The Church-Rosser Property for \( \beta\eta \)-reduction in Typed \( \lambda \)-Calculi

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Abstract

In this paper we investigate the Church-Rosser property (CR) for Pure Type Systems with \( \beta\eta \)-reduction. For Pure Type Systems with only \( \beta \)-reduction, CR on well typed terms follows immediately from CR on the so called 'pseudoterms' and subject reduction. For \( \beta\eta \)-reduction, CR on the set of pseudoterms is just false, as was shown by [Nederpelt 1973]. Here we prove that CR (for \( \beta\eta \)) on the well-typed terms of a fixed type holds, which is the maximum we can expect in view of Nederpelt's counterexample. The proof is given for a large class of Pure Type systems that contains e.g. LF (for which CR for \( \beta\eta \) was proved by [Salvesen 1989] and [Coquand 1991]), F, Fu and the Calculus of Constructions. In the proof, one key lemma (a very weak form of CR for \( \beta\eta \) on pseudoterms) takes a central position. It is remarkable that in the proof of this key lemma the counterexample to CR for \( \beta\eta \) is essentially used.

1 Introduction

Simply typed lambda calculus (from now on denoted by \( \lambda \rightarrow \)) and polymorphic lambda calculus are usually described by first giving the set of types and then the derivation rules for deriving typing judgements of the form \( \Gamma \vdash M : \sigma \), where \( \sigma \) is a type, \( \Gamma \) is a 'context' and \( M \) is an element of the set of so called 'pseudoterms'. For \( \lambda \rightarrow \) this amounts to the following.

\[
\text{Typ}_{\rightarrow} := \text{TVar} \upharpoonright \text{Typ} \rightarrow \text{Typ}_{\rightarrow},
\]

where TVar denotes the set of type variables. A context is a sequence of declarations \( x:\sigma \) with \( x \in \text{Typ}_{\rightarrow} \) and \( x \in \text{var} \), the set of term variables. The derivation rules are.

\[
\begin{align*}
(\text{var}) & \quad \Gamma \vdash x : \sigma \quad \text{if } x:\sigma \in \Gamma \\
(\lambda) & \quad \Gamma, x:\sigma \vdash M : \tau \quad \Gamma \vdash \lambda x:\sigma.M : \sigma \rightarrow \tau \\
(app) & \quad \Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma \quad \Gamma \vdash MN : \tau
\end{align*}
\]

For meta-theory it is convenient to introduce the set of pseudoterms of \( \lambda \rightarrow \),

\[
\text{T}_{\rightarrow} := \text{var} \upharpoonright (\text{Avar} : \text{Typ}_{\rightarrow} \rightarrow \text{T}_{\rightarrow}) \mid \text{T}_{\rightarrow} \rightarrow \text{T}_{\rightarrow},
\]

then one can see the derivation rules as singling out the well-typed terms from the set of pseudoterms. Let's write Term for the set of pseudoterms typable with some type in some context. Now, \( \beta \)-reduction on \( \text{T}_{\rightarrow} \) is Church-Rosser, so with the Subject Reduction property: \( \Gamma \vdash M : \sigma \rightarrow N \), then \( \Gamma \vdash M' : \sigma \rightarrow N' \), we obtain that \( \beta \)-reduction is CR on the well-typed terms: \( M \in \text{Term}_{\rightarrow} \rightarrow N \rightarrow M' \rightarrow N' \), then \( \exists N \in \text{Term}_{\rightarrow} \rightarrow M' \rightarrow N' \rightarrow M'' \rightarrow N'' \).

If one is interested in the combination of \( \beta \) and \( \eta \)-reduction, the situation is a little bit more complicated, because we have the following counterexample to CR for \( \beta\eta \)-reduction on \( \text{T}_{\rightarrow} \). (Originally due to [Nederpelt 1973].) Take

\[
M := \lambda x:\sigma.(\lambda y:\tau.y)x,
\]

with \( \sigma \neq \tau \). Then \( M \rightarrow \eta \lambda x:\sigma.x \) and \( M \rightarrow \eta \lambda x:\tau.x \) and both are in normal form. This problem can be overcome by noticing that the only problematic overlapping of redexes in \( \text{T}_{\rightarrow} \) is when we have a subterm \( \lambda x:\sigma.(\lambda y:\tau.M)x \) (with \( x \notin \text{FV}(M) \)) and by proving that in well-typed terms, this can only happen when \( \sigma \equiv \tau \). For the polymorphic lambda calculus the situation is quite similar; we now have polymorphic types and polymorphic terms (with abstraction and application), but for the problematic overlapping of redexes in a term \( \lambda x:\sigma.(\lambda y:\tau.M)x \) with \( x \notin \text{FV}(M) \), we still have that \( \sigma \equiv \tau \).

For the higher order lambda calculus (\( \text{Fu} \)), the situation is a bit more difficult, because we have reduction inside types. Now types are also formed inside a context and there is an extra rule, stating that convertible types have the same 'inhabitants'. The syntax is the following. There are two sets of variables, var, denoting the term variables, and \( \text{Var} \), denoting the so called constructor variables (the set of constructors will include the types.) We also distinguish two kinds of contexts, the constructor contexts, denoted by \( \Delta \), for declaring constructor variables, and the term contexts, denoted by \( \Gamma \), for declaring term variables. The term variables are declared to types and the constructor variables to so called kinds. 'Kind' represents the collection of kinds and 'Type' represents the kind of types. Kind is defined by \( \text{Kind} := \text{Type} \mid \text{Kind} \rightarrow \text{Kind} \). We now give the derivation rules. Let \( \Delta \) be a constructor context.

1. \( (\text{Var}) \quad \Delta \vdash \alpha : A \quad \text{if } \alpha : A \in \Delta \)
In the rules it has not yet been declared in r. It is of course reasonable to replace the rule (convq) by

certainty in the set of pseudoterms med that the newly declared variable is fresh, that is,

Here constructor variables are denoted by Greek characters and term variables by Roman characters.

This equality on pseudoterms as 

For systems with so called dependent types, that is types that have terms as subexpressions, like LF or the
Calcus of Constructions (CC), the Church-Rosser property is very complicated: if \( \text{Ax} : \text{Ax} (\lambda y : \text{B} . M) \text{Ax} \) is a well-typed term, we know \( A =_\beta B \), but now this doesn't bring us any further because the equality
may arise from an equality between terms. CR\(_\beta\) for LF was proved in [Salvesen 1989] and later by [Coquand 1991],
but for more complicated systems with dependent types the question of CR\(_\beta\) was still open. Here we shall prove CR\(_\beta\)
for normalizing, functional Pure Type Systems, which includes CC. (There is no proof of normalization for \( \beta\eta\)
reduction for CC in the literature. However, the strong normalization proof for \( \beta\eta\)-reduction for CC in
[Geuvers and Nederhof 1991] can easily be adapted to the case for \( \beta\eta\)-reduction.)

2 Pure Type Systems and the extension with \( \eta\)-reduction

Let's take a look at the definition of Pure Type Systems

(PTSs for short) and give some of the meta theory. (See [Geuvers and Nederhof 1991], [Barendregt 1994] for
more meta-theory and examples.)

Definition 2.1 For \( S \) a set, \( A \subset S \times S \) and \( R \subset S \times S \), \( \lambda (S, A, R) \) is the typed lambda calculus
with the following deduction rules.

\[
\begin{align*}
1. \text{(sort)} & \vdash s_1, s_2 : \text{if } (s_1, s_2) \in A. \\
2. \text{(var)} & \vdash A : s \\
& \Gamma, x : A \vdash x : A \\
3. \text{(weak)} & \vdash A : s \vdash M : C \\
& \Gamma, x : A \vdash M : C \\
4. \text{(II)} & \vdash A : s_1, s_2 \vdash \text{if } (s_1, s_2, s_3) \in R. \\
& \Gamma \vdash \Pi x : A. B : s_3 \\
& \Gamma, x : A \vdash M : B \vdash \Pi x : A. B : s \\
& \Gamma \vdash \lambda x : A. M : \Pi x : A. B
\end{align*}
\]
with equivalence classes of pseudoterms or representants of such classes. For practical reasons we shall denote the equality will be denoted by \( \equiv \). (So in fact we work with \( \equiv \)-equivalence classes of pseudoterms or representants of such classes. For practical reasons we shall only work with representants of \( \equiv \)-equivalence classes in which all bound variables are pairwise distinct and different from the free variables. (The variable convention) The \( \Gamma \) in the judgement \( \Gamma \vdash M : A \) is, as usual, called a context. For \( \Gamma = x_1 : A_1, \ldots, x_n : A_n \), \( FV(\Gamma) \) denotes the set \( FV(A_1, \ldots, A_n) \).

We say that \( M \) is of type \( A \) in \( \Gamma \) if \( \Gamma \vdash M : A \) is a conclusion of a deduction tree which uses only the rules stated above. A well-typed term is a pseudoterm \( A \) for which \( \Gamma \vdash A : B \) or \( \Gamma \vdash B : A \) for some \( \Gamma \) and \( B \). Term \( (\Gamma, A) \) denotes the set of well-typed terms in a PTS, and \( Term(\Gamma, A) \) denotes the well-typed terms of type \( A \) in \( \Gamma \). A PTS is normalizing, if every well-typed term in it reduces to a normal form.

The collection of PTSs contains well-known type systems like simply typed lambda calculus, \( \lambda \rightarrow \), \( \lambda \{ \text{Type, Kind} \}, \{ \text{Type, Kind} \}, \{ \text{Type, Type} \} \), Girard's systems \( E \) (\( \lambda \rightarrow \) extended with the rule \( \text{Kind, Kind} \)) and \( F \) (extended with \( \text{Kind, Kind} \)) and the Calculus of Constructions (\( F \) with the rule \( \text{Kind, Kind} \)). (It requires some meta-theory to see that this version of \( F \) is the same as the one introduced before.) The inconsistent type system \( \lambda_\eta \), where \( \lambda \rightarrow \) is a type is also a PTS (\( \lambda \{ \text{Type} \}, \{ \text{Type, Type} \} \)} \( \text{Type, Type} \} \)) and the notion of PTS also captures logical systems like first, second order and higher order predicate logic, as shown by [Berarducci 1988]. (See also [Barendregt 1990+].)

In this paper we want to study the extension with \( \eta \)-equality. First we give some motivations why this extension is interesting and why the Church-Rosser property of the combined \( \beta \eta \)-reduction is important. (This will hardly need any explanation.) Then we want to show what are the problems that arise when trying to prove CR for \( \beta \eta \). Section 2 gives an outline of the proof.

## 2.1 Properties of Pure Type Systems

Here we give some of the meta-theory for PTSs, mainly those theorems that will be encountered later in the Church-Rosser proof. For detailed proofs we refer to [Geuvers and Nederhof 1991] or [Barendregt 1990+]. In the following, unless noted otherwise, we work in an arbitrary PTS.

### Proposition 2.2 (CR\( \eta \))

The reduction relation \( \rightarrow_{\beta} \) is Church-Rosser on \( T \). (That is, if \( M \rightarrow_{\beta} M_1 \) and \( M \rightarrow_{\beta} M_2 \) then \( M_1 \rightarrow_{\beta} N \) and \( M_2 \rightarrow_{\beta} N \) for some \( N \in T \).

(The proof follows precisely the Church-Rosser proof of \( \beta \)-reduction on the untyped lambda terms in [Barendregt 1984].)

### Corollary 2.3 (Confluence of \( \beta \) on \( T \), CON\( \beta \))

If \( M \rightarrow_{\beta} M' \) then \( M \rightarrow_{\beta} N \) and \( M' \rightarrow_{\beta} N \) for some \( N \in T \).

We shall write \( M \rightarrow_{\beta}^+ M' \) for \( \exists N \in T.M \rightarrow_{\beta} N \) & \( M' \rightarrow_{\beta} N \). From the form of the derivation rules we can conclude the following.

### Lemma 2.4 (Stripping)

1. \( \Gamma \vdash \lambda x : A.M : C \Rightarrow \Gamma, x : A \vdash M : B, \Gamma \vdash \Pi x : A.B : s \) for some \( B \) and \( s \) such that \( \Pi x : A.B =_{\beta} C \).

2. \( \Gamma \vdash MN : C \Rightarrow \Gamma \vdash M : \Pi x : A.B, \Gamma \vdash N : A \) for some \( A \) and \( B \) such that \( B[N/x] =_{\beta} C \).

### Proposition 2.5 (Subject Reduction for \( \beta \), SR\( \beta \))

If \( \Gamma \vdash M : A \) and \( M \rightarrow_{\beta} M' \) then \( \Gamma \vdash M' : A \).

The proof of the proposition is by induction on the derivation, proving the statement simultaneously for a one step reduction in \( \Gamma \) or \( M \). It uses Stripping (2.4) and the fact that \( \Pi x : A.B =_{\beta} C \) if \( M \) implies \( \Pi x : A.B =_{\beta} C \) and \( B =_{\beta} D \), which holds by CON\( \beta \) on \( T \).

We don't have \( \eta \) in the (conv\( \beta \)) rule, but we do have subject reduction for \( \eta \).

### Proposition 2.6 (Subject Reduction for \( \eta \), SR\( \eta \))

If \( \Gamma \vdash M : A \) and \( M \rightarrow_{\eta} M' \) then \( \Gamma \vdash M' : A \).

The proof of this proposition is, just as for SR\( \beta \), by induction on the derivation, proving the statement simultaneously for a one step reduction in \( \Gamma \) or \( M \). In the proof one needs strengthening:

### Proposition 2.7 (Strengthening)

If \( \Gamma_1, x : A, \Gamma_2 \vdash M : B \) with \( x \notin FV(\Gamma_2, M, B) \), then \( \Gamma_1, \Gamma_2 \vdash M : B \).
The proof is in [van Benthem Jutting 199+]. In [Geuvers and Nederhof 1991] a proof is given for a subcollection of PTSs, the so called functional ones: A PTS is functional if the relation A is a function from S to S and the relation R is a function from S x S to S. (That is, if s : s' : s'' e A, then s' = s'' and if (s1, s2, s3) : (s1, s2, s3), R ) then s3 = s3.) Both proofs use CRp on T.

**Proposition 2.8 (Uniqueness of types, UT)** In a functional PTS: If \( \Gamma \vdash M : A \), \( \Gamma \vdash M : A' \), then \( A =_\beta A' \).

The proof is by induction on the derivation. It uses CONp in the sense that one needs \( \Pi x:A.B =_\beta \Pi x:C.D \Rightarrow B =_\beta D \) to hold. (The proposition is false for non-functional PTSs.)

**2.2 The extension with \( \eta \)**

Extending the PTSs with \( \eta \) means just replacing the conversion rule by

\[
\frac{\text{(conv}_{\beta\eta})}{\Gamma \vdash M : A \quad \Gamma \vdash B : s \quad A =_{\eta} B}{\Gamma \vdash M : B}
\]

To distinguish the two, we write PTS\(_{\eta}\) for the ones with (conv\(_{\beta\eta}\)) as conversion rule and PTS\(_{\beta\eta}\) for the ones with (conv\(_{\beta}\)).

There are many reasons for studying this extension. For one thing, the stronger \( \beta\eta \)-equivalence is quite natural, especially if we think about a PTS as representing a logic. The need for \( \beta\eta \)-equivalence also becomes apparent when we look at type systems that are used as frameworks, like LF. The completeness of the interpretation of a formal system in a signature relies on defining a mapping back, from terms in the signature to terms of the formal system, which is defined on (representants of) \( \beta\eta \)-equivalence classes. (See [Harper et al. 1987].) A nice consequence of CR\(_{\beta\eta}\) is that the unification algorithms for the Calculus of Constructions, as discussed in [Pfenning 1991] are complete. A semantical reason for strengthening the equality is that in almost all models of type systems the \( \eta \) rule is valid. Further CR\(_{\beta\eta}\) is essential if we relate the syntax of PTSs to the semantics: The systems that are really interpreted in e.g. [Streicher 1989] and [Jacobs 1991] are not the ones as formulated above (with equality on the pseudoterms as a side condition), but with an equality judgement inside the context. If \( \zeta \) is a PTS, write \( \zeta_{=} \) for the associated 'semantical' version of the system (with equality judgement), replacing \( \beta \) by \( =_\eta \). The conversion rule of \( \zeta_{=} \) is

\[
\frac{\text{(conv}_{\beta\eta})}{\Gamma \vdash M : A \quad \Gamma \vdash A = B : s \quad \Gamma \vdash M : B}
\]

where the judgement \( \Gamma \vdash A = B : s \) is generated by

\[
\begin{align*}
(\beta) & \quad \Gamma \vdash \lambda x:A.M : \Pi x:C.D \quad \Gamma \vdash N : C \\
(\eta) & \quad \Gamma \vdash (\lambda x:A.M)N = M[N/x] : D[N/x]
\end{align*}
\]

taking the transitive, reflexive, symmetric closure, made compatible with application, \( \lambda \) - and \( \Pi \)- abstraction. In this version of the system the conversion rule can only be applied to two equal types if they are equal via a path through the well-typed terms. In the original PTS version one only claims that the types are equal as pseudoterms. If we have only \( \beta \)-conversion (i.e. in \( \zeta_{=} \) we don't have (\( \eta \))), the two versions are equivalent: If \( M, M' \in \text{Term} \) with \( M =_\beta M' \), then there is a path between them through Term (by CR\(_{\beta}\) on \( \Gamma \) and SR\(_{\beta}\)). This is expressed by the following theorem for any PTS\(_{\beta}\).

**Theorem 2.9**

\[
\begin{align*}
\Gamma \vdash M : A & \quad \Gamma \vdash M' : A \quad M =_\beta M' \\
M =_\beta M & \quad \Leftrightarrow \quad \Gamma \vdash M = M' : A.
\end{align*}
\]

In fact, this theorem also states that the system with equality on the pseudoterms is a 'sound' system: there would really be something wrong if \( \Gamma \vdash M, M' : A \) and \( M =_\beta M' \) as pseudoterms without there being a path from \( M \) to \( M' \) through Term(\( \Gamma, A \)). If we extend both versions of PTSs with \( \eta \), we need some form of Church-Rosser to prove the equivalence of the two versions. It suffices to have that if \( M, M' \) are of the same type in some context \( \Gamma \) and \( M =_\beta M' \), then \( M \downarrow_{\beta\eta} M' \). This is precisely what will be proved, for normalizing functional PTSs. As consequences we find that \( \zeta_{=} \) and \( \zeta_{=} \) are equivalent (for \( \zeta \) functional and normalizing) and that CR\(_{\beta\eta}\) and CON\(_{\beta\eta}\) hold for \( \zeta_{=} \). (It is not clear how a proof of CR\(_{\beta\eta}\) for \( \zeta_{=} \) could be essentially simpler then via the proof for \( \zeta \).)

**2.3 Problems with proving CR\(_{\beta\eta}\)**

As already remarked, the main problem with proving CR\(_{\beta\eta}\) is that on the pseudoterms \( \Gamma \) it is just false. The counterexample \( (\lambda x:A.(\lambda y:B.y)x)z \) with \( A \neq \beta\eta B \) implies that also CON\(_{\beta\eta}\) on \( \Gamma \) (see 2.3) is not true, even if we restrict ourselves to well-typed terms. For the meta-theory for PTSs with \( \eta \) this has some serious consequences. UT is not immediate anymore for functional PTSs, with \( =_\eta \) replaced by \( =_{\beta\eta} \). (One needs that \( \Pi x:A.B =_{\beta\eta} \Pi x:C.D \Rightarrow B =_{\beta\eta} D \).) For SR\(_{\beta}\) and SR\(_{\eta}\) the situation is also problematic: The first requires fixing \( \Pi x:A.B =_{\beta\eta} \Pi x:C.D \Rightarrow A =_{\beta\eta} C \) & \( B =_{\beta\eta} D \). The second requires strengthening, the proof of which uses Church-Rosser. Our solution to this will be to prove a very weak form of Church-Rosser for \( \beta\eta \)-reduction on \( \Gamma \) (using the counterexample to CR\(_{\beta\eta}(\Gamma)\)), which will turn out to be sufficient to get UT and SR\(_{\eta}\). Also this key lemma will be sufficient to prove strengthening (and so SR\(_{\eta}\)) for functional, normalizing PTSs. Using all this, we shall prove (for functional, normalizing PTSs)

\[
\Gamma \vdash M, M' : A, M =_{\beta\eta} M' \Rightarrow nf(M) \equiv nf(M')
\]
Where we write \( nf(M) \) for the normal form obtained by some normalization procedure. We have as an immediate corollary \( CON_{\beta\eta} \) for \( \text{Term}(\Gamma, A) \).

3 The proof of \( CR_{\beta\eta} \) for functional, normalizing pure type systems

In the proof of Church-Rosser we shall relate the \( \beta\eta \)-reduction on typed terms to the reductions on untyped lambda terms. Properties of reduction on the untyped terms will be used to obtain results about reduction in \( \Gamma \). We therefore define an erasure mapping from \( \Gamma \) to \( \Lambda \) and give some properties for it. Then we give some properties of \( \beta\eta \)-reduction and equality on \( \Gamma \), which will enable us to prove \( SR_{\beta\eta} \) and \( SR_{\eta} \).

The counterexample of [Nederpelt 1973] shows that, if one tries to prove \( CR_{\beta\eta} \), there is a problem in Definition 3.1, where we write \( nf(M) \) for the normal form obtained by some normalization procedure. We have the types of the \( \Lambda \)-abstracted variables. We shall call if it occurs equal, we say that \( M \) and \( M' \) are all \( \Lambda \) domains: \( A \) and give some properties for it. Then we have the following proposition, \( \text{Proposition 3.2} \)

Let \( M \) and \( M' \) be in \( \Gamma \).

\[
M \rightarrow_{\beta\eta} M' \Rightarrow |M| \rightarrow_{\beta\eta} |M'|, \quad \text{and similar for } \rightarrow_{\eta} \text{ and so for } =_{\eta}.
\]

\[
|M| \rightarrow_{\beta\eta} |M'| \Rightarrow \exists N[M \rightarrow_{\beta\eta} N & |N| \equiv |M'|].
\]

(This doesn't hold for \( \rightarrow_{\beta\eta} \).)

Proof The first is trivial: If the redex is erased by \(||\), then \( |M| \equiv |M'| \) and otherwise the same redex can still be done in \( \Lambda^\Gamma \), so \( |M| \rightarrow_{\beta\eta} |M'| \). The second is almost trivial: we are done if we give the proof for a one-step \( \beta \)-reduction. As \(|| \) only erases domains, a \( \beta \)-redex in \( |M| \) is also a \( \beta \)-redex in \( M \), and by evaluating it we find \( N \in \Gamma \) with \( M \rightarrow_{\beta\eta} N \) and \( N \equiv M' \). This is not valid for \( \eta \), as is shown by the counterexample \( M \equiv \lambda x: y(x: P x. x) x \). (This term can even be well-typed in e.g. the Calculus of Constructions: Take \( P \equiv \lambda x: \eta, y: (x: \eta) \rightarrow \eta \rightarrow \eta \). In Lemma 3.14 we shall see that nevertheless, if \( M \) is well-typed in the Calculus of Constructions and \( |M| \) is in \( \beta\eta\text{-nf} \), then \( M \) is in \( \beta\eta\text{-nf} \).

The following is an immediate corollary of the counterexample to \( CR_{\beta\eta} \) on \( \Gamma \).

**Lemma 3.3 (Domain Lemma)** If \( C[\lambda x: A. M] \) and \( B \) are in \( \Gamma \) (i.e. \( C \) is a pseudoterm with subterm \( \lambda x: A. M \)), then

\[
C[\lambda x: A. M] =_{\beta\eta} C[\lambda x: B. M]
\]

Proof

\[
C[\lambda x: A. M] =_{\beta\eta} C[\lambda x: B. M]
\]

where \( y \) is any variable not occurring free in \( A \) or \( M \).

**Lemma 3.4 (Key Lemma)** Let \( c \) be a variable or a sort.

1. \( cP_1 \ldots P_n =_{\beta\eta} Q \Rightarrow Q \rightarrow_{\beta\eta} \lambda y: A. Q_1 \ldots Q_n y, \) with \( Q_i =_{\beta\eta} P_i (1 \leq i \leq n) \).

2. \( \Pi x: A. P_1 =_{\beta\eta} Q \rightarrow Q \rightarrow_{\beta\eta} \lambda x: A. (\Pi x: Q_1. Q_2) y, \) with \( P_i =_{\beta\eta} Q_i (1 \leq i \leq 2) \).

Proof We only prove the first case, since the second is totally similar. Some notation: For \( D \in \Gamma \) and \( M \in \Gamma \), \( M^D \in \Gamma \) is the pseudoterm obtained by replacing all domains in \( M \) by \( D \). For \( D \in \Gamma \) and \( t \in \Lambda^\Gamma \), \( t^D \in \Gamma \) is the pseudoterm obtained by adding \( D \) as domain to every \( \Lambda \) abstraction in \( t \). (So e.g. \( \lambda x: A. M \) is replaced by \( \lambda x: A.D.t \).) For reasons of readability we adapt here the convention to use capitals for pseudoterms and small characters for elements of \( \Lambda^\Gamma \).

Let \( cP_1 \ldots P_n \) and \( Q \) be as in the first case of the lemma. By \( CR_{\beta\eta} \) on \( \Lambda^\Gamma \) we find \( t_1, \ldots, t_n \in \Lambda^\Gamma \) with \( cP_1 \ldots P_n =_{\beta\eta} Q_{t_1} \ldots t_n \) and \( Q \rightarrow_{\beta\eta} cP_1 \ldots t_n \). Using postponement of \( \eta \)-reduction, we find that \( Q \rightarrow_{\beta\eta} \lambda y: A. cQ_1 \ldots cQ_n y \rightarrow_{\eta} cP_1 \ldots t_n \). (Doing as many \( \beta \)-reductions as possible, i.e. we \( \beta \)-reduce all the \( \eta \)-redexes that are also \( \beta \)-redexes.) By 3.2 we find a term \( \lambda y: A. cQ_1 \ldots cQ_n y \equiv \lambda y: cQ_1 \ldots cQ_n y \) and \( \lambda y: A. cQ_1 \ldots cQ_n y \equiv \lambda y: cQ_1 \ldots cQ_n y \). The situation is as follows.
Take for $D$ some closed pseudoterm (or fresh variable), then $\lambda y: D. Q y \overset{\beta_n}{\rightarrow} P$ and $c P \overset{\beta_n}{\rightarrow} Q$. Using Lemma 3.3 we obtain (for $1 \leq i \leq n$),

$$P_i = \beta_n P_i \quad P_i^D = \beta_n P_i$$

so $Q_i = \beta_n Q_i$ and we are done. □

Using the Key Lemma we get the following important properties.

**Proposition 3.5 (UT for functional PTSs)** In a functional PTS we have $\Gamma \vdash M : A \& \Gamma \vdash M : A' \Rightarrow A = \beta_n A'$.

**Theorem 3.6 (SRp)** For $\zeta$ a PTS$_{\beta_n}$, $\zeta \vdash SRp$.

The proofs of the proposition and the theorem are straightforward by redoing the proofs for PTS$_{\beta_n}$ in [Geuvers and Nederhof 1991], using the Key Lemma. As corollaries of SRp and the Key Lemma we find: If $\Gamma \vdash Q \Pi x: P_1 P_2 : s \in S$ with $Q = \beta_n \Pi x: P_1 P_2$, then $Q \overset{\beta_n}{\rightarrow} \Pi x: Q_1 Q_2$ with $P_i = \beta_n Q_i$ and similar for $x P_1 \ldots P_n$. Further, if $\Gamma \vdash M : Q$ with $Q = \beta_n s \in S$, then $Q \overset{\beta_n}{\rightarrow} s$. (These are proved using the fact that a type is not a $\lambda$-abstraction.)

We shall now turn to the proof of SRp for functional, normalizing systems. Again the Key Lemma is used and, as said, we also need some form of strengthening to prove SRp. First we state another lemma, which is required for strengthening.

**Lemma 3.7 In a functional normalizing PTS$_{\beta_n}$:**

$$\begin{align*}
\Gamma \vdash D : s \\
\Gamma \vdash D' : s' \\
D = \beta_n D'
\end{align*} \Rightarrow s \equiv s'.
$$

There are essentially two ways to prove this lemma: It can be proved directly (it requires some sublemmas), but also by first adding strengthening as a rule to the system, that is we have an extra derivation rule called (strengthening)

$$\Gamma_1, x:A, \Gamma_2 \vdash M : B \quad \text{if } x \notin \text{FV}(\Gamma_2, M) \Rightarrow \Gamma_1, \Gamma_2 \vdash M : B$$

In the second case one has to redo the proof for SRp (straightforward) and prove SRp$_{\beta_n}$ (using the rule (strengthen)). Then the proof of CR$_{\beta_n}$ and CON$_{\beta_n}$ can be done in the same way as will be done here (in 3.12 - 3.16.). From CON$_{\beta_n}$ one then proves that Lemma 3.7 holds in the system without the (strengthen) rule and so CR$_{\beta_n}$ and CON$_{\beta_n}$ hold in the system without (strengthen). The direct proof involves some sublemmas. We give them here without a detailed proof.

**Sublemma 3.8** $\Gamma, x:D \vdash M : C$, $\Gamma \vdash D' : s \in S$, $D = \beta_n D'$ then $\Gamma, x:D' \vdash M : C$.

**Sublemma 3.9** $\Gamma \vdash B : s$, $\Gamma \vdash B' : s'$, $B = \beta_n B'$, $B$ in $\beta_n$ then $s \equiv s'$.

The proof of the first is by changing everywhere in the derivation tree of $\Gamma, x:D \vdash M : C$ the declaration $x : D$ into $x : D'$. We introduce $x$ as $x : D'$ (with the rule (var) or (weak)) and if $x$ is introduced with the (var) rule, we immediately apply (conv$_n$) to conclude $\Gamma, x:D' \vdash x : D$. The proof of the second is by induction on $B$, using the first sublemma and the Key Lemma. Now Lemma 3.7 easily follows.

Using 3.7 we can prove a weak form of strengthening for functional normalizing PTS$_{\beta_n}$ which is sufficient to prove SRp and which also implies strengthening itself. The proof uses also the Key Lemma and its corollaries.

**Lemma 3.10 In a functional, normalizing PTS$_{\beta_n}$:**

$$\begin{align*}
\Gamma_1, x:A, \Gamma_2 \vdash M : B \\
x \notin \text{FV}(\Gamma_2, M)
\end{align*} \Rightarrow \Gamma_1, \Gamma_2 \vdash M : B'$$

**Proposition 3.11 (SRp)** For $\zeta$ a functional, normalizing PTS$_{\beta_n}$, $\zeta \vdash SRp$.

**Proof** The proof is by proving simultaneously the case for a one step reduction in the context or in the subject. The only interesting case is when $\Gamma :- \pi x:A.M : C$ with $x \notin \text{FV}(M)$ and we have to prove $\Gamma \vdash M : C$. By the stripping Lemma (2.4), we find a term $B$ with $\Pi x:A.B = \beta_n C$ and $\Gamma, x:A \vdash M : B$. Again with Stripping Lemma we find a term $\Pi y:D.E$ with $\Gamma, x:A \vdash M : \Pi y:D.E, E[x/y] = \beta_n B$ and $D = \beta_n A$. By Lemma 3.10 we find that $\Gamma, x:A \vdash M : F$ with $F = \beta_n \Pi y:D.E = \beta_n \Pi x:A.B = \beta_n C$. Applying (conv$_n$) we conclude $\Gamma, x:A \vdash M : C$. □

The next lemmas will establish the proof of CON$_{\beta_n}$ on Term($\Gamma, A$) for functional normalizing Pure Type Systems. So suppose we work in such a PTS$_{\beta_n}$.

**Lemma 3.12**

$$\begin{align*}
\Gamma \vdash M : A \\
\Gamma \vdash M' : A' \\
M = \beta_n M' \\
\Gamma \vdash M \equiv M'
\end{align*} \Rightarrow M \equiv M' \quad \text{if } M, M' \text{ in } \beta_n$$

(For the definition of $\equiv$, see the remarks after Definition 3.1.)

**Proof** $M$ and $M'$ have the same structure (apart from the domains), say $M = \lambda x_1: A_1 \ldots x_n: A_n.N$ and $M' = \lambda x_1: A_1' \ldots x_n: A_n'.N'$, with $N$ and $N'$ not abstractions. From the type of $M$ and $M'$ we conclude that $A_i = \beta_n A_i'$. Now compare from left to right all domains in $N$ and $N'$. Say $B$ occurs as $z R_1 \ldots R_q (\lambda y_1: E_1 \ldots \lambda y_p: E_p.xz:B.P)$ in $N$ and $B'$ occurs as $z R_1' \ldots R_q'(\lambda y_1: E_1' \ldots \lambda y_p: E_p'.\lambda z: B'.P')$ in $N'$ and for all domains to the left of $B$ (respectively $B'$) we are already done. (This implies that
Lemma 3.13

\[ \Gamma \vdash A : s \\
A \in \beta\eta-nf \\
x \notin FV(C) \]  
\[ \Rightarrow x \notin FV(A). \]

Proof The proof is by induction on the structure of \( A \). For \( A \) a variable or a sort it's trivial. For \( A \equiv \Pi x : A_1 . A_2 \), we are done by induction hypothesis. Suppose now \( A \) is an application term and \( x \in FV(A) \).

Then \( x \) is only free in domains of \( A \). (Note that \(|C| = \beta\eta[|A|] \rightarrow \rightarrow \eta nf(|A|)\), and in typed lambda calculus \( \eta \)-reductions do not remove any free variables, so \( x \notin FV(|A|)\)). Say \( B \) is the leftmost domain of \( A \) in which \( x \) occurs free, say in the subterm \( zR_1 \ldots R_n \). Then there is a type for \( x \) in which \( x \) is not free (\( x \) is declared in the context or left to \( B \)), so there is a type for \( zR_1 \ldots R_n \) in which \( x \) is not free, so \( B = \beta\eta E \), for some \( E \) in which \( x \) is not free. Now by induction hypothesis, \( x \notin FV(B) \).

Lemma 3.14 For \( M \in \text{Term} \), if \( M \) in \( \beta\eta\text{-nf} \), then \(|M| \) in \( \beta\eta\text{-nf} \).

Proof Suppose \(|M| \) is not in \( \beta\eta\text{-nf} \). Then there is a \( \eta \)-redex in \(|M|\), which is not an \( \eta \)-redex in \( M \), say \( \lambda x_1 . \ldots . x_n . [\ldots] \). Then \( x \notin FV(N) \), but \( x \notin FV(|N|) \), so \( x \) is only free in domains of \( N \). Say \( B \) is the leftmost domain in which \( x \) is free, and say \( B \) occurs in the subterm \( zR_1 \ldots R_n \). Then there is a type for \( x \) in which \( x \) is not free (\( x \) is declared in the context or left to \( B \)), so there is a type for \( zR_1 \ldots R_n \) in which \( x \) is not free, so \( B = \beta\eta E \), for some \( E \) in which \( x \) is not free. Now by induction hypothesis, \( x \notin FV(B) \).

Lemma 3.15 (CON\(\beta\eta \) for types) Let \( s, s' \in S \).

\[ \Gamma \vdash A : s \\
\Gamma \vdash B : s' \\
A = \beta\eta B \]  
\[ \Rightarrow A \equiv B. \]

Proof By induction on the structure of \( A \), using the Key Lemma, 3.12 and 3.14.

If \( A \equiv \Pi x : A_1 . A_2 \), then \( B = \beta\eta \Pi x : B_1 . B_2 \) with \( A_1 = \beta\eta B_1 \) and \( A_2 = \beta\eta B_2 \) (by Key Lemma). By induction hypothesis \( A_1 \equiv B_1 \) and \( A_2 \equiv B_2 \).

If \( A \equiv \lambda x : A_1 . A_2 \), then \( B = \beta\eta \lambda x : B_1 . B_2 \) with \( \lambda x : A_1 . A_2 \equiv \lambda x : B_1 . B_2 \) (by Lemma 3.13). By induction hypothesis \( A_1 \equiv B_1 \) and \( A_2 \equiv B_2 \) (by Key Lemma).

Theorem 3.16 (CON\(\beta\eta \))

\[ \Gamma \vdash M : A \\
M = \beta\eta M' \]  
\[ \Rightarrow M \downarrow_{\beta\eta} M'. \]

Proof Define \( N \equiv nf(M), N' \equiv nf(M') \). We prove \( N \equiv N' \) and we are done.

By \( SR_\eta \) we find \( \Gamma \vdash N : A \) and \( \Gamma \vdash N' : A \). By 3.14, \(|N| \) and \(|N'| \) are in normal form, so \(|N| \equiv |N'| \). By 3.12, \( N \equiv N' \): All respective domains in \( N \) and \( N' \) are \( \beta\eta \)-equal. By CON\(\beta\eta \) for types (3.15), all respective domains in \( N \) and \( N' \) are syntactically equal (\( \equiv \)), so \( N \equiv N' \).
to Term(Fw) is defined. This mapping can easily be extended to CC with (conv) such that also \( \eta \)-reductions are preserved and all the required properties of the mapping still hold (using the Key Lemma.) We conjecture here the general theorem that, if a PTS is (strongly) normalizing, then the PTS is.

If we look at the Church-Rosser property from a point of view as to how to compute the common reduct, we see that the situation is really a bit more complicated then for untyped lambda calculus. In untyped lambda calculus, if \( M \rightarrow M_1 \) and \( M \rightarrow M_2 \), a common reduct of \( M_1 \) and \( M_2 \) can be found using complete developments. (See [Barendregt 1984].) Here one has to do something more, namely reducing the domains: Consider e.g. \( M := \lambda x:A.x \) and \( M_1 := \lambda x:A.x \) and \( M_2 := \lambda y:B.y \). There are no residuals of the \( \beta \)-redex in \( M_2 \), nor are there any residuals of the \( \eta \)-redex in \( M_1 \), so we have a complete development of the set containing both redexes, but \( M_1 \not\rightarrow M_2 \). (They would have been in the untyped case.) We still have to unity \( A \) and \( B \).

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References


