MATHIEU SUBSPACES OF UNIVARIATE POLYNOMIAL ALGEBRAS

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Abstract. We first give a characterization for Mathieu subspaces of univariate polynomial algebras over fields in terms of their radicals. We then deduce that for some classes of classical univariate orthogonal polynomials the Image Conjecture is true. We also prove two special cases of the one-dimensional Image Conjecture for univariate polynomial algebras $A[t]$ over commutative $\mathbb{Q}$-algebras A.

1. Introduction

The Jacobian Conjecture has been the subject of much research over the last seven decades (see [K], [BCW] and [E1]). Various subcases of this still mysterious conjecture have been verified. Also, several attempts have been made to generalize the conjecture. However, most of these attempts failed. One of these attempts, which is still fully alive, is *the Mathieu Conjecture* posed by Olivier Mathieu [Ma] in 1995. More recently, based on a symmetric reduction of *the Jacobian conjecture* obtained independently by M. de Bondt and the first author [BE] and G. Meng [Me], the second author gave in [Z1] a new equivalent formulation of *the Jacobian Conjecture*. This new formulation is called *the Vanishing Conjecture* and was in turn generalized later in [Z2] to the so-called *Generalized Vanishing Conjecture*. Very recently, the second author posed in [Z3] an even stronger conjecture, namely, *the Image Conjecture*.

In a subsequent paper [Z4], both the *Image* and the *Mathieu* conjectures were embedded in a general framework, by introducing the notion of Mathieu subspaces of rings or algebras. This notion forms a generalization of the notion of ideals. Both *the Image Conjecture* and

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the Mathieu conjecture can be re-expressed as saying that certain subspaces are Mathieu subspaces of suitable algebras. For a full account and background of these new conjectures the reader is referred to the recent survey paper [E2].

Furthermore, the two-dimensional *Jacobian Conjecture* can also be expressed directly in terms of Mathieu subspaces: it has been shown in [EWZ1] that this conjecture is actually equivalent to saying that the image of any derivation of the polynomial algebra $\mathbb{C}[x, y]$ in two variables, whose divergence is zero, is a Mathieu subspace of $\mathbb{C}[x, y]$, if its image contains the constant polynomial 1.

The above connections make it clear that it is very desirable to get a better understanding of Mathieu subspaces in some general setting. This paper continues the study of these subspaces, as initiated by the second author in the papers [Z3]-[Z5].

Now we give a brief description of the content of this paper. The results discussed here mainly concern Mathieu subspaces of univariate polynomial algebras over fields or UFDs. First, in section 2 we recall the definition of a Mathieu subspace and that of the radical of an arbitrary subspace. We also describe some of their basic properties, which will be needed later in the paper.

In section 3, for commutative algebras A over fields k, we recall some results obtained in [Z5] concerning the radicals of k-subspaces of A, whose elements are algebraic over k. In section 4, we use these results to derive a characterization for Mathieu subspaces of the univariate polynomial algebra $k[t]$ over k (see Theorem 4.1). Some consequences of this characterization are also discussed in this section.

In section 5, we use the characterization derived in section 4 to obtain information about *the Integral Conjectures* (see Conjecture 5.1) posed by the second author in [Z4]. One of these conjectures, the *Image Conjecture* for the univariate Hermite and Jacobi orthogonal polynomials (see Conjecture 5.2), is studied in detail in section 6. Finally, in section 7 we prove two special cases of *the Image Conjecture* for the univariate polynomial algebra $A[t]$ over a commutative $\mathbb{Q}$-algebra A (see Theorems 7.1 and 7.6).

2. **Some Basic Properties of Mathieu Subspaces and Their Radicals**

We first recall the notions of *radicals* and *Mathieu subspaces* introduced respectively in [Z5] and [Z4] by the second author. Throughout this section R will denote a unital commutative ring and A a commutative algebra over R.

For any subset V of A, we define the *radical* of V, denoted by $\mathfrak{r}(V)$, to be the subset of all $a \in A$ such that $a^m \in V$ for all large m. In other words, a belongs to $\mathfrak{r}(V)$ if and only if there exists $N \in \mathbb{N}$ such that $a^m \in V$ for all $m \geq N$. Note that if V is an ideal of A, then the radical $\mathfrak{r}(V)$ of V is just the radical ideal of V. In particular, $\mathfrak{r}(V)$ itself in this case is a radical ideal of A.

Definition 2.1. *A subset V of A is said to be a Mathieu subspace of A if V is an R-subspace or R-submodule of A and for any $a \in \mathfrak{r}(V)$, the following holds: for any $b \in A$, there exists $N \in \mathbb{N}$ (depending on both a and b) such that $a^m b \in V$ for all $m \geq N$.*

Note that the above definition of Mathieu subspaces is slightly different from the one given in [Z4] and [Z5]. But, as shown in Proposition 2.1 in [Z5] these two definitions are actually equivalent to each other.

Note also that any ideal of A is automatically a Mathieu subspace of A. Therefore, the notion of Mathieu subspaces provides a natural generalization of the notion of ideals.

To study radicals of arbitrary R-subspaces V of A, we consider I_V, the *largest* ideal of A contained in V. Since V contains the zero-ideal, it is easy to see that I_V always exists and is actually the sum of all ideals of A contained in V. In particular, when V itself is an ideal of A, we have that $I_V = V$.

We denote by $\bar{A}$ the quotient algebra A/I_V and by $\pi : A \to \bar{A}$ the quotient map from A to $\bar{A}$. We also let $\bar{a}$ and $\overline{V}$ denote $\pi(a)$ and $\pi(V)$, respectively.

The first result of this section provides a different point of view for the fact that I_V is the largest ideal contained in V.

Lemma 2.2. *Let V be an R-subspace of A. Then for the R-subspace $\overline{V}$ of the quotient algebra $\bar{A}$, we have $I_{\overline{V}} = 0$.*

Proof: Assume otherwise. Then there exists some nonzero element $\bar{a} \in \bar{A}$ such that $\bar{a}\bar{A} \subset I_{\overline{V}} \subset \overline{V}$. Hence $aA + I_V \subset V + I_V = V$, for $I_V \subset V$. Since $\bar{a}$ is nonzero, we have $a \notin I_V$, whence the ideal $aA + I_V$ is strictly larger than I_V. But this contradicts the maximal choice of I_V. $\square$

The next result shows that radicals correspond nicely under the algebra homomorphism $\pi : A \to \bar{A}$.

Proposition 2.3. *i)* $\mathfrak{r}(V) = \pi^{-1}(\,\mathfrak{r}(\pi(V))\,)$.

ii) $\pi(\mathfrak{r}(V)) = \mathfrak{r}(\pi(V))$.

iii) $\mathfrak{r}(V)$ *is an ideal of* A *if and only if* $\mathfrak{r}(\pi(V))$ *is an ideal of* $\bar{A}$.

Proof: Let $a \in \mathfrak{r}(V)$. Then $a^m \in V$ for all large m. Applying the R-algebra homomorphism π to a^m gives the inclusions $\subset$ in both *i*) and *ii*).

Now let $\pi(a) \in \mathfrak{r}(\pi(V))$. Then $\pi(a^m) = \pi(a)^m \in \pi(V)$ for all large m, say, $\pi(a^m) = \pi(v_m)$ for some v_m in V. Then $a^m - v_m$ belongs to the kernel of π, which is equal to I_V. Since I_V is contained in V and V is an R-subspace, we have that $a^m \in V$ for all large m. Hence we have $a \in \mathfrak{r}(V)$, which gives respectively the inclusions $\supset$ in *i*) and *ii*).

Finally, *iii*) follows readily from *i*) and *ii*) and the surjectivity of the R-algebra homomorphism π. $\square$

Furthermore, Mathieu subspaces also behave nicely under the algebra homomorphism π, as can be seen from the next proposition whose proof is straightforward and is left to the reader (or see Proposition 2.7 in [Z5] for a more general statement).

Proposition 2.4. *V is a Mathieu subspace of A if and only if $\overline{V} = \pi(V)$ is a Mathieu subspace of $\bar{A}$.*

Remark 2.5. *In the proofs of Propositions 2.3 and 2.4 given above we only use the fact that I_V is an ideal contained in V. Therefore, the same proof gives the following more general result: if f is a surjective R-algebra homomorphism from A to B, and V is an R-subspace of A such that the kernel of f is contained in V, then with $\bar{A}$ replaced by B and π by f, the statements of Propositions 2.3 and 2.4 also hold.*

From now on, except in section 7, the last section of the paper, we assume: k is a field, A a k-algebra and V a k-subspace of A. The importance of Propositions 2.3 and 2.4 comes from the fact that in various situations the quotient algebra $\bar{A}$ turns out to be algebraic over k, i.e., each element of $\bar{A}$ is a root of a nonzero univariate polynomial with coefficients in k. For example, this is the case when A is the univariate polynomial algebra $k[t]$ and the ideal I_V is nonzero. We will return to this situation in section 4. It is therefore natural to consider first the case that A is algebraic over k. This will be done in the next section.

3. **Mathieu Subspaces with Algebraic Radicals**

Suppose that V is a Mathieu subspace of A. Then what can be said about the structure of the radical $\mathfrak{r}(V)$ of V?

In this section we discuss this question under the additional assumption that all elements of the radical $\mathfrak{r}(V)$ are algebraic over k. The next result, first obtained in [Z5], asserts that the Mathieu subspaces, whose radicals are algebraic over k, are completely characterized by their radicals. More precisely, by Theorems 4.10 and 4.12 in [Z5] we have the following theorem.

Theorem 3.1. *If all elements of* $\mathfrak{r}(V)$ *are algebraic over* k*, then* V *is a Mathieu subspace of* A *if and only if* $\mathfrak{r}(V)$ *is an ideal in* A*. In this case, we also have* $\mathfrak{r}(V) = \mathfrak{r}(I_V)$*.*

Corollary 3.2. *Let* V*,* $\bar{A} = A/I_V$ *and* $\overline{V}$ *as before. Assume that all elements of* $\mathfrak{r}(\overline{V})$ *are algebraic over* k*. Then* V *is a Mathieu subspace of* A *if and only of* $\mathfrak{r}(V)$ *is an ideal of* A*. In this case, we also have* $\mathfrak{r}(V) = \mathfrak{r}(I_V)$*.*

Proof: First, the equivalence follows readily from Proposition 2.4, Theorem 3.1 and Proposition 2.3 *iii*). Second, by Theorem 3.1 and Lemma 2.2 we have $\mathfrak{r}(\overline{V}) = \mathfrak{r}(I_{\overline{V}}) = \mathfrak{r}(\overline{0})$. It then follows from Proposition 2.3 *i*) that $\mathfrak{r}(V) = \pi^{-1}(\mathfrak{r}(\overline{0}))$. Since the latter set is also equal to $\mathfrak{r}(I_V)$ (for $\text{Ker}\ \pi = I_V$), we obtain that $\mathfrak{r}(V) = \mathfrak{r}(I_V)$. $\square$

4. **A Characterization of Mathieu Subspaces of Univariate Polynomial Algebras over Fields**

Throughout this section k is a field and $A = k[t]$, the univariate polynomial algebra over k. Although, apart from the constant polynomials, the algebra A has no elements which are algebraic over k, we will show that in this case Mathieu subspaces can also be characterized by their radicals.

Theorem 4.1. *For any* k*-subspace* V *of* A*,* V *is a Mathieu subspace of* A *if and only if* $\mathfrak{r}(V) = \mathfrak{r}(I_V)$*.*

Proof: First, if the ideal I_V is nonzero, $\bar{A} = A/I_V$ is finite dimensional and hence algebraic over k. Then the theorem follows immediately from Corollary 3.2. So from now on we assume $I_V = 0$.

If $\mathfrak{r}(V) = \mathfrak{r}(I_V)$, then $\mathfrak{r}(V) = \mathfrak{r}(0) = 0$, since A has no zero-divisors. By Definition 2.1 V obviously is a Mathieu subspace of A.

Conversely, assume that V is a Mathieu subspace of A (with $I_V = 0$). We must show that $\mathfrak{r}(V) = \{0\}$. Assume the contrary and let $a \in A = k[t]$ be a nonzero element in $\mathfrak{r}(V)$. Then $a^m \in V$ for all large m. If the polynomial a has degree 0, it is a nonzero constant, and hence so are all a^m $(m \geq 1)$. It follows that V contains 1. Since V is a Mathieu subspace of A, it is easy to check (or see Lemma 4.5 in [Z4]) that in this case we have $V = A$. Hence we have $I_V = A$, which is a contradiction since $I_V = 0$. Therefore, we have $d := \deg a \geq 1$.

Since V is a Mathieu subspace of $A = k[t]$, there exists $N \geq 1$ such that $t^i a^m \in V$ for all $0 \leq i \leq d-1$ and all $m \geq N$. In particular, we have

(4.1) $\quad a^m h \in V$ *for all* $m \geq N$ *and all* $h \in k[t]$ *with* $\deg h \leq d-1$.

Now, let $m \geq N$. Since $I_V = 0$, the nonzero ideal $a^m A$ cannot be contained in V, so there exists a nonzero $b_m \in A$ of the lowest degree such that $a^m b_m \notin V$. Let b_{m_0} have the smallest degree amongst all the b_m $(m \geq N)$. Note that by the property in Eq. (4.1) we have $\deg b_{m_0} \geq d$. Furthermore, by the Euclidean division there exist q and r in A such that $\deg r \leq d-1$ and

$$b_{m_0} = qa + r. \tag{4.2}$$

Since $\deg b_{m_0} \geq d > \deg r$, it follows that $\deg qa = \deg b_{m_0}$. Since $\deg a \geq 1$, we deduce that $\deg q < \deg b_{m_0}$. Furthermore, by multiplying a^{m_0} to Eq. (4.2), we have

$$b_{m_0} a^{m_0} = q a^{m_0+1} + r a^{m_0}. \tag{4.3}$$

Now, by the choices of b_m's with $m = m_0$, the left hand side of the equation above does not belong to V. By the property in Eq. (4.1), we see that $ra^{m_0} \in V$. Hence, $qa^{m_0+1} \notin V$. But then, by the choices of b_m's with $m = m_0 + 1$, we obtain that $\deg b_{m_0+1} \leq \deg q < \deg b_{m_0}$, which contradicts the choice of b_{m_0}, since b_{m_0} has the least degree among all the b_m $(m \geq N)$. $\square$

One immediate consequence of Theorem 4.1 is the following necessary condition for a k-subspace V of the univariate polynomial algebra $A = k[t]$ to be a Mathieu subspace of $k[t]$.

Corollary 4.2. *Let V be a k-subspace of $A = k[t]$. Assume that V is a Mathieu subspace of A. Then $\mathfrak{r}(V)$ is a radical ideal of A.*

Another consequence of Theorem 4.1 is that for Mathieu subspaces V of $A = k[t]$, the integer N in Definition 2.1 can actually be chosen in a way that depends only on the element $a \in \mathfrak{r}(a)$, i.e., N can be chosen to be independent with the element $b \in A$ in Definition 2.1. More precisely, we have the following corollary.

Corollary 4.3. *Let V be a k-subspace of $A = k[t]$. Then V is a Mathieu subspace of A if and only if for any $a \in \mathfrak{r}(V)$, there exists $N \geq 1$ such that for all $b \in A$, we have $a^m b \in V$ for all $m \geq N$.*

Proof: The ($\Leftarrow$) part is obvious. To show the ($\Rightarrow$), assume that V is a Mathieu subspace of A and let $a \in \mathfrak{r}(V)$. Then by Theorem 4.1, we have $a \in \mathfrak{r}(I_V)$. Since I_V is an ideal of A, there exists $N \geq 1$ such that

$a^m \in I_V$ and hence $a^m A \subset I_V$ for all $m \geq N$. Consequently, for any $b \in A$, we have $a^m b \in I_V \subset V$ for all $m \geq N$, whence the corollary follows. $\square$

Finally, we conclude this section with the following example, which shows that Theorem 4.1 does not hold in general for polynomial algebras in two or more variables.

Example 4.4. *Let $B = \mathbb{C}[x, y]$, $D = x\partial_x - y\partial_y$ and $V = \operatorname{Im} D = D(\mathbb{C}[x, y])$, i.e., the image of the derivation D. Then as shown in the proof of Lemma* 3.4 *in* [EWZ1], *V is a Mathieu subspace of B and $\mathfrak{r}(V) = W_1 \cup W_2$, where W_1 is the $\mathbb{C}$-span of all the monomials $x^i y^j$ with $i < j$ and W_2 is the $\mathbb{C}$-span of all the monomials $x^i y^j$ with $i > j$. So $\mathfrak{r}(V)$ is not even a $\mathbb{C}$-subspace of B and hence, not an ideal of B.*

5. **Integral and Image Conjectures in Dimension One**

The aim of this section is to show how the results of the previous section can be used to obtain some new results concerning several conjectures of the second author posed in [Z4]. To keep this paper as much self-contained as possible, here we briefly recall these conjectures for the one-dimensional case.

Conjecture 5.1. (**Integral Conjecture**) *Let $B \subset \mathbb{R}$ be an open subset and σ a positive measure such that $\int_B g(t)d\sigma$ exists and is finite for all $g \in \mathbb{C}[t]$. Set*

$$V_B(\sigma) := \Big\{ f \in \mathbb{C}[t] \,\Big|\, \int_B f \, d\sigma = 0 \Big\}. \tag{5.1}$$

Then $V_B(\sigma)$ is a Mathieu subspace of $\mathbb{C}[t]$.

In [Z4] this conjecture is proved for several special cases. One of them is the case when σ is an *atomic measure* supported at finitely many points r_i in B, i.e., $\sigma(r_i) > 0$ for each i and, for any subset $U \subset B$, $\sigma(U)$ is the sum of $\sigma(r_i)$ over all the r_i's that are contained in U. It is proved in Proposition 3.11, [Z4] that $V_B(\sigma)$ in this case is a Mathieu subspace of $\mathbb{C}[t]$ by showing that $\mathfrak{r}(V_B(\sigma))$ is the ideal of all the polynomials vanishing at all r_i's.

To describe the second conjecture, we need to recall some results on univariate orthogonal polynomials. Let B be an open interval of $\mathbb{R}$ and $w(t)$ a so-called *weight function* on B, i.e., $w(t)$ is non-negative over B and its integral over B is finite and positive. To such a pair (B, w) one can associate a *Hermitian inner product* on $\mathbb{C}[t]$ by defining

$$\langle f, g\rangle = \int_B f(t)\bar{g}(t)\, w(t)dt, \tag{5.2}$$

where $\bar{g}(t)$ is the complex conjugate of the polynomial of $g(t)$, i.e., the polynomial obtained by taking the complex conjugates of the coefficients of $g(t)$.

Applying the Gram-Schmidt process to the standard basis $1, t, t^2, \cdots$ of $\mathbb{C}[t]$, we obtain a set of orthogonal polynomials of $\mathbb{C}[t]$. Making the following special choices for B and $w(t)$, we get the following *classical univariate orthogonal polynomials*.

1) The *Hermite polynomials*: $B = \mathbb{R}$ and $w(t) = e^{-t^2}$.
2) The *generalized Laguerre polynomials*: $B = (0, \infty)$ and $w(t) = t^\alpha e^{-t}$ with $\alpha > -1$.
3) The *Jacobi polynomials*: $B = (-1, 1)$ and $w(t) = (1-t)^\alpha(1+t)^\beta$ with $\alpha, \beta > -1$.

In each of the three cases above, we can define the differential operator

$$\Lambda = w^{-1} \circ \partial_t \circ w. \tag{5.3}$$

This gives respectively the following related operators:

$$\partial_t - 2t, \;\; \partial_t + (\alpha t^{-1} - 1), \;\; \partial_t - \alpha(1-t)^{-1} + \beta(1+t)^{-1}. \tag{5.4}$$

In [Z4] the second author makes the following conjecture.

Conjecture 5.2. (**Image Conjecture for classical orthogonal polynomials**) *Let Λ be as defined in Eq. (5.4) with α, $\beta > -1$. Set*

$$\mathrm{Im}'\Lambda := \mathbb{C}[t] \cap \Lambda(\mathbb{C}[t]). \tag{5.5}$$

Then $\mathrm{Im}'\Lambda$ is a Mathieu subspace of $\mathbb{C}[t]$.

Furthermore, the following result has also been proved in Lemma 2.5 c) and Proposition 3.3 in [Z4].

Proposition 5.3. *With the same notations as in Conjectures 5.1 and 5.2, we have*

i) $1 \in \mathrm{Im}'\Lambda$ *if and only if* $\mathrm{Im}'\Lambda = \mathbb{C}[t]$;
ii) *if* $1 \notin \mathrm{Im}'\Lambda$, *then* $\mathrm{Im}'\Lambda = V_B(\sigma)$. *Consequently, in this case Conjecture 5.1 holds for the pair (B, σ) with $d\sigma = wdt$ if and only if Conjecture 5.2 holds for the related differential operator Λ in Eq. (5.4).*

Next, we show that Conjectures 5.1 and 5.2 are actually respectively equivalent to the following two formally stronger conjectures.

Conjecture 5.4. (**Strong Integral Conjecture**) *With the same notations as in Conjecture 5.1, assume that σ is not an atomic measure supported at finitely many points. Then $\mathfrak{r}(V_B(\sigma)) = \{0\}$.*

In other words, the conjecture above claims that when the measure σ is not an atomic measure supported at finitely many points, the only polynomial f with $\int_B f^m d\sigma = 0$ $(m \geq 1)$ should be the zero polynomial.

Conjecture 5.5. (**Strong Image Conjecture for classical orthogonal polynomials**) *With the same notations as in Conjecture 5.2, assume $1 \notin \mathrm{Im}'\Lambda$. Then $\mathfrak{r}(\mathrm{Im}'\Lambda) = \{0\}$.*

Theorem 5.6. *i*) *The Integral Conjecture* (*Conjecture 5.1*) *is equivalent to the Strong Integral Conjecture* (*Conjecture 5.4*)*.*

ii) *The Image Conjecture for classical univariate orthogonal polynomials* (*Conjecture 5.2*) *is equivalent to the Strong Image Conjecture for classical univariate orthogonal polynomials* (*Conjecture 5.5*)*.*

Proof: *i*) Note first that if σ is an atomic measure supported at finitely many points, then by Proposition 3.3 in [Z4], Conjecture 5.1 holds. When σ is not an atomic measure supported at finitely many points, it is easy to see from Definition 2.1 that Conjecture 5.1 follows directly from Conjecture 5.4. Therefore, in any case the ($\Leftarrow$) part of statement *i*) holds.

To show the ($\Rightarrow$) part, put $V = V_B(\sigma)$. We claim that the largest ideal I_V of $\mathbb{C}[t]$ contained in V is equal to 0, which combining with Theorem 4.1 will imply Conjecture 5.4, for the polynomial algebra $\mathbb{C}[t]$ has no zero-divisors.

Assume the contrary and let $0 \neq f \in I_V$. Then $g := \bar{f}f \in I_V \subset V$ (where $\bar{f}$ denotes the complex conjugate of f), whence $g \in V$. Then by definition of V, the integral of g over B is equal to zero. On the other hand, since g is continues and positive over B (except at the finitely many zeroes of f in B) and σ is not an atomic measure supported at finitely many points, the integral of g over B is positive, which is a contradiction.

ii) Note first that if $1 \in \mathrm{Im}'\Lambda$, then by Proposition 5.3 *i*) we have $\mathrm{Im}'\Lambda = \mathbb{C}[t]$, which is obviously a Mathieu subspace of $\mathbb{C}[t]$. If $1 \notin \mathrm{Im}'\Lambda$, then it is easy to see from Definition 2.1 and Proposition 5.3 *ii*) that Conjecture 5.2 follows directly from Conjecture 5.5. Therefore, in any case the ($\Leftarrow$) part of statement *ii*) holds.

To show the ($\Rightarrow$) part, assume Conjecture 5.2 and $1 \notin \mathrm{Im}'\Lambda$. Then by Proposition 5.3 *ii*), we have $\mathrm{Im}'\Lambda = V_B(\sigma)$, where $d\sigma = w dt$. So the radicals of the two subspaces are equal. Since by our hypothesis $V_B(\sigma)$ is a Mathieu subspace of $\mathbb{C}[t]$, by Theorem 4.1 we have $\mathfrak{r}(\mathrm{Im}'\Lambda) =$

$\mathfrak{r}(V_B(\sigma)) = \mathfrak{r}(I_V)$, where $V = V_B(\sigma)$ as above. But, as shown in the proof of statement *i*) above, we also have $I_V = 0$ and $\mathfrak{r}(I_V) = \{0\}$. Hence $\mathfrak{r}(\mathrm{Im}'\Lambda) = \{0\}$, as desired. $\square$

6. Some Cases of The Strong Image Conjecture for the Hermite and Generalized Laguerre Polynomials

In this section, we prove some cases of the Strong Image Conjecture (**SIC**), Conjecture 5.5, and also of the following conjecture, which is the one dimensional case of *the Image Conjecture* posed by the second author in [Z3].

Conjecture 6.1. *Let A be any $\mathbb{Q}$-algebra, $c \in A$ and $a(t) \in A[t]$. Set $D := c\partial_t - a(t)$ and $\mathrm{Im}\, D := D(A[t])$. Then $\mathrm{Im}\, D$ is a Mathieu subspace of $A[t]$.*

The main result of this section is the following theorem.

Theorem 6.2. *Let $d \in \mathbb{N}$ and $\alpha \in \mathbb{Q}$ such that $\alpha \notin -(1 + (d+1)\mathbb{N})$ and $(d, \alpha) \neq (0, 0)$. Let $D = \partial_t + \alpha t^{-1} - t^d$. Then $\mathfrak{r}(\mathrm{Im}'D) = \{0\}$, where $\mathrm{Im}'D := \mathbb{C}[t] \cap D(\mathbb{C}[t])$.*

One consequence of the theorem above is the following corollary on the case of the **SIC** for the Hermite and generalized Laguerre polynomials.

Corollary 6.3. *The* **SIC** *holds for the Hermite polynomials and the generalized Laguerre polynomials with $\alpha \in \mathbb{Q}$ (and $\alpha > -1$).*

Proof: For the Hermite polynomial case, we make the variable change $t = \sqrt{2}s$. Since $\partial_s + \alpha s^{-1} - 2s = \sqrt{2}(\partial_t + \alpha t^{-1} - t)$, by Theorem 6.2 with $d = 1$ we see that the **SIC** holds in this case.

The generalized Laguerre polynomial case follows from Theorem 6.2 by taking $d = 0$, in case $\alpha \neq 0$. If $\alpha = 0$, then $D = \partial_t - 1$, which is an invertible map with the inverse $D^{-1} = -\sum_{k\geq 0} \partial_t^k$. Hence $\mathrm{Im}'D = \mathbb{C}[t]$. In particular, $1 \in \mathrm{Im}'D$ and the condition of the **SIC** does not apply in this case. But, since $\mathbb{C}[t]$ is obviously a Mathieu subspace of $\mathbb{C}[t]$, we see that Conjecture 5.2 still holds in this case. $\square$

Another consequence of Theorem 6.2 is the following special case of Conjecture 6.1.

Corollary 6.4. *Conjecture 6.1 holds for the case that $A = \mathbb{C}$ and $a(t) = \lambda t^d$ for all $\lambda \in \mathbb{C}$ and $d \geq 0$.*

Proof: Note that in this case $D = c\partial_t - \lambda t^d$ is a differential operator of $\mathbb{C}[t]$ itself. Hence we have $\mathrm{Im}'D = \mathrm{Im}\,D$. If $c = 0$, then $D = -\lambda t^d$ and $\mathrm{Im}\,D$ is the ideal of $\mathbb{C}[t]$ generated by λt^d. If $c \neq 0$ but $\lambda = 0$, then $D = c\partial_t$, whence $\mathrm{Im}\,D = \mathbb{C}[t]$. Therefore, in either of these two cases, $\mathrm{Im}\,D$ is an ideal of $\mathbb{C}[t]$ and hence, is also a Mathieu subspace of $\mathbb{C}[t]$.

Assume that both c and λ are nonzero. If $d = 0$, then $D = c\partial_t - \lambda$, which is invertible with the inverse $D^{-1} = -\sum_{i\geq 0} \lambda^{-i+1} c^i \partial_t^i$. Hence $\mathrm{Im}\,D = \mathbb{C}[t]$, which is obviously a Mathieu subspace of $\mathbb{C}[t]$. Therefore, Conjecture 6.1 also holds in this case.

So, we may further $d \geq 1$. Let $\beta \in \mathbb{C}$ such that $\beta^{d+1} = c/\lambda$. By changing of variables $t = \beta s$, it is easy to check that $D = c\beta^{-1}(\partial_s - s^d)$. Then by the fact that $\mathrm{Im}'D = \mathrm{Im}\,D$ (as mentioned above), Conjecture 6.1 follows immediately from Theorem 6.2 (for the different operator $\partial_s - s^d$) and Definition 2.1. $\square$

Next, we give a proof for Theorem 6.2 starting with the following observations.

First, since $Dt^n = (n+\alpha)t^{n-1} - t^{d+n}$ for all $n \geq 1$, we have

$$t^{n+d} \equiv (n+\alpha)\, t^{n-1} \pmod{\mathrm{Im}'D}. \tag{6.1}$$

Applying the relation above repeatedly, it follows that for any $k \geq d+1$, we have

$$t^k \equiv c_k\, t^i \pmod{\mathrm{Im}'D}, \tag{6.2}$$

where $0 \leq i \leq d$ with $i \equiv k \mod (d+1)$ and c_k is given by

$$c_k = \big(k - (d+1) + 1 + \alpha\big)\big(k - 2(d+1) + 1 + \alpha\big) \cdots \big(i + 1 + \alpha\big).$$

Therefore, we can define a $\mathbb{C}$-linear map $\mathcal{L}$ from $\mathbb{C}[t]$ to the vector subspace S of polynomials of degree $\leq d$, by setting $\mathcal{L}(t^k) = c_k t^i$ for all $k \geq d+1$, and $\mathcal{L}(t^j) = t^j$ for all $0 \leq j \leq d$. Then $\mathcal{L}$ has the property that for any $h \in \mathbb{C}[t]$, $h \in \mathrm{Im}'D$ if and only if $\mathcal{L}(h) \in S \cap \mathrm{Im}'D$.

Furthermore, we also define the $\mathbb{C}$-linear functional $\mathcal{L}_0 : \mathbb{C}[t] \to \mathbb{C}$ by setting $\mathcal{L}_0(h) := \mathcal{L}(h)(0)$ for all $h \in \mathbb{C}[t]$, i.e., we set $\mathcal{L}_0(h)$ to be the constant term of the polynomial $\mathcal{L}(h) \in S$.

Lemma 6.5. *Let $d \in \mathbb{N}$, $\alpha \in \mathbb{C}$ and $D = \partial_t + \alpha t^{-1} - t^d$. Assume $(d, \alpha) \neq (0, 0)$. Then we have*

$$\mathrm{Im}'D \subseteq \mathrm{Ker}\ \mathcal{L}_0. \tag{6.3}$$

Proof: First, if $\alpha \neq 0$, then $\alpha t^{-1} - t^d = D \cdot 1 \notin \mathrm{Im}'D$. It is easy to see that in this case every nonzero element of $\mathrm{Im}'D$ has degree at least $d+1$, whence $S \cap \mathrm{Im}'D = \{0\}$. If $\alpha = 0$, then $d \geq 1$ by the assumption of the theorem. Since in this case $t^d = D(-1) \in \mathrm{Im}'D$, it is easy to see that $S \cap \mathrm{Im}'D = \mathbb{C}\, t^d$, the one-dimensional subspace spanned by t^d.

Hence, in any case we have that $f(0) = 0$ for all $f \in S \cap \mathrm{Im}'D$. Since for any $h \in \mathbb{C}[t]$, $h \in \mathrm{Im}'D$ if and only if $\mathcal{L}(h) \in S \cap \mathrm{Im}'D$, we see that the lemma follows. $\square$

Second, the $\mathbb{C}$-linear functional $\mathcal{L}_0$ can be described more explicitly by the next lemma. But, we need first to fix the following notation: for any $\alpha \in \mathbb{C}$ and positive integers $q, n \in \mathbb{N}$, we set

$$[qn, n]_\alpha! := ((q-1)n + 1 + \alpha)((q-2)n + 1 + \alpha) \cdots (1 + \alpha). \tag{6.4}$$

Furthermore, for convenience we also set (for the case $q = 0$)

$$[0,\, n]_\alpha! := 1 \tag{6.5}$$

for all $\alpha \in \mathbb{C}$ and integers $n \geq 1$.

Note that if $\alpha \notin -(1 + (d+1)\mathbb{N})$, then it follows from Eq. (6.4) that $[q(d+1),\, (d+1)]_\alpha \neq 0$ for all $q \in \mathbb{N}$.

Lemma 6.6. *Let d and α be as in Theorem 6.2. Then for any $0 \leq i \leq d$ and $q \in \mathbb{N}$, we have*

$$\mathcal{L}_0\big(t^{q(d+1)+i}\big) = \begin{cases} 0, & \text{if } i > 0, \\ [q(d+1),\, d+1]_\alpha! & \text{if } i = 0. \end{cases} \tag{6.6}$$

Proof: First, by definition of $\mathcal{L}$ we have that $\mathcal{L}(t^{q(d+1)+i}) = c_i t^i$ for some $c_i \in \mathbb{C}$. Consequently, if $i > 0$, the constant term of $\mathcal{L}(t^{q(d+1)+i})$ is zero, which gives the first case of Eq. (6.6).

Second, by Eq. (6.1) with $n = (q-1)(d+1)+1$ we have

$$t^{q(d+1)} \equiv \big((q-1)(d+1) + 1 + \alpha\big)\, t^{(q-1)(d+1)} \pmod{\mathrm{Im}'D}.$$

Then by applying the induction on q, the second case of Eq. (6.6) also follows. $\square$

<u>*Proof of Theorem 6.2*</u>: Assume otherwise. We fix a nonzero $f(t) \in \mathfrak{r}(\mathrm{Im}'D)$ and derive a contradiction as follows.

First, we write

$$f = c_s t^s + c_{s+1} t^{s+1} + \cdots + c_N t^N \tag{6.7}$$

for some integers $s \leq N$ and $c_i \in \mathbb{C}$ $(s \leq i \leq N)$ with $c_s,\, c_N \neq 0$.

Note that we may obviously assume that $c_s = 1$. Moreover, we may also assume that $s \geq 1$. To see this, suppose that $s = 0$ and that we have already proved the $s \geq 1$ case. Let $m_0 \geq 1$ be such that $f^m \in \mathrm{Im}'D$ for all $m \geq m_0$ and set $g := f^{m_0} - f^{m_0+1}$. Then it is easy to see that $g(0) = 0$ (for $f(0) = 1$) and $g^m \in \mathrm{Im}'D$ for all $m \geq 1$. By our assumption on the $s \geq 1$ case, we have $g = 0$. Since $f \neq 0$, we deduce that $f = 1$, whence $1 \in \mathrm{Im}'D$. But this is obviously impossible, since $(d, \alpha) \neq (0, 0)$.

Next, by a similar reduction used by M. Boyarchenko in his unpublished proof (but see [FPYZ]) for the case of Conjecture 5.4 with $B = [0, 1] \subset \mathbb{R}$ and $d\sigma = dt$ (see also [EWZ2] or [E2] for a similar reduction), we may also assume that all the coefficients c_i's of f in Eq. (6.7) belong to some algebraic number field K. Then by a well-known result in algebraic number theory (e.g., see [W]), we know that for each prime $p \geq 2$, there exists at least one extension of the p-valuation $v_p(\cdot)$ of $\mathbb{Q}$ to K, which we will still denote by $v_p(\cdot)$, such that for all but finitely many prime numbers p, we have $v_p(c_i) \geq 0$ for all $s \leq i \leq N$.

Now we consider $f^{m(d+1)}$ $(m \geq 1)$. Note that for all large m this element belongs to $\mathrm{Im}'D$, and hence by Eq. (6.3) it also belongs to $\mathrm{Ker}\, \mathcal{L}_0$, i.e., $\mathcal{L}_0(f^{m(d+1)}) = 0$ for all large m. From our reductions on f, we have $s \geq 1$ and $c_s = 1$. Hence we also have

$$f^{m(d+1)} = t^{sm(d+1)} + \sum_{k \geq sm(d+1)+1} \phi_k t^k, \tag{6.8}$$

where the ϕ_k's are polynomials in the c_i's with integer coefficients.

Now apply $\mathcal{L}_0$ to Eq. (6.8) and observe that by Lemma 6.6 the only powers of t on the right hand side of the equation, which contribute to $\mathcal{L}_0(f^{m(d+1)})$, are the powers t^k with k divisible by $d+1$. So for all $m \gg 0$ we have

$$\mathcal{L}_0\big(t^{sm(d+1)}\big) + \sum_{i \geq 1} \phi_{(sm+i)(d+1)} \mathcal{L}_0\big(t^{(sm+i)(d+1)}\big) = 0. \tag{6.9}$$

Then by Eq. (6.6), we get

$$[sm(d+1),\, d+1]_\alpha! + \sum_{i \geq 1} [(sm+i)(d+1),\, d+1]_\alpha!\, \phi_{(sm+i)(d+1)} = 0.$$

Dividing by $[sm(d+1),\, d+1]_\alpha!$ from the equation above, we get

$$1 + \sum_{i \geq 1} b_i\, \phi_{(sm+i)(d+1)} = 0,$$

$$\sum_{i \geq 1} b_i\, \phi_{(sm+i)(d+1)} = -1, \tag{6.10}$$

where the coefficients b_i's are given by

$$b_i = \big((sm+i-1)(d+1)+1+\alpha\big) \cdots \big(sm(d+1)+1+\alpha\big). \tag{6.11}$$

Assume first that $\alpha \neq 0$. Note also that $\alpha \neq -1$, since by assumption $\alpha \notin -(1+(d+1)\mathbb{N})$. Write $\alpha = r/q$ for some nonzero integers such that $q \geq 1$ and $\gcd(r, q) = 1$. Then

$$b_i = \big((sm+i-1)q(d+1)+q+r\big)\cdots\big(smq(d+1)+q+r\big)/q^i. \tag{6.12}$$

Observe that q and the numerator of b_i have no common factor. We claim that for any large enough m, there exists a prime number p_m which divides the numerators of all the b_i's.

Observe first that $smq(d+1)+q+r$ divides the numerators of all the b_i's and that $q+r \neq 0$ (since $\alpha \neq -1$). Let $s_0 = \gcd(s(d+1), q+r)$, $s(d+1) = s_0 s_*$ and $q+r = s_0 h$. Hence $\gcd(s_*, h) = 1$. Then we have

$$smq(d+1)+q+r = (s_0 s_* q)m + s_0 h = s_0\big((s_*q)m + h\big). \tag{6.13}$$

Since $\gcd(s_*q, h) = 1$, it follows from Dirichlet's prime number theorem (e.g., see Theorem 66 and Corollary 4.1, p. 297 in [FT]) that there exists infinitely many $m \geq 1$ such that $p_m := (s_*q)m + h$ is a prime number. Note that by Eqs. (6.12) and (6.13) any such a prime number p_m divides the numerators of all the b_i's in Eq. (6.10).

Now choose and fix any such large enough $m \geq 1$, and write p_m as p for short, such that an extension $v_p(\cdot)$ of p-valuation of $\mathbb{Q}$ to the number field K satisfies that $v_p(c_i) \geq 0$ for all $s \leq i \leq N$. Then we also have $v_p(\phi_j) \geq 0$ for all ϕ_j's in Eq. (6.10). Since $v_p(b_i) > 0$ for all b_i's in Eq. (6.10), it follows that the v_p-valuation of the left hand side of Eq. (6.10) is positive. Consequently, from Eq. (6.10) we have $v_p(-1) > 0$. But, this is a contradiction since $v_p(-1) = 0$.

Finally, we consider the case $\alpha = 0$. Note that in this case by Eq. (6.11), all the nonzero b_i's in Eq. (6.10) are positive integers that are divisible by $sm(d+1)+1$. Then by Dirichlet's prime number theorem again, there exist infinitely many $m \geq 1$ such that $p_m := sm(d+1)+1$ is a prime. Applying the same argument as above we will get a contradiction again. Therefore, the theorem follows. □

Next we consider the case when the condition $\alpha \notin -(1+(d+1)\mathbb{N})$ in Theorem 6.2 fails.

Theorem 6.7. *Let $d \in \mathbb{N}$, $\alpha \in -(1+(d+1)\mathbb{N})$ and $D = \partial_t + \alpha t^{-1} - t^d$. Then we have*

i) The statement of Theorem 6.2 does not hold. More precisely, we have $t^{d+1} \in \mathfrak{r}(\mathrm{Im}' D)$.

ii) If $d \geq 1$, then $(t^{d+1})^m t = t^{(d+1)m+1} \notin \mathrm{Im}' D$ for all $m \geq 0$. So $\mathrm{Im}' D$ is not a Mathieu subspace of $\mathbb{C}[t]$.

iii) *If* $d = 0$*, then* $\mathrm{Im}'D$ *is a Mathieu subspace of* $\mathbb{C}[t]$ *and* $\mathfrak{r}(\mathrm{Im}'D) = t\,\mathbb{C}[t]$.

Proof: *i)* Let $\alpha = -(1+q(d+1))$ for some $q \in \mathbb{N}$. Then for each $m \geq 0$, by Eq. (6.1) with $n = (m+q)(d+1)+1$ we get

$$t^{(m+q+1)(d+1)} \equiv m(d+1)t^{(m+q)(d+1)} \pmod{\mathrm{Im}'D}. \tag{6.14}$$

In particular, by choosing $m = 0$ we see that $t^{(q+1)(d+1)} \in \mathrm{Im}'D$. Then from this fact and Eq. (6.14), it is easy to see that for any $k \geq q+1$, we have $t^{k(d+1)} \in \mathrm{Im}'D$, whence $t^{d+1} \in \mathfrak{r}(\mathrm{Im}'D)$.

ii) In a similar way, for each $m \geq 1$, by Eq. (6.1) with $n = (m-1)(d+1)+2$ we get

$$t^{m(d+1)+1} \equiv \big((m-1-q)(d+1)+1\big)t^{(m-1)(d+1)+1} \pmod{\mathrm{Im}'D}. \tag{6.15}$$

Since $d \geq 1$ and the factor appearing on the right hand side of the equation above is equivalent 1 modulo $d+1$, we see that this factor can not be equal to zero. It then follows by applying Eq. (6.15) repeatedly that $t^{(d+1)m+1} \equiv c\,t \pmod{\mathrm{Im}'D}$ for some nonzero $c \in \mathbb{C}$. On the other hand, under the assumptions $d \geq 1$ and $\alpha \in -(1+(d+1)\mathbb{N})$ it is easy to verify directly that $\alpha \neq 0$ and $t \notin \mathrm{Im}'D$. Hence statement *ii)* follows.

iii) Since $d = 0$, we have $\alpha = -(1+q)$. Then for any $m \geq 0$, by Eq. (6.1) with $d = 0$ and $n = m+1$ we have

$$t^{m+1} \equiv (m-q)t^m \pmod{\mathrm{Im}'D}. \tag{6.16}$$

Applying this equation repeatedly we see that $t^n \in \mathrm{Im}'D$ for all $n \geq q+1$. Hence, $t^{q+1}\mathbb{C}[t] \subseteq \mathrm{Im}'D$ and $t\,\mathbb{C}[t] \subseteq \mathfrak{r}(\mathrm{Im}'D)$.

On the other hand, since $\alpha \leq -1$ in this case, it is easy to check that $1 \notin \mathrm{Im}'D$. Then from Corollary 7.12 in [Z5] with $\mathcal{A} = \mathbb{C}[t]$ and the maximal ideal $\mathfrak{m} = (t)$ it follows that $\mathrm{Im}'D$ is indeed a Mathieu subspace of $\mathbb{C}[t]$. Moreover, by Corollary 4.2 we see that $\mathfrak{r}(\mathrm{Im}'D)$ is an ideal of $\mathbb{C}[t]$. Since $t\,\mathbb{C}[t] \subseteq \mathfrak{r}(\mathrm{Im}'D)$ (as pointed out above) and $1 \notin \mathfrak{r}(\mathrm{Im}'D)$ (for $1 \notin \mathrm{Im}'D$), we also have $\mathfrak{r}(\mathrm{Im}'D) = t\,\mathbb{C}[t]$. Hence, statement *iii)* follows. $\square$

To conclude this section we point out that the **SIC** also holds for the following special Jacobi polynomials. But, let's first recall the following results related with the univariate Jacobi polynomials.

Theorem 6.8. *Let* $B = (-1,1) \subset \mathbb{R}$*,* $w(t) = (1-t)^\alpha(1+t)^\beta$ $(\alpha, \beta > -1)$*,* $d\sigma = w(t)dt$ *and* $V_B(\sigma)$ *as in Eq. (5.1). Then we have*

i) *If* $\alpha, \beta \in \mathbb{N}$*, then* $\mathfrak{r}(V_B(\sigma)) = 0$.

ii) *If* $\alpha = \beta = \lambda - \frac{1}{2}$ *and* $\lambda \in \frac{1}{2}\mathbb{N}$*, then* $\mathfrak{r}(V_B(\sigma)) = 0$.

Proof: i) follows from Theorem 3.4 and Corollary 3.5 in [P]. *ii*) is exactly the content of Proposition 4.2 in [FPYZ]. $\square$

From the theorem above, we see that the *Strong Integral Conjecture* (Conjecture 5.4) holds for all the *Gegenbauer polynomials* (i.e., the Jacobi polynomials with $\alpha = \beta = \lambda - \frac{1}{2}$) with $\lambda \in \frac{1}{2}\mathbb{N}$. Hence, *Strong Integral Conjecture* also holds for the *Chebyshev polynomials of the first and the second kind*, i.e., the *Gegenbauer polynomials* with $\lambda = 0, 1$, respectively, and also for the *Legendre polynomials*, i.e., the *Gegenbauer polynomials* with $\lambda = \frac{1}{2}$.

To see whether or not the **SIC** also holds for the differential operator $D = \partial_t - \alpha(1-t)^{-1} + \beta(1+t)^{-1}$ $(\alpha, \beta > -1)$ related with the Jacobi orthogonal polynomials, we need first to show the following lemma.

Lemma 6.9. *Let* $D = \partial_t - \alpha(1-t)^{-1} + \beta(1+t)^{-1}$ *with* $\alpha, \beta > -1$. *Then* $1 \in \mathrm{Im}'D$ *if and only if* $\alpha = 0$ *or* $\beta = 0$.

Proof: The ($\Leftarrow$) part can be easily checked. For example, if $\alpha \neq 0$ and $\beta = 0$, we have $D(t-1) = (1+\alpha) \neq 0$, for $\alpha > -1$, whence $1 \in \mathrm{Im}'D$. The other case is similar.

Conversely, let $1 = D(h)$ for some $h \in \mathbb{C}[t]$. Assume otherwise, i.e., both α and β are nonzero. Then multiplying the equation $D(h) = 1$ by $1 - t^2$ we obtain

$$(1-t^2)\partial_t h - \alpha(1+t)h + \beta(1-t)h = 1 - t^2. \tag{6.17}$$

It follows from the equation above that both $1+t$ and $1-t$ divide h, so $h = (1-t^2)g$ for some $g \in \mathbb{C}[t]$. Substituting this equality into Eq. (6.17) and then dividing by $1 - t^2$ gives

$$(1-t^2)\partial_t g - (2t)g - \alpha(1+t)g + \beta(1-t)g = 1. \tag{6.18}$$

Now, write $g = c_d t^d +$ *lower order terms*, with $0 \neq c_d \in \mathbb{C}$. Then by comparing the coefficients of t^{d+1} on both sides of Eq. (6.18), we get

$$-(d+2) - (\alpha + \beta) = 0.$$

It then follows that $\alpha + \beta = -(d+2) \leq -2$, which is a contradiction since both α and β are greater than -1. $\square$

Now, from Proposition 5.3, Theorem 6.8 and Lemma 6.9 we immediately get the following corollary.

Corollary 6.10. *Let* $D = \partial_t - \alpha(1-t)^{-1} + \beta(1+t)^{-1}$ *such that* α *and* β *are not both zero. Then the following statements hold.*

i) *If* $\alpha, \beta \in \mathbb{N}$, *then the* **SIC** *holds for* D, *i.e.,* $\mathfrak{r}(\mathrm{Im}'D) = 0$.

ii) *If* $\alpha = \beta = \lambda - \frac{1}{2}$ *with* $\lambda \in \frac{1}{2}\mathbb{N}$, *then* $\mathfrak{r}(\mathrm{Im}'D) = 0$.

In particular, the **SIC** holds for the differential operators related with the *Gegenbauer polynomials* with $\lambda \in \frac{1}{2}\mathbb{N}$ and $\lambda \geq 1$, and also for differential operators related with the *Chebyshev polynomials of the first and the second kind*. For the *Legendre polynomials*, i.e., the case $\lambda = \frac{1}{2}$, we have $D = \partial_t$ and $1 \in \operatorname{Im}' D = \mathbb{C}[t]$, the condition of the **SIC** is not satisfied. But, the *Strong Integral conjecture* obviously still holds in this case (as already pointed before).

7. The One Dimensional Image Conjecture over a Commutative Ring

In this section, we study more special cases of Conjecture 6.1 for commutative $\mathbb{Q}$-algebras A. It has been shown in Theorem 2.8 in [EWZ2] that the conjecture holds under the assumptions that a is a non-zero-divisor and that aA is a radical ideal of A. We first show in the next theorem that when A is a UFD, this radical hypothesis can actually be dropped.

Theorem 7.1. *Let A be a UFD and a $\mathbb{Q}$-algebra. Let $a \in A$ and $D := \partial_t - a$. Then $\operatorname{Im} D$ is a Mathieu subspace of $A[t]$.*

Remark 7.2. *In case that A is an $\mathbb{F}_p$-algebra (not necessarily a UFD), by Theorem* 2.2 *in* [EWZ2] *the theorem above also holds provided the element $a \in A$ is not a zero-divisor. Furthermore, it has also been shown in Corollary* 2.6 *in* [EWZ2] *that for all $f \in A[t]$ with $f^p \in \operatorname{Im} D$, the p-th power of each coefficient of f belongs to aA. From this result and the fact that each monomial $a^p t^n = D^p((-1)^p t^n)$ belongs to* $\operatorname{Im} D$, *one deduces easily that $\mathfrak{r}(\operatorname{Im} D) = \mathfrak{r}(aA[t])$. In particular, $\mathfrak{r}(\operatorname{Im} D)$ in this case is a (radical) ideal of $A[t]$.*

To prove Theorem 7.1, note that the case $a = 0$ is trivial. So, from now on we assume $a \neq 0$ and derive first a lemma as follows.

We denote by $\mathcal{L}$ the A-linear map from $A[t]$ to A defined by $\mathcal{L}(t^n) = n!$ for all $n \geq 0$ and by $\mathfrak{r}(a)$ the radical ideal of aA. Since a is not a zero-divisor of A, it is easy to see that:

$$(*) \qquad \text{for any } b \in A,\ b \in \operatorname{Im} D \text{ if and only if } b \in aA$$

Lemma 7.3. *With the setting above, the following statements hold.*

i) for any $n \geq 0$, $a^n t^n \equiv n! \pmod{\operatorname{Im} D}$. Furthermore, $a^{n+1} t^n A$ is contained in $\operatorname{Im} D$ for all $n \geq 0$.

ii) Let $f = p(at)$ for some $p(t) \in A[t]$. Then $f \in \operatorname{Im} D$ if and only if $\mathcal{L}(p) \in aA$.

iii) Let f be as in ii). Then $f \in \mathfrak{r}(\operatorname{Im} D)$ if and only if all coefficients of $p(t)$ belong to $\mathfrak{r}(a)$.

Proof: *i*) The first statement follows by induction on $n \geq 1$ from the equality $D(a^{n-1}t^n) = na^{n-1}t^{n-1} - a^n t^n$. The second statement follows from the first one and the equivalence in $(*)$.

ii) By *i*) and the definition of the A-linear map $\mathcal{L}$, we have that $f = p(at) \in \mathrm{Im}\, D$ if and only if $\mathcal{L}(p) \in \mathrm{Im}\, D$. Then the statement follows immediately from the equivalence in $(*)$.

iii) Write $p(t) = \sum c_i t^i$ with $c_i \in A$. It follows from *ii*) that $f \in \mathfrak{r}(\mathrm{Im}\, D)$ if and only if $\mathcal{L}(p^m) \in aA$ for all large m. Clearly, if all c_i's belong to $\mathfrak{r}(a)$, then for all large m, $\mathcal{L}(p^m)$ belongs to aA, whence $f \in \mathfrak{r}(\mathrm{Im}\, D)$.

Conversely, assume that $\mathcal{L}(p^m)$ belongs to aA for all large m and suppose that some c_i's, say c_{i_0}, does not belong to $\mathfrak{r}(a)$. Then by the fact that $\mathfrak{r}(a)$ is the intersection of all prime ideals containing aA, there exists a prime ideal $\mathfrak{p}$ of A which contains a but not c_{i_0}.

Now, let $R := A/\mathfrak{p}$ and K the field of fractions of R. Define $\mathcal{L}_1 : K[t] \to K$ to be the K-linear map such that $\mathcal{L}_1(t^n) = n!$ for all $n \geq 0$. Then by viewing the reduced polynomial $\bar{p} \in R[t]$ inside $K[t]$, we have $\mathcal{L}_1(\bar{p}^m) = 0$ for all large m. Then by Lefschetz's principle and Theorem 4.9 in [EWZ2], we have $\bar{p} = 0$, i.e., all the coefficients c_i's of p lie in the prime ideal $\mathfrak{p}$, which is a contradiction. $\Box$

Corollary 7.4. *Let $f = p(at)$ for some $p \in A[t]$. If $f \in \mathfrak{r}(\mathrm{Im}\, D)$, then for every $g \in A[t]$, we have $gf^m \in \mathrm{Im}\, D$ for all large m.*

Proof: Let $g \in A[t]$ with degree $d \geq 0$. By Lemma 7.3 *iii*), all coefficients of $p(t)$ belong to $\mathfrak{r}(a)$. Hence there exists $N \geq 1$ such that all coefficients of $p(t)^N$ belong to aA. Consequently, all coefficients of $p(t)^{N(d+1)}$ belong to $a^{d+1}A$ and the same holds for $p(t)^m$ whenever $m \geq N(d+1)$. Then for each $m \geq N(d+1)$ and each $i \geq 0$, the coefficient of t^i in $f^m = p(at)^m$ belongs to the ideal $a^{i+d+1}A$, whence for each $j \geq 0$ the coefficient of t^j in $f^m g$ lies in the ideal $a^{j+1}A$. It then follows from Lemma 7.3 *i*) that $gf^m \in \mathrm{Im}\, D$ for all $m \geq N(d+1)$. $\Box$

Proof of Theorem 7.1: First, if a is a unit in A, then since ∂_t is locally nilpotent on $A[t]$, the operator D as an A-linear map is invertible with the inverse map given by $D^{-1} = -\sum_{i\geq 0} a^{-i-1}\partial_t^i$, whence $\mathrm{Im}\, D = A[t]$ and obviously is a Mathieu subspace of $A[t]$.

So we may assume that aA is a proper ideal A. Hence, so is the ideal $I := \cap_{i=0}^{\infty} a^i A$. Then for each $c \in A \backslash I$, there exists a unique integer n such that $c \in a^n A$ but $c \notin a^{n+1}A$. We define $v_a(c) := n$ in this case, and set $v_a(c) := \infty$ for all $c \in I$. Furthermore, we extend $v_a(\cdot)$ to $A[t]$ by setting $v_a(c\, t^i) := v_a(c) - i$ for all $c \in A$ and $i \geq 0$.

Now let $f \in \mathfrak{r}(\mathrm{Im} D)$ and write $f = \sum_{i=0}^{d} c_i t^i$ with $c_i \in A$. If all the coefficients c_i's belong to I, then f certainly can be written in the form $p(at)$ for some $p(t) \in A[t]$. Then for any $g \in A[t]$, by Corollary 7.4 we have $f^m g \in \mathrm{Im} D$ when $m \gg 0$. So we may assume that not all coefficients of f belong to I. Let $s(f)$ be the minimum of all $v_a(c_i t^i)$ $(0 \leq i \leq d)$. If $s(f) \geq 0$, then by Corollary 7.4 again we are done.

So assume $s(f) \leq -1$. Then $f_1 := a^{-s(f)} f$ also belongs to $\mathfrak{r}(\mathrm{Im} D)$ and $s(f_1) = 0$. Write $f_1 = \sum b_i t^i$. Then it follows that each b_i is of the form $b_i = a^i d_i$ for some $d_i \in A$, and furthermore that $d_i \notin aA$ for some i. Let $p(t) = \sum d_i t^i$. Then $f_1(t) = p(at)$ and by Lemma 7.3 *iii)* $d_i \in \mathfrak{r}(a)$ for each i.

By Lemma 7.5 below, there exist $u \in A$ and $\tilde{d}_i \in A$ such that $ud_i = \tilde{d}_i a$ for all i, and $\tilde{d}_i \notin \mathfrak{r}(a)$ for some i. Observe that since $-s(f) \geq 1$, the polynomial $a^{-s(f)-1} f$ belongs to $\mathfrak{r}(\mathrm{Im} D)$. Hence, so does the polynomial $f_2 := ua^{-s(f)-1} f$.

Now we consider

$$\begin{aligned} f_2(t) &= a^{-1} u f_1(t) = a^{-1} u \sum b_i t^i = a^{-1} u \sum d_i a^i t^i \\ &= \sum a^{-1} (ud_i) a^i t^i = \sum \tilde{d}_i a^i t^i. \end{aligned}$$

Hence we have $f_2(t) = q(at)$ with $q(t) = \sum \tilde{d}_i t^i \in A[t]$. Then applying Lemma 7.3 to $f_2 \in \mathfrak{r}(\mathrm{Im} D)$, we see that all $\tilde{d}_i \in \mathfrak{r}(a)$. But this is a contradiction, since as pointed out above $\tilde{d}_i \notin \mathfrak{r}(a)$ for some i. $\Box$

Lemma 7.5. *Let $d_1, \cdots, d_n$ all be in $\mathfrak{r}(a)$ and $d_i \notin aA$ for some i. Then there exist elements $u \in A$ and $\tilde{d}_i \in A$, such that $ud_i = \tilde{d}_i a$ for all i, and $\tilde{d}_i \notin \mathfrak{r}(a)$ for some i.*

Proof: Let b be the greatest common divisor of a and all d_i's. Since some $d_i \notin aA$, there exists an irreducible factor p of a such that its multiplicity in b is smaller than its multiplicity in a. Let $u = a/b$ and $\tilde{d}_i = d_i/b$ for all i. Then $ud_i = a\tilde{d}_i$ for each i. Furthermore, if $\tilde{d}_i \in \mathfrak{r}(a)$ for all i, then p divides each $\tilde{d}_i$ and hence pb is a common divisor of a and all d_i's. But this contradicts the definition of b. $\Box$

Next, we consider the special case of Conjecture 6.1 under the further assumption that $1 \in \mathrm{Im} D$. Note that if Conjecture 6.1 holds under this assumption, then we will have $\mathrm{Im} D = A[t]$. This is because of the general fact that any Mathieu subspace V of an algebra B with $1 \in V$ must be the whole algebra B, as first noticed in Lemma 4.5 in [Z4]. Indeed, since $1^m = 1 \in V$ for all $m \geq 1$, then for any $u \in B$, by taking large enough m, we have $u = u1^m \in V$, whence $V = B$.

Theorem 7.6. *Let A be any $\mathbb{Q}$-algebra, $a(t) \in A[t]$ and $D := c\partial_t - a(t)$ for some $c \in A$. Assume $1 \in \operatorname{Im} D$. Then $\operatorname{Im} D = A[t]$.*

Before we prove this theorem we make some preparations. First, if $a(t) = 0$, the hypothesis that $1 \in \operatorname{Im} D$ implies that c is a unit in A, which in turn implies that $\operatorname{Im} D = A[t]$ (since A is a $\mathbb{Q}$-algebra). So we may assume that $a(t) \neq 0$. Furthermore, we can also reduce to the case that A is Noetherian. More precisely, let $1 = D(h)$ for some $h(t) \in A[t]$ and A_0 the Noetherian $\mathbb{Q}$-subalgebra of A generated by c and coefficients of $a(t)$ and $h(t)$. Then D as an operator on $A_0[t]$ (by restriction) also has the property $1 \in D(A_0[t]) = \operatorname{Im} D_{A_0[t]}$. So, if we can prove the Noetherian case, it follows that $A_0[t] = D(A_0[t])$. In particular, each monomial t^n belongs to $D(A_0[t]) \subset D(A[t])$, whence $\operatorname{Im} D = A[t]$.

To prove the Noetherian case, we will make a reduction to the domain case by using the following result in commutative algebra.

Lemma 7.7. *For any Noetherian ring A, the zero ideal (0) is a product of finitely many prime ideals of A.*

Proof: Since A is Noetherian, its nilradical $\mathfrak{n}$ can be written as $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ for some prime ideals $\mathfrak{p}_i$. Since $\mathfrak{n}^e = 0$ for some $e \geq 1$, it follows that

$$(0) \subset (\mathfrak{p}_1 \cdots \mathfrak{p}_r)^e \subset (\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r)^e = \mathfrak{n}^e = (0).$$

Hence we have $(0) = \mathfrak{p}_1^e \mathfrak{p}_2^e \cdots \mathfrak{p}_r^e$. $\square$

Proof of Theorem 7.6: As pointed above, we may assume that A is Noetherian and that $a(t)$ is nonzero. By Lemma 7.7 we write $(0) = \mathfrak{p}_1 \cdots \mathfrak{p}_s$ for some prime ideals $\mathfrak{p}_i$.

We next show that we may also assume that A is a domain. Namely, let $\bar{A} := A/\mathfrak{p}_1$ and $\overline{D}$ be the induced operator on $\bar{A}[t]$. If we can prove the domain case, it follows that $\bar{A}[t] = \overline{D}(\bar{A}[t])$. Then for any $f \in A[t]$, we have $f = D(b) + \sum_i p_i a_i$ for some $b \in A[t]$, $p_i \in \mathfrak{p}_1$ and $a_i \in A[t]$. By a similar result with $\mathfrak{p}_1$ replaced by $\mathfrak{p}_2$, we obtain that $a_i = D(b_i) + \sum p_{ij} a_{ij}$ for some $b_i \in A[t]$, $p_{ij} \in \mathfrak{p}_2$ and $a_{ij} \in A[t]$. Combining these two results we get

$$f \in D(A[t]) + \mathfrak{p}_1 \mathfrak{p}_2 A[t].$$

Repeating this argument we finally find that

$$f \in D(A[t]) + \mathfrak{p}_1 \cdots \mathfrak{p}_s A[t] = D(A[t]).$$

So it remains to prove the case that A is a domain. However, in this case by comparing the degrees of both sides of the equation $1 =$

$(c\partial_t - a(t))h$, it is easy to see that $\deg a(t) = \deg h(t) = 0$, i.e., both $a(t)$ and $h(t)$ actually belong to A. It then follows that $a(t)$ is a unit in A. So we may assume that $a(t) = 1$. Since $c\partial_t$ is locally nilpotent on $A[t]$, it is easy to see that D is invertible with the inverse $D^{-1} = -\sum_{i\geq 0} c^i \partial_t^i$. Hence $\operatorname{Im} D = A[t]$, and the theorem follows. □

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