Chaotic Polynomial Automorphisms; counterexamples to several conjectures

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Abstract

We give a polynomial counterexample to a discrete version of the Markus-Yamabe Conjecture and a conjecture of Deng, Meisters and Zampieri, asserting that if \( F : \mathbb{C}^n \rightarrow \mathbb{C}^n \) is a polynomial map with 
\[ \det(JF) \in \mathbb{C}^*, \]
then for all \( \lambda \in \mathbb{R} \) large enough \( \lambda F \) is global analytic linearizable. These counterexamples hold in any dimension \( \geq 4 \).

Introduction

In [4] a new approach to the Jacobian Conjecture is introduced. The authors conjecture that if \( F : \mathbb{C}^n \rightarrow \mathbb{C}^n \) is a polynomial map with \( F(0) = 0 \) and 
\[ JF(0) = I, \]
then for all \( \lambda > 1 \), \( \lambda \) large enough there exists an analytic automorphism \( \varphi_{\lambda} : \mathbb{C}^n \rightarrow \mathbb{C}^n \) such that 
\[ \varphi_{\lambda}^{-1} \circ \lambda F \circ \varphi_{\lambda} = \lambda I \]
i.e. \( \varphi_{\lambda} \) conjugates \( \lambda F \) to its linear part. We also say that \( \lambda F \) is analytic linearisable to its linear part. We call this conjecture the DMZ-conjecture (after Deng, Meisters and Zampieri). Of course this conjecture, if true, would imply the Jacobian Conjecture since it follows readily that \( \lambda F \) and hence \( F \) is injective. The local existence of \( \varphi_{\lambda} \) is guaranteed by the Poincaré-Siegel theorem (cf. [1, section 25, p. 193]) since if \( \lambda > 1 \) the eigenvalues of \( \lambda I \) are non-resonant.
Furthermore \( \varphi_{\lambda}(0) = 0 \) and \( \varphi_{\lambda} \) is unique if we assume that \( J\varphi_{\lambda}(0) = I \), which we can do without loss of generality. It was shown in [4] that \( \varphi_{\lambda}^{-1} \) is entire, however the convergence of \( \varphi_{\lambda} \) could only be proved in some neighbourhood of 0. Meisters in [8] restricted the problem to polynomial maps of the form 
\[ F = X + H \]
with \( H \) cubic homogeneous and \( \det(JF) = 1 \) (or equivalently \( JH \) nilpotent) and conjectured that for such maps \( \lambda F \) can be conjugated to its linear part \( \lambda I \) by means of polynomial automorphisms \( \varphi_{\lambda} \), for almost all \( \lambda \in \mathbb{C} \), except a finite number of roots of unity. In [5] the first author gave a
counterexample to this conjecture for any dimension $\geq 4$. On the other hand it was recently shown by Gorni and Zampieri in [7] that this example can be conjugated to its linear part for all $\lambda$ with $|\lambda| \neq 1$ by means of an analytic automorphism $\varphi_\lambda$! So the $DMZ$-conjecture remained open.

Another proof of the fact that the counterexample of [5] satisfies the $DMZ$-conjecture was even more recently given by Deng in [3]. In his very elegant and short paper he proves that an analytic map $F : \mathbb{C}^n \to \mathbb{C}^n$ with $F(0) = 0$ can be analytically conjugated to its linear part if and only if $F$ is an analytic automorphism of $\mathbb{C}^n$ and 0 is a global attractor of $F$ (i.e. for every $x \in \mathbb{C}$ the sequence $x, F(x), F^2(x), \ldots$ tends to 0). In the same paper he conjectured that if $F = X + H$ with $H$ cubic homogeneous and $JH$ nilpotent then 0 is a global attractor of $F \circ \lambda$ for all $\lambda$ with $|\lambda| < 1$. (In fact in the argument he gave to motivate this conjecture he does not use that $H$ is of degree 3.)

A similar kind of question was brought up independently by Cima, Gasull and Mañosa in [2]. They studied the problem that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map with $F(0) = 0$ and such that the eigenvalues of $JF(x)$ are smaller then 1 in absolute value for all $x \in \mathbb{R}^n$, then 0 is a global attractor of $F$. They call it the discrete Markus-Yamabe Question and show that this problem implies the Jacobian Conjecture and that it is true for triangular maps.

In this paper we give a counterexample to the $DMZ$-conjecture of the form $F = X + H$, where $H$ is homogeneous of degree 5 in any dimension $n \geq 4$. Furthermore we show that if $0 < \lambda < 1$ $\lambda F$ is a counterexample to the discrete Markus-Yamabe Question.

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1 A counterexample to the discrete Markus-Yamabe Question

Let $n \geq 4$ and consider the polynomial ring $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$. In $\mathbb{R}[X]$ define the element

$$d(X) := X_3X_1 + X_4X_2$$

**Theorem 1.1** Let $n \geq 4$ and $m \in \mathbb{N}, m \geq 1$. Define the polynomial automorphism

$$F = (X_1 + X_4d(X)^2, X_2 - X_3d(X)^2, X_3 + X_4^m, X_4, \ldots, X_n).$$

Then for each $0 < \lambda < 1$ $\lambda F$ is a counterexample to the discrete Markus-Yamabe Question. More precisely, if $0 < \lambda < 1$ and $a \in \mathbb{R}$ is such that $a\lambda > 1$ then the first component of $(\lambda F)^k(a, a, \ldots, a)$ tends to infinity if $k$ tends to infinity.

**Definition 1.2** For each $\lambda > 0$ and $a > 0$ we put $(\lambda F)^k(a) := (\lambda F)^k(a, a, \ldots, a)$ and denote the first component of this vector by $f_k(\lambda, a)$. So

$$f_k(\lambda, a) := ((\lambda F)^k(a))_1,$$

for all $k \geq 1$. Furthermore we put

$$d_k(\lambda, a) := d((\lambda F)^k(a)),$$

for all $k \geq 1$.

**Lemma 1.3**

i). $d(\lambda F(X)) = \lambda^2[X_4^{m+1}d(X)^2 + d(X) + X_4^mX_1]$

ii). $d_{k+1}(\lambda, a) \geq \lambda^2(\lambda^k a)^{m+1}(d_k(\lambda, a))^2$, for all $k \geq 1$.

iii). $f_{k+1}(\lambda, a) \geq \lambda^{k+1}a(d_k(\lambda, a))^2$, for all $k \geq 1$.

**Proof.** i) is easy to verify. Consequently, since all monomials in $d(\lambda F(X))$ have positive coefficients, we get

$$d_{k+1}(\lambda, a) = d((\lambda F)(\lambda F)^k(a))$$

$$\geq \lambda^2((\lambda F)^k(a))_4^{m+1}d((\lambda F)^k(a))^2$$

$$= \lambda^2(\lambda^k a)^{m+1}(d_k(\lambda, a))^2$$

3
since the fourth component of \((\lambda F)^k(a)\) equals \(\lambda^k a\). This proves ii). Finally
\[
f_{k+1}(\lambda, a) = (\lambda F)_1((\lambda F)^k(a)) \
\geq \lambda((\lambda F)^k(a))d((\lambda F)^k(a))
\]
(\text{using that } (\lambda F)_1 = \lambda X_4d(X)^2 + \lambda X_1). So \(f_{k+1}(\lambda, a) \geq \lambda^{k+1}a(d_k(\lambda, a))^2\), which proves iii). \(\square\)

**Proposition 1.4** We have:
\[
f_k(\lambda, a) \geq \lambda_{p_k}a^{p_k+(2m+1)(k-1)+4} \\
d_k(\lambda, a) \geq \lambda_{p_k+m(k-1)+1}a^{p_k+(2m+1)(k-1)+m+4}
\]
for all \(k \geq 1\), where \(p_1 = 1\) and \(p_{k+1} = 2p_k + (2m+1)(k-1) + 4\) for all \(k \geq 1\).

**Proof.** Use induction on \(k\). Details are left to the reader. \(\square\)

**Proof of theorem 1.1.** It follows immediately from the estimation of \(f_k(\lambda, a)\) in proposition 1.4 that \(\lim_{k \to \infty} f_k(\lambda, a) = \infty\) if \(\lambda a > 1\). Furthermore one easily verifies that \(\lambda F = \lambda X + H\) with \(JH\) nilpotent. So for all \(x \in \mathbb{R}^n\) the eigenvalues of \(JF(x)\) are equal to \(\lambda\). \(\square\)

**Corollary 1.5** Let \(m = 5\) and \(0 < \lambda < 1\). Put \(\tilde{F} := \lambda F\lambda^{-1}\). Then \(\tilde{F} = X + H\) with \(H\) homogeneous of degree 5 and \(JH\) nilpotent. However 0 is not a global attractor of \(\tilde{F} \circ \lambda = \lambda F\).

## 2 A counterexample to the DMZ-conjecture

Let \(n \geq 4\) and consider the polynomial ring \(\mathbb{C}[X] := \mathbb{C}[X_1, \ldots, X_n]\). In \(\mathbb{C}[X]\) define the element \(d(X) := X_3X_1 + X_4X_2\).

**Theorem 2.1** Let \(n \geq 4\) and \(m \geq 3, m\) odd. Define the polynomial automorphism
\[
F = (X_1 + X_4d(X))^2, X_2 - X_3d(X)^2, X_3 + X_4^m, X_4, \ldots, X_n).
\]
Then \(F\) is a counterexample to the DMZ-conjecture. More precisely, for every \(\lambda > 0, \lambda \neq 1\), \(\lambda F\) is not global analytic linearisable to \(\lambda X\).
The proof of this theorem is based on the following observation which is due to Bo Deng (cf [3]).

**Lemma 2.2** Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be an analytic map with $F(0) = 0$. Put $A := JF(0)$ and suppose that the eigenvalues of $A$ are smaller than 1 in absolute value. If $F$ is global analytic linearisable to its linear part $A$ then 0 is a global attractor of $F$.

**Proof.** Let $x \in \mathbb{C}^n$ and let $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ be the analytic automorphism of $\mathbb{C}^n$ such that $\varphi^{-1}F\varphi = A$. Then $F = \varphi A \varphi^{-1}$ and hence $F^k(x) = \varphi A^k \varphi^{-1}(x)$, for all $k \geq 1$. By the hypothesis on the eigenvalues of $A$ it follows that $A^k \varphi^{-1}(x) \to 0$ if $k \to \infty$. Consequently $F^k(x) = \varphi (A^k \varphi^{-1}(x)) \to 0$ if $k \to \infty$. \hfill $\Box$

**Proof of theorem 2.1.** i). From lemma 2.2 and theorem 1.1 it follows that $\lambda F$ is not analytic linearisable if $0 < \lambda < 1$.

ii). Now let $\lambda > 1$. Suppose that $\lambda F$ is analytic linearisable. We derive a contradiction. Then $(\lambda F)^{-1} = F^{-1} \circ \lambda^{-1}$ is also analytic linearisable. Put $\mu := \lambda^{-1}$ and $G := F^{-1}$. So $G \circ \mu$ is analytic linearisable. One easily verifies that

$$G = (X_1 - X_4 \tilde{d}(X)^2, X_2 + (X_3 - X_4^m) \tilde{d}(X)^2, X_3 - X_4^m, X_4, \ldots, X_n) \tag{1}$$

where

$$\tilde{d}(X) := d(X) - X_4^m X_1. \tag{2}$$

Since $0 < \mu < 1$ it follows from lemma 2.2 that 0 is a global attractor of $G \circ \mu$. However we will show below (corollary 2.6) that for every $0 < \mu < 1$ 0 is not a global attractor of $G \circ \mu$. Hence we have derived a contradiction. \hfill $\Box$

So it remains to show that 0 is not a global attractor of $G \circ \mu$. First we show that 0 is not a global attractor of $\mu G$ if $0 < \mu < 1$. To prove this we need some lemmas. So let $G$ and $\tilde{d}(X)$ be as in (1) resp. (2).

For each $a > 0$ let $a^* := (a, -a, a, -a, a, \ldots, a) \in \mathbb{R}^n$. Then we define for each $a > 0$ and $\mu > 0$:

$$g_k(\mu, a) := ((\mu G)^k(a^*))_1$$
$$\tilde{d}_k(\mu, a) := \tilde{d}((\mu G)^k(a^*))$$

for all $k \geq 1$. 

5
Lemma 2.3 \textbf{i).} \(d(G(X)) = \tilde{d}(X)\).

\textbf{ii).} \(\tilde{d}((\mu G)(X)) = \mu^2 \tilde{d}(X) - \mu^{m+1}X_1^m X_4 + \mu^{m+1}X_4^{m+1} \tilde{d}(X)^2\).

\textbf{iii).} \(\tilde{d}_{k+1}(\mu, a) = (\mu^{k+1}a)^{m+1}(\tilde{d}_k(\mu, a))^2 + \mu^2 \tilde{d}_k(\mu, a) + \mu^{k+1}a \cdot (\mu^{k+1}g_k(\mu, a) \text{ for all } k \geq 1\).

\textbf{iv).} \(g_{k+1}(\mu, a) = \mu^{k+1}a(\tilde{d}_k(\mu, a))^2 + mg_k(\mu, a) \text{ for all } k \geq 1\).

\textbf{Proof.} The proofs of \textbf{i}) and \textbf{ii}) are straightforward and left to the reader. From \textbf{ii}) we deduce that

\[\tilde{d}_{k+1}(\mu, a) = \tilde{d}((\mu G)^{k+1}(a^*)) = \tilde{d}((\mu G)((\mu G)^k(a^*))) = \mu^2 \tilde{d}((\mu G)^k(a^*)) - \mu^{m+1}((\mu G)^k(a^*))^m((\mu G)^k(a^*))_1 + \mu^{m+1}((\mu G)^k(a^*))_1^{m+1} \tilde{d}((\mu G)^k(a^*))^2\]

Now observe that \(((\mu G)^k(a^*))_1 = \mu^k(-a)\), hence since \(m\) is odd \(((\mu G)^k(a^*))_1^m = -(\mu^k a)^m\). So we get

\[\tilde{d}_{k+1}(\mu, a) = \mu^2 \tilde{d}_k(\mu, a) + \mu^{m+1}(\mu^ka)^m g_k(\mu, a) + \mu^{m+1}(\mu^ka)^{m+1}(\tilde{d}_k(\mu, a))^2\]

which proves \textbf{iii}). Finally

\[g_{k+1}(\mu, a) = ((\mu G)^{k+1}(a^*))_1 = (\mu G)_1((\mu G)^k(a^*)) = \mu((\mu G)^k(a^*))_1 - \mu((\mu G)^k(a^*))_4(\tilde{d}((\mu G)^k(a^*))^2) = \mu g_k(\mu, a) - \mu \cdot \mu^k(-a)(\tilde{d}_k(\mu, a))^2 = \mu g_k(\mu, a) + \mu^{k+1}a(\tilde{d}_k(\mu, a))^2\]

which proves \textbf{iv}).

\textbf{Corollary 2.4} \textbf{i).} \(\tilde{d}_{k+1}(\mu, a) \geq (\mu^{k+1}a)^{m+1}(\tilde{d}_k(\mu, a))^2 \text{ for all } k \geq 1\).

\textbf{ii).} \(g_{k+1}(\mu, a) \geq \mu^{k+1}a(\tilde{d}_k(\mu, a))^2 \text{ for all } k \geq 1\).

\textbf{Proof.} By induction on \(k\) one readily verifies that for all \(k \geq 1\) both \(\tilde{d}_k(\mu, a)\) and \(g_k(\mu, a)\) are polynomials in \(\mu\) and \(a\) with coefficients in \(\mathbb{N}\). Then the result follows from lemma 2.3 \textbf{iii}) and \textbf{iv}).
Proposition 2.5 We have:

\[ g_k(\mu, a) \geq \mu^{q_k(m+1)+k} a^{q_k+2k(m+1)+1} \]
\[ d_k(\mu, a) \geq \mu^{q_k+k(m+1)} a^{q_k+2k+1(m+1)} \]

for all \( k \geq 1 \), where \( q_1 = 0 \) and \( q_{k+1} = 2q_k + 2k \) for all \( k \geq 1 \).

Proof. Use induction on \( k \). \[ \square \]

Corollary 2.6 If \( \mu a > 1 \) and \( a > 1 \) then \( \lim_{k \to \infty} ((G \circ \mu)^k(G(a^*)))_1 = \infty \). So 0 is not a global attractor of \( G \circ \mu \).

Proof. Observe that \( (G\mu)^k(G(a^*)) = \mu^{-1}(\mu G)^{k+1}(a^*) \). Then apply proposition 2.5. \[ \square \]

References


