A remark on the Dunkl differential-difference operators

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§1. Introduction
Let $E$ be a Euclidean vector space of dimension $n$ with inner product $(\cdot, \cdot)$. For $\alpha \in E$ with $(\alpha, \alpha) = 2$ we write

$$(1.1) \quad r_\alpha(\lambda) = \lambda - (\alpha, \lambda)\alpha, \ \lambda \in E$$

for the orthogonal reflection in the hyperplane perpendicular to $\alpha$.

**Definition 1.1.** A normalized root system $R$ in $E$ is a finite set of non zero vectors in $E$, normalized by $(\alpha, \alpha) = 2 \ \forall \alpha \in R$, such that $r_\alpha(\beta) \in R \ \forall \alpha, \beta \in R$.

Let $R \subset E$ be a normalized root system. We write $W = W(R)$ for the group generated by the reflections $r_\alpha, \alpha \in R$. Denote by $\mathbb{C}[E]$ the algebra of $\mathbb{C}$-valued polynomial functions on $E$. For $w \in W, \xi \in E, \alpha \in R$ introduce the operators

$$(1.2) \quad w, \partial_\xi, \Delta_\alpha : \mathbb{C}[E] \longrightarrow \mathbb{C}[E]$$

by

$$(1.3) \quad (wp)(\lambda) = p(w^{-1}\lambda)$$

$$(1.4) \quad (\partial_\xi p)(\lambda) = \frac{d}{dt}\{p(\lambda + t\xi)\}_{t=0}$$

$$(1.5) \quad (\Delta_\alpha p)(\lambda) = \frac{p(\lambda) - p(r_\alpha \lambda)}{(\alpha, \lambda)}.$$

**Remark 1.2.** The operators $\Delta_\alpha, \alpha \in R$ were studied by Bernstein, Gel’fand and Gel’fand and are related to the Schubert cells and the cohomology of $G/P$ [BGG]. They are the infinitesimal analogues of the Demazure operators [De 1,2].

Let $R_+ = \{\alpha \in R; (\alpha, \lambda) > 0\}$ for some fixed generic $\lambda \in E$ be a positive subsystem of $R$. 

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Definition 1.3. Suppose for $\alpha \in R$ we have given $k_\alpha \in \mathbb{C}$ with $k_{w\alpha} = k_\alpha \ \forall w \in W, \forall \alpha \in R$. For $\xi \in E$ the operator

\begin{equation}
D_\xi = \partial_\xi + \sum_{\alpha \in R_+} k_\alpha (\alpha, \xi) \Delta_\alpha : \mathbb{C}[E] \rightarrow \mathbb{C}[E]
\end{equation}

is called the Dunkl differential-difference operator.

Remark 1.4. It is easy to see that $D_\xi$ is independent of the choice of the positive subsystem $R_+ \subset R$. If we write $q_\alpha = e^{2\pi i k_\alpha}$ then one can think of the operator $D_\xi$ as a $q$-analogue (corresponding to the case $k_\alpha \rightarrow 0$) of the directional derivative $\partial_\xi$. We also write $D_\xi = D_\xi(k)$ to indicate the dependence on $k \in K = \{ k = (k_\alpha)_{\alpha \in R} \in \mathbb{C}^R; k_{w\alpha} = k_\alpha \ \forall w \in W, \forall \alpha \in R \}$.

Theorem 1.5 (Dunkl [Du]): We have $D_\xi D_\eta = D_\eta D_\xi \ \forall \xi, \eta \in E$.

Let $\mathbb{C}[E^*]$ be the symmetric algebra on $E$. For $\pi \in \mathbb{C}[E^*]$ we write $\partial_\pi$ when we think of $\pi$ as a constant coefficient differential operator on $E$ (rather than a polynomial function on $E^*$). In view of Theorem 1.5 the constant coefficient differential operator $\partial_\pi$ has a well defined $q$-analogue

\begin{equation}
D_\pi : \mathbb{C}[E] \rightarrow \mathbb{C}[E]
\end{equation}

defined for a monomial $\pi = \xi_1^{d_1} \ldots \xi_n^{d_n}$ by

\begin{equation}
D_\pi = D_\pi (k) = D_{\xi_1}^{d_1} \ldots D_{\xi_n}^{d_n}
\end{equation}

and extended by linearity.

Theorem 1.6 (Dunkl [Du]): Suppose $\xi_1, \ldots, \xi_n$ is an orthonormal basis for $E$. The $q$-analogue of the Laplacian is given by

\begin{equation}
\sum_{j=1}^n D_{\xi_j}^2 = \sum_{j=1}^n \partial_{\xi_j}^2 + 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{\langle \alpha, \cdot \rangle} \{ \partial_\alpha - \Delta_\alpha \}.
\end{equation}

In Section 2 we review the proofs of both theorems as given by Dunkl. We write $\mathbb{C}[E]^W$ and $\mathbb{C}[E^*]^W$ for the space of $W$-invariants in $\mathbb{C}[E]$ and $\mathbb{C}[E^*]$ respectively. We denote by $\mathbb{A}$ the associative algebra of endomorphisms of $\mathbb{C}[E]$ generated by $\langle \xi, \cdot \rangle$ and $D_\eta$ for $\xi, \eta \in E$. Let $\mathbb{A}^W = \{ D \in \mathbb{A}; wD = Dw \ \forall w \in W \}$ be the subalgebra of $W$-invariant operators in $\mathbb{A}$, and denote by

\begin{equation}
\text{Res} (D) : \mathbb{C}[E]^W \rightarrow \mathbb{C}[E]^W, \ D \in \mathbb{A}^W
\end{equation}
the restriction of $D$ to $\mathbb{C}[E]^W$. Clearly $\text{Res}: \mathbb{A}^W \rightarrow \text{End}(\mathbb{C}[E]^W)$ is a homomorphism of algebras. Since $wD\xi w^{-1} = D_{w\xi} \forall w \in W, \forall \xi \in E$ we have $D_{\pi} \in \mathbb{A}^W \forall \pi \in \mathbb{C}[E^*]^W$.

**Theorem 1.7.** Suppose by the Chevalley theorem that $\mathbb{C}[E]^W = \mathbb{C}[p_1, \ldots, p_n]$ with $p_1, \ldots, p_n$ homogeneous of degrees $d_1 \leq \ldots \leq d_n$. Then the set

\begin{equation}
\{\text{Res}(D_{\pi}); \pi \in \mathbb{C}[E^*]^W\}
\end{equation}

is a commuting family of differential operators in the Weyl algebra $\mathbb{C}[k, p_1, \ldots, p_n, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}]$ containing the operator

\begin{equation}
\text{Res}(\sum_{j=1}^{n} D_{\xi_j}^2) = \sum_{j=1}^{n} \partial_{\xi_j}^2 + 2 \sum_{\alpha \in \mathbb{R}^+} k_\alpha \frac{1}{(\alpha, \cdot)} \partial_\alpha.
\end{equation}

**Remark 1.8.** The proof of this theorem is a triviality. However it can be reformulated as the complete integrability for the generalized non periodic Calogero-Moser system (both on the quantum mechanical level of differential operators and on the classical mechanical level of symbols). For root systems $R$ of type $A$ the complete integrability of the Calogero-Moser system was first established by Moser by realizing the system as a Lax pair [Mo]. The method of Moser was extended by Olshanetsky and Perelomov to cover the root systems $R$ of classical type [OP]. In the crystallographic case $(\alpha, \beta)^2 \in \mathbb{Z}$ \(\forall \alpha, \beta \in R\) the above theorem has been obtained before by Opdam using transcendental methods [HO, Hel1, Op 1,2, He 2].

Suppose $S \subset R$ is a set of roots in $R$ invariant under $W$. Let $S_+ = S \cap R_+$ and put

\begin{equation}
p_S(\cdot) = \prod_{\alpha \in S_+} (\alpha, \cdot) \in \mathbb{C}[E]
\end{equation}

\begin{equation}
\pi_S = \prod_{\alpha \in S_+} \alpha \in \mathbb{C}[E^*].
\end{equation}

Clearly we have

\begin{equation}
w p_S = \chi(w)p_S, w \pi_S = \chi(w)\pi_S \ \forall w \in W
\end{equation}

for some one dimensional character $\chi = \chi_S$ of $W$, and conversely every $p \in \mathbb{C}[E]$ with $wp = \chi(w)p \ \forall w \in W$ is divisible in $\mathbb{C}[E]$ by $p_S$. Although $p_S^{-1}D_{\pi_S}(k)$ need not be an endomorphism of $\mathbb{C}[E]$ it follows that $p_S^{-1}D_{\pi_S}(k)(p) \in \mathbb{C}[E]^W \ \forall p \in \mathbb{C}[E]^W$, and hence

\begin{equation}
G(1_S, k) := \text{Res}(p_S^{-1}D_{\pi_S}(k)) \in \text{End}(\mathbb{C}[E]^W)
\end{equation}
is a well defined endomorphism of $\mathbb{C}[E]^{W}$. We also write
\[
G(-1, k) := \text{Res}(D_{\pi S}(k - 1_{S}) \cdot p_{S}) \in \text{End}(\mathbb{C}[E]^{W})
\]
where $k - 1_{S} \in K$ is the multiplicity function by $(k - 1_{S})_{\alpha} = k_{\alpha} - 1$ for $\alpha \in S$ and $(k - 1_{S})_{\alpha} = k_{\alpha}$ for $\alpha \in R \setminus S$.

**Theorem 1.9.** The operators (1.16) and (1.17) are differential operators in the Weyl algebra $\mathbb{C}[k, p_{1}, \ldots, p_{n}, \frac{\partial}{\partial p_{1}}, \ldots, \frac{\partial}{\partial p_{n}}]$ and satisfy the shift relations
\[
(1.18) \quad G(1, k)\text{Res}(D_{\pi}(k)) = \text{Res}(D_{\pi}(k + 1_{S}))G(1, k)
\]
\[
(1.19) \quad G(-1, k)\text{Res}(D_{\pi}(k)) = \text{Res}(D_{\pi}(k - 1_{S}))G(-1, k)
\]
$\forall \pi \in \mathbb{C}[E^{*}]^{W}$. Here $(k \pm 1_{S})_{\alpha} = k_{\alpha} \pm 1 \forall \alpha \in S$ and $(k \pm 1_{S})_{\alpha} = k_{\alpha} \forall \alpha \in R \setminus S$.

The proofs of both Theorem 1.7 and 1.9 will be given in Section 3.

**Remark 1.10.** In the terminology of Opdam the operator (1.16) is a raising operator and the operator (1.17) a lowering operator for the commuting family (1.11). Again in the crystallographic case the above theorem was obtained by Opdam [Op 2]. Recall Macdonald’s (infinitesimal) constant term conjecture, which says that for $R(s) > 0$
\[
(1.20) \quad \int_{E} \prod_{\alpha \in R_{+}} |(\alpha, \lambda)|^{2s} d\gamma(\lambda) = \prod_{j=1}^{n} \left(\frac{(sd_{j})!}{s!}\right),
\]
where $d\gamma(\lambda) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}(\lambda, \lambda)} d\lambda$ is the Gaussian measure on $E$ [Ma].

The same arguments as given in [Op 3, Section 6] show that the evaluation of this integral is equivalent with
\[
(1.21) \quad G(-1, k)(1) = |W| \cdot \prod_{i=1}^{n} \prod_{j=1}^{m_{i}} (d_{i}k - j),
\]
where $-1 = -1_{R}$ and $k = k_{\alpha} \forall \alpha \in R$. In turn this latter formula is related to the normalization of the “multivariable Bessel function associated with $R$” at $\xi = 0$. This normalization problem has been analyzed by Opdam, and the desired formula (1.21) can be obtained [Op 4]. After this one can proceed as in [Op 3, Section 7] to compute the Bernstein-Sato polynomial of the discriminant without the crystallographic restriction in accordance with a conjecture of Yano and Sekiguchi [YS].
§2. The Dunkl differential-difference operators.

Using the bracket $\cdot, \cdot$ for the commutator of endomorphisms of $\mathbb{C}[E]$ we can write for $\xi, \eta \in E$

\begin{equation}
[D_\xi, D_\eta] = I + II + III
\end{equation}

with

\begin{align}
I &= [\partial_\xi, \partial_\eta] = 0 \\
II &= \sum_{\alpha \in R_+} k_\alpha \{(\alpha, \xi)[\Delta_\alpha, \partial_\eta] + (\alpha, \eta)[\partial_\xi, \Delta_\alpha]\} \\
III &= \sum_{\alpha, \beta \in R_+} k_\alpha k_\beta (\alpha, \xi)(\beta, \eta)[\Delta_\alpha, \Delta_\beta].
\end{align}

Lemma 2.1. For $\xi \in E, \alpha \in R$ we have

\begin{equation}
[\partial_\xi, \Delta_\alpha] = \frac{(\alpha, \xi)}{(\alpha, \cdot)} \{r_\alpha \partial_\alpha - \Delta_\alpha\}.
\end{equation}

Proof: Using the definition $\Delta_\alpha = \frac{1}{(\alpha, \cdot)} (1 - r_\alpha)$ we get

\begin{align*}
[\partial_\xi, \Delta_\alpha] &= [\partial_\xi, \frac{1}{(\alpha, \cdot)} (1 - r_\alpha)] + \frac{1}{(\alpha, \cdot)} [\partial_\xi, 1 - r_\alpha] \\
&= -\frac{(\alpha, \xi)}{(\alpha, \cdot)} (1 - r_\alpha) + \frac{1}{(\alpha, \cdot)} r_\alpha (\partial_\xi - r_\alpha \xi) \\
&= -\frac{(\alpha, \xi)}{(\alpha, \cdot)} \Delta_\alpha + \frac{(\alpha, \xi)}{(\alpha, \cdot)} r_\alpha \partial_\alpha. \quad \text{Q.E.D}
\end{align*}

Using (2.5) the second term (2.3) can be rewritten as

\begin{equation}
II = \sum_{\alpha \in R_+} k_\alpha \frac{(\alpha, \xi)(\alpha, \eta)}{(\alpha, \cdot)} \{r_\alpha \partial_\alpha - \Delta_\alpha\}(-1 + 1) = 0.
\end{equation}

The third term (2.4) can be written as

\begin{equation}
III = \sum_{\alpha, \beta \in R_+} k_\alpha k_\beta \{(\alpha, \xi)(\beta, \eta) - (\alpha, \eta)(\beta, \xi)\} \Delta_\alpha \Delta_\beta
\end{equation}

and for the proof of Theorem 1.5 it remains to verify the vanishing of this third term.

Proposition 2.2. Suppose $B(\cdot, \cdot)$ is a bilinear form on $E$ such that

\begin{equation}
B(r_\alpha \lambda, r_\alpha \mu) = B(\mu, \lambda) \quad \forall \lambda, \mu \in E, \forall \alpha \in R \cap \text{span} \langle \lambda, \mu \rangle.
\end{equation}
If $w \in W$ is a pure rotation (i.e. $\dim \text{Im}(w - \text{Id}) = 2$) then

$$
\sum_{\alpha, \beta \in R_+, r\alpha r\beta = w} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(\alpha, \cdot)(\beta, \cdot)} = 0
$$

and

$$
\sum_{\alpha, \beta \in R_+, r\alpha r\beta = w} k_\alpha k_\beta B(\alpha, \beta) \Delta_\alpha \Delta_\beta = 0.
$$

**Proof:** Using the definition $\Delta_\alpha = \frac{1}{(\alpha, \cdot)}(1 - r_\alpha)$ the left hand side of (2.10) can be written as a sum of the following three terms

$$
A = \sum_{\alpha, \beta \in R_+, r\alpha r\beta = w} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(\alpha, \cdot)(\beta, \cdot)}
$$

$$
B = - \sum_{\alpha, \beta \in R_+, r\alpha r\beta = w} k_\alpha k_\beta B(\alpha, \beta) \left\{ \frac{1}{(\alpha, \cdot)(r_\alpha r_\beta, \cdot)} r_\alpha + \frac{1}{(\alpha, \cdot)(\beta, \cdot)} r_\beta \right\}
$$

$$
C = \sum_{\alpha, \beta \in R_+, r\alpha r\beta = w} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(\alpha, \cdot)(r_\alpha r_\beta, \cdot)} r_\alpha r_\beta
$$

with the summations over the same index set as in (2.9) and (2.10).

Let $S = R \cap \text{Im}(w - \text{Id})$ be the normalized root system of the largest dihedral group $W(S)$ containing $w$. If $w = r_\alpha r_\beta$ then for $\gamma \in S$ we have $r_\gamma w r_\gamma = w^{-1}$ and hence $r_{r_\gamma r_\alpha r_\beta} = r_\beta r_\alpha$. We claim that $r_\gamma A = A \forall \gamma \in S$. Indeed we have

$$
r_\gamma A = \sum_{\alpha, \beta \in R_+, r\alpha r\beta = w} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(r_\gamma \alpha, \cdot)(r_\gamma \beta, \cdot)}
$$

$$
= \sum_{\alpha, \beta \in R_+, r\alpha r\beta = w} k_\alpha k_\beta B(r_\gamma \alpha, r_\gamma \beta) \frac{1}{(\alpha, \cdot)(\beta, \cdot)}
$$

$$
= \sum_{\alpha, \beta \in R_+, r\alpha r\beta = w} k_\alpha k_\beta B(\beta, \alpha) \frac{1}{(\alpha, \cdot)(\beta, \cdot)}
$$

$$
= A
$$

since the summation in (2.9) is independent of the choice of $R_+$. Let $S_+ = R_+ \cap S$ and put $p_S = \prod_{\alpha \in S_+} (\alpha, \cdot)$. Then $p_S$ transforms under the group $W(S)$ according to the sign character and every polynomial in $\mathbb{C}[E]$ transforming under $W(S)$ according to the sign character is divisible in $\mathbb{C}[E]$ by $p_S$. Now observe that $p_S A \in \mathbb{C}[E]$ transforms
under $W(S)$ according to the sign character. Hence $A \in \mathbb{C}[E]$. Since $A$ is homogeneous of degree minus two we have $A = 0$. This proves (2.9).

Since $w = r_\alpha r_\beta = r_{r_\alpha r_\alpha}$ and $B(\alpha, \beta) = B(r_\alpha \beta, r_\alpha \alpha) = -B(r_\alpha \beta, \alpha)$ the vanishing of the term (2.12) is clear, and for the term (2.13) we can write $C = -Aw = 0$. Q.E.D.

**Lemma 2.3.** For $\xi, \eta \in E$ fixed the bilinear form

\[(2.14) \quad B(\lambda, \mu) = (\lambda, \xi)(\mu, \eta) - (\lambda, \eta)(\mu, \xi)\]

on $E$ satisfies condition (2.8).

**Proof:** Clearly $B(\mu, \lambda) = -B(\lambda, \mu)$ is an alternating form. For $\lambda \in E, \lambda \neq 0$ we write $\lambda' = \sqrt{2|\lambda|^{-1}} \lambda$ and get

\[B(r_\lambda \lambda', r_\lambda' \mu) = B(-\lambda, \mu - (\lambda', \mu)\lambda') = B(-\lambda, \mu) = B(\mu, \lambda).\]

Hence for $\lambda, \mu \in E$ generic we get by continuity

\[B(r_\nu \lambda, r_\nu \mu) = B(\mu, \lambda) \quad \forall \nu \in \text{span} \{\lambda, \mu\}, (\nu, \nu) = 2. \quad \text{Q.E.D.}\]

The proof of Theorem 1.5 now follows by regrouping the terms in (2.7) as a sum over $\{\alpha, \beta \in R_+; r_\alpha r_\beta = w\}$ where $w \in W$ runs over the pure rotations in $W$ and by applying (2.10).

The proof of Theorem 1.6 is just an easy calculation.

\[
\sum_{j=1}^{n} D_{\xi_j}^2 = \sum_{j=1}^{n} (\partial \xi_j + \sum_{\alpha \in R_+} k_\alpha (\alpha, \xi_j) \Delta_\alpha)^2
\]

\[= \sum_{j=1}^{n} \left\{ \partial^2 \xi_j + \sum_{\alpha \in R_+} k_\alpha (\alpha, \xi_j) (\partial \xi_j \Delta_\alpha + \Delta_\alpha \partial \xi_j) + \sum_{\alpha, \beta \in R_+} k_\alpha k_\beta (\alpha, \xi_j) (\beta, \xi_j) \Delta_\alpha \Delta_\beta \right\}
\]

\[= \sum_{j=1}^{n} \partial^2 \xi_j + \sum_{\alpha \in R_+} k_\alpha (\partial \Delta_\alpha + \Delta_\alpha \partial) + \sum_{\alpha, \beta \in R_+} k_\alpha k_\beta (\alpha, \beta) \Delta_\alpha \Delta_\beta.
\]

The third term vanishes by Proposition 2.2 and because $\Delta^2_\alpha = 0$. Using Lemma 2.1 we get

\[\partial_\alpha \Delta_\alpha + \Delta_\alpha \partial_\alpha = [\partial_\alpha, \Delta_\alpha] + 2\Delta_\alpha \partial_\alpha
\]

\[= \frac{(\alpha, \alpha)}{(\alpha, \cdot)} \left\{ r_\alpha \partial_\alpha - \Delta_\alpha \right\} + \frac{2}{(\alpha, \cdot)} (1 - r_\alpha) \partial_\alpha
\]

\[= \frac{2}{(\alpha, \cdot)} \left\{ \partial_\alpha - \Delta_\alpha \right\}.
\]
§3. The Opdam shift operators.
Recall that \( D \in \text{End}(\mathbb{C}[p_1, \ldots, p_m]) \) is a differential operator of degree \( \leq d \) if and only if
\[
(3.1) \quad \text{ad}(p)^{d+1}(D) = 0 \quad \forall p \in \mathbb{C}[p_1, \ldots, p_m].
\]
Hence the fact that the operators (1.11), (1.16) and (1.17) are differential operators is clear from
\[
(3.2) \quad \text{ad}(p)(D_\xi) = \text{ad}(p)(\partial_\xi) = -\partial_\xi(p)
\]
\[
(3.3) \quad \text{ad}(p)^2(D_\xi) = 0
\]
\( \forall p \in \mathbb{C}[E]^W, \forall \xi \in E. \) Hence Theorem 1.7 is an immediate consequence of Theorem 1.5 and Theorem 1.6.

**Theorem 3.1.** For the \( q \)-analogue of the Laplacian we have
\[
(3.4) \quad \text{Res}(p_S^{-1} \circ \left\{ \sum_{j=1}^{n} D_{\xi_j}^2(k) \right\} \circ p_S) = \text{Res}(\sum_{j=1}^{n} D_{\xi_j}^2(k + 1_S)).
\]

**Proof:** First we observe that the left hand side of (3.4) is a well defined endomorphism of \( \mathbb{C}[E]^W. \) We now use Theorem 1.6 and just calculate term by term. For the first term we get
\[
p_S^{-1} \circ \left\{ \sum_{j=1}^{n} \partial_{\xi_j}^2 \right\} \circ p_S = \sum_{j=1}^{n} \partial_{\xi_j}^2 + 2 \sum_{\alpha \in S_+} \frac{1}{(\alpha, \cdot)} \partial_\alpha + p_S^{-1}(\sum_{j=1}^{n} \partial_{\xi_j}^2)(p_S)
\]
\[
= \sum_{j=1}^{n} \partial_{\xi_j}^2 + 2 \sum_{\alpha \in S_+} \frac{1}{(\alpha, \cdot)} \partial_\alpha.
\]
For the second term we get
\[
p_S^{-1} \circ \left\{ 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_\alpha \right\} \circ p_S = 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_\alpha + p_S^{-1} \cdot \left( 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_\alpha \right)(p_S)
\]
\[
= 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_\alpha + 2 \sum_{\alpha \in R_+, \beta \in S_+} k_{\alpha} \frac{(\alpha, \beta)}{(\alpha, \cdot)(\beta, \cdot)}
\]
\[
= 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_\alpha + 2 \sum_{\beta \in S_+} k_{\beta} \frac{2(\beta, \cdot)^2}{(\beta, \cdot)^2}
\]
\[
+ 2 \sum_{\alpha \in R_+, \beta \in S_+} k_{\alpha} \frac{(\alpha, \beta)}{(\alpha, \cdot)(\beta, \cdot)}
\]
\[
= 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_\alpha + 2 \sum_{\beta \in S_+} k_{\beta} \frac{2}{(\beta, \cdot)^2}.
\]
by the same argument as in the proof of Proposition 2.2. Finally for the third term we have
\[
p^{-1}_S \circ \left\{ 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)} \Delta_\alpha \right\} \circ p_S = 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)^2} \{ 1 - p^{-1}_S \circ r_\alpha \circ p_S \}
\]
\[
= 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)^2} \{ 1 - \chi_S(r_\alpha) r_\alpha \}
\]
\[
= 2 \sum_{\alpha \in S_+} k_\alpha \frac{1}{(\alpha, \cdot)^2} \{ 1 + r_\alpha \} + 2 \sum_{\alpha \in R_+ \setminus S_+} k_\alpha \frac{1}{(\alpha, \cdot)} \Delta_\alpha
\]
\[
= 2 \sum_{\alpha \in S_+} k_\alpha \frac{2}{(\alpha, \cdot)^2} - 2 \sum_{\alpha \in S_+} k_\alpha \frac{1}{(\alpha, \cdot)} \Delta_\alpha
\]
\[
+ 2 \sum_{\alpha \in R_+ \setminus S_+} k_\alpha \frac{1}{(\alpha, \cdot)} \Delta_\alpha.
\]
Taking all three terms together yields
\[
p^{-1}_S \circ \left\{ \sum_{j=1}^n D_{\xi_j}^2(k) \right\} \circ p_S = \sum_{j=1}^n \partial_{\xi_j}^2 + 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)} \partial_\alpha + 2 \sum_{\alpha \in S_+} \frac{1}{(\alpha, \cdot)} \partial_\alpha
\]
\[
+ 2 \sum_{\alpha \in S_+} k_\alpha \frac{1}{(\alpha, \cdot)} \Delta_\alpha - 2 \sum_{\alpha \in R_+ \setminus S_+} k_\alpha \frac{1}{(\alpha, \cdot)} \Delta_\alpha. \quad \text{Q.E.D.}
\]

**Corollary 3.2.** We have the shift relations
\[
G(1_S, k) \text{Res} \left( \sum_{j=1}^n D_{\xi_j}^2(k) \right) = \text{Res} \left( \sum_{j=1}^n D_{\xi_j}^2(k + 1_S) \right) G(1_S, k)
\]
\[
G(-1_S, k) \text{Res} \left( \sum_{j=1}^n D_{\xi_j}^2(k) \right) = \text{Res} \left( \sum_{j=1}^n D_{\xi_j}^2(k - 1_S) \right) G(-1_S, k).
\]

**Proof:** Indeed we have
\[
\text{Res} \left( p^{-1}_S D_{\pi_S}(k) \right) \text{Res} \left( \sum_{j=1}^n D_{\xi_j}^2(k) \right) = \text{Res} \left( \sum_{j=1}^n p^{-1}_S D_{\pi_S}(k) D_{\xi_j}^2(k) \right)
\]
\[
= \text{Res} \left( \sum_{j=1}^n p^{-1}_S D_{\xi_j}^2(k) D_{\pi_S}(k) \right)
\]
\[
= \text{Res} \left( \sum_{j=1}^n p^{-1}_S D_{\xi_j}^2(k) p_S \right) \text{Res} \left( p^{-1}_S D_{\pi_S}(k) \right)
\]
\[
= \text{Res} \left( \sum_{j=1}^n D_{\xi_j}^2(k + 1_S) \right) \text{Res} \left( p^{-1}_S D_{\pi_S}(k) \right)
\]

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which proves (3.5). The relation (3.6) is proved similarly. Q.E.D.

**Theorem 3.3.** As endomorphisms of $\mathbb{C}[E]$ the operators

\[
E = \frac{1}{2} \sum_{j=1}^{n} (\xi_j, \cdot)^2
\]

(3.7)

\[
H = \sum_{j=1}^{n} (\xi_j, \cdot) \partial_{\xi_j} + \left( \frac{n}{2} + \sum_{\alpha \in R_+} k_\alpha \right)
\]

(3.8)

\[
F = -\frac{1}{2} \sum_{j=1}^{n} D_{\xi_j}^2
\]

(3.9)

satisfy the commutation relations of $sl(2)$:

\[
\]

(3.10)

**Proof:** The Euler operator $\sum_{j=1}^{n} (\xi_j, \cdot) \partial_{\xi_j}$ acts as multiplication by $d$ on the space of homogeneous polynomials in $\mathbb{C}[E]$ of degree $d$. Hence the commutation relations $[H, E] = 2E, \ [H, F] = -2F$ rephrase that $E$ and $F$ are homogeneous of degree plus and minus two respectively.

Since $[p, \Delta_\alpha] = 0 \ \forall p \in \mathbb{C}[E]^W, \ \forall \alpha \in R$ we get

\[
[E, D_\xi] = [E, \partial_\xi] = -(\xi, \cdot) \ \forall \xi \in E,
\]

and therefore

\[
[E, F] = -\frac{1}{2} \sum_{j=1}^{n} [E, D_{\xi_j}^2]
\]

\[
= \frac{1}{2} \sum_{j=1}^{n} \{ (\xi_j, \cdot) D_{\xi_j} + D_{\xi_j} (\xi_j, \cdot) \}
\]

\[
= \sum_{j=1}^{n} (\xi_j, \cdot) D_{\xi_j} + \frac{1}{2} \sum_{j=1}^{n} [D_{\xi_j}, (\xi_j, \cdot)]
\]

\[
= \sum_{j=1}^{n} (\xi_j, \cdot) D_{\xi_j} + \frac{n}{2} + \frac{1}{2} \sum_{j=1}^{n} \sum_{\alpha \in R_+} k_\alpha (\alpha, \xi_j) [\Delta_\alpha, (\xi_j, \cdot)]
\]

\[
= \sum_{j=1}^{n} (\xi_j, \cdot) \partial_{\xi_j} + \sum_{\alpha \in R_+} k_\alpha (\alpha, \cdot) \Delta_\alpha + \frac{n}{2} + \sum_{\alpha \in R_+} k_\alpha r_\alpha
\]

\[
= \sum_{j=1}^{n} (\xi_j, \cdot) \partial_{\xi_j} + \left( \frac{n}{2} + \sum_{\alpha \in R_+} k_\alpha \right).
\]

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Here we have used that for $\xi \in E$

$$[\Delta_\alpha, (\xi, \cdot)] = \frac{1}{(\alpha, \cdot)} [r_\alpha, (\xi, \cdot)]$$

$$= \frac{1}{(\alpha, \cdot)} \{(r_\alpha \xi, \cdot) - (\xi, \cdot)\} r_\alpha$$

$$= (\alpha, \xi) r_\alpha.$$  \hspace{1cm} \text{Q.E.D.}

**Proposition 3.4.** Using the inner product $(\cdot, \cdot)$ on $E$ we have an isomorphism between $\mathbb{C}[E]$ and $\mathbb{C}[E^*]$. For $p \in \mathbb{C}[E]$ we write $\pi \in \mathbb{C}[E^*]$ for the corresponding element. For $p \in \mathbb{C}[E]$ homogeneous of degree $d$ we have

$$(3.12) \quad D_\pi = (-1)^d \frac{1}{d!} \text{ad}(F)^d(p).$$

**Proof:** Clearly $\text{ad}(H)D_\pi = -dD_\pi$ and by Theorem 1.5 we have $\text{ad}(F)D_\pi = 0$. Using (3.11) and induction on $d$ (assuming $\pi$ to be a monomial as in (1.9) with $d = d_1 + \cdots + d_n$) it is easy to see that

$$(-1)^d \frac{1}{d!} \text{ad}(E)^d(D_\pi) = p$$

and hence

$$\text{ad}(E)^{d+1}(D_\pi) = 0.$$  

By standard representation theory of $sl(2)$ we conclude (3.12).  \hspace{1cm} \text{Q.E.D.}

**Corollary 3.5.** For $\pi \in \mathbb{C}[E^*]^W$ we have

$$(3.13) \quad \text{Res} \left( p_S^{-1} \circ D_\pi(k) \circ p_S \right) = \text{Res} \left( D_\pi(k + 1_s) \right).$$

**Proof:** This is easily derived from Theorem 3.1 and Proposition 3.4.  \hspace{1cm} \text{Q.E.D.}

The proof of Theorem 1.9 now goes along the same lines as the proof of Corollary 3.2.

**Remark 3.6.** The above type of arguments to use an $sl(2)$ to reduce the computation of higher order operators to those of the second order one go back to Harish-Chandra [Ha].

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References


