A counterexample to Meisters’ cubic-linear linearization conjecture

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Abstract

We show that the cubic linear polynomial automorphism $F : \mathbb{C}^5 \to \mathbb{C}^5$ given by Drużkowski in [2] has the property that for every $\lambda > 0$ the dilation $\lambda F$ is not global analytic linearizable to its linear part.

Introduction

In [1] Deng, Meisters and Zampieri proposed a new approach to prove the Jacobian Conjecture. They conjectured that for every polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ with $\det(JF) \in \mathbb{C}^*$, $F(0) = 0$ and $JF(0) = I_n$, there exists an integer $N$ such that for all $\lambda \in \mathbb{C}$ with $|\lambda| > N$ the dilation $\lambda F$ is global analytic linearizable to its linear part i.e. there exists an analytic automorphism $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ such that $\varphi^{-1}\lambda F \varphi = \lambda JF(0)X$. They observed that this conjecture, if true, implies the Jacobian Conjecture. However in [3] the authors showed that the conjecture is false for all $n \geq 4$ by giving a counterexample of the form $F = X + H$ where $H$ is homogeneous of degree five.

In spite of this counterexample Meisters conjectured (and offered a $200 reward until February 17, 1996) that every cubic linear map satisfies the linearization conjecture stated above (see [6] and also [5] page 85).

In this paper we show that also this conjecture is false. In fact we use the cubic linear map given by Drużkowski in [2, example 7.8], which was studied by Meisters in [4]. The method to prove that this map is a counterexample to the conjecture mentioned above is similar to the one used in [3].

1 Drużkowski’s cubic linear map

In [2] Drużkowski describes the following cubic linear polynomial automorphism $F = X - H(X_1, X_2, s_1, s_2, s_3)$ where $X = X_1, \ldots, X_{15}$ and

$$H(X_1, X_2, s_1, s_2, s_3) := (0, 0, s_1^3, s_2^3, s_3^2, (-X_1 + s_3)^3, (-X_2 + s_3)^3, (-X_1 + s_2)^3, (-X_1 + s_1)^3, (-X_2 + s_1)^3, (-X_2 + s_2)^3,$$
\((-X_1 - s_2)^3, (-X_2 - s_3)^3, (-X_2 - s_1)^3, (-X_1 - X_2 + s_3)^3, (-X_1 - X_2 + s_1)^3\)

and

\[
\begin{align*}
s_1 &:= 2X_4 + X_5 - X_6 - X_7 - X_8 + X_{11} + X_{14} \\
s_2 &:= X_3 + X_5 - \frac{1}{2}X_7 - \frac{1}{2}X_{10} + \frac{1}{2}X_{12} + \frac{1}{2}X_{13} \\
s_3 &:= -X_3 + 2X_4 - X_8 + X_9 + X_{10} + X_{11} - X_{15}
\end{align*}
\]

In the proof of the main theorem we need the inverse of \(F\). To describe this inverse we follow the notes of \([4]\), in which he describes the inverse by the following formulas. Define in \(\mathbb{Q}[Y] := \mathbb{Q}[Y_1, \ldots, Y_{13}]\) the polynomials

\[
\begin{align*}
g_1 &:= 2Y_4 + Y_5 - Y_6 - Y_7 - Y_8 + Y_{11} + Y_{14} - 3Y_1^2 Y_2 - 3Y_1 Y_2^2 \\
g_2 &:= Y_3 + Y_5 - \frac{1}{2}Y_7 - \frac{1}{2}Y_{10} + \frac{1}{2}Y_{12} + \frac{1}{2}Y_{13} \\
g_3 &:= -Y_3 + 2Y_4 - Y_8 + Y_9 + Y_{10} + Y_{11} - Y_1 5 + 3Y_1^2 Y_2 + 3Y_1 Y_2^2
\end{align*}
\]

Put

\[
B := \begin{pmatrix}
1 & 6Y_1^2 & -6Y_1 Y_2 \\
0 & 3Y_2^2 & 3Y_1^2 \\
6Y_1 Y_2 & 6Y_1^2 & 1
\end{pmatrix}
\]

Then \(\det(B) = 1\). So \(B \in \text{GL}_3(\mathbb{Q}[Y_1, Y_2])\). Finally put

\[
\begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix} := B^{-1} \begin{pmatrix}
g_1 \\
g_2 \\
g_3
\end{pmatrix}
\]

**Lemma 1.1 (Meisters, [4])** The inverse \(G\) of \(F\) is given by

\[
G(Y) = Y + H(Y_1, Y_2, S_1, S_2, S_3).
\]

**Proof.** Left to the reader. \(\Box\)

**Corollary 1.2**

\[
\begin{align*}
G_3(0, 0, Y_3, Y_4, 0, \ldots, 0) &= Y_3 + (2Y_4)^3 \\
G_4(0, 0, Y_3, Y_4, 0, \ldots, 0) &= Y_4 + Y_5^3
\end{align*}
\]

**Proof.** Just observe that \(B_{(0,0,Y_3,0,\ldots,0)} = I_3\). So \(S_i(0, 0, Y_3, Y_4, 0, \ldots, 0)\) equals \(g_i(0, 0, Y_3, Y_4, 0, \ldots, 0)\) for \(i = 1, 2, 3\), which implies the result. \(\Box\)
2 The main theorem

Theorem 2.1 Let $F$ be as described in section 1. Then for all $\lambda > 0 \lambda F$ is not global analytic linearizable to its linear part $\lambda X$.

Proof. 1. First we consider the case that $0 < \lambda < 1$. Suppose that $\lambda F$ is analytic linearizable. Then it is well-known (cf.[3, lemma 2.2]) that 0 is a global attractor of $\lambda F$. Therefore let $a > 0$ be such that $a\lambda > 1$. Denote the $i$-th component of $(\lambda F)^k$ by $(\lambda F)^{k}_i$. Observe that

$$
(\lambda F)_3(0,0,X_3,0,0,X_6,0,\ldots,0) = X_3 + X_6^3 \\
(\lambda F)_4(0,0,X_3,0,0,X_6,0,\ldots,0) = X_6 + X_3^3
$$

So if we put $A := (0,0,a,0,0,a,0,\ldots,0)$ then it follows from lemma 2.2 below that $\lim_{n \to \infty} (\lambda F)^n_3(A) = \infty$, so 0 is not a global attractor of $\lambda F$.

2. Now let $\lambda > 1$ and suppose that $\lambda F$ is analytic linearizable. Then $(\lambda F)^{-1} = F^{-1} \circ \lambda^{-1}$ is also analytic linearizable. Put $\mu := \lambda^{-1}$ and $G := F^{-1}$. So $G \circ \mu$ is analytic linearizable and again we conclude that 0 is a global attractor of $G \circ \mu$ (since $0 < \mu < 1$). Let $a > 0$ be such that $a\mu > 1$. Put $A' := (0,0,a,a,0,\ldots,0)$. Since $\lim_{n \to \infty} (G \circ \mu)^n(G(A')) = 0$ and $(G \circ \mu)^n(G(A')) = \mu^{-1}(\mu G)^{n+1}(A')$ it follows that $\lim_{n \to \infty} (\mu G)^n(A') = 0$. However by corollary 1.2 and lemma 2.2 below we deduce that $\lim_{n \to \infty} (\mu G)_3^n(A') = \infty$, a contradiction.

So also for $\lambda > 1$, $\lambda F$ is not analytic linearizable. Since obviously for $\lambda = 1$ $\lambda F$ is not linearizable to its linear part, the proof of the theorem is complete. 

\[\Box\]

Lemma 2.2 Let $c \geq 1$, $0 < \lambda < 1$ and $a > 0$ such that $a\lambda > 1$. Let $f = (f_1,f_2) \in \mathbb{R}[X,Y]^2$ given by $f_1 = X + cY^3$ and $f_2 = Y + X^3$. Then

$$
(\lambda f)_i^n(a,a) > a^{p[n]}
$$

for $i = 1, 2$ and for all $n \geq 1$ where $p(1) = 1$ and $p(n + 1) = 3p(n) - 1$, for all $n \geq 1$.

Proof. By induction on $n$. 

\[\Box\]

References


