THE MATRIX OF A CONNECTION HAVING REGULAR SINGULARITIES ON A VECTOR BUNDLE OF RANK 2 ON $\mathbb{P}^1(\mathbb{C})$

by

W. Dekkers

The theorem in section 4 is valid in the complex analytic situation as well as in the algebraic situation. For simplicity we shall restrict ourselves to the complex analytic situation, but in fact the proof is purely algebraic.

1. $X = \mathbb{P}^1(\mathbb{C})$, $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X , $\mathcal{E}$ is a holomorphic vector bundle of rank 2 on X and Ω^1 is the sheaf of holomorphic differentials on X .

S is a finite non-void set of points of X and $\mathcal{O}(S)$ is the sheaf of germs of functions on X holomorphic on $X-S$ meromorphic in S .

Let 0 and ∞ be distinct points of X . Then on $U_0 = X - \{\infty\}$ and on $U_\infty = X - \{0\}$ we have holomorphic co-ordinate functions z and x respectively, such that $z(0) = 0$, $x(\infty) = 0$ and $x = \frac{1}{z}$ on $U_0 \cap U_\infty$.

2. Bases meromorphic in S .

Let $(e) = (e_1, e_2)$ be a basis of $\mathcal{E}/_{X-S}$.

(e) is called a basis of $\mathcal{E}$ meromorphic in S if for each $s \in S$ there exists a neighbourhood U of s , a basis (f) of $\mathcal{E}/_U$ and a $T \in GL(2, \Gamma(U, \mathcal{O}(S)))$ such that $(e) = (f)T$ on $U-S$.

For $n \in \mathbb{Z}$ let $\mathcal{O}(n)$ be the locally free sheaf of rank 1 in the sense of [2] no 54. There exist integers n and m such that $\mathcal{E}$ is isomorphic to $\mathcal{O}(n) \oplus \mathcal{O}(m)$ (cf. [3]). Now $S \neq \emptyset$, hence $X-S$ is different from X . Therefore there exists an isomorphism

$$\mathcal{O}(n) \oplus \mathcal{O}(m)/_{X-S} \to \mathcal{O}^2/_{X-S} .$$

Moreover this isomorphism can be chosen in such a way that the bases of $\mathcal{O}(n) \oplus \mathcal{O}(m)$ meromorphic in S correspond to the bases of $\mathcal{O}^2$ meromorphic in S . In particular $\mathcal{E}$ has a global basis meromorphic in S .

3. Connections.

Let ∇ be a (holomorphic) connection on $\mathcal{E}$.

∇ induces $\mathbb{C}$-linear morphisms

$$\nabla_{\frac{d}{dz}} , \nabla_{\frac{d}{dx}} : \mathcal{E} \to \mathcal{E}$$

(cf. [1], chap. I, 2.4 and 2.5).

Instead of $\nabla_{\frac{d}{dz}}$ and $\nabla_{\frac{d}{dx}}$ we shall write ${}_z\nabla$ and ${}_x\nabla$ respectively. One has

$${}_z\nabla = -\frac{1}{z^2}\, {}_x\nabla .$$

Let $(e) = (e_1, e_2)$ be a basis of $\mathcal{E}$ meromorphic in S . Then

$${}_z\nabla\, e_1 = \gamma_{11}\, e_1 + \gamma_{21}\, e_2$$

$${}_z\nabla\, e_2 = \gamma_{12}\, e_1 + \gamma_{22}\, e_2$$

$(\gamma_{ij} \in \Gamma(S, \mathcal{O}(S)))$.

$$\Gamma = \Gamma({}_z\nabla,(e)) = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$$

is called <u>the matrix of</u> ${}_z\nabla$ <u>with respect to the basis</u> (e). If (e') is another basis of $\mathcal{E}$ meromorphic in S then $(e') = (e)T$ for some $T \in GL(2,\Gamma(X,\mathcal{O}(S)))$. We have

$$\Gamma' = T^{-1}\Gamma T + T^{-1}\frac{dT}{dz}.$$

Now let ∇ be a connection on $\mathcal{E}/_{X-S}$. ∇ is called a <u>connection on</u> $\mathcal{E}$ <u>having regular singularities in</u> S if for each $s \in U_0 \cap S$ $(s \in U_\infty \cap S)$ there exist a neighbourhood U of s and a basis (e) of $\mathcal{E}/_U$ meromorphic in $\{s\}$ such that the coefficients of $\Gamma({}_z\nabla/_{U-S},(e))$ $(\Gamma({}_x\nabla/_{U-S},(e)))$ have poles of order at most 1 in s. This definition agrees with that of [1], page 52.

4. THEOREM. <u>Let</u> $S = \{s_1,\dots,s_n\}$ <u>be a finite collection of points of</u> $X = \mathbb{P}^1(\mathbb{C})$, $s_i \neq s_j$ <u>if</u> $i \neq j$, $n \geq 2$, $z(s_i) = a_i$. <u>Let</u> $\mathcal{E}$ <u>be a vector bundle of rank 2 on</u> X <u>and</u> ∇ <u>a connection on</u> $\mathcal{E}$ <u>having regular singularities in</u> S. <u>Then there exists a basis</u> (e) <u>of</u> $\mathcal{E}$ <u>meromorphic in</u> S <u>such that</u>:

i) <u>if one of the points, say</u> s_n, <u>is</u> ∞ <u>then</u>

$$\Gamma({}_z\nabla,(e)) = \frac{A_1}{z-a_1} + \dots + \frac{A_{n-1}}{z-a_{n-1}}$$

<u>where</u> $A_1,\dots,A_{n-1} \in M_2(\mathbb{C})$;

ii) <u>if</u> $S \subset U_0$ <u>then</u>

$$\Gamma({}_z\nabla,(e)) = \frac{A_1}{z-a_1} + \dots + \frac{A_n}{z-a_n}$$

<u>where</u> $A_1,\dots,A_n \in M_2(\mathbb{C})$, $A_1 + \dots + A_n = 0$.

($M_2(\mathbb{C})$ denotes the set of 2×2-matrices with complex coefficients.)

The proof of the theorem takes the rest of this article.

5. Reduction to the case where $\mathcal{E} = \mathcal{O}^2$.

Let

$$\varphi : \mathcal{E}/_{X-S} \to \mathcal{O}^2/_{X-S}$$

be an isomorphism such that the bases of $\mathcal{E}$ meromorphic in S correspond to the bases of $\mathcal{O}^2$ meromorphic in S . Let ∇' be the connection induced by φ on $\mathcal{O}^2/_{X-S}$. Then it's easy to prove that ∇' is a connection on $\mathcal{O}^2$ having regular singularities in S . Moreover if (g) is a basis of $\mathcal{O}^2$ meromorphic in S then $\varphi^{-1}(g)$ is a basis of $\mathcal{E}$ meromorphic in S and

$$\Gamma({}_z\nabla, \varphi^{-1}(g)) = \Gamma({}_z\nabla', (g)) \text{ .}$$

This reduces the proof of the theorem to the case where $\mathcal{E} = \mathcal{O}^2$.

6. Case ii) in the theorem is an easy consequence of case i) (change of co-ordinate functions). Moreover in case i) we may assume $a_1 = 0$. Hence we only need to prove the theorem for the case

$$\underline{\mathcal{E} = \mathcal{O}^2 \text{ , } a_1 = 0 \text{ , } a_n = \infty \text{ .}}$$

Remark : $\Gamma(X,\mathcal{O}(S)) = \mathbb{C}[z,\frac{1}{z},\frac{1}{z-a_2},\dots,\frac{1}{z-a_{n-1}}]$. Let (e) be a basis of $\mathcal{O}^2$ meromorphic in S and let $\Gamma = \Gamma({}_z\nabla,(e))$.
Then $\Gamma \in M_2(\Gamma(X,\mathcal{O}(S))) = M_2(\mathbb{C}[z,\frac{1}{z},\frac{1}{z-a_2},\dots,\frac{1}{z-a_{n-1}}])$ and we must prove the existence of $T \in GL(2,\mathbb{C}[z,\frac{1}{z},\frac{1}{z-a_2},\dots,\frac{1}{z-a_{n-1}}])$ such that

$$\Gamma' = T^{-1}\Gamma T + T^{-1}\frac{dT}{dz} = \frac{A_1}{z} + \frac{A_2}{z-a_2} + \dots + \frac{A_{n-1}}{z-a_{n-1}}$$

where $A_1,\dots,A_{n-1} \in M_2(\mathbb{C})$.

7. Réduction of the poles in $z = 0$ to simple poles.

Let f be a function meromorphic in 0 , f not identically equal to zero. Then $f = z^n g$ where g is a function holomorphic in 0 , $g(0) \neq 0$, $n \in \mathbb{Z}$. We define

$$v(f) = n \qquad (v(0) = \infty) .$$

Let $\Gamma = \Gamma({}_z\nabla, (e)) = (\gamma_{ij}) = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$. ∇ has a regular singularity in $z=0$. Now from lemma 1.9.6 in [1], one easily deduces :

LEMMA.

$$-v(\gamma_{11}) = n > 1 \Rightarrow -v(\gamma_{22}) = n \;,\; -v(\gamma_{12}\,\gamma_{21}) = 2n$$

$$-v(\gamma_{11}) \leq 1 \Rightarrow -v(\gamma_{22}) \leq 1 \;,\; -v(\gamma_{12}\,\gamma_{21}) \leq 2 \;.$$

<u>Moreover if</u> $-v(\gamma_{11}) = n > 1$ <u>then</u>

(*)
$$\gamma_{ij} = a_{ij}\, z^{-n_{ij}}(1 + z\, c_{ij})$$

<u>where</u> $a_{ij} \in \mathbb{C}$, $a_{ij} \neq 0$, c_{ij} <u>holomorphic in</u> $z = 0$

$$n = n_{11} = n_{22} = \frac{n_{12} + n_{21}}{2}$$

<u>and</u>

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ is nilpotent.}$$

<u>Suppose that $-v(\gamma_{11}) \leq 1$.</u>

Let

$$T = \begin{pmatrix} z^m & 0 \\ 0 & 1 \end{pmatrix}$$

where $m \in \mathbb{Z}$. Then

$$\Gamma' = T^{-1}\Gamma T + T^{-1}\frac{dT}{dz} = \begin{pmatrix} \gamma_{11} + \frac{m}{z} & z^{-m}\,\gamma_{12} \\ z^m\,\gamma_{21} & \gamma_{22} \end{pmatrix} .$$

From the lemma immediately follows that we can choose m in such a way that $-v(\gamma'_{ij}) \le 1$ for all $i,j \in \{1,2\}$ and then we are ready.

So we may assume that $-v(\gamma_{11}) = n > 1$ and that (γ_{ij}) has the form (*) of the lemma. Let

$$T = \begin{pmatrix} 1 & \lambda z^{\ell} \\ 0 & 1 \end{pmatrix}$$

with $\lambda \in \mathbb{C}$ and $\ell \in \mathbb{Z}$.

Then

$$\Gamma' = \begin{pmatrix} \gamma_{11} - \lambda z^{\ell}\gamma_{21} & \gamma_{12} + \lambda z^{\ell}(\gamma_{11} - \gamma_{22}) - \lambda^2 z^{2\ell}\gamma_{21} + \lambda \ell z^{\ell-1} \\ \gamma_{21} & \gamma_{22} + \lambda z^{\ell}\gamma_{21} \end{pmatrix}.$$

Choose $\lambda = \frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}}$ and $\ell = n_{21} - n_{11} = n_{22} - n_{12}$. Look at the pole of $\gamma'_{11} = \gamma_{11} - \lambda z^{\ell}\gamma_{21}$ in $z = 0$. We have

$$a_{11} z^{-n_{11}} - \frac{a_{11}}{a_{21}} z^{n_{21}-n_{11}} . a_{21} z^{-n_{21}} = 0 .$$

Hence $-v(\gamma'_{11}) < n$.

Proceeding in this way we finally get a matrix Γ' such that $-v(\gamma'_{11}) \le 1$.

8. For the points s_i $(i = 2,\dots,n-1)$ we proceed in a similar way, using T's $\in GL(2,\mathbb{C}[z,\frac{1}{z-a_i}])$ of the form

$$\begin{pmatrix} (z-a_i)^m & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \lambda(z-a_i)^{\ell} \\ 0 & 1 \end{pmatrix}.$$

Doing this we don't introduce new poles of order >1 in s_j, $j = 1,\dots,n-1$, $j \neq i$.

9. We now have

$$\Gamma({}_z\nabla,(e)) = \frac{A_1}{z} + \frac{A_2}{z-a_2} + \dots + \frac{A_{n-1}}{z-a_{n-1}} + C(z)$$

where (e) is a basis of $\mathcal{O}^2$ meromorphic in S, $A_i \in M_2(\mathbb{C})$, $C(z) \in M_2(\mathbb{C}[z])$.

Let $x(s_i) = b_i = \frac{1}{a_i}$. Then $b_n = 0$ and $b_1 = \infty$. Replacing z by x we may equally well suppose that we have reached

$$\Gamma({}_x\nabla,(e)) = \frac{B_1}{x} + \frac{B_2}{x-b_2} + \dots + \frac{B_{n-1}}{x-b_{n-1}} + D(x) .$$

Let $\Gamma = \Gamma({}_z\nabla,(e)) = -\frac{1}{z^2}\,\Gamma({}_x\nabla,(e))$ (cf. 3.). Then Γ has the form

$$(**) \qquad \frac{A_1}{z} + \frac{A_2}{z-a_2} + \dots + \frac{A_{n-1}}{z-a_{n-1}} + \frac{C_2}{z^2} + \dots + \frac{C_{m+1}}{z^{m+1}}$$

where A_i, $C_i \in M_2(\mathbb{C})$.

We try to reduce the poles in $z=0$ to simple poles. We don't want to spoil the situation outside $z=0$ during the reduction process, so after each step Γ should still have the form (**).

We distinguish two cases :

(I) $\underline{-v(\gamma_{11}) \le 1}$

(II) $\underline{-v(\gamma_{11}) > 1}$.

In case (I) again we distinguish two cases :

(Ia) $\underline{-v(\gamma_{11}) \le 1 ,\ \gamma_{21} = 0 ,\ -v(\gamma_{12}) \ge 2}$

(Ib) $\underline{-v(\gamma_{11}) \le 1 ,\ \gamma_{21} \ne 0 ,\ -v(\gamma_{21}) \le 0 ,\ -v(\gamma_{12}) \ge 2}$.

10. Case (Ia) : $-v(\gamma_{11}) \le 1$, $\gamma_{21} = 0$, $-v(\gamma_{12}) \ge 2$.

Now $-v(\gamma_{22}) \le 1$ too. We may replace Γ by $\Gamma - \gamma_{22} I_2$ where I_2 denotes the identity element in $M_2(\mathbb{C})$. So we may assume $\gamma_{22} = 0$,

$$\Gamma = \begin{pmatrix} \frac{p_1}{z} + \frac{p_2}{z-a_2} + \dots + \frac{p_{n-1}}{z-a_{n-1}} & \frac{q_1}{z} + \frac{q_2}{z-a_2} + \dots + \frac{q_{n-1}}{z-a_{n-1}} + \frac{c_2}{z^2} + \dots + \frac{c_{m+1}}{z^{m+1}} \\ 0 & 0 \end{pmatrix}$$

where $m \geq 1$, $c_{m+1} \neq 0$.

We show that Γ can be transformed into a matrix Γ' with $-v(\gamma'_{11}) \leq 1$, $\gamma'_{21} = 0$, $-v(\gamma'_{12}) \leq m$. We have

1) $p_1 \neq m$

2) $p_i \neq 0$ for some $i \geq 2$

or

3) $p_1 + \dots + p_{n-1} \neq 0$.

<u>Case 1) : $p_1 \neq m$</u> .

$$T_1 = \begin{pmatrix} 1 & \alpha z^{-m} \\ 0 & 1 \end{pmatrix}$$

with $\alpha \in \mathbb{C}$ transforms Γ into

$$\Gamma' = \begin{pmatrix} \gamma_{11} & \gamma_{12} + \alpha z^{-m} \gamma_{11} - m\alpha z^{-m-1} \\ 0 & 0 \end{pmatrix} .$$

The coefficient of z^{-m-1} in γ'_{12} is

$$c_{m+1} + \alpha p_1 - m\alpha .$$

Take $\alpha = -\dfrac{c_{m+1}}{p_1 - m}$. Then $-v(\gamma'_{12}) \leq m$.

<u>Case 2) : $p_i \neq 0$ for some $i \geq 2$</u> .

Take

$$T_2 = \begin{pmatrix} z-a_i & \alpha \\ 0 & z \end{pmatrix}$$

with $\alpha \in \mathbb{C}$. Then

$$\Gamma' = \begin{pmatrix} \gamma_{11} + \dfrac{1}{z-a_i} & \dfrac{z}{z-a_i} \gamma_{12} + \dfrac{\alpha}{z-a_i} \gamma_{11} - \dfrac{\alpha}{z(z-a_i)} \\ 0 & \dfrac{1}{z} \end{pmatrix}$$

$$\gamma'_{12} = \frac{a_i q_i + \alpha p_i}{(z-a_i)^2} + \frac{q'_1}{z} + \frac{q'_2}{z-a_2} + \dots + \frac{q'_{n-1}}{z-a_{n-1}} + \frac{c'_2}{z^2} + \dots + \frac{c'_m}{z^m} .$$

Choose $\alpha = -\frac{a_i q_i}{p_i}$.

<u>Case 3) : $p_1 + \dots + p_{n-1} \neq 0$.</u>

$$T_3 = \begin{pmatrix} 1 & \alpha z \\ 0 & z \end{pmatrix}$$

with $\alpha \in \mathbb{C}$ transforms Γ into

$$\Gamma' = \begin{pmatrix} \gamma_{11} & z(\gamma_{12} + \alpha\gamma_{11}) \\ 0 & \frac{1}{z} \end{pmatrix}$$

$$\gamma'_{12} = q_1 + \dots + q_{n-1} + \alpha(p_1 + \dots + p_{n-1}) + \frac{q'_1}{z} + \dots + \frac{q'_{n-1}}{z-a_{n-1}} + \frac{c'_2}{z^2} + \dots + \frac{c'_m}{z^m} .$$

Take $\alpha = -\frac{q_1 + \dots + q_{n-1}}{p_1 + \dots + p_{n-1}}$.

11. <u>Case (Ib) : $-v(\gamma_{11}) \le 1$, $\gamma_{21} \neq 0$, $-v(\gamma_{21}) \le 0$, $-v(\gamma_{12}) \ge 2$.</u>

As in case (Ia) here again we may assume $\gamma_{22} = 0$.

$$\Gamma = \begin{pmatrix} \frac{p_1}{z} + \frac{p_2}{z-a_2} + \dots + \frac{p_{n-1}}{z-a_{n-1}} & \frac{q_1}{z} + \frac{q_2}{z-a_2} + \dots + \frac{q_{n-1}}{z-a_{n-1}} + \frac{c_2}{z^2} + \dots + \frac{c_{m+1}}{z^{m+1}} \\ \frac{r_2}{z-a_2} + \dots + \frac{r_{n-1}}{z-a_{n-1}} & 0 \end{pmatrix}$$

$m \ge 1$, $c_{m+1} \neq 0$, $r_j \neq 0$ for at least one j .

Let $r_i \neq 0$. Then take

$$T_2 = \begin{pmatrix} z-a_i & \alpha \\ 0 & z \end{pmatrix} , \quad \alpha = \frac{a_i}{2r_i} \left(p_i + \sqrt{p_i^2 + 4r_i q_i}\right) .$$

Then Γ' again has the form (**) and $-v(\gamma'_{12}) \le m$.

12. Case II : $-v(\gamma_{11}) = n > 1$.

Γ has the form (*) of the lemma in 7. We may suppose that $n_{21} - n_{11} \le 0$ for if $n_{21} - n_{11} > 0$ then $n_{12} - n_{22} = n_{11} - n_{21} < 0$ and then we replace Γ by $T^{-1}\Gamma T$ where

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Here we proceed in exactly the same way as at the end of 7. Taking

$$T = \begin{pmatrix} 1 & \lambda z^{\ell} \\ 0 & 1 \end{pmatrix}$$

with $\lambda = \dfrac{a_{11}}{a_{21}}$ and $\ell = n_{21} - n_{11}$ we get $-v(\gamma'_{11}) < n$. $\ell \le 0$ hence Γ' has the form (**) (cf. the end of 7.). Proceeding in this way we finally get $-v(\gamma_{11}) \le 1$ and then we have case (I) again.

This concludes the proof of the theorem.

13. Remark.

The theorem doesn't remain valid if one replaces $\mathbb{C}$ by an arbitrary field K of characteristic zero, K not closed.

If $S = \{0,1,\infty\}$ and Γ is of type (Ib)

$$\Gamma = \begin{pmatrix} \dfrac{p_1}{z} + \dfrac{p_2}{z-1} & \dfrac{q_1}{z} + \dfrac{q_2}{z-1} + \dfrac{c_2}{z^2} \\ \dfrac{r_2}{z-1} & 0 \end{pmatrix}$$

then Γ can be transformed into a matrix of the form $\dfrac{A_1}{z} + \dfrac{A_2}{z-1}$ if and only if at least one of the roots

$$\sqrt{(p_1-1)^2-4r_2c_2}\ ,\ \sqrt{p_2^2+4r_2q_2}\ ,\ \sqrt{(p_1+p_2)^2+4r_2(q_1+q_2)}$$

exists in K .

REFERENCES

[1] DELIGNE P. Equations différentielles à points singuliers réguliers. Lecture Notes in Math. 163 (1970).

[2] SERRE J.P. Faisceaux algébriques cohérents. Annals of Math. 61 (1955), pp. 197-278.

[3] GROTHENDIECK A. Sur la classification des fibrés holomorphes su la sphère de Riemann. Amer. Jour. of Math. 79 (1957), pp. 121-138.