A remark on the Dunkl differential-difference operators

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§1. Introduction

Let $E$ be a Euclidean vector space of dimension $n$ with inner product $(\cdot, \cdot)$. For $\alpha \in E$ with $(\alpha, \alpha) = 2$ we write

\begin{equation}
(1.1) \quad r_{\alpha}(\lambda) = \lambda - (\alpha, \lambda)\alpha, \quad \lambda \in E
\end{equation}

for the orthogonal reflection in the hyperplane perpendicular to $\alpha$.

**Definition 1.1.** A normalized root system $R$ in $E$ is a finite set of non zero vectors in $E$, normalized by $(\alpha, \alpha) = 2$ for all $\alpha \in R$, such that $r_{\alpha}(\beta) \in R$ for all $\alpha, \beta \in R$.

Let $R \subset E$ be a normalized root system. We write $W = W(R)$ for the group generated by the reflections $r_{\alpha}, \alpha \in R$. Denote by $\mathbb{C}[E]$ the algebra of $\mathbb{C}$-valued polynomial functions on $E$. For $w \in W$, $\xi \in E$, $\alpha \in R$ introduce the operators

\begin{equation}
(1.2) \quad w, \partial_\xi, \Delta_\alpha : \mathbb{C}[E] \rightarrow \mathbb{C}[E]
\end{equation}

by

\begin{equation}
(1.3) \quad (wp)(\lambda) = p(w^{-1}\lambda)
\end{equation}

\begin{equation}
(1.4) \quad (\partial_\xi p)(\lambda) = \frac{d}{dt}\{p(\lambda + t\xi)\}_{t=0}
\end{equation}

\begin{equation}
(1.5) \quad (\Delta_\alpha p)(\lambda) = \frac{p(\lambda) - p(r_\alpha\lambda)}{(\alpha, \lambda)}.
\end{equation}

**Remark 1.2.** The operators $\Delta_\alpha, \alpha \in R$ were studied by Bernstein, Gel’fand and Gel’fand and are related to the Schubert cells and the cohomology of $G/P$ [BGG]. They are the infinitesimal analogues of the Demazure operators [De 1,2].

Let $R_+ = \{\alpha \in R; (\alpha, \lambda) > 0\}$ for some fixed generic $\lambda \in E$ be a positive subsystem of $R$. 

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Definition 1.3. Suppose for $\alpha \in R$ we have given $k_\alpha \in C$ with $k_{w\alpha} = k_\alpha \forall w \in W, \forall \alpha \in R$. For $\xi \in E$ the operator

\begin{equation}
D_\xi = \partial_\xi + \sum_{\alpha \in R_+} k_\alpha (\alpha, \xi) \Delta_\alpha : C[E] \longrightarrow C[E]
\end{equation}

is called the Dunkl differential-difference operator.

Remark 1.4. It is easy to see that $D_\xi$ is independent of the choice of the positive subsystem $R_+ \subset R$. If we write $q_\alpha = e^{2\pi ik_\alpha}$ then one can think of the operator $D_\xi$ as a $q$-analogue (corresponding to the case $k_\alpha \rightarrow 0$) of the directional derivative $\partial_\xi$.

We also write $D_\xi = D_\xi(k)$ to indicate the dependence on $k \in K = \{k = (k_\alpha)_{\alpha \in R} \in C^R; k_{w\alpha} = k_\alpha \forall w \in W, \forall \alpha \in R\}$.

Theorem 1.5 (Dunkl [Du]): We have $D_\xi D_\eta = D_\eta D_\xi \forall \xi, \eta \in E$.

Let $C[E^*]$ be the symmetric algebra on $E$. For $\pi \in C[E^*]$ we write $\partial_\pi$ when we think of $\pi$ as a constant coefficient differential operator on $E$ (rather than a polynomial function on $E^*$). In view of Theorem 1.5 the constant coefficient differential operator $\partial_\pi$ has a well defined $q$-analogue

\begin{equation}
D_\pi : C[E] \longrightarrow C[E]
\end{equation}

defined for a monomial $\pi = \xi_1^{d_1} \cdots \xi_n^{d_n}$ by

\begin{equation}
D_\pi = D_\pi(k) = D_{\xi_1}^{d_1} \cdots D_{\xi_n}^{d_n}
\end{equation}

and extended by linearity.

Theorem 1.6 (Dunkl [Du]): Suppose $\xi_1, \ldots, \xi_n$ is an orthonormal basis for $E$. The $q$-analogue of the Laplacian is given by

\begin{equation}
\sum_{j=1}^n D_{\xi_j}^2 = \sum_{j=1}^n \partial_{\xi_j}^2 + 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)} \{\partial_\alpha - \Delta_\alpha\}.
\end{equation}

In Section 2 we review the proofs of both theorems as given by Dunkl.

We write $C[E]^W$ and $C[E^*]^W$ for the space of $W$-invariants in $C[E]$ and $C[E^*]$ respectively. We denote by $A$ the associative algebra of endomorphisms of $C[E]$ generated by (multiplication by) $(\xi, \cdot)$ and $D_\eta$ for $\xi, \eta \in E$. Let $A^W = \{D \in A; wD = Dw \forall w \in W\}$ be the subalgebra of $W$-invariant operators in $A$, and denote by

\begin{equation}
\text{Res}(D) : C[E]^W \longrightarrow C[E]^W, \ D \in A^W
\end{equation}
the restriction of $D$ to $\mathbb{C}[E]^W$. Clearly $\text{Res}: \mathbb{A}^W \rightarrow \text{End}(\mathbb{C}[E]^W)$ is a homomorphism of algebras. Since $wD_{\xi}w^{-1} = D_{w\xi} \forall w \in W$, $\forall \xi \in E$ we have $D_\pi \in \mathbb{A}^W \forall \pi \in \mathbb{C}[E^*]^W$.

**Theorem 1.7.** Suppose by the Chevalley theorem that $\mathbb{C}[E]^W = \mathbb{C}[p_1, \ldots, p_n]$ with $p_1, \ldots, p_n$ homogeneous of degrees $d_1 \leq \ldots \leq d_n$. Then the set

\begin{equation}
\{\text{Res}(D_\pi); \pi \in \mathbb{C}[E^*]^W\}
\end{equation}

is a commuting family of differential operators in the Weyl algebra $\mathbb{C}[k, p_1, \ldots, p_n, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}]$ containing the operator

\begin{equation}
\text{Res}\left(\sum_{j=1}^{n} D_{\xi_j}^2\right) = \sum_{j=1}^{n} \partial_{\xi_j}^2 + 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha}.
\end{equation}

**Remark 1.8.** The proof of this theorem is a triviality. However it can be reformulated as the complete integrability for the generalized non periodic Calogero-Moser system (both on the quantum mechanical level of differential operators and on the classical mechanical level of symbols). For root systems $R$ of type $A$ the complete integrability of the Calogero-Moser system was first established by Moser by realizing the system as a Lax pair [Mo]. The method of Moser was extended by Olshanetsky and Perelomov to cover the root systems $R$ of classical type [OP]. In the crystallographic case $(\alpha, \beta)^2 \in \mathbb{Z} \forall \alpha, \beta \in R$ the above theorem has been obtained before by Opdam using transcendental methods [HO, He1, Op 1,2, He 2].

Suppose $S \subset R$ is a set of roots in $R$ invariant under $W$. Let $S_+ = S \cap R_+$ and put

\begin{equation}
p_S(\cdot) = \prod_{\alpha \in S_+} (\alpha, \cdot) \in \mathbb{C}[E]\end{equation}

\begin{equation}\pi_S = \prod_{\alpha \in S_+} \alpha \in \mathbb{C}[E^*].\end{equation}

Clearly we have

\begin{equation}wp_S = \chi(w)p_S, w\pi_S = \chi(w)\pi_S \quad \forall w \in W\end{equation}

for some one dimensional character $\chi = \chi_S$ of $W$, and conversely every $p \in \mathbb{C}[E]$ with $wp = \chi(w)p \forall w \in W$ is divisible in $\mathbb{C}[E]$ by $p_S$. Although $p_S^{-1}D_{\pi_S}(k)$ need not be an endomorphism of $\mathbb{C}[E]$ it follows that $p_S^{-1}D_{\pi_S}(k)(p) \in \mathbb{C}[E]^W \forall p \in \mathbb{C}[E]^W$, and hence

\begin{equation}G(1_S, k) := \text{Res}(p_S^{-1}D_{\pi_S}(k)) \in \text{End}(\mathbb{C}[E]^W)\end{equation}
is a well defined endomorphism of $\mathbb{C}[E]^W$. We also write

\[(1.17) \quad G(-1_S, k) := \text{Res}(D_{\pi_S}(k - 1_S) \cdot p_S) \in \text{End}(\mathbb{C}[E]^W)\]

where $k - 1_S \in K$ is the multiplicity function by $(k - 1_S)_\alpha = k_\alpha - 1$ for $\alpha \in S$ and $(k - 1_S)_\alpha = k_\alpha$ for $\alpha \in R \setminus S$.

**Theorem 1.9.** The operators (1.16) and (1.17) are differential operators in the Weyl algebra $\mathbb{C}[k, p_1, \ldots, p_n, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}]$ and satisfy the shift relations

\[(1.18) \quad G(1_S, k) \text{Res}(D_{\pi}(k)) = \text{Res}(D_{\pi}(k + 1_S))G(1_S, k)\]

\[(1.19) \quad G(-1_S, k) \text{Res}(D_{\pi}(k)) = \text{Res}(D_{\pi}(k - 1_S))G(-1_S, k)\]

$\forall \pi \in \mathbb{C}[E^*]^W$. Here $(k \pm 1_S)_\alpha = k_\alpha \pm 1 \forall \alpha \in S$ and $(k \pm 1_S)_\alpha = k_\alpha \forall \alpha \in R \setminus S$.

The proofs of both Theorem 1.7 and 1.9 will be given in Section 3.

**Remark 1.10.** In the terminology of Opdam the operator (1.16) is a raising operator and the operator (1.17) a lowering operator for the commuting family (1.11). Again in the crystallographic case the above theorem was obtained by Opdam [Op 2]. Recall Macdonald’s (infinitesimal) constant term conjecture, which says that for $\mathcal{R}(s) > 0$

\[(1.20) \quad \int_E \prod_{\alpha \in R^+} |(\alpha, \lambda)|^{2s} d\gamma(\lambda) = \prod_{j=1}^{n} \frac{(sd_j)!}{s!},\]

where $d\gamma(\lambda) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{4}(\lambda, \lambda)} d\lambda$ is the Gaussian measure on $E$ [Ma].

The same arguments as given in [Op 3, Section 6] show that the evaluation of this integral is equivalent with

\[(1.21) \quad G(-1, k)(1) = |W| \cdot \prod_{i=1}^{n} \prod_{j=1}^{m_i} (d_ik - j),\]

where $-1 = -1_R$ and $k = k_\alpha \forall \alpha \in R$. In turn this latter formula is related to the normalization of the “multivariable Bessel function associated with $R$” at $\xi = 0$. This normalization problem has been analyzed by Opdam, and the desired formula (1.21) can be obtained [Op 4]. After this one can proceed as in [Op 3, Section 7] to compute the Bernstein-Sato polynomial of the discriminant without the crystallographic restriction in accordance with a conjecture of Yano and Sekiguchi [YS].
§2. The Dunkl differential-difference operators.

Using the bracket $[\cdot, \cdot]$ for the commutator of endomorphisms of $\mathbb{C}[E]$ we can write for $\xi, \eta \in E$

\[
[D_\xi, D_\eta] = I + II + III
\]

with

\[
I = [\partial_\xi, \partial_\eta] = 0
\]

\[
II = \sum_{\alpha \in R_+} k_\alpha \{(\alpha, \xi)[\Delta_\alpha, \partial_\eta] + (\alpha, \eta)[\partial_\xi, \Delta_\alpha]\}
\]

\[
III = \sum_{\alpha, \beta \in R_+} k_\alpha k_\beta (\alpha, \xi)(\beta, \eta)[\Delta_\alpha, \Delta_\beta].
\]

**Lemma 2.1.** For $\xi \in E$, $\alpha \in R$ we have

\[
[\partial_\xi, \Delta_\alpha] = \frac{(\alpha, \xi)}{(\alpha, \cdot)}\{r_\alpha \partial_\alpha - \Delta_\alpha\}.
\]

**Proof:** Using the definition $\Delta_\alpha = \frac{1}{(\alpha, \cdot)}(1 - r_\alpha)$ we get

\[
[\partial_\xi, \Delta_\alpha] = [\partial_\xi, \frac{1}{(\alpha, \cdot)}(1 - r_\alpha)] + \frac{1}{(\alpha, \cdot)}[\partial_\xi, 1 - r_\alpha]
\]

\[
= -\frac{(\alpha, \xi)}{(\alpha, \cdot)}(1 - r_\alpha) + \frac{1}{(\alpha, \cdot)}r_\alpha(\partial_\xi - r_\alpha \xi)
\]

\[
= -\frac{(\alpha, \xi)}{(\alpha, \cdot)}\Delta_\alpha + \frac{(\alpha, \xi)}{(\alpha, \cdot)}r_\alpha \partial_\alpha. \quad \text{Q.E.D}
\]

Using (2.5) the second term (2.3) can be rewritten as

\[
II = \sum_{\alpha \in R_+} k_\alpha \frac{(\alpha, \xi)(\alpha, \eta)}{(\alpha, \cdot)}\{r_\alpha \partial_\alpha - \Delta_\alpha\}(-1 + 1) = 0.
\]

The third term (2.4) can be written as

\[
III = \sum_{\alpha, \beta \in R_+} k_\alpha k_\beta \{((\alpha, \xi)(\beta, \eta) - (\alpha, \eta)(\beta, \xi))\Delta_\alpha \Delta_\beta
\]

and for the proof of Theorem 1.5 it remains to verify the vanishing of this third term.

**Proposition 2.2.** Suppose $B(\cdot, \cdot)$ is a bilinear form on $E$ such that

\[
B(r_\alpha \lambda, r_\alpha \mu) = B(\mu, \lambda) \quad \forall \lambda, \mu \in E, \forall \alpha \in R \cap \text{span} \langle \lambda, \mu \rangle.
\]
If $w \in W$ is a pure rotation (i.e. $\dim \text{Im}(w - \text{Id}) = 2$) then

\begin{equation}
\sum_{\alpha, \beta \in R_+, r_\alpha r_\beta = w} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(\alpha, \cdot)(\beta, \cdot)} = 0 \tag{2.9}
\end{equation}

and

\begin{equation}
\sum_{\alpha, \beta \in R_+, r_\alpha r_\beta = w} k_\alpha k_\beta B(\alpha, \beta) \Delta_\alpha \Delta_\beta = 0. \tag{2.10}
\end{equation}

**Proof:** Using the definition $\Delta_\alpha = \frac{1}{(\alpha, \cdot)}(1 - r_\alpha)$ the left hand side of (2.10) can be written as a sum of the following three terms

\begin{equation}
A = \sum_{\alpha, \beta \in R_+, r_\alpha r_\beta = w} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(\alpha, \cdot)(\beta, \cdot)} \tag{2.11}
\end{equation}

\begin{equation}
B = -\sum_{\alpha, \beta \in R_+, r_\alpha r_\beta = w} k_\alpha k_\beta B(\alpha, \beta) \left\{ \frac{1}{(\alpha, \cdot)(r_\alpha \beta, \cdot)} r_\alpha + \frac{1}{(\alpha, \cdot)(\beta, \cdot)} r_\beta \right\} \tag{2.12}
\end{equation}

\begin{equation}
C = \sum_{\alpha, \beta \in R_+, r_\alpha r_\beta = w} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(\alpha, \cdot)(r_\alpha \beta, \cdot)} r_\alpha r_\beta \tag{2.13}
\end{equation}

with the summations over the same index set as in (2.9) and (2.10).

Let $S = R \cap \text{Im}(w - \text{Id})$ be the normalized root system of the largest dihedral group $W(S)$ containing $w$. If $w = r_\alpha r_\beta$ then for $\gamma \in S$ we have $r_\gamma w r_\gamma = w^{-1}$ and hence $r_{r_\gamma \alpha} r_{r_\beta \gamma} = r_\beta r_\alpha$. We claim that $r_\gamma A = A \ \forall \gamma \in S$. Indeed we have

\[
r_\gamma A = \sum_{\alpha, \beta \in R_+, r_\alpha r_\beta = w} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(r_\gamma \alpha, \cdot)(r_\gamma \beta, \cdot)}
\]

\[
= \sum_{\alpha, \beta \in r_\gamma R_+, r_\beta r_\alpha = w} k_\alpha k_\beta B(r_\gamma \alpha, r_\gamma \beta) \frac{1}{(\alpha, \cdot)(\beta, \cdot)}
\]

\[
= \sum_{\alpha, \beta \in r_\gamma R_+, r_\beta r_\alpha = w} k_\alpha k_\beta B(\beta, \alpha) \frac{1}{(\alpha, \cdot)(\beta, \cdot)}
\]

\[
= A
\]

since the summation in (2.9) is independent of the choice of $R_+$. Let $S_+ = R_+ \cap S$ and put $p_S = \prod_{\alpha \in S_+} (\alpha, \cdot)$. Then $p_S$ transforms under the group $W(S)$ according to the sign character and every polynomial in $\mathbb{C}[E]$ transforming under $W(S)$ according to the sign character is divisible in $\mathbb{C}[E]$ by $p_S$. Now observe that $p_S A \in \mathbb{C}[E]$ transforms
under $W(S)$ according to the sign character. Hence $A \in \mathbb{C}[E]$. Since $A$ is homogeneous of degree minus two we have $A = 0$. This proves (2.9).

Since $w = r_{\alpha}r_{\beta} = r_{r_{\alpha}r_{\alpha}}$ and $B(\alpha, \beta) = B(r_{\alpha}r_{\beta}, r_{\alpha}r_{\alpha}) = -B(r_{\alpha}r_{\beta}, \alpha)$ the vanishing of the term (2.12) is clear, and for the term (2.13) we can write $C = -Aw = 0$. Q.E.D.

**Lemma 2.3.** For $\xi, \eta \in E$ fixed the bilinear form

$$B(\lambda, \mu) = (\lambda, \xi)(\mu, \eta) - (\lambda, \eta)(\mu, \xi) \quad (2.14)$$

on $E$ satisfies condition (2.8).

**Proof:** Clearly $B(\mu, \lambda) = -B(\lambda, \mu)$ is an alternating form. For $\lambda \in E, \lambda \neq 0$ we write $\lambda' = \sqrt{2}|\lambda|^{-1}\lambda$ and get

$$B(r_{\lambda'}\lambda, r_{\lambda'}\mu) = B(-\lambda, \mu - (\lambda', \mu)\lambda') = B(-\lambda, \mu) = B(\mu, \lambda).$$

Hence for $\lambda, \mu \in E$ generic we get by continuity

$$B(r_{\nu}\lambda, r_{\nu}\mu) = B(\mu, \lambda) \quad \forall \nu \in \text{span} \langle \lambda, \mu \rangle, (\nu, \nu) = 2. \quad \text{Q.E.D.}$$

The proof of Theorem 1.5 now follows by regrouping the terms in (2.7) as a sum over $\{\alpha, \beta \in R_+; r_{\alpha}r_{\beta} = w\}$ where $w \in W$ runs over the pure rotations in $W$ and by applying (2.10).

The proof of Theorem 1.6 is just an easy calculation.

$$\sum_{j=1}^{n} D_{\xi_j}^2 = \sum_{j=1}^{n} \left( \partial \xi_j + \sum_{\alpha \in R_+} k_{\alpha}(\alpha, \xi_j)\Delta_{\alpha} \right)^2$$

$$= \sum_{j=1}^{n} \left\{ \partial_{\xi_j}^2 + \sum_{\alpha \in R_+} k_{\alpha}(\alpha, \xi_j)(\partial_{\xi_j} \Delta_{\alpha} + \Delta_{\alpha} \partial_{\xi_j}) + \sum_{\alpha, \beta \in R_+} k_{\alpha}k_{\beta}(\alpha, \xi_j)(\beta, \xi_j)\Delta_{\alpha}\Delta_{\beta} \right\}$$

$$= \sum_{j=1}^{n} \partial_{\xi_j}^2 + \sum_{\alpha \in R_+} k_{\alpha}(\partial_{\alpha} \Delta_{\alpha} + \Delta_{\alpha} \partial_{\alpha}) + \sum_{\alpha, \beta \in R_+} k_{\alpha}k_{\beta}(\alpha, \beta)\Delta_{\alpha}\Delta_{\beta}.$$
§3. The Opdam shift operators.

Recall that $D \in \text{End}(\mathbb{C}[p_1, \ldots, p_m])$ is a differential operator of degree $\leq d$ if and only if
\begin{equation}
(3.1) \quad \text{ad}(p)^{d+1}(D) = 0 \quad \forall p \in \mathbb{C}[p_1, \ldots, p_n].
\end{equation}
Hence the fact that the operators (1.11), (1.16) and (1.17) are differential operators is clear from
\begin{equation}
(3.2) \quad \text{ad}(p)(D\xi) = \text{ad}(p)(\partial\xi) = -\partial\xi(p)
\end{equation}
\begin{equation}
(3.3) \quad \text{ad}(p)^2(D\xi) = 0 \quad \forall p \in \mathbb{C}[E]^W, \forall \xi \in E.
\end{equation}
Hence Theorem 1.7 is an immediate consequence of Theorem 1.5 and Theorem 1.6.

**Theorem 3.1.** For the $q$-analogue of the Laplacian we have
\begin{equation}
(3.4) \quad \text{Res}(p_S^{-1} \circ \left\{ \sum_{j=1}^{n} D_{\xi_j}^2(k) \right\} \circ p_S) = \text{Res}(\sum_{j=1}^{n} D_{\xi_j}^2(k + 1_S)).
\end{equation}

**Proof:** First we observe that the left hand side of (3.4) is a well defined endomorphism of $\mathbb{C}[E]^W$. We now use Theorem 1.6 and just calculate term by term. For the first term we get
\begin{equation}
p_S^{-1} \circ \left\{ \sum_{j=1}^{n} \partial_{\xi_j}^2 \right\} \circ p_S = \sum_{j=1}^{n} \partial_{\xi_j}^2 + 2 \sum_{\alpha \in S_+} \frac{1}{(\alpha, \cdot)} \partial_\alpha + p_S^{-1}(\sum_{j=1}^{n} \partial_{\xi_j}^2)(p_S)
\end{equation}
\begin{equation}
= \sum_{j=1}^{n} \partial_{\xi_j}^2 + 2 \sum_{\alpha \in S_+} \frac{1}{(\alpha, \cdot)} \partial_\alpha.
\end{equation}
For the second term we get
\begin{equation}
p_S^{-1} \circ \left\{ 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)} \partial_\alpha \right\} \circ p_S = 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)} \partial_\alpha + p_S^{-1} \cdot \left( \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)} \partial_\alpha \right)(p_S)
\end{equation}
\begin{equation}
= 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)} \partial_\alpha + 2 \sum_{\alpha, \beta \in S_+} \frac{k_\alpha}{(\alpha, \cdot)}(\alpha, \beta) \cdot \frac{1}{(\beta, \cdot)}
\end{equation}
\begin{equation}
= 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)} \partial_\alpha + 2 \sum_{\beta \in S_+} \frac{k_\beta}{(\beta, \cdot)} \cdot \frac{1}{(\beta, \cdot)}
\end{equation}
\begin{equation}
+ 2 \sum_{\alpha, \beta \in S_+} \frac{k_\alpha}{(\alpha, \cdot)}(\alpha, \beta) \cdot \frac{1}{(\beta, \cdot)}
\end{equation}
\begin{equation}
= 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)} \partial_\alpha + 2 \sum_{\beta \in S_+} \frac{2}{(\beta, \cdot)}.
\end{equation}
by the same argument as in the proof of Proposition 2.2. Finally for the third term we have

\[ p_S^{-1} \circ \left\{ 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \Delta_{\alpha} \right\} \circ p_S = 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)^2} \left\{ 1 - p_S^{-1} \circ r_{\alpha} \circ p_S \right\} \]

\[ = 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)^2} \left\{ 1 - \chi_S(r_{\alpha}) r_{\alpha} \right\} \]

\[ = 2 \sum_{\alpha \in S_+} k_{\alpha} \frac{1}{(\alpha, \cdot)^2} \left\{ 1 + r_{\alpha} \right\} + 2 \sum_{\alpha \in R_+ \setminus S_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \Delta_{\alpha} \]

\[ = 2 \sum_{\alpha \in S_+} k_{\alpha} \frac{2}{(\alpha, \cdot)^2} - 2 \sum_{\alpha \in S_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \Delta_{\alpha} \]

\[ + 2 \sum_{\alpha \in R_+ \setminus S_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \Delta_{\alpha}. \]

Taking all three terms together yields

\[ p_S^{-1} \circ \left\{ \sum_{j=1}^{n} D_{\xi_j}^2(k) \right\} \circ p_S = \sum_{j=1}^{n} \partial^2_{\xi_j} + 2 \sum_{\alpha \in R_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} + 2 \sum_{\alpha \in S_+} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} \]

\[ + 2 \sum_{\alpha \in S_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \Delta_{\alpha} - 2 \sum_{\alpha \in R_+ \setminus S_+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \Delta_{\alpha}. \quad \text{Q.E.D.} \]

**Corollary 3.2.** We have the shift relations

\begin{align*}
(3.5) \quad & G(1_S, k) \text{Res} \left( \sum_{j=1}^{n} D_{\xi_j}^2(k) \right) = \text{Res} \left( \sum_{j=1}^{n} D_{\xi_j}^2(k + 1_S) \right) G(1_S, k) \\
(3.6) \quad & G(-1_S, k) \text{Res} \left( \sum_{j=1}^{n} D_{\xi_j}^2(k) \right) = \text{Res} \left( \sum_{j=1}^{n} D_{\xi_j}^2(k - 1_S) \right) G(-1_S, k).
\end{align*}

**Proof:** Indeed we have

\[ \text{Res}(p_S^{-1} D_{\pi_S}(k)) \text{Res}(\sum_{j=1}^{n} D_{\xi_j}^2(k)) = \text{Res}(\sum_{j=1}^{n} p_S^{-1} D_{\pi_S}(k) D_{\xi_j}^2(k)) \]

\[ = \text{Res}(\sum_{j=1}^{n} p_S^{-1} D_{\xi_j}^2(k) D_{\pi_S}(k)) \]

\[ = \text{Res}(\sum_{j=1}^{n} p_S^{-1} D_{\xi_j}^2(k) p_S) \text{Res}(p_S^{-1} D_{\pi_S}(k)) \]

\[ = \text{Res}(\sum_{j=1}^{n} D_{\xi_j}^2(k + 1_S) \text{Res}(p_S^{-1} D_{\pi_S}(k)). \]

\[ 9 \]
which proves (3.5). The relation (3.6) is proved similarly. Q.E.D.

**Theorem 3.3.** As endomorphisms of \( \mathbb{C}[E] \) the operators

\[
E = \frac{1}{2} \sum_{j=1}^{n} (\xi_j,)^2
\]

(3.7)

\[
H = \sum_{j=1}^{n} (\xi_j, \partial_{\xi_j}) + \left( \frac{n}{2} + \sum_{\alpha \in R_+} k_{\alpha} \right)
\]

(3.8)

\[
F = -\frac{1}{2} \sum_{j=1}^{n} D_{\xi_j}^2
\]

(3.9)

satisfy the commutation relations of \( sl(2) \):

\[
\]

(3.10)

**Proof:** The Euler operator \( \sum_{j=1}^{n} (\xi_j, \partial_{\xi_j}) \) acts as multiplication by \( d \) on the space of homogeneous polynomials in \( \mathbb{C}[E] \) of degree \( d \). Hence the commutation relations \( [H, E] = 2E, \ [H, F] = -2F \) rephrase that \( E \) and \( F \) are homogeneous of degree plus and minus two respectively.

Since \( [p, \Delta_\alpha] = 0 \forall p \in \mathbb{C}[E]^W, \forall \alpha \in R \) we get

\[
[E, D_{\xi}] = [E, \partial_{\xi}] = -(\xi, \cdot) \quad \forall \xi \in E,
\]

(3.11)

and therefore

\[
[E, F] = -\frac{1}{2} \sum_{j=1}^{n} [E, D_{\xi_j}^2]
\]

\[
= \frac{1}{2} \sum_{j=1}^{n} \{(\xi_j, D_{\xi_j}) + D_{\xi_j}(\xi_j, \cdot)\}
\]

\[
= \sum_{j=1}^{n} (\xi_j, D_{\xi_j}) + \frac{1}{2} \sum_{j=1}^{n} [D_{\xi_j}, (\xi_j, \cdot)]
\]

\[
= \sum_{j=1}^{n} (\xi_j, D_{\xi_j}) + \frac{n}{2} + \frac{1}{2} \sum_{j=1}^{n} \sum_{\alpha \in R_+} k_{\alpha}(\alpha, \xi_j)[\Delta_\alpha, (\xi_j, \cdot)]
\]

\[
= \sum_{j=1}^{n} (\xi_j, \partial_{\xi_j}) + \sum_{\alpha \in R_+} k_{\alpha}(\alpha, \cdot)\Delta_\alpha + \frac{n}{2} + \sum_{\alpha \in R_+} k_{\alpha}r_\alpha
\]

\[
= \sum_{j=1}^{n} (\xi_j, \partial_{\xi_j}) + \left( \frac{n}{2} + \sum_{\alpha \in R_+} k_{\alpha} \right).
\]
Here we have used that for $\xi \in E$

\[
[\Delta_\alpha, (\xi, \cdot)] = -\frac{1}{(\alpha, \cdot)}[r_\alpha, (\xi, \cdot)] \\
= -\frac{1}{(\alpha, \cdot)}\{(r_\alpha \xi, \cdot) - (\xi, \cdot)\}r_\alpha \\
= (\alpha, \xi)r_\alpha.
\]

Q.E.D.

**Proposition 3.4.** Using the inner product $(\cdot, \cdot)$ on $E$ we have an isomorphism between $\mathbb{C}[E]$ and $\mathbb{C}[E^*]$. For $p \in \mathbb{C}[E]$ we write $\pi \in \mathbb{C}[E^*]$ for the corresponding element. For $p \in \mathbb{C}[E]$ homogeneous of degree $d$ we have

\[(3.12) \quad D_\pi = (-1)^d \frac{1}{d!} \text{ad}(F)^d(p).\]

**Proof:** Clearly $\text{ad}(H)D_\pi = -dD_\pi$ and by Theorem 1.5 we have $\text{ad}(F)D_\pi = 0$. Using (3.11) and induction on $d$ (assuming $\pi$ to be a monomial as in (1.9) with $d = d_1 + \cdots + d_n$) it is easy to see that

\[-(-1)^d \frac{1}{d!} \text{ad}(E)^d(D_\pi) = p\]

and hence

\[-\text{ad}(E)^{d+1}(D_\pi) = 0.\]

By standard representation theory of $sl(2)$ we conclude (3.12). Q.E.D.

**Corollary 3.5.** For $\pi \in \mathbb{C}[E^*]^W$ we have

\[(3.13) \quad \text{Res}\left(p_S^{-1} \circ D_\pi(k) \circ p_S\right) = \text{Res}\left(D_\pi(k + 1_S)\right).\]

**Proof:** This is easily derived from Theorem 3.1 and Proposition 3.4. Q.E.D.

The proof of Theorem 1.9 now goes along the same lines as the proof of Corollary 3.2.

**Remark 3.6.** The above type of arguments to use an $sl(2)$ to reduce the computation of higher order operators to those of the second order one go back to Harish-Chandra [Ha].

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References


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