mu^k a)^m (\tilde{d}_k(\mu, a))^2 \\
= (\mu^{k+1} a)^m + (\mu^{k+1} a)^m g_k(\mu, a) + \mu^2 d_k(\mu, a) 
\]

which proves iii). Finally

\[
g_{k+1}(\mu, a) = ((\mu G)^{k+1}(a^*))_1 \\
= (\mu G)_1((\mu G)^k(a^*)) \\
= \mu((\mu G)^k(a^*))_1 - \mu(\mu G)^k(a^*)_1 (\tilde{d}(\mu G)^k(a^*))^2 \\
= g_k(\mu, a) - \mu \cdot \mu^k (-a)(\tilde{d}_k(\mu, a))^2 \\
= g_k(\mu, a) + \mu^{k+1} a (\tilde{d}_k(\mu, a))^2 
\]

which proves iv). \( \square \)

Corollary 2.4  

i). \( \tilde{d}_{k+1}(\mu, a) \geq (\mu^k a)^m (\tilde{d}_k(\mu, a))^2 \) for all \( k \geq 1 \).

ii). \( g_{k+1}(\mu, a) \geq \mu^{k+1} a (\tilde{d}_k(\mu, a))^2 \) for all \( k \geq 1 \).

Proof. By induction on \( k \) one readily verifies that for all \( k \geq 1 \) both \( \tilde{d}_k(\mu, a) \) and \( g_k(\mu, a) \) are polynomials in \( \mu \) and \( a \) with coefficients in \( \mathbb{N} \). Then the result follows from lemma 2.3 iii) and iv). \( \square \)
Proposition 2.5 We have:
\[ g_k(\mu, a) \geq \mu^{q_k(m+1)+k}a^{(q_k+2k)(m+1)+1} \]
\[ d_k(\mu, a) \geq \mu^{(q_k+k)(m+1)}a^{(q_k+2k+1)(m+1)} \]
for all \( k \geq 1 \), where \( q_1 = 0 \) and \( q_{k+1} = 2q_k + 2k \) for all \( k \geq 1 \).

Proof. Use induction on \( k \). \( \square \)

Corollary 2.6 If \( \mu a > 1 \) and \( a > 1 \) then \( \lim_{k \to \infty} ((G \circ \mu)^k(G(a^*))_1 = \infty. \)
So 0 is not a global attractor of \( G \circ \mu \).

Proof. Observe that \((G\mu)^k(G(a^*)) = \mu^{-1}(\mu G)^{k+1}(a^*)\). Then apply proposition 2.5. \( \square \)

References


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