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\[ D_n(A) \] for a class of polynomial automorphisms and stably tameness

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Abstract

In this paper we introduce a set, denoted by \( D_n(A) \), for every commutative ring \( A \) and every positive integer \( n \). It is shown that the elements of this set can be used to give an explicit description of the class \( H_n(A) \) introduced in [5]. We deduce that each polynomial map of the form \( F = X + H \) with \( H \in H_n(A) \) can be written as a finite product of automorphisms of the form \( \exp(D) \), where each \( D \) is a locally nilpotent derivation satisfying \( D^2(X_i) = 0 \) for all \( i \). Furthermore we deduce that all such \( F \)'s are stably tame.

1 Notations, definitions and an explicit description of the class \( H_n(A) \)

1.1 Notations

Throughout this paper \( A \) denotes an arbitrary commutative ring and \( A[X] := A[X_1, \ldots, X_n] \) denotes the polynomial ring in \( n \) variables over \( A \). Furthermore if \( G = (G_1, \ldots, G_n) \in A[X]^n \) and \( S = (S_{ij}(X)) \in M_{p \times q}(A[X]) \) then \( S(G) \) or \( S|_G \) denotes the \( p \times q \) matrix \((S_{ij}(G_1, \ldots, G_n))_{i,j} \). In particular if \( F \in A[X]^n \) \((= M_{n,1}(A[X]))\) then the composition of the polynomial maps \( F \) and \( G \), denoted \( F \circ G \), is equal to \( F(G) \).

Matrix multiplication will be denoted by the symbol ‘\(*\)’. So if \( S, T \in M_n(A[X]) \) then the matrix product of \( S \) and \( T \) is denoted by \( S \ast T \). By \( X \) we denote the column vector \((X_1, \ldots, X_n)' \). In the sequel we also need another multiplication in \( M_n(A[X]) \), which we denote by ‘\( \triangle \)’. This multiplication is defined as follows:

\[ S \triangle T := S(T \ast X) \ast T \]

for all \( S, T \in M_n(A[X]) \).

One easily verifies that this multiplication is associative, so it makes sense to write

\[ S_1 \triangle S_2 \triangle \cdots \triangle S_n \]
for each n-tuple \(S_1, \ldots, S_n\) in \(M_n(A[X])\). Sometimes we need to extend a vector of length 1 \(\leq p \leq n-1\) or a \(p \times p\) matrix to a vector of length \(n\) respectively an \(n \times n\) matrix. This is done as follows: let 1 \(\leq p \leq n-1\), \(c \in A[X]^p\) and \(T \in M_p(A[X])\). Then \(\vec{c}^n\) denotes the vector

\[
\vec{c}^n = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in A[X]^n,
\]

obtained by extending \(c\) by \(n-p\) zeros and \(\tilde{T}^n\) denotes the matrix

\[
\tilde{T}^n = \begin{pmatrix} T \\ 0 \\ I_{n-p} \end{pmatrix} \in M_n(A[X]),
\]

obtained by extending \(T\) with the \(n-p \times n-p\) identity matrix. To simplify the notations we drop the superscript ‘\(n\)’ and write \(\vec{c}\) and \(\tilde{T}\), even sometimes when it is clear from the context that we mean \(\vec{c}^{n-1}\) respectively \(\tilde{T}^{n-1}\) instead of \(\vec{c}^n\) respectively \(\tilde{T}^n\).

Finally the adjoint of a matrix \(T\) is denoted by \(\text{Adj}(T)\) and if \(a_1, \ldots, a_p\) are elements of a (non-necessary commutative) ring then \(\prod_{i=1}^p a_i\) denotes the element \(a_1 \cdots a_p\).

### 1.2 \(\mathcal{D}_n(A)\) and the class \(\mathcal{H}_n(A)\)

In [5] we introduced a new class of polynomial maps, denoted by \(\mathcal{H}_n(A)\), and showed that for each \(H \in \mathcal{H}_n(A)\) the Jacobian matrix \(JH\) is nilpotent and that the polynomial map \(F = X + H\) is invertible over \(A\) with \(\det(JF) = 1\).

Let us recall the definition of \(\mathcal{H}_n(A)\).

**Definition 1.1** First if \(n = 1\) we define \(\mathcal{H}_1(A) = A\). If \(n \geq 2\) we define \(\mathcal{H}_n(A)\) inductively as follows: let \(H \in A[X]^n\), then \(H \in \mathcal{H}_n(A)\) if and only if there exist \(T \in M_n(A)\), \(c \in A^n\) and \(H_\ast \in \mathcal{H}_{n-1}(A[X_n])\) such that

\[
H = \text{Adj}(T) \ast \begin{pmatrix} H_\ast \\ 0 \end{pmatrix}_{|T \ast X} + c. \tag{1}
\]

The main aim of this section is to give an explicit description of the elements of \(\mathcal{H}_n(A)\). Therefore we introduce some useful objects.

**Definition 1.2** Let \(n \geq 2\). Then \(\mathcal{D}_n(A)\) is the set of \((2n-1)\)-tuples

\[
(T, c) := (T_2, \ldots, T_n, c_1, \ldots, c_n)
\]

where \(T_n \in M_n(A)\), \(T_i \in M_i(A[X_{i+1}, \ldots, X_n])\) for all \(2 \leq i \leq n-1\), \(c_n \in A^n(= M_{n,1}(A))\) and \(c_i \in M_{i,1}(A[X_{i+1}, \ldots, X_n])\) for all \(1 \leq i \leq n-1\).
If \( n \geq 3 \) we get a natural map \( \pi : D_n(A) \to D_{n-1}(A[X_n]) \) defined by
\[
\pi((T_2, \ldots, T_n, c_1, \ldots, c_n)) = (T_2, \ldots, T_{n-1}, c_1, \ldots, c_{n-1}).
\]
Instead of \( \pi((T, c)) \) we often write \( (T', c') \).

**Definition 1.3** Let \( n \geq 2 \) and \( 0 \leq p \leq n - 2 \). Then
\[
E_{n,p} : D_n(A) \to A[X]^n
\]
is given by
1. \( E_{n,0}((T, c)) := \text{Adj}(T_n) * c_{n-1} |_{T_n \star X} \) for all \( (T, c) \in D_n(A) \).
2. If \( n \geq 3 \) and \( 1 \leq p \leq n - 2 \), then inductively (with respect to \( n \))
\[
E_{n,p}((T, c)) := \text{Adj}(T_n) * \left( E_{n-1,p-1}((T', c')) |_{T_n \star X} \right)
\]
Instead of \( E_{n,p}((T, c)) \) we simply write \( E_{n,p}(T, c) \).

Now we are able to give the main result of this section.

**Proposition 1.4** Let \( n \geq 2 \) and \( H \in A[X]^n \). Then \( H \in H_n(A) \) if and only if there exists \( (T, c) \in D_n(A) \) such that
\[
H = \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n.
\]

**Proof.** By induction on \( n \). The case \( n = 2 \) is obvious, so let \( n \geq 3 \). Then
\[
H = \text{Adj}(T_n) * \left( H_+ |_{T_n \star X} \right) + c_n
\]
where \( T_n \in M_n(A) \), \( c_n \in A^n \) and \( H_+ \in H_{n-1}(A[X_n]) \). So by the induction hypothesis we have
\[
H_+ = \sum_{p=0}^{n-3} E_{n-1,p}(T^*, c^*) + c^*_{n-1}
\]
for some \( (T^*, c^*) \in D_{n-1}(A[X_n]) \). Put \( (T, c) := (T^*, T_n, C^*, c_n) \) and observe that \( (T, c) \in D_n(A) \) and \( (T', c') = (T^*, c^*) \). So
\[
E = \sum_{p=0}^{n-2} \text{Adj}(T_n) * E_{n-1,p}(T', c') |_{T_n \star X} + \text{Adj}(T_n) * E_{n,p}(T', c') |_{T_n \star X} + c_n
\]
\[
= \sum_{p=1}^{n-2} E_{n,p}(T, c) + E_{n,0}(T, c) + c_n
\]
\[
= \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n.
\]
\[\square\]
Proposition 1.5 Let $n \geq 2$, $0 \leq p \leq n - 2$ and $(T, c) \in \mathcal{D}_n(A)$. Then

$$E_{n,p}(T, c) = \text{Adj}(\tilde{T}_{n-p} \cdots \tilde{T}_{n-1} \tilde{T}_n) * \tilde{c}_{n-p-1} |(\tilde{T}_{n-p} \cdots \tilde{T}_{n-1} \tilde{T}_n)^*x \rangle$$

Proof. By induction on $p$. The case $p = 0$ is obvious. So let $p \geq 1$. Then

$$E_{n,p}(T, c) = \text{Adj}(T_n) * \left( \begin{array}{c} E_{n-1,p-1}(T', c') \\ 0 \end{array} \right) \big|_{T_n * x}$$

$$= \text{Adj}(T_n) * \left[ \text{Adj}(\tilde{T}_{n-p} \cdots \tilde{T}_{n-1})|_{T_n * x} * \left( \tilde{c}_{n-p-1} |(\tilde{T}_{n-p} \cdots \tilde{T}_{n-1})^*x \rangle \big|_{T_n * x} \right) \right]$$

$$= \text{Adj}(\tilde{T}_{n-p} \cdots \tilde{T}_{n-1} \tilde{T}_n) * \tilde{c}_{n-p-1} |(\tilde{T}_{n-p} \cdots \tilde{T}_{n-1} \tilde{T}_n)^*x \rangle$$

Example 1.6 Consider the polynomial map $F := X + H : \mathbb{C}^4 \rightarrow \mathbb{C}^4$ where $H$ equals

$$\left( \begin{array}{c} -X_2X_4^2 - c_4X_3^2X_4 - 2m_4X_2X_3X_4 - g_4X_1X_3X_4 - k_4X_3^3 - \frac{m_4}{g_4}X_2X_3^2 - m_4X_1X_3^2 \\ -X_3X_4^2 - c_3X_3^2X_4 + g_4X_2X_3X_4 - k_3X_3^3 + m_4X_2X_3^2 + g_4^2X_1X_3^2 \\ -\frac{1}{3}X_4^3 \\ 0 \end{array} \right)$$

and $c_3, k_3, c_4, g_4, k_4, m_4 \in \mathbb{C}$ and $g_4 \neq 0$. This $F$ is invertible. In fact if we take $P = P^{-1} = (X_4, X_3, X_2, X_1)$, we have that $PFP$ is one of the eight representatives of the cubic homogeneous maps in dimension four as given by the second author in [6], also published in [4, Theorem 2.10].

Now consider the following element $(\tilde{T}, c)$ of $\mathcal{D}_4(\mathbb{C})$ where

$$T = \left( \begin{array}{c} 1 \\ g_4^2X_3 \\ g_4X_4 + m_4X_3 \end{array} \right) \cdot \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right)$$

and

$$c = \left( \begin{array}{c} \frac{m_4}{g_4}X_2 \\ -X_2X_4^2(\epsilon_4X_3 + k_4X_3) \\ -X_3X_4^2 - \epsilon_3X_3X_4^2 - k_3X_3^3 \\ \frac{1}{3}X_4^3 \end{array} \right)$$

Our claim is that

$$H = \sum_{p=0}^{2} E_{4,p}(T, c) + c_4.$$
To prove this we will compute $E_{4,0}$, $E_{4,1}$ and $E_{4,2}$ by the method of proposition 1.5. Note that $c_4 = 0$. Since $T_4 = T_3 = I_4$, $E_{4,0}$ and $E_{4,1}$ are easy:

\[
E_{4,0} = \text{Adj}(T_4) \ast \tilde{c}_3 \mid_{T_4 \ast X} = \tilde{c}_3 = \begin{pmatrix}
0 \\
0 \\
\frac{-1}{3} X_4^3 \\
0
\end{pmatrix}
\]

\[
E_{4,1} = \text{Adj}(\bar{T}_3 \triangle T_4) \ast \tilde{c}_2 \mid_{(\bar{T}_3 \triangle T_4) \ast X} = \tilde{c}_2 = \begin{pmatrix}
-X_3^2(e_4X_4 + k_4X_3) \\
-X_3^2X_4^2 - e_3X_4X_3^3 - k_3X_3^3 \\
0 \\
0
\end{pmatrix}
\]

Before we compute $E_{4,2}$ we present the following identities:

\[
\bar{T}_2 \triangle \bar{T}_3 \triangle T_4 = \bar{T}_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
g_4 X_3 & g_4 X_4 + m_4 X_3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\text{Adj}(\bar{T}_2 \triangle \bar{T}_3 \triangle T_4) = \begin{pmatrix}
g_4 X_4 + m_4 X_3 & 0 & 0 & 0 \\
-g_4^2 X_3 & 1 & 0 & 0 \\
0 & 0 & g_4 X_4 + m_4 X_3 & 0 \\
0 & 0 & 0 & g_4 X_4 + m_4 X_3
\end{pmatrix}
\]

\[
(\bar{T}_2 \triangle \bar{T}_3 \triangle T_4) \ast X = \begin{pmatrix}
X_1 \\
g_4^2 X_1 X_3 + g_4 X_2 X_4 + m_4 X_2 X_3 \\
X_3 \\
X_4
\end{pmatrix}
\]

\[
\tilde{c}_1 \mid_{(\bar{T}_2 \triangle \bar{T}_3 \triangle T_4) \ast X} = \begin{pmatrix}
-X_1 X_3 - \frac{1}{g_4} X_2 X_4 - \frac{m_4}{g_4} X_2 X_3 \\
0 \\
0 \\
0
\end{pmatrix}
\]

and finally

\[
E_{4,2} = \text{Adj}(\bar{T}_2 \triangle \bar{T}_3 \triangle T_4) \ast \tilde{c}_1 \mid_{(\bar{T}_2 \triangle \bar{T}_3 \triangle T_4) \ast X}
\]

\[
= \begin{pmatrix}
-X_1 X_3 - \frac{1}{g_4} X_2 X_4 - \frac{m_4}{g_4} X_2 X_3 \\
X_3(g_4^2 X_1 X_3 + g_4 X_2 X_4 + m_4 X_2 X_3) \\
0 \\
0
\end{pmatrix}
\]

It is easy to verify that $H = E_{4,0} + E_{4,1} + E_{4,2} + c_4$, which was our claim.

2 Nice derivations

Let $B := A[x_1, \ldots, x_n]$ be a finitely generated $A$-algebra and $D$ a subset of $\text{Der}_A(B)$. By $B^D$ we denote the set of all $b \in B$ such that $d(b) = 0$ for all $d \in D$. 
Definition 2.1 Let \( D \subseteq \text{Der}_A(B) \) a finite subset and \( \tau \in \text{Der}_A(B) \).

1. We say that \( \tau \) is derived from \( D \) in at most one step if \( \tau \) is of the form \( \tau = \sum_{d \in D} b_d d \), where \( b_d \in B^D \) for all \( d \in D \).

2. Let \( m \geq 2 \). We say that \( \tau \) is derived from \( D \) in at most \( m \) steps if there exists a sequence of finite subsets

\[
D = D_0, D_1, D_2, \ldots, D_m
\]

of \( \text{Der}_A(B) \) such that \( \tau \in D_m \) and all elements of \( D_i \) are derived from \( D_{i-1} \) in at most one step, for all \( 1 \leq i \leq m \). If furthermore the elements of \( D \) satisfy \( d_1 d_2(x_i) = 0 \) for all \( d_1, d_2 \in D \) and all \( i \), then \( \tau \) is called nice of order \( \leq m \), with respect to \( x_1, \ldots, x_n \) and \( D \).

Proposition 2.2 Notations as in definition 2.1. If \( d_1 d_2(x_i) = 0 \) for all \( d_1, d_2 \in D \) and all \( i \), then \( d_1 d_2(x_i) = 0 \) for all \( d_1, d_2 \in D_m \) and all \( i \). In particular \( d^2(x_i) = 0 \) for every nice derivation.

Proof. We use induction on \( m \). The case \( m = 0 \) is obvious since \( D_0 = D \). Now let \( m \geq 1 \). Then \( d_1 = \sum_{d \in D_{m-1}} b_d d \), \( d_2 = \sum_{d' \in D_{m-1}} b_{d'} d' \) with \( b_d, b_{d'} \in B^{D_{m-1}} \). Then

\[
d_1 d_2(x_i) = \sum_{d, d'} b_d b_{d'} d(x_i) + \sum_{d, d'} b_d b_{d'} d'(x_i).
\]

Now observe that \( d(b_{d'}) = 0 \) since \( b_{d'} \in B^{D_{m-1}} \) and \( d \in D_{m-1} \). Finally the induction hypothesis gives \( d'(x_i) = 0 \) for all \( d, d' \in D_{m-1} \) and all \( i \), so (2) implies \( d_1 d_2(x_i) = 0 \).

We demonstrate these aspects by the so-called Winkelmann derivation. See [10].

Example 2.3 Let \( \tau = (1 + X_4 X_2 - X_5 X_3) \partial_{X_1} + X_5 \partial_{X_2} + X_4 \partial_{X_3} \), a derivation on

\( B := A[X_1, X_2, X_3, X_4, X_5] \). Let \( D = \{ \partial_{X_1}, \partial_{X_2}, \partial_{X_3} \} \). Then \( \tau \) is nice of order two with respect to \( X_1, X_2, X_3, X_4, X_5 \) and \( D \). To show that this is true, we present a sequence of finite subsets of \( \text{Der}_A(B) \),

\[
D = D_0, D_1, D_2
\]

Take \( D_1 := \{ \partial_{X_1}, X_5 \partial_{X_2} + X_4 \partial_{X_3} \} \) and \( D_2 := \{ \tau \} \). Note that in definition 2.1 it is not demanded that the set \( D_i \) of this sequence is a subset of \( D_{i+1} \). The only demand is that each \( D_i \) is a finite subset of \( \text{Der}_A(B) \). Since \( X_4, X_5 \in B^D \) it follows immediately that \( \partial_{X_1} \) and \( X_5 \partial_{X_2} + X_4 \partial_{X_3} \) are derived from \( D \) in one step. And from \( 1 + X_4 X_2 - X_5 X_3 \in B^{D_1} \) it follows that \( \tau \) is derived from \( D_1 \) in one step. Obviously we have \( d_1 d_2(x_i) = 0 \) for all \( d_1, d_2 \in D \) and hence with proposition 2.2 also \( \tau^2(x_i) = 0 \).
3 Derivations associated to polynomial maps

The main aim of this section is to show that for each $0 \leq p \leq n - 2$ the polynomial map $X + E_p(T, c)$ (where $(T, c) \in \mathcal{D}_n(A)$) is of the form $\exp(d)$, for some nice $A$-derivation $d$ of $A[X]$. Observe that $d$ is locally nilpotent if $d$ is nice with respect to $X_1, \ldots, X_n$ since $d^2(X_i) = 0$ for all $i$, by proposition 2.2.

In order to prove this result (see theorem 3.3), we need to generalise some of the notions of section 1 to arbitrary finitely generated $A$-algebras. So let $B := A[x_1, \ldots, x_n]$ be a finitely generated $A$-algebra and $\varphi : A[X_1, \ldots, X_n] \to B$ the $A$-ringhomomorphism defined by $\varphi(X_i) = x_i$ for all $i$. For each $p, q \geq 1$ consider the natural extension

$$\varphi : M_{p, q}(A[X_1, \ldots, X_n]) \to M_{p, q}(B).$$

Then for each $(T, c) \in \mathcal{D}_n(A)$ we define

$$E_{n, p}(T, c)(x) := \varphi(E_{n, p}(T, c)) \in B^n.$$

Now let $\partial_1, \ldots, \partial_n$ be an $n$-tuple of $A$-derivations of $B$. To each vector $b = (b_1, \ldots, b_n)^t \in B^n$ we associate the following $A$-derivation of $B$:

$$D(b; \partial_1, \ldots, \partial_n) := b_1 \partial_1 + \cdots + b_n \partial_n \quad (= b^t \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}).$$

To formulate the next lemma we need some more notations: let $(T, c) \in \mathcal{D}_n(A)$. Put

$$\begin{align*}
(x'_1, \ldots, x'_n)^t &:= T_n * (x_1, \ldots, x_n)^t \\
(\partial'_1, \ldots, \partial'_n) &:= (\text{Adj}(T_n))^t * (\partial_1, \ldots, \partial_n)^t \\
x'' &:= (x'_1, \ldots, x'_{n-1}) \\
(T'', c'') &:= (T'(X_n = x''_n), c'(X_n = x''_n)) \in \mathcal{D}_{n-1}(A[x''_n])
\end{align*}$$

Lemma 3.1 Let $n \geq 3$ and $1 \leq p \leq n - 2$. Then

$$D(E_{n, p}(T, c)(x); \partial_1, \ldots, \partial_n) = D(E_{n-1, p-1}(T'', c'')(x''); \partial'_1, \ldots, \partial'_{n-1}).$$

Proof.

$$D(E_{n, p}(T, c)(x); \partial_1, \ldots, \partial_n)$$

$$= (E_{n, p}(T, c)(x))^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}$$

$$= \left( (E_{n-1, p-1}(T', c')|_{T_n \cdot x'})^t \begin{pmatrix} 0 \\ \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \right) * (\text{Adj}(T_n))^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}.$$
Derivations associated to polynomial maps

\[
\begin{pmatrix}
\partial'_1 \\
\vdots \\
\partial'_{n-1}
\end{pmatrix}
= \left( (E_{n-1,p-1}(T^n, e^p)(x^n))^t \ 0 \right) \cdot \\
= D(E_{n-1,p-1}(T^n, e^p)(x^n); \partial'_1, \ldots, \partial'_{n-1})
\]

\[\square\]

Lemma 3.2 Notations as above. Let \( a \in A \) and let \( \partial_1, \ldots, \partial_n \) be \( A \)-derivations of \( B \) such that \( \partial_i(x_j) = a \delta_{ij} \) for all \( i,j \). Then

\[\partial'_i(x'_j) = a \det(T_n) \delta_{ij}\]

for all \( i,j \).

Proof. Denote the \( i \)-th column of \( \text{Adj}(T_n) \) by \( (t^*_i, \ldots, t^*_n)^t \) and the \( j \)-th row of \( T_n \) by \( (t_j, \ldots, t_n_n) \). Then

\[
\partial'_i(x'_j) = \left( \sum_{s=1}^n t^*_s \partial_s \right) \left( \sum_{s=1}^n t_{js} x_s \right)
= \sum_{s=1}^n a t^*_s t_{js}
= a(T_n \ast \text{Adj}(T_n))_{ji}
= a \det(T_n) \delta_{ij}
\]

Now we are able to prove:

Theorem 3.3 Let \( \partial_1, \ldots, \partial_n \) be \( A \)-derivations on \( A[x_1, \ldots, x_n] \) such that there exists an element \( a \in A \) such that \( \partial_i(x_j) = a \delta_{ij} \) for all \( i,j \). Let \( (T, c) \in D_n(A) \). Then the \( A \)-derivation \( d := D(E_{n,p}(T, c)(x); \partial_1, \ldots, \partial_n) \) is nice with respect to \( x_1, \ldots, x_n \) and \( D_0 := \{ \partial_1, \ldots, \partial_n \} \), for all \( n \geq 2 \) and all \( 0 \leq p \leq n-2 \).

Proof.

1. The hypothesis on the \( \partial_i \) imply that \( dd'(x_i) = 0 \) for all \( d, d' \in D_0 \) and all \( i \).

2. First we consider the case \( p = 0 \). Then

\[E_{n,0}(T, c) = \text{Adj}(T_n) \ast \tilde{e}_{n-1} \ast_{\tau_{n} \ast X}.\]

So

\[d = (\tilde{e}_{n-1} \ast_{\tau_{n} \ast X})^t \ast (\text{Adj}(T_n))^t \ast \left( \begin{array}{c} \partial_1 \\ \vdots \\ \partial_n \end{array} \right).\]
Write $\gamma(x_n, \ldots) = (\gamma_1(X_n), \ldots, \gamma_{n-1}(X_n), 0)$. Then the definition of $x_n'$ and the $\partial_j'$ imply that

$$d = \gamma(x_n', \ldots, \gamma_{n-1}(x_n'), 0) * (\partial'_1, \ldots, \partial'_{n-1}) = \sum_{i=1}^{n-1} \gamma_i(x_n') \partial_{i}'$$

Put $D_1 := \{\partial'_1, \ldots, \partial'_{n-1}\}$ and observe that $D_1 \subseteq \text{Der}_A(B)$ and that each element of $D_1$ is derived from $D_0$ in at most one step. Finally since $\partial'_i(x_n') = 0$ for all $1 \leq i \leq n-1$ (by lemma 3.2) we get that $\gamma_i(x_n') \in B^{D_i}$ for all $1 \leq i \leq n-1$. So (3) implies that $d$ is derived from $D_1$ in at most one step. Consequently $d$ is derived from $D_0$ in at most two steps. So $d$ is nice with respect to $x_1, \ldots, x_n$ and $D_0$ by case 1.

3. Now we prove the theorem by induction on $n$. If $n = 2$, then $p = 0$ and we are in case 2. So let $n \geq 3$. By case 2 we may assume that $p \geq 1$. Then by lemma 3.1 we have

$$d = D(E_{n-1,p-1}(T', c') (x''); \partial'_1, \ldots, \partial'_{n-1})$$

with $(T', c') \in D_{n-1}(A[x_n'])$. By lemma 3.2 we can apply the induction hypothesis to the ring $A[x_n']$ and the $(n-1)$-tuple of $A[x_n']$-derivations $\partial'_1, \ldots, \partial'_{n-1}$ on the $A[x_n']$-algebra $B' := A[x_n'][x_1', \ldots, x_{n-1}']$. So the $A[x_n']$-derivation $d$ on $B'$ is nice with respect to $D_0' := \{\partial'_1, \ldots, \partial'_{n-1}\}$ and $x_1', \ldots, x_{n-1}'$. So there exists a sequence

$$D_0', D_1', \ldots, D_m'$$

of finite subsets of $\text{Der}_A(A[x_n'])(B')$ such that $d \in D_m'$ and $D_i'$ is derived from $D_{i-1}'$ in at most one step for all $1 \leq i \leq m$. Now observe that $D_0' \subset \text{Der}_A(B)$ and that $B' \subseteq B$ since by definition obviously $x_i' \in B$ for all $i$. Consequently if $d'$ is an $A[x_n']$-derivation of $B'$ derived from $D_0'$ in at most one step, then $d' \in \text{Der}_A(B)$. Hence $D_i' \subset \text{Der}_A(B)$. Arguing in a similar way we conclude by induction on $i$ that $D_i' \subset \text{Der}_A(B)$ for all $0 \leq i \leq m$. Since as remarked in case 2 above, all elements of $D_0'$ ($= D_1$ in case 2) are derived from $D_0$ in at most one step we deduce that $d$ is derived from $D_0$ in at most $m+1$ steps. Just define $D_i := D_{i-1}'$ for all $1 \leq i \leq m+1$. Hence $d$ is nice with respect to $x_1, \ldots, x_n$ and $D_0$ by 1.

Corollary 3.4 Let $(T, c) \in \mathcal{D}_n(A)$ and $0 \leq p \leq n-2$. Put

$$D := D \left( E_{n,p}(T, c); \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \right).$$

Then $D$ is nice with respect to $X_1, \ldots, X_n$ and $\{\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\}$. Furthermore we have $\exp(D) = X + E_{n,p}(T, c)$ and the inverse map is given by $\exp(-D) = X - E_{n,p}(T, c)$.

Proof. The first part is an immediate consequence of theorem 3.3. Furthermore $D^2(X_i) = 0$ by proposition 2.2. So $\exp(D)(X) = X + E_{n,p}(T, c)$ and the inverse map is given by $\exp(-D)(X) = X - E_{n,p}(T, c)$. □
4 The main theorem

In this section we show that for every $H \in \mathcal{H}_n(A)$ the polynomial map $F = X + H$ is a product of $n$ polynomial automorphisms of the form $\exp(D)$, where each $D$ is a nice derivation on $A[X]$. More precisely

**Theorem 4.1** Let $F = X + H$, where $H = \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n$, for some $(T, c) \in D_n(A)$. Then

$$F = \exp(D \left( c_n; \frac{\partial}{\partial X^1}, \ldots, \frac{\partial}{\partial X^n} \right)) \prod_{p=0}^{n-2} \exp(D \left( E_{n,p}(T, c); \frac{\partial}{\partial X^1}, \ldots, \frac{\partial}{\partial X^n} \right)).$$

**Proof.** Observe that

$$\exp(-D \left( c_n; \frac{\partial}{\partial X^1}, \ldots, \frac{\partial}{\partial X^n} \right)) \circ F = \sum_{p=0}^{n-2} E_{n,p}(T, c).$$

So the case $n = 2$ follows from corollary 3.4. Hence we may assume that $n \geq 3$. Now theorem 4.1 follows directly from proposition 4.2 below and corollary 3.4. □

**Proposition 4.2** Let $n \geq 3$, $0 \leq p \leq n - 3$ and $(T, c) \in D_n(A)$. Then

$$\exp(-D(E_{n,p}(T, c))) \circ (X + \sum_{q=p}^{n-2} E_{n,q}(T, c)) = X + \sum_{q=p+1}^{n-2} E_{n,q}(T, c).$$

**Proof.** Put $G := \exp(-D(E_{n,p}(T, c)))$. So $G = X - E_{n,p}(T, c)$ (by corollary 3.4). Hence if we put

$$U := \tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_1 \Delta T_n$$

then by proposition 1.4 we get

$$G = X - \text{Adj}(U) \ast \tilde{c}_{n-p-1} |_{U^*X}.$$ 

So if we put

$$f := X + \sum_{q=p}^{n-2} E_{n,q}(T, c)$$

then

$$G \circ f = f - \text{Adj}(U(f)) \ast \tilde{c}_{n-p-1} |_{U^*f}.$$ 

Since $U(f) = f$ (by corollary 4.4 below, with $j = 0$) we get

$$G \circ f = f - \text{Adj}(U) \ast \tilde{c}_{n-p-1} |_{U^*f}.$$ 

Now observe that each component of $\tilde{c}_{n-p-1}$ belongs to $A[X_{n-p}, \ldots, X_n]$ and that for each $i \geq n - p$ $(U \ast f)_i = (U \ast X)_i$ (by lemma 4.3 below). So $\tilde{c}_{n-p-1} |_{U^*f} = \tilde{c}_{n-p-1} |_{U^*X}$ and hence

$$G \circ f = f - \text{Adj}(U) \ast \tilde{c}_{n-p-1} |_{U^*X}$$

(by proposition 1.4). □
Lemma 4.3 Let \( n \geq 3 \), \( 0 \leq p \leq n - 2 \), \( 0 \leq j \leq p \) and \((T, c) \in \mathcal{D}_n(A)\). Put \( f := X + \sum_{q=p}^{n-2} E_{n,q}(T, c) \). Then
\[
[(T_{n-p+j} \triangle \cdots \triangle T_{n-1} \triangle T_n) * f]_i = [(T_{n-p+j} \triangle \cdots \triangle T_{n-1} \triangle T_n) * X]_i
\]
for all \( i \geq n - p + j \).

Proof. Put \( U := T_{n-p+j} \triangle \cdots \triangle T_{n-1} \triangle T_n \). It suffices to show that for each \( q \geq p \)
\[
[U * E_{n,q}(T, c)]_i = 0
\]
for all \( i \geq n - p + j \). So let \( q \geq p \), then \( q \geq p - j \).

1. We first treat the case that \( q = p - j \). Then \( j = 0 \) and \( q = p \). Consequently \( U = T_{n-p} \triangle \cdots \triangle T_{n-1} \triangle T_n \), \( E_{n,q}(T, c) = E_{n,p}(T, c) \) and hence by proposition 1.4
\[
U * E_{n,q}(T, c) = U * \text{Adj}(U) * \tilde{c}_{n-p-1} |_U X
= \det(U) * \tilde{c}_{n-p-1} |_U X
\]
Since the last \( p+1 \) coordinates of \( \tilde{c}_{n-p-1} \) are zero, we obtain that
\[
[U * E_{n,q}(T, c)]_i = 0
\]
for all \( i \geq n - p \), which proves the case that \( q = p - j \).

2. Now assume that \( q \geq p - j + 1 \). So \( n - q \leq n - p + j - 1 \). Put \( V := T_{n-q} \triangle \cdots \triangle T_{n-p+j-1} \). Then by proposition 1.4 we can write
\[
E_{n,q}(T, c) = \text{Adj}(V \triangle U) * \tilde{c}_{n-q-1} |_{(V \triangle U)X}
= \text{Adj}(V |_{U X} \triangle U) * \tilde{c}_{n-q-1} |_{(V \triangle U)X}
= \text{Adj}(U) * \text{Adj}(V |_{U X} \triangle U) * \tilde{c}_{n-q-1} |_{(V \triangle U)X}
\]
Consequently
\[
U * E_{n,q}(T, c) = \det(U) * \text{Adj}(V |_{U X} \triangle U) * \tilde{c}_{n-q-1} |_{(V \triangle U)X}
\]  
(5)
Note that \( V \), and hence \( V |_{U X} \), is of the form \( \tilde{B} \) for some \( B \in M_{n-p+j-1}(A[X]) \). Furthermore \( (\tilde{c}_{n-q-1})_i = 0 \) if \( i \geq n - q \) which implies that \( (\tilde{c}_{n-q-1} |_{(V \triangle U)X})_i = 0 \) if \( i \geq n - p + j \) (since \( n - p + j > n - q \)). Now the desired result (4) follows from (5).

\[ \Box \]

Corollary 4.4 Notations as in lemma 4.3. Then
\[
(T_{n-p+j} \triangle \cdots \triangle T_{n-1} \triangle T_n)(f) = T_{n-p+j} \triangle \cdots \triangle T_{n-1} \triangle T_n.
\]
Proof. By induction on $N := p - j$. If $N = 0$ the result is obvious. So let $N \geq 1$. Then
\[
(T_{n-p+j} \Delta \cdots \Delta T_{n-1} \Delta T_n)(f)
\]
\[
= T_{n-p+j} (t_{n-p+j+1} \Delta \cdots \Delta T_{n-1} \Delta T_n)(f) * (T_{n-p+j+1} \Delta \cdots \Delta T_{n-1} \Delta T_n)(f)
\]
by the induction hypothesis. Finally observe that the matrix elements of $T_{n-p+j}$ depend only on $X_{n-p+j+1}, \ldots, X_n$. The result follows immediately from lemma 4.3 (with $j + 1$ instead of $j$).

5 Stably tameness

With theorem 4.1 we are now able to prove the stably tame generators conjecture for all maps in our class $\mathcal{H}_n(A)$. And will also show that this result is ‘sharp’: we give an example of an element of our class which is not tame, so in general we cannot get a better result than this stable tameness.

First let us recall the conjecture (it was already mentioned in [1], [2], [3], [4] and [7]):

Conjecture 5.1 For every invertible polynomial map $F : k^n \to k^n$ over a field $k$ there exist $t_1, \ldots, t_m$ such that
\[
F^{[m]} = (F, t_1, \ldots, t_m) : k^{n+m} \to k^{n+m}
\]
is tame, i.e. $F$ is stably tame.

Theorem 5.2 Let $F = X + H$ with $H \in \mathcal{H}_n(A)$. Then $F$ is stably tame.

To do this we use the following result due to Martha Smith in [9]:

Proposition 5.3 Let $D$ be a locally nilpotent derivation of $A[X]$. Let $a \in \text{ker}(D)$. Extend $D$ to $A[X][t]$ by setting $D(t) = 0$. Note that $tD$ is locally nilpotent. Define $\rho \in \text{Aut}_A A[X][t]$ by $\rho(X_i) = X_i, i = 1, \ldots, n$ and $\rho(t) = t + a$. Then
\[
(\exp(aD), t) = \rho^{-1} \exp(-tD) \rho \exp(tD).
\]

Corollary 5.4 Let $D, a$ be as in proposition 5.3. If $D$ is conjugate by a tame automorphism to a triangular derivation, then $(\exp(aD), t)$ is tame.

Lemma 5.5 Let $\tau$ be a nice derivation of order $m$ with respect to $X_1, \ldots, X_n$ and $D := \{\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\}$ on $A[X]$. Then $\exp(a\tau)$ is stably tame for all $a \in \text{ker}(\tau)$.
Proof. We use induction on $m$. Consider the case that $m = 1$. Then $\tau = \sum_{d \in D} b_d d$ with $b_d \in A[X]^D = \cap_{d \in D} \ker(d) = A$. And hence $\tau(X_i) \in A$ and clearly $\tau$ is on triangular form. So now we can apply corollary 5.4 and find that $\exp(a \tau)$ is stably tame.

Now consider the case $m > 1$. We may assume that for all nice derivations $\sigma \in \text{Der}_X(A[X])$ of order $m - 1$ with respect to $D$ and $X_1, \ldots, X_n$ and for any commutative ring $A$ we have that $\exp(a \sigma)$ is stably tame for all $a \in \ker(\sigma)$. Let $\tau$ be nice of order $m$. Define $\rho$ and extend $\tau$ to $A[X][t]$ as in proposition 5.3 (in fact we extend all derivations of $D_1$ to $A[X][t]$ in this way). Now from

$$\exp(a \tau), t) = \rho^{-1} \exp(-t \tau) \rho \exp(t \tau)$$

it follows that it suffices to see that $\exp(t \tau)$ is stably tame. Now we see that $t \tau = \sum_{d \in D_{m-1}} t b_d d$ with $tb_d \in A[X][t]^{D_{m-1}}$. But from this it follows that

$$\exp(t \tau) = \exp(\sum_{d \in D_{m-1}} t b_d d) = \prod_{d \in D_{m-1}} \exp(t b_d d)$$

This last equation follows from proposition 1.5. Obviously it suffices to prove that each $\exp(tb_d d)$ is stably tame to conclude that $\exp(t \tau)$ is stably tame. But $d$ is a nice derivation of order $m - 1$, $tb_d \in \ker(d)$ and hence we can apply the induction hypothesis to the ring $A[t]$ and find that $\exp(t \tau)$ is stably tame and hence $\exp(a \tau)$ is stably tame. \hfill $\Box$

Proof of theorem 5.2. Now if we look at theorem 4.1 we see that each $F = X + H$ with $H \in \mathcal{H}_2(A)$ can be written as the product of a finite number of $\exp(a; D_i)^s$ where each $D_i$ is a nice derivation with respect to $X_1, \ldots, X_n$ and $\{ \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \}$ and $a_i \in \ker(D_i)$. Applying lemma 5.5 $n$ times gives us the desired result: $F$ is stably tame. \hfill $\Box$

Remark 5.6 Note that we don’t give an indication of the value of $m$ in conjecture 5.1. As can be seen from the proof above, this $m$ can be very high. At the highest level we have $n \exp(a_i; D_i)^s$, but each of these factors can give rise to a great number of extra variables, depending on the ‘order of niceness’ of each $D_i$.

To conclude this paper we show that in general the automorphisms $F = X + H$ with $H \in \mathcal{H}_2(A)$ need not be tame. Actually, this idea was already presented by Nagata in [8].

Example 5.7 Let $A$ be a domain, but not a principal ideal domain. Let $a, b \in A$ such that $Aa + Ab$ is not a principal ideal. Let $f(T) \in A[T]$ with $\deg(f) \geq 2$ and let $F = X + H$ with

$$H = \left( \begin{array}{c} bf(ax_1 + bx_2) \\ -af(ax_1 + bx_2) \end{array} \right)$$

Since $H \in \mathcal{H}_2(A)$ $F$ is an automorphism of $A[X_1, X_2]$. However, it is shown in [8] that $F$ is not tame.
References


