BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 50, Number 2, April 2013, Pages 337–345
S 0273-0979(2012)01363-0
Article electronically published on January 9, 2012

Quantum bounded symmetric domains, by Leonid L. Vaksman, Translated from the Russian manuscript and with a Foreword by Olga Bershtein and Sergey D. Sinel'shchikov, Translations of Mathematical Monographs, Vol. 238, American Mathematical Society, Providence, RI, 2010, xii+256 pp., ISBN 978-0-8218-4909-5, hardcover, US $105.00

Bounded symmetric domains were first investigated, and immediately classified, by E. Cartan in 1935 [Car35]. Ever since, bounded symmetric domains have been intensely studied as objects of complex analysis, differential geometry, functional analysis, and representation theory. They form an important class of homogeneous spaces and are therefore natural examples in harmonic analysis. The book under review by L. Vaksman adds yet another facet to this colorful picture. The combination of bounded symmetric domains with the theory of quantum groups takes the subject into the realm of deformation quantization and Poisson geometry, on the one hand, and of noncommutative algebra on the other. Vaksman's motivation for the construction and investigation of quantum bounded symmetric domains, however, stems from the interest in locally compact quantum groups, as will be illustrated below.

Vaksman studied quantum groups with V. Drinfeld in Kharkov, Ukraine, in the 1980s and early 1990s. Jointly with Y. Soibelman, he did pioneering work on compact quantum groups, in particular on quantum SU(2) and its relations to q-special functions [VS88]. With his student L. Korogodskiĭ, he was among the first to investigate examples of noncompact quantum groups [VK91]. In the late 1980s these topics were at the forefront of research on quantum groups. Vaksman continued to run a seminar and an active research group on the subject until his untimely death after a severe illness in 2007. He followed a broad research program which aimed at the development of a comprehensive theory of quantum group analogs of bounded symmetric domains. Vaksman and his collaborators developed this theory largely on their own. Since 1997, Vaksman coauthored about twenty papers on the subject and an extensive set of "lecture notes" [Vak01]. The monograph under review is his mathematical legacy in that it presents the beginning of the theory he envisioned. Originally he had intended to cover significantly more material. Sadly, however, this plan was "not accessible due to some nonmathematical reasons", as he writes in the introduction.

2010 *Mathematics Subject Classification.* Primary 32M15, 20G42; Secondary 17B37, 32A50, 33D80.

1. Bounded symmetric domains

A bounded domain is a bounded, open, connected subset of $\mathbb{C}^n$. It is natural to ask for classification of bounded domains up to isomorphism (that is, a holomorphic bijection with holomorphic inverse). In the simplest case, $n = 1$, Riemann's mapping theorem implies that any simply connected bounded domain is isomorphic to the unit disk $\mathbb{D}^1 = \{z \in \mathbb{C} \mid |z| < 1\}$. To obtain classification results in higher dimensions, Cartan assumed additionally that the bounded domain $\mathbb{D}$ is symmetric; that is, that every point of $\mathbb{D}$ is the isolated fixed point of an involutory automorphism. In this case $\mathbb{D}$ is simply connected and Cartan's answer to the classification problem is surprisingly simple: there exist four classical series of irreducible bounded symmetric domains and two exceptional cases of dimensions 16 and 27, respectively. The first of the four series is given by the matrix balls

$$\{Z \in \mathrm{Mat}_{n,m}(\mathbb{C}) \mid \mathrm{id}_n - Z\bar{Z}^t \text{ is positive definite}\},$$

which for $n = m = 1$ does indeed coincide with $\mathbb{D}^1$. Several classical textbooks describe the structure of bounded symmetric domains ([Bor98], [Hel78], see also the survey [Kor99]).

Any bounded domain $\mathbb{D}$ has a natural Kähler metric, the Bergman metric, which is invariant under automorphisms. Hence, if $\mathbb{D}$ is symmetric, then it is a Hermitian symmetric space. In this way one identifies the irreducible bounded symmetric domains with nonflat noncompact irreducible Hermitian symmetric spaces. In the simplest case of the unit disk $\mathbb{D} = \mathbb{D}^1$, the Bergman metric coincides with the hyperbolic (Poincaré) metric.

Moreover, there is a one-to-one correspondence between irreducible nonflat noncompact and compact Hermitian symmetric spaces. The unit disk $\mathbb{D}^1$, for example, corresponds to the one-dimensional complex projective space $\mathbb{C}P^1$. In general, compact Hermitian symmetric spaces are certain generalized flag manifolds G/P where G is a complex semisimple Lie group and P a parabolic subgroup. The flag manifold G/P possesses a cell decomposition with exactly one cell in maximal dimension, called the "big cell". Via the exponential map, the big cell is identified with a commutative Lie subalgebra $\mathfrak{u}^-$ of $\mathfrak{g}$, which is a prehomogeneous vector space of commutative parabolic type [Rub92]. In this way $\mathbb{D}$ is realized as a centrally symmetric convex bounded domain inside $\mathfrak{u}^-$. This embedding of $\mathbb{D}$ into $\mathfrak{u}^-$ is called the "Harish-Chandra embedding" and forms the basis for Vaksman's construction of quantum bounded symmetric domains.

2. Analysis on bounded symmetric domains

The fundamental problems of harmonic analysis on homogeneous spaces are formulated for example in the first chapter of Helgason's book [Hel84]. We restrict here to the example of the unit disk $\mathbb{D}^1$. The group

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \middle| \, |a|^2 - |b|^2 = 1 \right\}$$

acts transitively on $\mathbb{D}^1$ by means of

$$z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$$

and allows the identification $\mathbb{D}^1$ with the homogeneous space $\mathrm{SU}(1,1)/\mathrm{SO}(2)$. The algebra of $\mathrm{SU}(1,1)$-invariant differential operators on $\mathbb{D}^1$ is generated by the Laplacian $\Box = -(1-|z|^2)^2 \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$, and the $\mathrm{SU}(1,1)$ invariant integral is given by $d\nu = (1-|z|^2)^{-2}dxdy$ for $z = x+iy$. Harmonic analysis aims to solve the spectral problem for $\Box$ on the space $L^2(d\nu)$ of square integrable functions on $\mathbb{D}^1$. It turns out that the spectrum is purely continuous and equals the semiaxis $(-\infty, -1/4]$. Similar to the situation on the real line, the eigenfunction expansion is obtained via the Fourier transform. To make this precise, let

$$P(z,\zeta) = \frac{1-|z|^2}{|1-z\bar{\zeta}|^2}, \qquad z \in \mathbb{D}^1, \zeta \in \partial\mathbb{D}^1,$$

denote the Poisson kernel for $\mathbb{D}^1$. For any C^∞-function f on $\mathbb{D}^1$ with compact support, one has

$$f(z) = \int_0^\infty \left\{ \int_0^{2\pi} P(z,e^{i\theta})^{i\rho+\frac{1}{2}} \tilde{f}(\rho, e^{i\theta}) \frac{d\theta}{2\pi} \right\} s(\rho) d\rho, \tag{1}$$

where

$$\tilde{f}(\rho, e^{i\theta}) = \int_{\mathbb{D}^1} P(z,e^{i\theta})^{-i\rho+\frac{1}{2}} f(z) d\nu(z)$$

is the hyperbolic Fourier transform and $s(\rho)d\rho = 2\rho \tanh(\pi\rho)d\rho$ is the Plancherel measure. Equation (1) can now indeed be seen as a continuous decomposition of f into eigenvectors of $\Box$ because the integrand

$$u(z) = \int_0^{2\pi} P(z,e^{i\theta})^{i\rho+\frac{1}{2}} \tilde{f}(\rho, e^{i\theta}) \frac{d\theta}{2\pi}$$

is a solution of the differential equation

$$\Box u = -\left(\rho^2 + \frac{1}{4}\right) u.$$

Details can be found in [Hel84, Introduction §4].

We now move on to another aspect of analysis on the unit disc $\mathbb{D}^1$, namely its relation to the representation theory of the group $\mathrm{SU}(1,1)$ via so-called Bergman spaces. For $\lambda \in (1,\infty)$, one defines $L^2(d\nu_\lambda)$ by the measure

$$d\nu_\lambda = (1-|z|^2)^\lambda d\nu = (1-|z|^2)^{\lambda-2}dxdy.$$

The weighted Bergman space $L^2_a(d\nu_\lambda)$ is the completion of the space of polynomials $\mathbb{C}[z]$ inside $L^2(d\nu_\lambda)$. The spaces $L^2_a(d\nu_\lambda)$ consist of holomorphic functions. Moreover, for $\lambda \in \mathbb{N}$, they carry a representation T_λ of $\mathrm{SU}(1,1)$ via the action

$$(T_\lambda(g)f)(z) = f(g^{-1}z)(-\bar{b}z+a)^{-\lambda} \qquad \text{for } g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}.$$

The representations T_λ, $\lambda \geq 2$, are irreducible and unitary with respect to the natural Hermitian inner product on $L^2_a(d\nu_\lambda)$. They form the so-called holomorphic discrete series of representations of $\mathrm{SU}(1,1)$; see [Kna86, Chap. VI]. We have arrived at an instance of the general principle that important classes of representations can be realized geometrically as function spaces on the unit disk or its boundary. This principle holds in more generality and establishes the relation between analysis on bounded symmetric domains and the representation theory of real reductive groups.

In Vaksman's monograph all of the above results and much more is deformed into the setting of quantum groups. Before we describe his approach, however, we should first recall what quantum groups are all about.

3. Locally compact quantum groups

The story of quantum groups has been told many times. Quantum groups arose in the early 1980s as the mathematical underpinning of the quantum inverse scattering method developed by the school of L. Faddeev. The main objects are certain Hopf-algebra deformations $U_q(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ of a semisimple complex Lie algebra $\mathfrak{g}$, which were first discovered by V. Drinfeld and M. Jimbo ([Dri87] and [Jim85]). Quantum groups, or quantized enveloping algebras, owe their name to the fact that they can be obtained via deformation quantization from $U(\mathfrak{g})$. In their early days, quantum groups provided breathtaking new techniques and results in representation theory and beyond. As an example we just mention the theory of canonical and crystal bases, which on a basic level extends the classical Littlewood–Richardson rule for tensor product decompositions of $\mathfrak{sl}_n(\mathbb{C})$-modules to general $\mathfrak{g}$ ([Lus90], [Kas91]). Other fundamental contributions concern modular representation theory and conformal field theory.

Dually to $U_q(\mathfrak{g})$ the theory of quantum groups provides noncommutative analogs of algebras of functions on groups and homogeneous spaces. We write $\mathbb{C}_q[G]$ to denote the quantum group analog of the algebra of regular functions on a complex algebraic group G with Lie algebra $\mathfrak{g}$. Real forms of G and $\mathfrak{g}$ are encoded by Hopf-$*$-algebra structures on $\mathbb{C}_q[G]$ and $U_q(\mathfrak{g})$, respectively. One of the simplest examples, the quantum group version of $\mathrm{SU}(2)$, was independently discovered in the pioneering work by S. Woronowicz [Wor87b]. His approach might be called "noncommutative topology" and is motivated by the Gelfand representation, which identifies any commutative C^*-algebra with the algebra of continuous functions on a locally compact Hausdorff space. In Woronowicz's program the goal was to find a general operator algebraic framework of quantum groups as noncommutative C^*-algebras, which includes known noncommutative analogs of functions on groups such as $\mathrm{SU}(2)$, $\mathrm{SU}(1,1)$, and the group $E(2)$ of plane motions. The group structure should be reflected in a Hopf-algebra structure of the C^*-algebra. The framework of C^*-algebras can be seen as the starting point to carry out the program of harmonic analysis in greater generality [Dix77].

There exists a satisfactory theory of compact quantum groups [Wor87a] which encompasses C^*-algebra versions of all compact real forms obtained in Drinfeld's theory. In Drinfeld's approach to quantum groups, on the other hand, much of harmonic analysis, for instance the Peter–Weyl decomposition, is already built into the definition of $\mathbb{C}_q[G]$. In this setting for quantum $\mathrm{SU}(2)$, the relation between zonal spherical functions and q-special functions was pioneered by Vaksman and Soibelman in [VS88], T. Koornwinder in [Koo89], and T. Masuda et al. in [MMN+91] and [MMN+88]. By now there exists a well developed q-analog of harmonic analysis on compact Riemannian symmetric spaces ([Nou96], [Let04]). It results in an interpretation of multivariable Macdonald polynomials as q-analogs of zonal spherical functions. For the simplest case of the quantum $\mathbb{C}P^1$, also referred to as the "Podleś sphere", this program was performed by Koornwinder [Koo93]. In view of Vaksman's monograph, it is relevant to note that most of harmonic analysis on compact versions of Drinfeld–Jimbo quantum groups and their homogeneous spaces can be done on a purely algebraic level.

In the noncompact setting much less has so far been achieved. Noncompact real forms of $\mathfrak{g}$ result in different $*$-algebra structures on the Hopf algebra $\mathbb{C}_q[G]$. The simplest example, the quantum group analog of $\mathrm{SU}(1,1)$, was obtained right at the beginning of the development of quantum groups as a Hopf-$*$-algebra. It has been enigmatic within Woronowicz's approach. He proved in [Wor91] that the comultiplication of the Hopf-algebra cannot be lifted to a comultiplication on the level of unbounded operators on a Hilbert space satisfying the given $*$-algebra relations. This "no-go theorem" resulted in a major setback in the development of noncompact quantum groups. The problem, however, was partially resolved when Korogodskiĭ [Kor94] introduced an additional generator and discussed the quantum group analog of the normalizer of $\mathrm{SU}(1,1)$ in $\mathrm{SL}(2,\mathbb{C})$. His idea was used in the approach by E. Koelink and J. Kustermans [KK03] to formulate quantum $\mathrm{SU}(1,1)$ in the operator algebraic framework for locally compact quantum groups which had meanwhile been introduced by Kustermans and S. Vaes [KV00]. On the algebraic side there are virtually no results about harmonic analysis on noncompact Drinfeld–Jimbo quantum groups.

4. Vaksman's hidden agenda

The origin of Vaksman's work on quantum group analogs of bounded symmetric domains lies in his early work with Korogodskiĭ on noncompact quantum groups. Vaksman takes Woronowicz's general program into account, but his focus is on the concrete examples provided by Drinfeld's theory. He aims at the development of harmonic analysis for noncompact quantum groups. As in the classical case, significant milestones are provided by finding the Fourier transform and the Plancherel measure. But there is ample opportunity for diversion, studying Toeplitz operators, Berezin quantization, and other ramifications of complex analysis. Vaksman takes the point of view that such problems can be solved even in the absence of a general notion of noncompact quantum groups. Instead he takes guidance from the representation theory of real reductive groups and proposes the investigation of quantum Harish-Chandra modules [SSSV01]. In [SSV01] he suggests the construction and investigation of q-analogs of nondegenerate spherical principal series representations. This goal was achieved in [BSV07] which was among his last papers. The program suggested by Vaksman is vast, not least because it encompasses more or less the classical theory of harmonic analysis for real reductive groups. From this perspective the restriction to bounded symmetric domains turned out to be a wise choice. They form a very important class of examples, but even better, their compact counterparts have generally proved amenable to the extension of classical mathematical concepts to the quantum group setting.

5. Content of the book

The first half of Vaksman's monograph is devoted to the quantum group analog of complex analysis on $\mathbb{D}^1$ as indicated in Section 2. The quantum disc $\mathrm{Pol}_q(\mathbb{C})$ was first introduced by S. Klimek and A. Lesniewski [KL93] as the noncommutative $*$-algebra with generators z and z^* and the single relation

$$z^* z - q^2 z z^* = 1 - q^2, \tag{2}$$

where $q \in (0,1)$ is a fixed deformation parameter. This definition conforms with the general idea of noncommutative geometry that noncommutative versions of a space are given by deformations of the algebra of functions on the space. Indeed, if we set $q = 1$, then we obtain the algebra of polynomial functions on $\mathbb{D}^1$. The explicit choice of the parameters in (2) allows one to endow $\mathrm{Pol}_q(\mathbb{C})$ with the quantum group analog of the action of $\mathrm{SU}(1,1)$, or more explicitly with a module algebra structure over the Hopf-$*$-algebra $U_q(\mathfrak{su}(1,1))$.

To be able to formulate results similar to (1) in the quantum group setting, Vaksman needs a quantum group analog of functions with compact support. He constructs these finite functions $\mathcal{D}(\mathbb{D})_q$ using an operator algebraic approach. More explicitly, he first shows that the quantum disc $\mathrm{Pol}_q(\mathbb{C})$ has a unique faithful representation by bounded operators on a Hilbert space. This representation has a basis $\{e_0, e_1, e_2, \dots\}$ of eigenvectors for the element $y = 1 - zz^*$ with corresponding eigenvalues $\{1, q^2, q^4, \dots\}$. In a suitable completion he then defines $\mathcal{D}(\mathbb{D})_q$ to be the subalgebra of operators which act nontrivially only on finitely many of the basis vectors e_i. This corresponds to the fact that the function vanishes for small $1 - zz^*$; in other words, the function has compact support in $\mathbb{D}^1$.

For the space of functions $\mathcal{D}(\mathbb{D})_q$ Vaksman manages to define an invariant integral corresponding to $d\nu$ from Section 2 and develops the theory in impressive analogy to the classical case. To give a flavor of this, we highlight just one of the many interesting constructions. He defines q-analogs $L^2_a(d\nu_\lambda)_q$ of weighted Bergman spaces inside a Hilbert space $L^2(d\nu_\lambda)_q$ which is a q-analog of $L^2(d\nu_\lambda)$. He then defines Toeplitz operators $\mathcal{T}_f^{(\lambda)} = \hat{f} : L^2_a(d\nu_\lambda)_q \to L^2_a(d\nu_\lambda)_q$ by $\mathcal{T}_f^{(\lambda)}(\psi) = P_\lambda(f\psi)$, where $P_\lambda : L^2(d\nu_\lambda)_q \to L^2_a(d\nu_\lambda)_q$ denotes the orthogonal projection. In Example 1.129 Vaksman shows that

$$\hat{z}^* \hat{z} = q^2 \hat{z}\hat{z}^* + 1 - q^2 + q^{2(\lambda-1)} \frac{1-q^2}{1-q^{2(\lambda-1)}} (1 - \hat{z}\hat{z}^*)(1 - \hat{z}^*\hat{z}).$$

The above relation was obtained originally by Klimek and Lesniewski [KL93] via deformation quantization of a two-parameter family of Poisson structures on $\mathbb{D}^1$. Vaksman has constructed this algebra by completely different means.

In the second half of his monograph Vaksman begins to develop the theory of quantum bounded symmetric domains in full generality. After a reminder on Drinfeld–Jimbo quantum groups Vaksman addresses the construction of the algebra $\mathrm{Pol}_q(\mathfrak{u}^-)$ of polynomial functions which is a q-analog of the algebra of complex valued polynomials on $\mathfrak{u}^-$ considered as a real vector space. In the simplest case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, the algebra $\mathrm{Pol}_q(\mathfrak{u}^-)$ coincides with the quantum disc $\mathrm{Pol}_q(\mathbb{C})$. To obtain $\mathrm{Pol}_q(\mathfrak{u}^-)$, Vaksman first considers Harish-Chandra modules $N(\mathfrak{q}^+, 0)$ with a $U_q(\mathfrak{g})$-module coalgebra structure and defines a q-analog $\mathbb{C}_q[\mathfrak{u}^-]$ of the algebra of polynomials on the complex vector space $\mathfrak{u}^-$ by duality. The algebra $\mathrm{Pol}_q(\mathfrak{u}^-)$ is then constructed by imposing commutation relations between $\mathbb{C}_q[\mathfrak{u}^-]$ and its opposite algebra $\mathbb{C}_q[\mathfrak{u}^+]$ by means of the universal R-matrix of $U_q(\mathfrak{g})$. In private conversation, Vaksman occasionally called this construction, which first appeared in [SV98], his "best mathematical result". Not only is $\mathrm{Pol}_q(\mathfrak{u}^-)$ the starting point of the general theory of quantum bounded symmetric domains, it is also an example of the relevance of Harish-Chandra modules within this theory. The algebra $\mathbb{C}_q[\mathfrak{u}^-]$ is a quantum analog of the prehomogeneous vector space of commutative parabolic type and has been constructed independently by other authors [Jak96], [KMT98]. These constructions, however, are purely algebraic, while Vaksman immediately ventures into the construction of the Fock representation and of the quantum algebra $\mathcal{D}(\mathbb{D})_q$ of functions with compact support, which finally allow the definition of a quantum analog of the invariant integral for any bounded symmetric domain $\mathbb{D}$. In the remaining sections, he constructs analogs of the Borel embedding and discusses invariant differential operators on quantum bounded symmetric domains. Fourier transform and Plancherel measure in the general case, however, remain projects for the future. The final third chapter gives a brief outlook on directions of possible future work. Keywords include the Shilov boundary, spherical principal series representations, and the Penrose transform.

Vaksman's book provides a stunning example of the extent to which classical mathematics can be q-deformed within the theory of quantum groups. His writing is not always easy to read and would have benefited from broader exposition and more editorial work, had this been possible. The reader, however, who is willing to work through the details, and who is always prepared to consult external references, will be rewarded by a real sense of discovery. We can only follow Drinfeld's words on the book cover and strongly recommend Vaksman's book to mathematicians interested in noncommutative geometry, quantum groups, C^*-algebras, and operator theory.

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