Reducibility of Types in Typed Lambda Calculus*

Comment on a Paper by Richard Statman

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Consider types built up from a base type 0 using the operation \( \rightarrow \). A type \( \sigma \) is reducible to a type \( \tau \), notation \( \sigma \leq \tau \), iff there exists a closed term \( M \) in \( \sigma \rightarrow \tau \) such that for all closed \( N_1, N_2 \) in \( \sigma \) we have \( N_1 = \mu N_2 \iff MN_1 = \mu MN_2 \). Two types are equivalent iff each is reducible to the other. In (Statman, 1980, in "To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism" (J. P. Seldin and J. R. Hindley, Eds.), pp. 511-534, Academic Press, New York/London) is shown that the equivalence classes of types are well ordered in type \( \omega + 2 \) or \( \omega + 3 \). The paper does not decide if it is \( \omega + 2 \) or \( \omega + 3 \) because it is not clear whether \( \mu \equiv ((0 \rightarrow 0) \rightarrow 0) \rightarrow 0 \rightarrow 0 \) and \( \nu \equiv (0 \rightarrow 0) \rightarrow (0 \rightarrow 0) \rightarrow 0 \rightarrow 0 \) are equivalent. We show that \( \mu \) and \( \nu \) are not equivalent and conclude that the equivalence classes are ordered in type \( \omega + 3 \).

\* One of the referees pointed out that a solution for the problem was also found, but not published, by M. Zaionc of the University of Krakow, Poland.

1. INTRODUCTION

DEFINITION-NOTATION 1. Type, the set of types, is inductively defined as follows: (1) \( 0 \in \text{Type} \); (2) \( \sigma, \tau \in \text{Type} \Rightarrow \sigma \rightarrow \tau \in \text{Type} \). \( \Lambda^\tau \) is the set of all typed \( \lambda \)-terms.

A type \( \sigma \) is reducible to a type \( \tau \), notation \( \sigma \leq \tau \), iff there exists a closed term \( M \) in \( \sigma \rightarrow \tau \) such that for all closed terms \( N_1, N_2 \) in \( \sigma \)
\[
N_1 = \mu N_2 \iff MN_1 = \mu MN_2.
\]

\( \sigma \) and \( \tau \) are equivalent if each is reducible to the other.

We denote the equivalence class of \( \sigma \) by \([\sigma]\) and define
\[
[\sigma] < [\tau] \quad \text{iff} \quad \sigma \leq \tau \text{ but not } \tau \leq \sigma.
\]

\[
\mu \equiv ((0 \rightarrow 0) \rightarrow 0) \rightarrow 0 \rightarrow 0, \quad \nu \equiv (0 \rightarrow 0) \rightarrow (0 \rightarrow 0) \rightarrow 0 \rightarrow 0.
\]

In \([\text{Statman, 1980}]\) the following theorem is proved:

STATMAN'S THEOREM.  The equivalence classes of types are well ordered in type $\omega + 2$ or $\omega + 3$. A system of representatives is the following:

0. 0
1. $0 \rightarrow 0$

\[
\begin{array}{l}
n. \quad 0 \rightarrow (0 \rightarrow \cdots (0 \rightarrow 0) \cdots) \\
\end{array}
\]

$k$ times

\[
\begin{array}{l}
\omega. \quad (0 \rightarrow 0) \rightarrow 0 \rightarrow 0 \\
\omega + 1. \quad (0 \rightarrow 0) \rightarrow (0 \rightarrow 0) \rightarrow 0 \rightarrow 0 \equiv v \\
\omega + 2 (?) \quad (((0 \rightarrow 0) \rightarrow 0) \rightarrow 0) \rightarrow 0 \rightarrow 0 \equiv \mu \\
\omega + 3. \quad (0 \rightarrow (0 \rightarrow 0)) \rightarrow 0 \rightarrow 0.
\end{array}
\]

Note that 0 represents the types with no closed terms. After $\omega + 2$ there is a question mark in Statman's theorem because it is not clear whether the reducibility "$v \leq \mu$" is strict. We shall show that indeed $v < \mu$, i.e., not $\mu \leq v$. As a consequence, the question mark may be omitted and we can conclude that the types are well ordered in type $\omega + 3$.

THEOREM 2. $\mu$ is not reducible to $v$.

2. THE PROOF OF THE THEOREM

We start with some notations and definitions. Then in Lemmas 7 and 8 we determine the syntactic form of closed terms of type $\mu \rightarrow v$. Lemma 9 is a technical but central lemma in the proof. From these three lemmas we deduce Proposition 12 by a rather simple induction argument and the theorem follows as Corollary 13.

Notation 3. (i) $1 \equiv 0 \rightarrow 0$; $2 \equiv (0 \rightarrow 0) \rightarrow 0$.

(ii) $u$ is a fixed variable of type $\mu$.

(iii) $\mathcal{V}$ is a fixed collection of variables of types 0, 1, and 2, infinitely many variables of each type:

\[
\begin{array}{l}
n, n', n_1, n_2, \cdots \quad \text{range over variables of type 0 in } \mathcal{V}, \\
f, f', f_1, f_2, \cdots \quad \text{range over variables of type 1 in } \mathcal{V}, \text{ and} \\
F, F', F_1, F_2, \cdots \quad \text{range over variables of type 2 in } \mathcal{V}.
\end{array}
\]

(iv) $g, g', g_1, g_2, \cdots$ also denote variables of type 1 (either or not in $\mathcal{V}$).
(v) \( \mathcal{P} \) is the set of all terms of type 1 without free variables of type 0.
(vi) \( M \in \sigma \) denotes "\( M \) is a term of type \( \sigma \)."

**Definition 4.** \( M \in A^\dagger \) is in long \( \beta\eta \)-normal form iff

\[
M \equiv \lambda x_1 \cdots x_n \cdot x M_1 \cdots M_m,
\]

where \( x M_1 \cdots M_m \) has type 0 and each \( M_i \) is in long \( \beta\eta \)-normal form. (Note that each \( M \in A^\dagger \) has a (unique) long \( \beta\eta \)-normal form.)

**Example 5.** Let \( x \) be a variable of type \( (0 \to 0) \to ((0 \to 0) \to 0) \to 0 \). Then \( x \) is in \( \beta\eta \)-normal form, but \( x \) is not in long \( \beta\eta \)-normal form.

\[
x = \beta\eta \cdot sF \cdot x F = \beta\eta \cdot sF \cdot x (\lambda n \cdot fn)(\lambda f' \cdot F f') = \beta\eta \cdot sF \cdot x (\lambda n \cdot fn)(\lambda f' \cdot F(\lambda n' \cdot f' n'))
\]

and this last term is the long \( \beta\eta \)-normal form of \( x \).

**Definition 6.** \( \mathcal{A} \) is the collection of terms defined by:

1. \( n \in \mathcal{V} \Rightarrow n \in \mathcal{A} \);
2. \( M \in \mathcal{A}, f \in \mathcal{V} \Rightarrow f M \in \mathcal{A} \);
3. \( M \in \mathcal{A}, F, n \in \mathcal{V} \Rightarrow F(\lambda n \cdot M) \in \mathcal{A} \).

\( \mathcal{B} \) is the collection of terms defined by:

1. \( n \in \mathcal{V} \Rightarrow n \in \mathcal{B} \);
2. \( M \in \mathcal{B}, f \in \mathcal{V} \Rightarrow f M \in \mathcal{B} \);
3. \( M \in \mathcal{B}, F, n \in \mathcal{V} \Rightarrow F(\lambda n \cdot M) \in \mathcal{B} \);
4. \( M_1, M_2 \in \mathcal{B}, F \in \mathcal{V} \Rightarrow u(\lambda F \cdot M_1) M_2 \in \mathcal{B} \).

**Lemma 7.** Let \( M \in 0 \) be in long \( \beta\eta \)-normal form. Then

- a. \( FV(M) \subset \mathcal{V} \Leftrightarrow M \in \mathcal{A} \);
- b. \( FV(M) \subset (\mathcal{V} \cup \{u\}) \Leftrightarrow M \in \mathcal{B} \).

**Proof:**

\( (\Rightarrow) \) Trivial in both cases.

\( (\Leftarrow) \) By a simple induction on the length of \( M \). We only give the proof for b: Let \( M \in 0 \) be in long \( \beta\eta \)-normal form with \( FV(M) \subset (\mathcal{V} \cup \{u\}) \). Then there are 4 possibilities

1. \( M \equiv n \)
2. \( M \equiv f M_1 \)
3. \( M \equiv F N_{0 \to 0} \equiv F(\lambda n \cdot M_1) \)
4. \( M \equiv u N_{((0 \to 0) \to 0) \to 0} M_2 \equiv u(\lambda F \cdot M_1) M_2 \)
with $M_i \in \mathcal{B}$, $M_i$ in long $\beta\eta$-normal form, and $FV(M_i) \subseteq (\mathcal{V} \cup \{u\})$ for $i = 1, 2$.

In case (1) the result is immediate. In cases (2)–(4) we have $M_i \in \mathcal{B}$ by the induction hypothesis, so $M \in \mathcal{B}$. [1]

**Lemma 8.** Let $U \in \mu \rightarrow \nu$, $U$ closed. Then for some $L \in \mathcal{B}$,

$$U = _{\beta\eta} \mu f_1 f_2 n \cdot L.$$  

**Proof.** $U = _{\beta\eta} \mu f_1 f_2 n \cdot L$ with $L \in \mathcal{B}$ and $FV(L) \subseteq \{u, f_1, f_2, n\} \subseteq (\mathcal{V} \cup \{u\})$. We may choose $L$ in long $\beta\eta$-normal form and the assertion follows directly from Lemma 7. [1]

**Lemma 9.** Let $L \in \mathcal{A}$. Then one of the following two cases holds:

(i) There exist $P \in \mathcal{P}$ and $n \in \mathcal{V}$ such that for each $H \in 2$,

$$(\lambda F \cdot L)(\lambda g \cdot g(Hg)) = Pn.$$  

(ii) There exist $P_1 \in \mathcal{P}$ and $P_2 \in \mathcal{P}$ such that for each $H \in 2$,

$$(\lambda F \cdot L)(\lambda g \cdot g(Hg)) = P_1(HP_2).$$

(In this case $g$ should be taken outside $FV(H)$.)

**Proof.** By induction on the generation of $L$ in $\mathcal{A}$:

$L \equiv n$. Then $(\lambda F \cdot L)(\lambda g \cdot g(Hg)) = n$, so (i) holds with $P \equiv \lambda n' \cdot n'$.

$L \equiv fL'$. We distinguish two cases for $L'$:

Case (i) $(\lambda F \cdot L')(\lambda g \cdot g(Hg)) = Pn$ for some $P \in \mathcal{P}$ and each $H \in 2$. Then $(\lambda F \cdot fL')(\lambda g \cdot g(Hg)) = f(Pn) = (\lambda n' \cdot f(Pn'))n$ with $\lambda n' \cdot f(Pn') \in \mathcal{P}$.

Case (ii) $(\lambda F \cdot L')(\lambda g \cdot g(Hg)) = P_1(HP_2)$ for each $H \in 2$. Then $(\lambda F \cdot fL')(\lambda g \cdot g(Hg)) = f(P_1(HP_2)) = (\lambda n' \cdot f(P_1n')(HP_2))$.

$L \equiv F'(\lambda n' \cdot L')$. We may suppose $n' \notin FV(H)$. Again we distinguish two cases for $L'$:

Case (i) $(\lambda F \cdot L')(\lambda g \cdot g(Hg)) = Pn$ for each $H \in 2$.

(a) Suppose $F' \neq F$. Then

$$(\lambda F \cdot L)(\lambda g \cdot g(Hg)) = (\lambda F \cdot F'(\lambda n' \cdot L'))(\lambda g \cdot g(Hg))$$

$$= F'(\lambda n' \cdot (\lambda F \cdot L')(\lambda g \cdot g(Hg)))$$

$$= F'(\lambda n' \cdot Pn) = (\lambda n'' \cdot F'(\lambda n' \cdot Pn''))n.$$
Suppose \( F' \equiv F \).
(b1) Suppose \( n' \neq n \). Then
\[
(\lambda F \cdot L)(\lambda g \cdot g(Hg)) = (\lambda F \cdot F(\lambda n' \cdot L'))(\lambda g \cdot g(Hg)) \\
= (\lambda g \cdot g(Hg))(\lambda n' \cdot (\lambda F \cdot L')(\lambda g \cdot g(Hg))) \\
= (\lambda g \cdot g(Hg))(\lambda n' \cdot Pn) = Pn.
\]

(b2) Suppose \( n' \equiv n \). We show that (ii) in the lemma holds for \( L \). Let \( H \in 2 \). Then
\[
(\lambda F \cdot L)(\lambda g \cdot g(Hg)) = (\lambda F \cdot F(\lambda n \cdot L'))(\lambda g \cdot g(Hg)) \\
= (\lambda g \cdot g(Hg))(\lambda n \cdot (\lambda F \cdot L')(\lambda g \cdot g(Hg))) \\
= (\lambda g \cdot g(Hg))(\lambda n \cdot Pn) \\
= (\lambda n \cdot Pn)(H(\lambda n \cdot Pn)) = P(HP).
\]

Case (ii) \((\lambda F \cdot L')(\lambda g \cdot g(Hg)) = P_1(HP_2)\) for each \( H \in 2 \).

(a) \( F' \neq F \). This case is trivial again.

(b) \( F' \equiv F \). Then
\[
(\lambda F \cdot L)(\lambda g \cdot g(Hg)) = (\lambda F \cdot F(\lambda n' \cdot L'))(\lambda g \cdot g(Hg)) \\
= (\lambda g \cdot g(Hg))(\lambda n' \cdot (\lambda F \cdot L')(\lambda g \cdot g(Hg))) \\
= (\lambda g \cdot g(Hg))(\lambda n' \cdot Pn) \\
= (\lambda n \cdot Pn)(H(\lambda n \cdot Pn)) = P(HP_2).
\]

**Notation-Definition 10.** \( y_1 \in ((0 \to 0) \to 0) \to 0 \) and \( y_2 \in 0 \) are variables.
\[
M_i = \lambda y_1 y_2 y_1(\lambda g_1 \cdot g_1(y_1(\lambda g_2 \cdot g_2(g_1, y_2)))) \in \mu \quad \text{for} \quad i = 1, 2.
\]
(Note that \( M_i \) is a closed term.)

We are going to prove that for each closed \( U \in \mu \to \nu \) the term \( UM_i \) does not depend on \( i \) (modulo \( \beta\eta \)-conversion). We start with a lemma on \( M_i \).

**Lemma 11.** For each \( L_1, L_2 \in \mathcal{A} \) and \( F \in \mathcal{V} \) the term \( M_1(\lambda F \cdot L_1) L_2 \) does not depend on \( i \) (modulo \( \beta\eta \)-conversion).

**Proof.** Let \( G_i = \lambda f_1 f_2 n \cdot f_i n \) for \( i = 1, 2 \). Then
\[
M_i = \lambda y_1 y_2 y_1(\lambda g_1 \cdot g_1(y_1(\lambda g_2 \cdot g_2(G_1, g_1 g_2 y_2))))
\]
and
\[
M_i(\lambda F \cdot L_1) L_2 = (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1((\lambda F \cdot L_1)(\lambda g_2 \cdot g_2(G_1 g_1 g_2 L_2)))).
\]
We apply Lemma 9 to \( L = L_1 \).
Case 1. 9(i) holds for $L_1$: There exist $P \in \mathcal{P}$ and $n \in V$ such that for each $H \in 2$

$$(\lambda F \cdot L_1)(\lambda g \cdot g(Hg)) = Pn.$$ 

Take $g = g_1$ and $H = \lambda g'. (\lambda F \cdot L_1)(\lambda g_2 \cdot g_2(G_i g' g_2 L_2))$. Then it follows that $M_i(\lambda F \cdot L_1) L_2 = Pn$ for $i = 1, 2$.

Case 2. 9(ii) holds for $L_1$: There exist $P_1 \in \mathcal{P}$ and $P_2 \in \mathcal{P}$ such that for each $H \in 2$,

$$(\lambda F \cdot L_1)(\lambda g \cdot g(Hg)) = P_1(HP_2).$$ 

Let $H_{1i} = \lambda g' \cdot P_1(G_i g' P_2 L_2)$ and $H_{2i} = \lambda g' \cdot G_i g' L_2$, where $g_1$ is already bound in $M_i$.

We may assume that $g_j \notin FV(H_{ji})$ for $j = 1, 2$ (if necessary, replace $g_j$ in the definition of $M_i$ by a fresh variable). Now

$$M_i(\lambda F \cdot L_1) L_2 = (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1((\lambda F \cdot L_1)(\lambda g_2 \cdot g_2(G_i g_1 g_2 L_2))))$$

$$= (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1((\lambda F \cdot L_1)(\lambda g_2 \cdot g_2(H_{2i} g_2))))$$

$$= (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1(P_1(H_{2i} P_2)))$$

$$= (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1(P_1(G_i g_1 P_2 L_2)))$$

$$= P_1(P_1(G_i P_2 P_2 L_2))$$

$$= P_1(P_1(P_2 L_2)).$$

**Proposition 12.** For each closed $U \in \mu \rightarrow v$ the term $UM_i$ does not depend on $i$ (modulo $\beta\eta$-conversion).

**Corollary 13.** A closed term $U \in \mu \rightarrow v$ cannot be injective (for closed terms w.r.t. $\beta\eta$-conversion). In particular, not $\mu \leq v$.

**Proof.** Immediate. 

**Proof of Proposition 12.** By Lemma 8 we have $U = \lambda f_1 f_2 n \cdot L$ for some $L \in \mathcal{B}$. Then $UM_i = \lambda f_1 f_2 n \cdot L[u := M_i]$.

We show by induction on the generation of $L$ in $\mathcal{B}$ that $L[u := M_i]$ does not depend on $i$:

$L \equiv n'$. This case is trivial.

$L \equiv fL'$. $L'[u := M_i]$ does not depend on $i$ (induction hypothesis) so $(fL')[u := M_i] = fL'[u := M_i]$ does not depend on $i$. 
$L \equiv F(\lambda n' \cdot L')$. This case is also trivial because $M_i$ has no free variables, so

$$(F(\lambda n' \cdot L'))[u := M_i] = F(\lambda n' \cdot L'[u := M_i]).$$

$L \equiv u(\lambda F \cdot L_1) L_2$, with $L_1, L_2 \in \mathcal{B}$. Now $L[u := M_i] = M_i(\lambda F \cdot L_1[u := M_i])(L_2[u := M_i])$, where $L_1[u := M_i]$ and $L_2[u := M_i]$ do not depend on $i$ (induction hypothesis). Moreover, $L_i[u := M_i] \in \mathcal{V}$ and $FV(L_j[u := M_i]) \in \mathcal{V}$ for $j = 1, 2$. So the long $\beta\eta$-normal form of $L_i[u := M_i]$ is in $\mathcal{A}$ (by Lemma 7). Now it follows from Lemma 11 that $L[u := M_i]$ does not depend on $i$.

**Remark 14.** Proposition 12, and its proof, remain valid if, for a fixed $k \geq 0$, we replace $i, g_i$, and $M_i$ by

$$i = (i_1, i_2, \ldots, i_k) \quad \text{with} \quad i_1, i_2, \ldots, i_k \in \{1, 2\},$$

$$g_i = \lambda n' \cdot g_{i_1}(g_{i_2}(\cdots (g_{i_k}(n') \cdots)),$$

$$M_i = \lambda y_1 y_2 y_3 (\lambda g_1 \cdot g_{i_1}(y_1(\lambda g_2 \cdot g_{i_2}(g_{i_3}(y_2))))).$$

The definitions of $G_i, H_{1i}$ and $H_{2i}$ in the proof of Lemma 11 are obvious. At the end we get $P_1(P_1 P_2 \cdots (P_2(L_2) \cdots) (k \text{ times } P_2)$.

**ACKNOWLEDGMENT**

The author would like to thank Henk Barendregt for critical comments and for several hints resulting in essential improvements of the paper.

**REFERENCE**