A remark on the Dunkl differential-difference operators

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§1. Introduction

Let $E$ be a Euclidean vector space of dimension $n$ with inner product $(\cdot, \cdot)$. For $\alpha \in E$ with $(\alpha, \alpha) = 2$ we write

$$r_\alpha(\lambda) = \lambda - (\alpha, \lambda)\alpha, \ \lambda \in E$$

for the orthogonal reflection in the hyperplane perpendicular to $\alpha$.

**Definition 1.1.** A normalized root system $R$ in $E$ is a finite set of non zero vectors in $E$, normalized by $(\alpha, \alpha) = 2 \ \forall \alpha \in R$, such that $r_\alpha(\beta) \in R \ \forall \alpha, \beta \in R$.

Let $R \subset E$ be a normalized root system. We write $W = W(R)$ for the group generated by the reflections $r_\alpha, \alpha \in R$. Denote by $\mathbb{C}[E]$ the algebra of $\mathbb{C}$-valued polynomial functions on $E$. For $w \in W, \xi \in E, \alpha \in R$ introduce the operators

$$w, \partial_\xi, \Delta_\alpha : \mathbb{C}[E] \longrightarrow \mathbb{C}[E]$$

by

$$wp(\lambda) = p(w^{-1}\lambda)$$

$$\partial_\xi p(\lambda) = \frac{d}{dt}\{p(\lambda + t\xi)\}_{t=0}$$

$$\Delta_\alpha p(\lambda) = \frac{p(\lambda) - p(r_\alpha\lambda)}{(\alpha, \lambda)}.$$ 

**Remark 1.2.** The operators $\Delta_\alpha, \alpha \in R$ were studied by Bernstein, Gel’fand and Gel’fand and are related to the Schubert cells and the cohomology of $G/P$ [BGG]. They are the infinitesimal analogues of the Demazure operators [De 1,2].

Let $R_+ = \{\alpha \in R; (\alpha, \lambda) > 0\}$ for some fixed generic $\lambda \in E$ be a positive subsystem of $R$. 

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**Definition 1.3.** Suppose for $\alpha \in \mathbb{R}$ we have given $k_{\alpha} \in \mathbb{C}$ with $k_{w\alpha} = k_{\alpha} \forall w \in W, \forall \alpha \in R$. For $\xi \in E$ the operator

\begin{equation}
D_\xi = \partial_\xi + \sum_{\alpha \in R_+} k_{\alpha}(\alpha, \xi)\Delta_\alpha : \mathbb{C}[E] \rightarrow \mathbb{C}[E]
\end{equation}

is called the Dunkl differential-difference operator.

**Remark 1.4.** It is easy to see that $D_\xi$ is independent of the choice of the positive subsystem $R_+ \subset \mathbb{R}$. If we write $q_{\alpha} = e^{2\pi ik_{\alpha}}$ then one can think of the operator $D_\xi$ as a $q$-analogue (corresponding to the case $k_{\alpha} \rightarrow 0$) of the directional derivative $\partial_\xi$. We also write $D_\xi = D_\xi(k)$ to indicate the dependence on $k \in K = \{k = (k_{\alpha})_{\alpha \in R} \in \mathbb{C}^R; k_{w\alpha} = k_{\alpha} \forall w \in W, \forall \alpha \in R\}$.

**Theorem 1.5 (Dunkl [Du]):** We have $D_\xi D_\eta = D_\eta D_\xi \forall \xi, \eta \in E$.

Let $\mathbb{C}[E^*]$ be the symmetric algebra on $E$. For $\pi \in \mathbb{C}[E^*]$ we write $\partial_\pi$ when we think of $\pi$ as a constant coefficient differential operator on $E$ (rather than a polynomial function on $E^*$). In view of Theorem 1.5 the constant coefficient differential operator $\partial_\pi$ has a well defined $q$-analogue

\begin{equation}
D_\pi : \mathbb{C}[E] \rightarrow \mathbb{C}[E]
\end{equation}

defined for a monomial $\pi = \xi_1^{d_1}\ldots\xi_n^{d_n}$ by

\begin{equation}
D_\pi = D_\pi(k) = D_{\xi_1}^{d_1}\ldots D_{\xi_n}^{d_n}
\end{equation}

and extended by linearity.

**Theorem 1.6 (Dunkl [Du]):** Suppose $\xi_1, \ldots, \xi_n$ is an orthonormal basis for $E$. The $q$-analogue of the Laplacian is given by

\begin{equation}
\sum_{j=1}^n D_{\xi_j}^2 = \sum_{j=1}^n \partial_{\xi_j}^2 + 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \{\partial_\alpha - \Delta_\alpha\}.
\end{equation}

In Section 2 we review the proofs of both theorems as given by Dunkl.

We write $\mathbb{C}[E]^W$ and $\mathbb{C}[E^*]^W$ for the space of $W$-invariants in $\mathbb{C}[E]$ and $\mathbb{C}[E^*]$ respectively. We denote by $\mathbb{A}$ the associative algebra of endomorphisms of $\mathbb{C}[E]$ generated by (multiplication by) $(\xi, \cdot)$ and $D_\eta$ for $\xi, \eta \in E$. Let $\mathbb{A}^W = \{D \in \mathbb{A}; wD = Dw \forall w \in W\}$ be the subalgebra of $W$-invariant operators in $\mathbb{A}$, and denote by

\begin{equation}
\text{Res}(D) : \mathbb{C}[E]^W \rightarrow \mathbb{C}[E]^W, \quad D \in \mathbb{A}^W
\end{equation}
the restriction of $D$ to $\mathbb{C}[E]^W$. Clearly $\text{Res} : A^W \to \text{End}(\mathbb{C}[E]^W)$ is a homomorphism of algebras. Since $wD_\xi w^{-1} = D_{w\xi} \forall w \in W, \forall \xi \in E$ we have $D_\pi \in A^W \forall \pi \in \mathbb{C}[E^*]^W$.

**Theorem 1.7.** Suppose by the Chevalley theorem that $\mathbb{C}[E]^W = \mathbb{C}[p_1, \ldots, p_n]$ with $p_1, \ldots, p_n$ homogeneous of degrees $d_1 \leq \ldots \leq d_n$. Then the set

$$\{\text{Res}(D_\pi); \pi \in \mathbb{C}[E^*]^W\}$$

is a commuting family of differential operators in the Weyl algebra $\mathbb{C}[k, p_1, \ldots, p_n, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}]$ containing the operator

$$\text{Res}(\sum_{j=1}^n D_{\xi_j}^2) = \sum_{j=1}^n \partial_{\xi_j}^2 + 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)} \partial_\alpha.$$  

**Remark 1.8.** The proof of this theorem is a triviality. However it can be reformulated as the complete integrability for the generalized non periodic Calogero-Moser system (both on the quantum mechanical level of differential operators and on the classical mechanical level of symbols). For root systems $R$ of type $A$ the complete integrability of the Calogero-Moser system was first established by Moser by realizing the system as a Lax pair [Mo]. The method of Moser was extended by Olshanetsky and Perelomov to cover the root systems $R$ of classical type [OP]. In the crystallographic case $(\alpha, \beta)^2 \in \mathbb{Z}$ $\forall \alpha, \beta \in R$ the above theorem has been obtained before by Opdam using transcendental methods [HO, He1, Op 1,2, He 2].

Suppose $S \subset R$ is a set of roots in $R$ invariant under $W$. Let $S_+ = S \cap R_+$ and put

$$p_S(\cdot) = \prod_{\alpha \in S_+} (\alpha, \cdot) \in \mathbb{C}[E]$$

$$\pi_S = \prod_{\alpha \in S_+} \alpha \in \mathbb{C}[E^*].$$

Clearly we have

$$wp_S = \chi(w)p_S, w\pi_S = \chi(w)\pi_S \forall w \in W$$

for some one dimensional character $\chi = \chi_S$ of $W$, and conversely every $p \in \mathbb{C}[E]$ with $wp = \chi(w)p \forall w \in W$ is divisible in $\mathbb{C}[E]$ by $p_S$. Although $p_S^{-1}D_{\pi_S}(k)$ need not be an endomorphism of $\mathbb{C}[E]$ it follows that $p_S^{-1}D_{\pi_S}(k)(p) \in \mathbb{C}[E]^W \forall p \in \mathbb{C}[E]^W$, and hence

$$G(1_S, k) := \text{Res}(p_S^{-1}D_{\pi_S}(k)) \in \text{End}(\mathbb{C}[E]^W)$$
is a well defined endomorphism of $\mathbb{C}[E]^W$. We also write
\begin{equation}
G(-1_S, k) := \text{Res}(D_{\pi_S}(k - 1_S) \cdot p_S) \in \text{End}(\mathbb{C}[E]^W)
\end{equation}
where $k - 1_S \in K$ is the multiplicity function by $(k - 1_S)_\alpha = k_\alpha - 1$ for $\alpha \in S$ and $(k - 1_S)_\alpha = k_\alpha$ for $\alpha \in R \setminus S$.

**Theorem 1.9.** The operators (1.16) and (1.17) are differential operators in the Weyl algebra $\mathbb{C}[k, p_1, \ldots, p_n, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}]$ and satisfy the shift relations
\begin{align}
& (1.18) \quad G(1_S, k)\text{Res}(D_{\pi}(k)) = \text{Res}(D_{\pi}(k + 1_S))G(1_S, k) \\
& (1.19) \quad G(-1_S, k)\text{Res}(D_{\pi}(k)) = \text{Res}(D_{\pi}(k - 1_S))G(-1_S, k)
\end{align}
\forall \pi \in \mathbb{C}[E^*]^W$. Here $(k \pm 1_S)_\alpha = k_\alpha \pm 1 \forall \alpha \in S$ and $(k \pm 1_S)_\alpha = k_\alpha \forall \alpha \in R \setminus S$.

The proofs of both Theorem 1.7 and 1.9 will be given in Section 3.

**Remark 1.10.** In the terminology of Opdam the operator (1.16) is a raising operator and the operator (1.17) a lowering operator for the commuting family (1.11). Again in the crystallographic case the above theorem was obtained by Opdam [Op 2]. Recall Macdonald’s (infinitesimal) constant term conjecture, which says that for $R(s) > 0$
\begin{equation}
\int_{E} \prod_{\alpha \in R_+} |(\alpha, \lambda)|^{2s}d\gamma(\lambda) = \prod_{j=1}^{n} \frac{(sd_j)!}{s!},
\end{equation}
where $d\gamma(\lambda) = (2\pi)^{-\frac{n}{2}}e^{-\frac{1}{2}(\lambda, \lambda)}d\lambda$ is the Gaussian measure on $E$ [Ma].

The same arguments as given in [Op 3, Section 6] show that the evaluation of this integral is equivalent with
\begin{equation}
G(-1, k)(1) = |W| \cdot \prod_{i=1}^{n} \prod_{j=1}^{m_i} (d_i k - j),
\end{equation}
where $-1 = -1_R$ and $k = k_\alpha \forall \alpha \in R$. In turn this latter formula is related to the normalization of the “multivariable Bessel function associated with $R$” at $\xi = 0$. This normalization problem has been analyzed by Opdam, and the desired formula (1.21) can be obtained [Op 4]. After this one can proceed as in [Op 3, Section 7] to compute the Bernstein-Sato polynomial of the discriminant without the crystallographic restriction in accordance with a conjecture of Yano and Sekiguchi [YS].
§2. The Dunkl differential-difference operators.

Using the bracket $[\cdot, \cdot]$ for the commutator of endomorphisms of $\mathbb{C}[E]$ we can write for $\xi, \eta \in E$

(2.1) \[ [D_\xi, D_\eta] = I + II + III \]

with

(2.2) \[ I = [\partial_\xi, \partial_\eta] = 0 \]

(2.3) \[ II = \sum_{\alpha \in R_+} k_\alpha \{(\alpha, \xi)[\Delta_\alpha, \partial_\eta] + (\alpha, \eta)[\partial_\xi, \Delta_\alpha]\} \]

(2.4) \[ III = \sum_{\alpha, \beta \in R_+} k_\alpha k_\beta (\alpha, \xi)(\beta, \eta)[\Delta_\alpha, \Delta_\beta]. \]

**Lemma 2.1.** For $\xi \in E$, $\alpha \in R$ we have

(2.5) \[ [\partial_\xi, \Delta_\alpha] = \frac{(\alpha, \xi)}{(\alpha, \cdot)} \{r_\alpha \partial_\alpha - \Delta_\alpha\}. \]

**Proof:** Using the definition $\Delta_\alpha = \frac{1}{(\alpha, \cdot)}(1 - r_\alpha)$ we get

\[
[\partial_\xi, \Delta_\alpha] = [\partial_\xi, \frac{1}{(\alpha, \cdot)}(1 - r_\alpha)] + \frac{1}{(\alpha, \cdot)}[\partial_\xi, 1 - r_\alpha] \\
= -\frac{(\alpha, \xi)}{(\alpha, \cdot)}(1 - r_\alpha) + \frac{1}{(\alpha, \cdot)}r_\alpha(\partial_\xi - r_\alpha \xi) \\
= -\frac{(\alpha, \xi)}{(\alpha, \cdot)}\Delta_\alpha + \frac{(\alpha, \xi)}{(\alpha, \cdot)}r_\alpha \partial_\alpha. \]

Q.E.D

Using (2.5) the second term (2.3) can be rewritten as

(2.6) \[ II = \sum_{\alpha \in R_+} k_\alpha \frac{(\alpha, \xi)(\alpha, \eta)}{(\alpha, \cdot)} \{r_\alpha \partial_\alpha - \Delta_\alpha\}(-1 + 1) = 0. \]

The third term (2.4) can be written as

(2.7) \[ III = \sum_{\alpha, \beta \in R_+} k_\alpha k_\beta \{(\alpha, \xi)(\beta, \eta) - (\alpha, \eta)(\beta, \xi)\} \Delta_\alpha \Delta_\beta \]

and for the proof of Theorem 1.5 it remains to verify the vanishing of this third term.

**Proposition 2.2.** Suppose $B(\cdot, \cdot)$ is a bilinear form on $E$ such that

(2.8) \[ B(r_\alpha \lambda, r_\alpha \mu) = B(\mu, \lambda) \quad \forall \lambda, \mu \in E, \forall \alpha \in R \cap \text{span} \langle \lambda, \mu \rangle. \]
If \( w \in W \) is a pure rotation (i.e. \( \dim \text{Im}(w - \text{Id}) = 2 \)) then

\[
\sum_{\alpha, \beta \in R_+ \cap \text{Im}(w - \text{Id})} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(\alpha, \cdot)(\beta, \cdot)} = 0
\]

and

\[
\sum_{\alpha, \beta \in R_+ \cap \text{Im}(w - \text{Id})} k_\alpha k_\beta B(\alpha, \beta) \Delta_\alpha \Delta_\beta = 0.
\]

**Proof:** Using the definition \( \Delta_\alpha = \frac{1}{(\alpha, \cdot)} (1 - r_\alpha) \) the left hand side of (2.10) can be written as a sum of the following three terms

\[
A = \sum_{\alpha, \beta \in R_+ \cap \text{Im}(w - \text{Id})} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(\alpha, \cdot)(\beta, \cdot)}
\]

\[
B = - \sum_{\alpha, \beta \in R_+ \cap \text{Im}(w - \text{Id})} k_\alpha k_\beta B(\alpha, \beta) \left\{ \frac{1}{(\alpha, \cdot)(r_\alpha \beta, \cdot)} r_\alpha + \frac{1}{(\alpha, \cdot)(\beta, \cdot)} r_\beta \right\}
\]

\[
C = \sum_{\alpha, \beta \in R_+ \cap \text{Im}(w - \text{Id})} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(\alpha, \cdot)(r_\alpha \beta, \cdot)} r_\alpha r_\beta
\]

with the summations over the same index set as in (2.9) and (2.10).

Let \( S = R \cap \text{Im}(w - \text{Id}) \) be the normalized root system of the largest dihedral group \( W(S) \) containing \( w \). If \( w = r_\alpha r_\beta \) then for \( \gamma \in S \) we have \( r_\gamma w r_\gamma = w^{-1} \) and hence \( r_{r_\alpha r_\beta} = r_\beta r_\alpha \). We claim that \( r_\gamma A = A \forall \gamma \in S \). Indeed we have

\[
r_\gamma A = \sum_{\alpha, \beta \in R_+ \cap \text{Im}(w - \text{Id})} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(r_\gamma \alpha, \cdot)(r_\gamma \beta, \cdot)}
\]

\[
= \sum_{\alpha, \beta \in r_\gamma R_+ \cap \text{Im}(w - \text{Id})} k_\alpha k_\beta B(r_\gamma \alpha, r_\gamma \beta) \frac{1}{(\alpha, \cdot)(\beta, \cdot)}
\]

\[
= \sum_{\alpha, \beta \in r_\gamma R_+ \cap \text{Im}(w - \text{Id})} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(\alpha, \cdot)(\beta, \cdot)}
\]

\[
= A
\]

since the summation in (2.9) is independent of the choice of \( R_+ \). Let \( S_+ = R_+ \cap S \) and put \( p_S = \prod_{\alpha \in S_+} (\alpha, \cdot) \). Then \( p_S \) transforms under the group \( W(S) \) according to the sign character and every polynomial in \( \mathbb{C}[E] \) transforming under \( W(S) \) according to the sign character is divisible in \( \mathbb{C}[E] \) by \( p_S \). Now observe that \( p_S A \in \mathbb{C}[E] \) transforms
under \( W(S) \) according to the sign character. Hence \( A \in \mathbb{C}[E] \). Since \( A \) is homogeneous of degree minus two we have \( A = 0 \). This proves (2.9).

Since \( w = r_\alpha r_\beta = r_{r_\alpha r_\beta} \) and \( B(\alpha, \beta) = B(r_\alpha r_\beta, r_\alpha \alpha) = -B(r_\alpha r_\beta, \alpha) \) the vanishing of the term (2.12) is clear, and for the term (2.13) we can write \( C = -Aw = 0 \). Q.E.D.

**Lemma 2.3.** For \( \xi, \eta \in E \) fixed the bilinear form

\[
(2.14) \quad B(\lambda, \mu) = (\lambda, \xi)(\mu, \eta) - (\lambda, \eta)(\mu, \xi)
\]
on \( E \) satisfies condition (2.8).

**Proof:** Clearly \( B(\mu, \lambda) = -B(\lambda, \mu) \) is an alternating form. For \( \lambda \in E, \lambda \neq 0 \) we write

\[
\lambda' = \sqrt{2} |\lambda|^{-1} \lambda
\]
and get

\[
B(r_\lambda \lambda', r_\lambda' \mu) = B(-\lambda, \mu - (\lambda', \mu) \lambda') = B(-\lambda, \mu) = B(\mu, \lambda).
\]

Hence for \( \lambda, \mu \in E \) generic we get by continuity

\[
B(r_\nu \lambda, r_\nu \mu) = B(\mu, \lambda) \quad \forall \nu \in \text{span} \langle \lambda, \mu \rangle, (\nu, \nu) = 2.
\]
Q.E.D.

The proof of Theorem 1.5 now follows by regrouping the terms in (2.7) as a sum over \( \{\alpha, \beta \in R_+; r_\alpha r_\beta = w\} \) where \( w \in W \) runs over the pure rotations in \( W \) and by applying (2.10).

The proof of Theorem 1.6 is just an easy calculation.

\[
\sum_{j=1}^{n} D^2_{\xi_j} = \sum_{j=1}^{n} \left( \partial \xi_j + \sum_{\alpha \in R_+} k_{\alpha}(\alpha, \xi_j) \Delta_\alpha \right)^2
\]

\[
= \sum_{j=1}^{n} \left\{ \partial^2 \xi_j + \sum_{\alpha \in R_+} k_{\alpha}(\alpha, \xi_j) (\partial \xi_j \Delta_\alpha + \Delta_\alpha \partial \xi_j) + \sum_{\alpha, \beta \in R_+} k_{\alpha} k_{\beta}(\alpha, \xi_j)(\beta, \xi_j) \Delta_\alpha \Delta_\beta \right\}
\]

\[
= \sum_{j=1}^{n} \partial^2 \xi_j + \sum_{\alpha \in R_+} k_{\alpha}(\partial \Delta_\alpha + \Delta_\alpha \partial_\alpha) + \sum_{\alpha, \beta \in R_+} k_{\alpha} k_{\beta}(\alpha, \beta) \Delta_\alpha \Delta_\beta.
\]

The third term vanishes by Proposition 2.2 and because \( \Delta^2_\alpha = 0 \). Using Lemma 2.1 we get

\[
\partial_\alpha \Delta_\alpha + \Delta_\alpha \partial_\alpha = [\partial_\alpha, \Delta_\alpha] + 2 \Delta_\alpha \partial_\alpha
\]

\[
= \frac{(\alpha, \alpha)}{(\alpha, \cdot)} \left\{ r_\alpha \partial_\alpha - \Delta_\alpha \right\} + \frac{2}{(\alpha, \cdot)} (1 - r_\alpha) \partial_\alpha
\]

\[
= \frac{2}{(\alpha, \cdot)} \left\{ \partial_\alpha - \Delta_\alpha \right\}.
\]
§3. The Opdam shift operators.
Recall that $D \in \text{End}(\mathbb{C}[p_1, \ldots, p_m])$ is a differential operator of degree $\leq d$ if and only if
\begin{equation}
\text{ad}(p)^{d+1}(D) = 0 \quad \forall p \in \mathbb{C}[p_1, \ldots, p_n].
\end{equation}
Hence the fact that the operators (1.11), (1.16) and (1.17) are differential operators is clear from
\begin{equation}
\text{ad}(p)(D_{\xi}) = \text{ad}(p)(\partial_{\xi}) = -\partial_{\xi}(p)
\end{equation}
\begin{equation}
\text{ad}(p)^2(D_{\xi}) = 0
\end{equation}
\[\forall p \in \mathbb{C}[E]^W, \forall \xi \in E.\] Hence Theorem 1.7 is an immediate consequence of Theorem 1.5 and Theorem 1.6.

**Theorem 3.1.** For the $q$-analogue of the Laplacian we have
\begin{equation}
\text{Res}(p_S^{-1} \circ \left\{ \sum_{j=1}^{n} D_{\xi_j}^2(k) \right\} \circ p_S) = \text{Res}(\sum_{j=1}^{n} D_{\xi_j}^2(k + 1_S)).
\end{equation}

**Proof:** First we observe that the left hand side of (3.4) is a well defined endomorphism of $\mathbb{C}[E]^W$. We now use Theorem 1.6 and just calculate term by term. For the first term we get
\[p_S^{-1} \circ \left\{ \sum_{j=1}^{n} D_{\xi_j}^2 \right\} \circ p_S = \sum_{j=1}^{n} D_{\xi_j}^2(p_S) + 2 \sum_{\alpha \in S_+} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} + p_S^{-1}(\sum_{j=1}^{n} D_{\xi_j}^2(p_S))
\]
\[= \sum_{j=1}^{n} D_{\xi_j}^2(p_S) + 2 \sum_{\alpha \in S_+} \frac{1}{(\alpha, \cdot)} \partial_{\alpha}.
\]
For the second term we get
\[p_S^{-1} \circ \left\{ 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} \right\} \circ p_S = 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} + p_S^{-1} \left( 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} \right)(p_S)
\]
\[= 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} + 2 \sum_{\alpha \in R_+, \beta \in S_+} k_{\alpha} \frac{(\alpha, \beta)}{(\alpha, \cdot)(\beta, \cdot)}
\]
\[= 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} + 2 \sum_{\beta \in S_+} k_{\beta} \frac{(\beta, \beta)}{(\beta, \cdot)^2}
\]
\[+ 2 \sum_{\alpha \in R_+, \beta \in S_+} k_{\alpha} \frac{(\alpha, \beta)}{(\alpha, \cdot)(\beta, \cdot)}
\]
\[= 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} + 2 \sum_{\beta \in S_+} k_{\beta} \frac{2}{(\beta, \cdot)^2}.
\]
by the same argument as in the proof of Proposition 2.2. Finally for the third term we have
\[ p_S^{-1} \circ \left\{ \sum_{\alpha \in R_+} \frac{1}{\alpha} \Delta_\alpha \right\} \circ p_S = 2 \sum_{\alpha \in R_+} \frac{1}{\alpha \cdot \alpha} \{1 - p_S^{-1} \circ r_\alpha \circ p_S\} \]
\[ = 2 \sum_{\alpha \in R_+} \frac{1}{\alpha \cdot \alpha} \{1 - \chi_S(r_\alpha) r_\alpha\} \]
\[ = 2 \sum_{\alpha \in S_+} \frac{1}{\alpha \cdot \alpha} \{1 + r_\alpha\} + 2 \sum_{\alpha \in R_+ \setminus S_+} \frac{1}{\alpha \cdot \alpha} \Delta_\alpha \]
\[ = 2 \sum_{\alpha \in S_+} \frac{2}{\alpha \cdot \alpha} - 2 \sum_{\alpha \in S_+} \frac{1}{\alpha \cdot \alpha} \Delta_\alpha \]
\[ + 2 \sum_{\alpha \in R_+ \setminus S_+} \frac{1}{\alpha \cdot \alpha} \Delta_\alpha. \]
Taking all three terms together yields
\[ p_S^{-1} \circ \left\{ \sum_{j=1}^n D_{\xi_j}^2(k) \right\} \circ p_S = \sum_{j=1}^n \partial_{\xi_j}^2 + 2 \sum_{\alpha \in R_+} \frac{1}{\alpha \cdot \alpha} \partial_\alpha + 2 \sum_{\alpha \in S_+} \frac{1}{\alpha \cdot \alpha} \partial_\alpha \]
\[ + 2 \sum_{\alpha \in S_+} \frac{1}{\alpha \cdot \alpha} \Delta_\alpha - 2 \sum_{\alpha \in R_+ \setminus S_+} \frac{1}{\alpha \cdot \alpha} \Delta_\alpha. \] Q.E.D.

**Corollary 3.2.** We have the shift relations
\[ G(1_s, k) \text{Res} \left( \sum_{j=1}^n D_{\xi_j}^2(k) \right) = \text{Res} \left( \sum_{j=1}^n D_{\xi_j}^2(k + 1_s) \right) G(1_s, k) \] (3.5)
\[ G(-1_s, k) \text{Res} \left( \sum_{j=1}^n D_{\xi_j}^2(k) \right) = \text{Res} \left( \sum_{j=1}^n D_{\xi_j}^2(k - 1_s) \right) G(-1_s, k). \] (3.6)

**Proof:** Indeed we have
\[ \text{Res} \left( p_S^{-1} D_{\pi_S}(k) \right) \text{Res} \left( \sum_{j=1}^n D_{\xi_j}^2(k) \right) = \text{Res} \left( \sum_{j=1}^n p_S^{-1} D_{\pi_S}(k) D_{\xi_j}^2(k) \right) \]
\[ = \text{Res} \left( \sum_{j=1}^n p_S^{-1} D_{\xi_j}(k) D_{\pi_S}(k) \right) \]
\[ = \text{Res} \left( \sum_{j=1}^n p_S^{-1} D_{\xi_j}(k) p_S \right) \text{Res} \left( p_S^{-1} D_{\pi_S}(k) \right) \]
\[ = \text{Res} \left( \sum_{j=1}^n D_{\xi_j}(k + 1_s) \right) \text{Res} \left( p_S^{-1} D_{\pi_S}(k) \right) \]
\[ = \text{Res} \left( \sum_{j=1}^n D_{\xi_j}(k + 1_s) \right) \text{Res} \left( p_S^{-1} D_{\pi_S}(k) \right) \]
which proves (3.5). The relation (3.6) is proved similarly. Q.E.D.

**Theorem 3.3.** As endomorphisms of $\mathbb{C}[E]$ the operators

\begin{align}
E &= \frac{1}{2} \sum_{j=1}^{n} (\xi_{j,\cdot})^{2} \\
H &= \sum_{j=1}^{n} (\xi_{j,\cdot}) \partial \xi_{j} + \left( \frac{n}{2} + \sum_{\alpha \in \mathbb{R}^{+}} k_{\alpha} \right) \\
F &= -\frac{1}{2} \sum_{j=1}^{n} D_{\xi_{j}}^{2}
\end{align}

satisfy the commutation relations of $sl(2)$:

\begin{align}
\end{align}

**Proof:** The Euler operator $\sum_{j=1}^{n} (\xi_{j,\cdot}) \partial \xi_{j}$ acts as multiplication by $d$ on the space of homogeneous polynomials in $\mathbb{C}[E]$ of degree $d$. Hence the commutation relations $[H, E] = 2E$, $[H, F] = -2F$ rephrase that $E$ and $F$ are homogeneous of degree plus and minus two respectively.

Since $[p, \Delta_{\alpha}] = 0$ $\forall p \in \mathbb{C}[E]^{W}$, $\forall \alpha \in R$ we get

\begin{align}
[E, D_{\xi}] &= [E, \partial \xi] = - (\xi, \cdot) \quad \forall \xi \in E,
\end{align}

and therefore

\[ [E, F] = -\frac{1}{2} \sum_{j=1}^{n} [E, D_{\xi_{j}}^{2}] \]

\[ = \frac{1}{2} \sum_{j=1}^{n} \{(\xi_{j,\cdot}) D_{\xi_{j}} + D_{\xi_{j}} (\xi_{j,\cdot})\} \]

\[ = \sum_{j=1}^{n} (\xi_{j,\cdot}) D_{\xi_{j}} + \frac{1}{2} \sum_{j=1}^{n} [D_{\xi_{j}}, (\xi_{j,\cdot})] \]

\[ = \sum_{j=1}^{n} (\xi_{j,\cdot}) D_{\xi_{j}} + \frac{n}{2} + \sum_{j=1}^{n} \sum_{\alpha \in \mathbb{R}^{+}} k_{\alpha}(\alpha, \xi_{j})[\Delta_{\alpha}, (\xi_{j,\cdot})] \]

\[ = \sum_{j=1}^{n} (\xi_{j,\cdot}) \partial \xi_{j} + \sum_{\alpha \in \mathbb{R}^{+}} k_{\alpha}(\alpha, \cdot) \Delta_{\alpha} + \frac{n}{2} + \sum_{\alpha \in \mathbb{R}^{+}} k_{\alpha} r_{\alpha} \]

\[ = \sum_{j=1}^{n} (\xi_{j,\cdot}) \partial \xi_{j} + \left( \frac{n}{2} + \sum_{\alpha \in \mathbb{R}^{+}} k_{\alpha} \right). \]
Here we have used that for $\xi \in E$

$$[\Delta_\alpha, (\xi, \cdot)] = -\frac{1}{(\alpha, \cdot)} [r_\alpha, (\xi, \cdot)]$$

$$= -\frac{1}{(\alpha, \cdot)} \{(r_\alpha \xi, \cdot) - (\xi, \cdot)\} r_\alpha$$

$$= (\alpha, \xi) r_\alpha.$$  \hspace{1cm} Q.E.D.

**Proposition 3.4.** Using the inner product $(\cdot, \cdot)$ on $E$ we have an isomorphism between $\mathbb{C}[E]$ and $\mathbb{C}[E^*]$. For $p \in \mathbb{C}[E]$ we write $\pi \in \mathbb{C}[E^*]$ for the corresponding element. For $p \in \mathbb{C}[E]$ homogeneous of degree $d$ we have

$$(3.12) \quad D_\pi = (-1)^d \frac{1}{d!} \text{ad}(F)^d(p).$$

**Proof:** Clearly $\text{ad}(H)D_\pi = -dD_\pi$ and by Theorem 1.5 we have $\text{ad}(F)D_\pi = 0$. Using (3.11) and induction on $d$ (assuming $\pi$ to be a monomial as in (1.9) with $d = d_1 + \cdots + d_n$) it is easy to see that

$$( -1)^d \frac{1}{d!} \text{ad}(E)^d(D_\pi) = p$$

and hence

$$\text{ad}(E)^{d+1}(D_\pi) = 0.\hspace{1cm}$$

By standard representation theory of $sl(2)$ we conclude (3.12). Q.E.D.

**Corollary 3.5.** For $\pi \in \mathbb{C}[E^*]^W$ we have

$$(3.13) \quad \text{Res}\left(p_S^{-1} \circ D_\pi(k) \circ p_S\right) = \text{Res}\left(D_\pi(k + 1_s)\right).$$

**Proof:** This is easily derived from Theorem 3.1 and Proposition 3.4. Q.E.D.

The proof of Theorem 1.9 now goes along the same lines as the proof of Corollary 3.2.

**Remark 3.6.** The above type of arguments to use an $sl(2)$ to reduce the computation of higher order operators to those of the second order one go back to Harish-Chandra [Ha].

**Acknowledgements:** The author would like to thank the organizers of this conference for the invitation, and in particular Bill Barker for the hospitality at Bowdoin College.
References


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