Quantum integrability for the Kovalevsky top

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1 Introduction

Let N be a differentiable manifold of dimension n. Let $M = T^*N$ be the cotangent bundle of N, and denote by σ the natural symplectic form on M. In local coordinates $q = (q_1, \ldots q_n)$ on N and $p_i = \frac{\partial}{\partial q_i}$ (viewed as function on M) we have $\sigma = \sum dp_i \wedge dq_i$. For $f \in C^\infty(M)$ the hamiltonian vector field v_f on M is defined by $df(\cdot) = -\sigma(v_f, \cdot)$. For this sign convention we have

$$[v_f, v_g] = v_{\{f,g\}} \tag{1.1}$$

with the Poisson bracket $\{f, g\}$ of $f, g \in C^\infty(M)$ defined by $\{f, g\} = v_f(g) = \sigma(v_f, v_g)$, or in local coordinates

$$\{f, g\} = \sum \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right). \tag{1.2}$$

Let us write $P^k(M)$ for the subspace of $C^\infty(M)$ consisting of those functions which are homogeneous of degree k ($k \in \mathbb{N}$) in p. The subspace $P(M) = \oplus P^k(M)$ of $C^\infty(M)$ is closed under Poisson bracket. In fact

$$\{\cdot, \cdot\} : P^k(M) \otimes P^l(M) \to P^{k+l-1}(M) \tag{1.3}$$

making $P(M) = \oplus P^k(M)$ into a graded Poisson algebra.

Let $D_k(N)$ denote the space of smooth linear differential operators on N of order at most k. The space $D(N) = \cup D_k(N)$ of all differential operators on N becomes a filtered associative algebra with respect to composition of differential operators, and the associated commutator bracket $[\cdot, \cdot]$ satisfies

$$[\cdot, \cdot] : D_k(N) \otimes D_l(N) \to D_{k+l-1}(N). \tag{1.4}$$

Of course the Poisson algebra $P(M) = \oplus P^k(M)$ is just the associated graded of the filtered associative algebra $D(N) = \cup D_k(N)$ with

$$gr_k : D_k(N) \to P^k(M)$$

being the symbol map of order k.

Let g be a Riemannian metric on N. This gives rise to an isomorphism $TN \cong T^*N$, and therefore g can also be viewed as a quadratic form on T^*N. So by abuse of notation $g \in P^2(M)$. Let L be the Laplace operator on N associated with g. This means that $L \in D_2(N)$ with $gr_2(L) = g$, and in addition L is self adjoint with respect to the Riemannian volume element and $L(1) = 0$. If V is a potential function on N, then $V \in P^0(M)$ via pull back under the natural projection $M \to N$. With g and V given as above consider the hamiltonian function

$$h = \tfrac{1}{2}g + cV \in P^2(M) \tag{1.5}$$

and the associated Schrödinger operator

$$H = \tfrac{1}{2}L + cV \in D_2(N). \tag{1.6}$$

Here c is a real parameter with $\deg(c) = 2$ making indeed $h = \frac{1}{2}g + cV$ homogeneous of degree 2. From now on it is understood that elements of $P(M)$ and $D(N)$ are allowed to depend polynomially on this parameter c as well. With these notations it is obvious that

$$gr_2(H) = h. \tag{1.7}$$

Definition 1.1 *The Schrödinger operator $H_1 = H$ is called quantum completely integrable if there exist differential operators $H_i \in D_{d_i}(N)$ for $i = 1, 2, \ldots, n$ such that*

$$[H_i, H_j] = 0 \qquad \forall i, j \tag{1.8}$$

and the Poisson commuting symbols $h_i := gr_{d_i}(H_i) \in P^{d_i}(M)$ make the hamiltonian function h completely integrable (in the sense of classical mechanics).

So by the very definition a quantum completely integrable Schrödinger operator gives rise to a completely integrable hamiltonian function. Our concern here is if a converse statement holds.

Question 1.2 Is for any completely integrable hamiltonian function $h = \frac{1}{2}g + cV \in P^2(M)$ the associated Schrödinger operator $H = \frac{1}{2}L + cV \in D_2(N)$ quantum completely integrable?

The answer to this question ought to be yes, but I have no idea about a proof. On the other hand too much optimism about the correspondence between classical and quantum mechanics leads to inaccuracies, as can be learned for example from the theorem of Groenewold and van Hove [GS].

Example 1.3 For the rotation of a free rigid body the length squared of the angular momentum vector (which is a classical integral) lifts to the biinvariant Laplace operator on $SO(3, \mathbb{R})$ (which is the corresponding quantum integral). This fundamental observation of Casimir [C] played an important role in the development of semisimple Lie theorie [FdV],[S].

Example 1.4 For the Toda system associated with root systems a partial answer to the above question is known from the work of Goodman and Wallach [GW]. In the non periodic case the answer is always yes. However in the periodic case a positive answer is only known for the classical root systems and E_6.

Example 1.5 For the Calogero-Moser system associated with root systems the answer to the above question is fully yes. The complete integrability for the classical system was proved by Moser for type A_n by realizing the system via a Lax pair [M]. This method was extended by Olshanetsky and Perelomov in the case of classical root systems [OP]. The quantum complete integrability was obtained by Opdam for all root systems using analytic methods [O1]. After the work of Dunkl [D] the quantum complete integrability could be proved in an elementary way [H],[HS],[O2]. The only proof known at the moment of complete integrability of the classical system in case of exceptional root systems goes by deducing it from quantum complete integrability! The problem of quantum integrability for the Calogero-Moser system with doubly periodic potential was studied by Ochiai, Oshima and Sekiguchi [OOS].

Example 1.6 The C. Neumann system and the geodesic flow on an ellipsoid have been shown to be quantum completely integrable by Toth [T1], [T2]. The classical integrals are all quadratic and can be lifted to second order commuting differential operators.

The purpose of this paper is to answer the above question for the case of the Kovalevsky and Goryachev-Chaplygin tops.

Theorem 1.7 *For the Kovalevsky top the biquadratic classical integral and for the Goryachev-Chaplygin top the cubic classical integral can be lifted to quantum integrals.*

The proof will be given in the next section. Making a natural guess for the quantum integrals by analogy with the classical integrals it follows by straightforward algebra that only quadratic correction terms are needed. After this note was written I learned that the problem of quantum integrability of the Kovalevsky top had been studied before by Ramani, Grammaticos and Dorizzi [RGD]. Their method is to remain in the Poisson algebra, and work with the Moyal bracket instead of the Poisson bracket. Their final computation is performed using computer algebra. Our computation (Proposition 2.5 and Theorem 2.6) is done (by hand) directly in the universal envelopping algebra.

This work was done in the fall of 1995 while I enjoyed the pleasant atmosphere of the Mittag Leffler Institute at Stockholm. I am also indebted to V. Guillemin and A. Weinstein for useful discussions.

2 The rotation of a heavy top around a fixed point

Consider $\mathbb{R}^3$ with standard orthonormal basis e_1, e_2, e_3. Let E_1, E_2, E_3 be the endomorphisms of $\mathbb{R}^3$ given by $E_i(e_i) = 0$, $E_i(e_{i+1}) = e_{i+2}$, $E_i(e_{i+2}) = -e_{i+1}$ with i an index in $\mathbb{Z}/3\mathbb{Z}$. Then E_1, E_2, E_3 is a basis of the Lie algebra $so(3, \mathbb{R})$ with commutation relations $[E_i, E_{i+1}] = E_{i+2}$. Let L_1, L_2, L_3 be the left invariant vector fields on $SO(3, \mathbb{R})$ defined by

$$L_i(f)(x) = \frac{d}{dt}\{f(x \exp tE_i)\}_{t=0}. \tag{2.1}$$

Fix some unit vector $e \in \mathbb{R}^3$, and define smooth functions Q_1, Q_2, Q_3 on $SO(3, \mathbb{R})$ by

$$Q_i(x) = (xe_i, e). \tag{2.2}$$

Lemma 2.1 *The subalgebra of $D(SO(3,\mathbb{R}))$ generated by $L_1, L_2, L_3, Q_1, Q_2, Q_3$ is isomorphic to the universal enveloping algebra of $so(3,\mathbb{R}) \ltimes \mathbb{R}^3$, i.e. we have commutation relations*

$$\begin{array}{l} [L_i, L_{i+1}] = L_{i+2}, [L_i, L_{i+2}] = -L_{i+1}, [Q_i, Q_j] = 0 \\ [L_i, Q_i] = 0, [L_i, Q_{i+1}] = Q_{i+2}, [L_i, Q_{i+2}] = -Q_{i+1}. \end{array} \tag{2.3}$$

Proof: This is a direct computation using (2.1) and (2.2). □

In the associated Poisson algebra $P(T^*SO(3,\mathbb{R}))$ we shall use small roman letters instead of capitals. So the Poisson brackets of $l_1, l_2, l_3, q_1, q_2, q_3$ are given by

$$\begin{array}{l} \{l_i, l_{i+1}\} = l_{i+2}, \{l_i, l_{i+2}\} = -l_{i+1}, \{q_i, q_j\} = 0 \\ \{l_i, q_i\} = 0, \{l_i, q_{i+1}\} = q_{i+2}, \{l_i, q_{i+2}\} = -q_{i+1}. \end{array} \tag{2.4}$$

Now let $h \in P(T^*SO(3,\mathbb{R}))$ be the hamiltonian function defined by

$$h = \tfrac{1}{2}(I_1^{-1}l_1^2 + I_2^{-1}l_2^2 + I_3^{-1}l_3^2) - (c_1q_1 + c_2q_2 + c_3q_3) \tag{2.5}$$

for certain parameters $I_1, I_2, I_3 > 0$ and $c_1, c_2, c_3 \in \mathbb{R}$.

Lemma 2.2 *The Hamilton equation $\dot{f} = \{h, f\}$ for $f \in \mathbb{C}[l_1, l_2, l_3, q_1, q_2, q_3]$ amounts to the system*

$$\begin{array}{ll} \dot{l}_1 = (I_3^{-1} - I_2^{-1})l_2l_3 + (c_2q_3 - c_3q_2), & \dot{q}_1 = I_3^{-1}l_3q_2 - I_2^{-1}l_2q_3 \\ \dot{l}_2 = (I_1^{-1} - I_3^{-1})l_3l_1 + (c_3q_1 - c_1q_3), & \dot{q}_2 = I_1^{-1}l_1q_3 - I_3^{-1}l_3q_1 \\ \dot{l}_3 = (I_2^{-1} - I_1^{-1})l_1l_2 + (c_1q_2 - c_2q_1), & \dot{q}_3 = I_2^{-1}l_2q_1 - I_1^{-1}l_1q_2 \end{array} \tag{2.6}$$

Proof: This is a direct computation. □

The system (2.6) are the Euler equations describing the rotation of a heavy rigid body around a fixed point [A]. These equations are known to be completely integrable precisely in the following cases [AKN], [DKN], [BRS], [Au].

Euler case: $c_1 = c_2 = c_3 = 0$.
Lagrange case: $I_1 = I_2$, $c_1 = c_2 = 0$.
Kovalevsky case: $I_1 = I_2 = 2I_3$, $c_3 = 0$.
Goryachev-Chaplygin case: $I_1 = I_2 = 4I_3$, $c_3 = 0$.

In the last case an integral can only be found under the additional constraint $q_1l_1 + q_2l_2 + q_3l_3 = 0$. The question of quantum integrability requires only a further analysis in the last two cases. Without loss of generality we can take the hamiltonian function in the Kovalevsky case of the form

$$4h = l_1^2 + l_2^2 + 2l_3^2 - 4cq_1 \tag{2.7}$$

and in the Goryachev-Chaplygin case of the form

$$8h = l_1^2 + l_2^2 + 4l_3^2 - 8cq_1. \tag{2.8}$$

The explicit form of the additional integrals is given in the following propositions.

Proposition 2.3 If in the Kovalevsky case we put

$$k = (l_1 + il_2)^2 + 4c(q_1 + iq_2) \tag{2.9}$$

then $\{h, k\} = -il_3 k$ and therefore $\{h, k\,\overline{k}\} = 0$.

Proposition 2.4 [K]. If in the Goryachev-Chaplygin case we put

$$g = (l_1^2 + l_2^2)l_3 + 4cq_3 l_1 \tag{2.10}$$

then $\{h, g\} = cl_2(q_1 l_1 + q_2 l_2 + q_3 l_3)$ and therefore $\{h, g\} = 0$ under the restriction $q_1 l_1 + q_2 l_2 + q_3 l_3 = 0$.

Both propositions follow from a direct computation using (2.4). For the corresponding Schrödinger operator H we take in the Kovalevsky case

$$4H = L_1^2 + L_2^2 + 2L_3^2 - 4cQ_1 \tag{2.11}$$

and in the Goryachev-Chaplygin case

$$8H = L_1^2 + L_2^2 + 4L_3^2 - 8cQ_1. \tag{2.12}$$

If we introduce operators K and G in analogy with (2.9) and (2.10) respectively by

$$K = (L_1 + iL_2)^2 + 4c(Q_1 + iQ_2) \tag{2.13}$$

$$G = (L_1^2 + L_2^2)L_3 + 4cQ_3 L_1 \tag{2.14}$$

then the next proposition is obtained from a direct computation using (2.3).

Proposition 2.5 In the Kovalevsky case we have the commutation relations

$$[2H, K] = -i(L_3 K + K L_3) \tag{2.15}$$

$$[2H, K\,\overline{K} + \overline{K}\,K] = -16c(Q_3 L_2 + L_2 Q_3) \tag{2.16}$$

and in the Goryachev-Chaplygin case we have the commutation relation

$$[2H, G] = 2cL_2(Q_1 L_1 + Q_2 L_2 + Q_3 L_3) - 4cQ_1 L_3 + cQ_3 L_1 - 2cQ_2. \tag{2.17}$$

Theorem 2.6 *In the Kovalevsky case we have the commutation relation*

$$[H, K\,\overline{K} + \overline{K}\,K - 8(L_1^2 + L_2^2)] = 0 \tag{2.18}$$

and in the Goryachev-Chaplygin case we have

$$[H, G - 2cQ_2 - \tfrac{1}{4}L_3] = cL_2(Q_1 L_1 + Q_2 L_2 + Q_3 L_3). \tag{2.19}$$

Proof: A straighforward calculation shows that in the Kovalevsky case

$$[2H, L_1^2 + L_2^2] = -2c(Q_3 L_2 + L_2 Q_3), \tag{2.20}$$

and in the Goryachev-Chaplygin case

$$[8H, Q_2] = -8Q_1 L_3 + 2Q_3 L_1 - 5Q_2, [8H, L_3] = 8cQ_2. \tag{2.21}$$

Combination of these relations with (2.16) and (2.17) yields the desired result. □

So both in the Kovalevsky and the Goryachev-Chaplygin case we have to add to our initial guess a quadratic correction term in order to arrive at the correct quantum integral.

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