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Reducibility of Types in Typed Lambda Calculus*

Comment on a Paper by Richard Statman

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Consider types built up from a base type 0 using the operation →. A type σ is reducible to a type τ, notation σ ≤ τ, iff there exists a closed term M in σ → τ such that for all closed N₁, N₂ in σ we have N₁ = μN₂ ⇔ MN₁ = μMN₂. Two types are equivalent iff each is reducible to the other. In (Statman, 1980, in "To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism" (J. P. Seldin and J. R. Hindley, Eds.), pp. 511–534, Academic Press, New York/London) is shown that the equivalence classes of types are well ordered in type ω + 2 or ω + 3. The paper does not decide if it is ω + 2 or ω + 3 because it is not clear whether μ(→)(0 → 0 → 0) → 0 → 0 and ν(→)(0 → 0) → (0 → 0) → 0 → 0 are equivalent.

We show that μ and ν are not equivalent and conclude that the equivalence classes are ordered in type ω + 3. © 1988 Academic Press, Inc.

1. INTRODUCTION

DEFINITION-NOTATION 1. Type, the set of types, is inductively defined as follows: (1) 0 ∈ Type; (2) σ, τ ∈ Type ⇒ σ → τ ∈ Type. A is the set of all typed λ-terms.

A type σ is reducible to a type τ, notation σ ≤ τ, iff there exists a closed term M in σ → τ such that for all closed terms N₁, N₂ in σ

N₁ = μN₂ ⇔ MN₁ = μMN₂.

σ and τ are equivalent if each is reducible to the other.

We denote the equivalence class of σ by [σ] and define

[σ] < [τ]  iff  σ ≤ τ but not τ ≤ σ.

μ ≡ ((0 → 0) → 0) → 0 → 0,  ν ≡ (0 → 0) → (0 → 0) → 0 → 0.

In [Statman, 1980] the following theorem is proved:

* One of the referees pointed out that a solution for the problem was also found, but not published, by M. Zaionc of the University of Krakow, Poland.
STATMAN'S THEOREM. The equivalence classes of types are well ordered in type $\omega + 2$ or $\omega + 3$. A system of representatives is the following:

- $0$. $0$
- $1$. $0 \to 0$
- $n$. $0 \to (0 \to \cdots (0 \to 0) \cdots)$, $n$ times
- $\omega$. $(0 \to 0) \to 0 \to 0$
- $\omega + 1$. $(0 \to 0) \to (0 \to 0) \to 0 \to 0 \equiv v$
- $\omega + 2$ (?). $((0 \to 0) \to 0) \to 0 \to 0 \equiv \mu$
- $\omega + 3$. $(0 \to (0 \to 0)) \to 0 \to 0$.

Note that 0 represents the types with no closed terms. After $\omega + 2$ there is a question mark in Statman's theorem because it is not clear whether the reducibility "$v \leq \mu$" is strict. We shall show that indeed $v < \mu$, i.e., not $\mu \leq v$. As a consequence, the question mark may be omitted and we can conclude that the types are well ordered in type $\omega + 3$.

**Theorem 2.** $\mu$ is not reducible to $v$.

2. THE PROOF OF THE THEOREM

We start with some notations and definitions. Then in Lemmas 7 and 8 we determine the syntactic form of closed terms of type $\mu \to v$. Lemma 9 is a technical but central lemma in the proof. From these three lemmas we deduce Proposition 12 by a rather simple induction argument and the theorem follows as Corollary 13.

**Notation 3.**
(i) $1 \equiv 0 \to 0$; $2 \equiv (0 \to 0) \to 0$.
(ii) $u$ is a fixed variable of type $\mu$.
(iii) $\gamma$ is a fixed collection of variables of types 0, 1, and 2, infinitely many variables of each type:

- $n, n', n_1, n_2 \cdots$ range over variables of type 0 in $\gamma$.
- $f, f', f_1, f_2 \cdots$ range over variables of type 1 in $\gamma$, and
- $F, F', F_1, F_2 \cdots$ range over variables of type 2 in $\gamma$.

(iv) $g, g', g_1, g_2 \cdots$ also denote variables of type 1 (either or not in $\gamma$).
(v) $\mathcal{P}$ is the set of all terms of type 1 without free variables of type 0.

(vi) $M \in \sigma$ denotes "$M$ is a term of type $\sigma$.

**Definition 4.** $M \in \Lambda^\tau$ is in long $\beta\eta$-normal form iff

$$M \equiv \lambda x_1 \cdots x_n \cdot xM_1 \cdots M_m,$$

where $xM_1 \cdots M_m$ has type 0 and each $M_i$ is in long $\beta\eta$-normal form. (Note that each $M \in \Lambda^\tau$ has a (unique) long $\beta\eta$-normal form.)

**Example 5.** Let $x$ be a variable of type $(0 \to 0) \to ((0 \to 0) \to 0) \to 0$. Then $x$ is in $\beta\eta$-normal form, but $x$ is not in long $\beta\eta$-normal form. $x = \beta\eta F \cdot xF = \beta\eta F \cdot x(\lambda n \cdot fn)(\lambda f' \cdot F f') = \beta\eta F \cdot x(\lambda n \cdot fn)(\lambda f' \cdot F(\lambda n' \cdot f'n))$ and this last term is the long $\beta\eta$-normal form of $x$.

**Definition 6.** $\mathcal{A}$ is the collection of terms defined by:

1. $n \in \forall \Rightarrow n \in \mathcal{A}$;
2. $M \in \mathcal{A}, f \in \forall \Rightarrow FM \in \mathcal{A}$;
3. $M \in \mathcal{A}, F, n \in \forall \Rightarrow F(\lambda n \cdot M) \in \mathcal{A}$.

$\mathcal{B}$ is the collection of terms defined by:

1. $n \in \forall \Rightarrow n \in \mathcal{B}$;
2. $M \in \mathcal{B}, f \in \forall \Rightarrow FM \in \mathcal{B}$;
3. $M \in \mathcal{B}, F, n \in \forall \Rightarrow F(\lambda n \cdot M) \in \mathcal{B}$;
4. $M_1, M_2 \in \mathcal{B}, F \in \forall \Rightarrow u(\lambda F \cdot M_1) M_2 \in \mathcal{B}$.

**Lemma 7.** Let $M \in \forall$ be in long $\beta\eta$-normal form. Then

a. $FV(M) \subset \forall \Rightarrow M \in \mathcal{A}$;

b. $FV(M) \subset (\forall \cup \{u\}) \Leftrightarrow M \in \mathcal{B}$.

**Proof:** ($\Rightarrow$) Trivial in both cases.

($\Leftarrow$) By a simple induction on the length of $M$. We only give the proof for b: Let $M \in \forall$ be in long $\beta\eta$-normal form with $FV(M) \subset (\forall \cup \{u\})$. Then there are 4 possibilities

1. $M \equiv n$
2. $M \equiv FM_1$
3. $M \equiv FN^{0 \to 0} \equiv F(\lambda n \cdot M_1)$
4. $M \equiv uN^{(0 \to 0) \to 0} M_2 \equiv u(\lambda F \cdot M_1) M_2$
with $M_i \in \mathcal{B}$, in long $\beta\eta$-normal form, and $FV(M_i) \subseteq (\mathcal{V} \cup \{u\})$ for $i = 1, 2$.

In case (1) the result is immediate. In cases (2)–(4) we have $M_i \in \mathcal{B}$ by the induction hypothesis, so $M \in \mathcal{B}$.

**Lemma 8.** Let $U \in \mu \rightarrow \nu$, $U$ closed. Then for some $L \in \mathcal{B}$,

$$U = \nu \mu f_1 f_2 n \cdot L.$$  

**Proof.** $U = \nu \mu f_1 f_2 n \cdot L$ with $L \in \mathcal{B}$ and $FV(L) \subseteq \{u, f_1, f_2, n\} \subseteq (\mathcal{V} \cup \{u\})$. We may choose $L$ in long $\beta\eta$-normal form and the assertion follows directly from Lemma 7.

**Lemma 9.** Let $L \in \mathcal{A}$. Then one of the following two cases holds:

(i) There exist $P \in \mathcal{P}$ and $n \in \mathcal{V}$ such that for each $H \in \mathcal{E}$,

$$(\lambda F \cdot L)(\lambda g \cdot g(\text{Hg})) = Pn.$$

(ii) There exist $P_1 \in \mathcal{P}$ and $P_2 \in \mathcal{P}$ such that for each $H \in \mathcal{E}$,

$$(\lambda F \cdot L)(\lambda g \cdot g(\text{Hg})) = P_1(HP_2).$$

(In this case $g$ should be taken outside $FV(H)$.)

**Proof.** By induction on the generation of $L$ in $\mathcal{A}$:

$L \equiv n$. Then $(\lambda F \cdot L)(\lambda g \cdot g(\text{Hg})) = n$, so (i) holds with $P \equiv \lambda n \cdot n'$.

$L \equiv fL'$. We distinguish two cases for $L'$:

Case (i) $(\lambda F \cdot L')(\lambda g \cdot g(\text{Hg})) = Pn$ for some $P \in \mathcal{P}$ and each $H \in \mathcal{E}$. Then $(\lambda F \cdot fL')(\lambda g \cdot g(\text{Hg})) = f(Pn) = (\lambda n' \cdot f(Pn')) n$ with $\lambda n \cdot f(Pn') \in \mathcal{P}$.

Case (ii) $(\lambda F \cdot L')(\lambda g \cdot g(\text{Hg})) = P_1(HP_2)$ for each $H \in \mathcal{E}$. Then $(\lambda F \cdot fL')(\lambda g \cdot g(\text{Hg})) = f(P_1(HP_2)) = (\lambda n' \cdot f(P_1n')(HP_2))$.

$L \equiv F'(\lambda n \cdot L')$. We may suppose $n' \notin FV(H)$. Again we distinguish two cases for $L'$:

Case (i) $(\lambda F \cdot L')(\lambda g \cdot g(\text{Hg})) = Pn$ for each $H \in \mathcal{E}$.

(a) Suppose $F' \neq F$. Then

$$(\lambda F \cdot L)(\lambda g \cdot g(\text{Hg})) = (\lambda F \cdot F'(\lambda n' \cdot L'))(\lambda g \cdot g(\text{Hg}))$$

$$= F'(\lambda n' \cdot (\lambda F \cdot L')(\lambda g \cdot g(\text{Hg})))$$

$$= F'(\lambda n' \cdot Pn) = (\lambda n'' \cdot F'(\lambda n' \cdot Pn')) n.$$
(b) Suppose $F' \equiv F$.

(b1) Suppose $n' \neq n$. Then

$$\frac{\lambda F \cdot L}{\lambda g \cdot g(Hg)} = (\lambda F \cdot F(\lambda n' \cdot L'))(\lambda g \cdot g(Hg))
= (\lambda g \cdot g(Hg))(\lambda n' \cdot (\lambda F \cdot L')(\lambda g \cdot g(Hg)))
= (\lambda g \cdot g(Hg))(\lambda n' \cdot Pn) = Pn.$$

(b2) Suppose $n' \equiv n$. We show that (ii) in the lemma holds for

$$\frac{\lambda F \cdot L}{\lambda g \cdot g(Hg)} = (\lambda F \cdot L')(\lambda g \cdot g(Hg))
= (\lambda g \cdot g(Hg))(\lambda n \cdot (\lambda F \cdot L')(\lambda g \cdot g(Hg)))
= (\lambda g \cdot g(Hg))(\lambda n \cdot Pn)
= (\lambda n \cdot Pn)(H(\lambda n \cdot Pn)) = P(HP).$$

Case (ii) $(\lambda F \cdot L')(\lambda g \cdot g(Hg)) = P_1(HP_2)$ for each $H \in 2$.

(a) $F' \neq F$. This case is trivial again.

(b) $F' \equiv F$. Then

$$\frac{\lambda F \cdot L}{\lambda g \cdot g(Hg)} = (\lambda F \cdot F(\lambda n' \cdot L'))(\lambda g \cdot g(Hg))
= (\lambda g \cdot g(Hg))(\lambda n \cdot (\lambda F \cdot L')(\lambda g \cdot g(Hg)))
= (\lambda g \cdot g(Hg))(\lambda n \cdot Pn)
= (\lambda n \cdot Pn)(H(\lambda n \cdot Pn)) = P(HP).$$

Notation-Definition 10. $y_1 \in ((0 \to 0) \to 0) \to 0$ and $y_2 \in 0$ are variables.

$$M_i = \lambda y_1 \cdot y_2 \cdot y_1(\lambda g_1 \cdot g_1(\lambda g_2 \cdot g_2(y_1(\lambda g_2 \cdot g_2(g_1, g_2, y_2))))) \in \mu \quad \text{for} \quad i = 1, 2.$$

(Note that $M_i$ is a closed term.)

We are going to prove that for each closed $U \in \mu \to v$ the term $UM_i$ does not depend on $i$ (modulo $\beta\eta$-conversion). We start with a lemma on $M_i$.

Lemma 11. For each $L_1, L_2 \in A$ and $F \in V$ the term $M_i(\lambda F \cdot L_1) L_2$ does not depend on $i$ (modulo $\beta\eta$-conversion).

Proof. Let $G_i = \lambda f_1 f_2 n \cdot f_1 n$ for $i = 1, 2$. Then

$$M_i = \lambda y_1 \cdot y_2 \cdot y_1(\lambda g_1 \cdot g_1(\lambda g_2 \cdot g_2(y_1(\lambda g_2 \cdot g_2(G_i, g_1, g_2, y_2)))))$$

and

$$M_i(\lambda F \cdot L_1) L_2 = (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1(\lambda F \cdot L_1)(\lambda g_2 \cdot g_2(G_i, g_1, g_2, L_2))).$$

We apply Lemma 9 to $L = L_1$. 
Case 1. 9(i) holds for $L_1$: There exist $P \in \mathcal{P}$ and $n \in \mathcal{V}$ such that for each $H \in 2$

$$(\lambda F \cdot L_1)(\lambda g \cdot g(Hg)) = Pn.$$ 

Take $g = g_1$ and $H = \lambda g' \cdot (\lambda F \cdot L_1)(\lambda g_2 \cdot g_2(G_1 g' g_2 L_2))$. Then it follows that $M_i(\lambda F \cdot L_1) L_2 = Pn$ for $i = 1, 2$.

Case 2. 9(ii) holds for $L_1$: There exist $P_1 \in \mathcal{P}$ and $P_2 \in \mathcal{P}$ such that for each $H \in 2$,

$$(\lambda F \cdot L_1)(\lambda g \cdot g(Hg)) = P_1(HP_2).$$

Let $H_{1i} = \lambda g' \cdot P_1(G_i g' P_2 L_2)$ and $H_{2i} = \lambda g' \cdot G_i g_1 g' L_2$, where $g_1$ is already bound in $M_i$.

We may assume that $g_j \notin FV(H_{ji})$ for $j = 1, 2$ (if necessary, replace $g_j$ in the definition of $M_i$ by a fresh variable). Now

$$M_i(\lambda F \cdot L_1) L_2 = (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1((\lambda F \cdot L_1)(\lambda g_2 \cdot g_2(G_1 g_1 g_2 L_2))))$$

$$= (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1((\lambda F \cdot L_1)(\lambda g_2 \cdot g_2(H_{2i} g_2))))$$

$$= (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1(P_1(H_{2i} P_2)))$$

$$= (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1(P_1(G_i g_1 P_2 L_2)))$$

$$= P_1(P_1(G_i P_2 P_2 L_2))$$

$$= P_1(P_1(P_2 L_2)).$$

**Proposition 12.** For each closed $U \in \mu \rightarrow \nu$ the term $UM_i$ does not depend on $i$ (modulo $\beta \eta$-conversion).

**Corollary 13.** A closed term $U \in \mu \rightarrow \nu$ cannot be injective (for closed terms w.r.t. $\beta \eta$-conversion). In particular, not $\mu \leq \nu$.

**Proof.** Immediate. 

**Proof of Proposition 12.** By Lemma 8 we have $U = \lambda f_1 f_2 n \cdot L$ for some $L \in \mathcal{B}$. Then $UM_i = \lambda f_1 f_2 n \cdot L[u := M_i]$.

We show by induction on the generation of $L$ in $\mathcal{B}$ that $L[u := M_i]$ does not depend on $i$:

$L \equiv n'$. This case is trivial.

$L \equiv fL'$. $L'[u := M_i]$ does not depend on $i$ (induction hypothesis) so $(fL')[u := M_i] = fL'[u := M_i]$ does not depend on $i$. 


$L \equiv F(\lambda n' \cdot L')$. This case is also trivial because $M_i$ has no free variables, so

$$(F(\lambda n' \cdot L'))[u := M_i] = F(\lambda n' \cdot L'[u := M_i]).$$

$L \equiv u(\lambda F \cdot L_1) L_2$, with $L_1, L_2 \in \mathcal{B}$. Now $L[u := M_i] = M_i(\lambda F \cdot L_1[u := M_i])(L_2[u := M_i])$, where $L_1[u := M_i]$ and $L_2[u := M_i]$ do not depend on $i$ (induction hypothesis). Moreover, $L_j[u := M_i] \in 0$ and $FV(L_j[u := M_i]) \in \mathcal{V}$ for $j = 1, 2$. So the long $\beta\eta$-normal form of $L_j[u := M_i]$ is in $\mathcal{A}$ (by Lemma 7). Now it follows from Lemma 11 that $L[u := M_i]$ does not depend on $i$.

Remark 14. Proposition 12, and its proof, remain valid if, for a fixed $k \geq 0$, we replace $i$, $g_i$, and $M_i$ by

$$i = (i_1, i_2, \ldots, i_k) \quad \text{with} \quad i_1, i_2, \ldots, i_k \in \{1, 2\},$$

$$g_i = \lambda n' \cdot g_i(g_{i_1} \cdots (g_{i_k}(n') \cdots),$$

$$M_i = \lambda y_1 y_2 \cdot y_1(\lambda g_1 \cdot g_1(y_1(\lambda g_2 \cdot g_2(g_1 y_2)))).$$

The definitions of $G_i$, $H_{1i}$ and $H_{2i}$ in the proof of Lemma 11 are obvious. At the end we get $P_1(P_1P_2 \cdots (P_2(L_2) \cdots) (k \text{ times } P_2)$.

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Reference