A counterexample to Meisters' cubic-linear linearization conjecture

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Abstract

We show that the cubic linear polynomial automorphism $F : \mathbb{C}^5 \to \mathbb{C}^5$ given by Drużkowski in [2] has the property that for every $\lambda > 0$ the dilation $\lambda F$ is not global analytic linearizable to its linear part.

Introduction

In [1] Deng, Meisters and Zampieri proposed a new approach to prove the Jacobian Conjecture. They conjectured that for every polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ with $\det(JF) \in \mathbb{C}^\times$, $F(0) = 0$ and $JF(0) = I_n$, there exists an integer $N$ such that for all $\lambda \in \mathbb{C}$ with $|\lambda| > N$ the dilation $\lambda F$ is global analytic linearizable to its linear part i.e. there exists an analytic automorphism $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ such that $\varphi^{-1}\lambda F \varphi = \lambda JF(0)X$. They observed that this conjecture, if true, implies the Jacobian Conjecture. However in [3] the authors showed that the conjecture is false for all $n \geq 4$ by giving a counterexample of the form $F = X + H$ where $H$ is homogeneous of degree five.

Inspite of this counterexample Meisters conjectured (and offered a $200 reward until February 17, 1996) that every cubic linear map satisfies the linearization conjecture stated above (see [6] and also [5] page 85).

In this paper we show that also this conjecture is false. In fact we use the cubic linear map given by Drużkowski in [2, example 7.8], which was studied by Meisters in [4]. The method to prove that this map is a counterexample to the conjecture mentioned above is similar to the one used in [3].

1 Drużkowski’s cubic linear map

In [2] Drużkowski describes the following cubic linear polynomial automorphism $F = X - H(X_1, X_2, s_1, s_2, s_3)$ where $X = X_1, \ldots, X_15$ and

$$H(X_1, X_2, s_1, s_2, s_3) := (0, 0, s_1^3, s_2^3, s_3^3, (-X_1 + s_3)^3, (-X_2 + s_3)^3, (-X_1 + s_2)^3, (-X_1 + s_1)^3, (-X_2 + s_1)^3, (-X_2 + s_2)^3, \ldots)$$
Lemma (Meisters, [4]) The inverse $G$ of $F$ is given by

$$G(Y) = Y + H(Y_1, Y_2, S_1, S_2, S_3).$$

Proof. Left to the reader. \qed

Corollary 1.2

$$G_3(0, 0, Y_3, Y_4, 0, \ldots, 0) = Y_3 + (2Y_4)^3$$

$$G_4(0, 0, Y_3, Y_4, 0, \ldots, 0) = Y_4 + Y_3^3$$

Proof. Just observe that $B_{(0,0,Y_3,Y_4,0,\ldots,0)} = I_3$. So $S_i(0, 0, Y_3, Y_4, 0, \ldots, 0)$ equals $g_i(0, 0, Y_3, Y_4, 0, \ldots, 0)$ for $i = 1, 2, 3$, which implies the result. \qed
2 The main theorem

Theorem 2.1 Let $F$ be as described in section 1. Then for all $\lambda > 0 \lambda F$ is not global analytic linearizable to its linear part $\lambda X$.

Proof. 1. First we consider the case that $0 < \lambda < 1$. Suppose that $\lambda F$ is analytic linearizable. Then it is well-known (cf. [3, lemma 2.2]) that $0$ is a global attractor of $\lambda F$. Therefore let $a > 0$ be such that $a\lambda > 1$. Denote the $i$-th component of $(\lambda F)^k$ by $(\lambda F)^k_i$. Observe that

$$(\lambda F)_3(0,0,X_3,0,0,X_6,0,\ldots,0) = X_3 + X_6^3$$

$$(\lambda F)_4(0,0,X_3,0,0,X_6,0,\ldots,0) = X_6 + X_3^3$$

So if we put $A := (0,0,a,0,0,a,\ldots,0)$ then it follows from lemma 2.2 below that $\lim_{n \to \infty} (\lambda F)^n_3(A) = \infty$, so $0$ is not a global attractor of $\lambda F$.

2. Now let $\lambda > 1$ and suppose that $\lambda F$ is analytic linearizable. Then $(\lambda F)^{-1} = F^{-1} \circ \lambda^{-1}$ is also analytic linearizable. Put $\mu := \lambda^{-1}$ and $G := F^{-1}$. So $G \circ \mu$ is analytic linearizable and again we conclude that $0$ is a global attractor of $G \circ \mu$ (since $0 < \mu < 1$). Let $a > 0$ be such that $a\mu > 1$. Put $A' := (0,0,a,a,0,\ldots,0)$. Since $\lim_{n \to \infty} (G \circ \mu)^n(G(A')) = 0$ and $(G \circ \mu)^n(G(A')) = \mu^{-1}(\mu G)^{n+1}(A')$ it follows that $\lim_{n \to \infty} (\mu G)^n(A') = 0$. However by corollary 1.2 and lemma 2.2 below we deduce that $\lim_{n \to \infty} (\mu G)^n_3(A') = \infty$, a contradiction. So also for $\lambda > 1$, $\lambda F$ is not analytic linearizable. Since obviously for $\lambda = 1$ $\lambda F$ is not linearizable to its linear part, the proof of the theorem is complete.

Lemma 2.2 Let $c \geq 1$, $0 < \lambda < 1$ and $a > 0$ such that $a\lambda > 1$. Let $f = (f_1,f_2) \in \mathbb{R}[X,Y]^2$ given by $f_1 = X + cY^3$ and $f_2 = Y + X^3$. Then

$$(\lambda f)_i^n(a,a) > a^{p(n)}$$

for $i = 1,2$ and for all $n \geq 1$ where $p(1) = 1$ and $p(n+1) = 3p(n) - 1$, for all $n \geq 1$.

Proof. By induction on $n$. \qed

References


