A counterexample to Meisters' cubic-linear linearization conjecture

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Abstract

We show that the cubic linear polynomial automorphism $F : \mathbb{C}^{15} \to \mathbb{C}^{15}$ given by Drużkowski in [2] has the property that for every $\lambda > 0$ the dilation λF is not global analytic linearizable to its linear part.

Introduction

In [1] Deng, Meisters and Zampieri proposed a new approach to prove the Jacobian Conjecture. They conjectured that for every polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ with $\det(JF) \in \mathbb{C}^*$, $F(0) = 0$ and $JF(0) = I_n$, there exists an integer N such that for all $\lambda \in \mathbb{C}$ with $|\lambda| > N$ the dilation λF is global analytic linearizable to its linear part i.e. there exists an analytic automorphism $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ such that $\varphi^{-1}\lambda F\varphi = \lambda JF(0)X$. They observed that this conjecture, if true, implies the Jacobian Conjecture. However in [3] the authors showed that the conjecture is false for all $n \geq 4$ by giving a counterexample of the form $F = X + H$ where H is homogeneous of degree five.

Inspite of this counterexample Meisters conjectured (and offered a $200 reward until February 17, 1996) that every cubic linear map satisfies the linearization conjecture stated above (see [6] and also [5] page 85).

In this paper we show that also this conjecture is false. In fact we use the cubic linear map given by Drużkowski in [2, example 7.8], which was studied by Meisters in [4]. The method to prove that this map is a counterexample to the conjecture mentioned above is similar to the one used in [3].

1 Drużkowski's cubic linear map

In [2] Drużkowski describes the following cubic linear polynomial automorphism $F = X - H(X_1, X_2, s_1, s_2, s_3)$ where $X = X_1, \ldots, X_{15}$ and

$$\begin{aligned} H(X_1, X_2, s_1, s_2, s_3) \;:=\; & (0, 0, s_1^3, s_2^3, s_3^3, (-X_1 + s_3)^3, (-X_2 + s_3)^3, \\ & (-X_1 + s_2)^3, (-X_1 + s_1)^3, (-X_2 + s_1)^3, \end{aligned}$$

$$(-X_1 - s_2)^3, (-X_2 - s_3)^3, (-X_2 - s_1)^3,$$
$$(-X_1 - X_2 + s_3)^3, (-X_1 - X_2 + s_1)^3)$$

and

$$\begin{aligned} s_1 &:= 2X_4 + X_5 - X_6 - X_7 - X_8 + X_{11} + X_{14} \\ s_2 &:= X_3 + X_5 - \tfrac{1}{2}X_7 - \tfrac{1}{2}X_{10} + \tfrac{1}{2}X_{12} + \tfrac{1}{2}X_{13} \\ s_3 &:= -X_3 + 2X_4 - X_8 + X_9 + X_{10} + X_{11} - X_{15} \end{aligned}$$

In the proof of the main theorem we need the inverse of F. To describe this inverse we follow the notes of [4], in which he describes the inverse by the following formulas. Define in $\mathbb{Q}[Y] := \mathbb{Q}[Y_1, \ldots, Y_{15}]$ the polynomials

$$\begin{aligned} g_1 &:= 2Y_4 + Y_5 - Y_6 - Y_7 - Y_8 + Y_{11} + Y_{14} - 3Y_1^2Y_2 - 3Y_1Y_2^2 \\ g_2 &:= Y_3 + Y_5 - \tfrac{1}{2}Y_7 - \tfrac{1}{2}Y_{10} + \tfrac{1}{2}Y_{12} + \tfrac{1}{2}Y_{13} \\ g_3 &:= -Y_3 + 2Y_4 - Y_8 + Y_9 + Y_10 + Y_11 - Y_15 + 3Y_1^2Y_2 + 3Y_1Y_2^2 \end{aligned}$$

Put

$$B := \begin{pmatrix} 1 & 6Y_1^2 & -6Y_1Y_2 \\ 3Y_2^2 & 1 & 3Y_2^2 \\ 6Y_1Y_2 & 6Y_1^2 & 1 \end{pmatrix}$$

Then $\det(B) = 1$. So $B \in \mathrm{GL}_3(\mathbb{Q}[Y_1, Y_2])$. Finally put

$$\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} := B^{-1} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

Lemma 1.1 (Meisters, [4]) *The inverse G of F is given by*

$$G(Y) = Y + H(Y_1, Y_2, S_1, S_2, S_3).$$

Proof. Left to the reader. □

Corollary 1.2

$$\begin{aligned} G_3(0, 0, Y_3, Y_4, 0, \ldots, 0) &= Y_3 + (2Y_4)^3 \\ G_4(0, 0, Y_3, Y_4, 0, \ldots, 0) &= Y_4 + Y_3^3 \end{aligned}$$

Proof. Just observe that $B_{|(0,0,Y_3,Y_4,0,\ldots,0)} = I_3$. So $S_i(0, 0, Y_3, Y_4, 0, \ldots, 0)$ equals $g_i(0, 0, Y_3, Y_4, 0, \ldots, 0)$ for $i = 1, 2, 3$, which implies the result. □

2 The main theorem

Theorem 2.1 *Let F be as described in section 1. Then for all $\lambda > 0$ λF is not global analytic linearizable to its linear part λX.*

Proof. 1. First we consider the case that $0 < \lambda < 1$. Suppose that λF is analytic linearizable. Then it is well-known (cf.[3, lemma 2.2]) that 0 is a global attractor of λF. Therefore let $a > 0$ be such that $a\lambda > 1$. Denote the i-th component of $(\lambda F)^k$ by $(\lambda F)_i^k$. Observe that

$$\begin{aligned}(\lambda F)_3(0,0,X_3,0,0,X_6,0,\ldots,0) &= X_3 + X_6^3 \\ (\lambda F)_4(0,0,X_3,0,0,X_6,0,\ldots,0) &= X_6 + X_3^3\end{aligned}$$

So if we put $A := (0,0,a,0,0,a,0,\ldots,0)$ then it follows from lemma 2.2 below that $\lim_{n\to\infty}(\lambda F)_3^n(A) = \infty$, so 0 is not a global attractor of λF.

2. Now let $\lambda > 1$ and suppose that λF is analytic linearizable. Then $(\lambda F)^{-1} = F^{-1} \circ \lambda^{-1}$ is also analytic linearizable. Put $\mu := \lambda^{-1}$ and $G := F^{-1}$. So $G \circ \mu$ is analytic linearizable and again we conclude that 0 is a global attractor of $G \circ \mu$ (since $0 < \mu < 1$). Let $a > 0$ be such that $a\mu > 1$. Put $A' := (0,0,a,a,0,\ldots,0)$. Since $\lim_{n\to\infty}(G\circ\mu)^n(G(A')) = 0$ and $(G\circ\mu)^n(G(A')) = \mu^{-1}(\mu G)^{n+1}(A')$ it follows that $\lim_{n\to\infty}(\mu G)^n(A') = 0$. However by corollary 1.2 and lemma 2.2 below we deduce that $\lim_{n\to\infty}(\mu G)_3^n(A') = \infty$, a contradiction. So also for $\lambda > 1$, λF is not analytic linearizable. Since obviously for $\lambda = 1$ λF is not linearizable to its linear part, the proof of the theorem is complete. □

Lemma 2.2 *Let $c \geq 1$, $0 < \lambda < 1$ and $a > 0$ such that $a\lambda > 1$. Let $f = (f_1, f_2) \in \mathbb{R}[X,Y]^2$ given by $f_1 = X + cY^3$ and $f_2 = Y + X^3$. Then*

$$(\lambda f)_i^n(a,a) > a^{p(n)}$$

for $i = 1,2$ and for all $n \geq 1$ where $p(1) = 1$ and $p(n+1) = 3p(n) - 1$, for all $n \geq 1$.

Proof. By induction on n. □

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