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Equivalences between Pure Type Systems and Systems of Illative Combinatory Logic

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Abstract Pure Type Systems, PTSs, were introduced as a generalization of the type systems of Barendregt's lambda cube and were designed to provide a foundation for actual proof assistants which will verify proofs. Systems of illative combinatory logic or lambda calculus, ICLs, were introduced by Curry and Church as a foundation for logic and mathematics. In an earlier paper we considered two changes to the rules of the PTSs which made these rules more like ICL rules. This led to four kinds of PTSs. Most importantly PTSs are about statements of the form \( M : A \), where \( M \) is a term and \( A \) a type. In ICLs there are no explicit types and the statements are terms. In this paper we show that for each of the four forms of PTS there is an equivalent form of ICL, sometimes if certain conditions hold.

1 Introduction

The similarity between rules of a generalized type theory (that of Martin-Löf [16]) and those of illative combinatory logic was first noted in Bunder [8]. When Pure Type Systems (PTSs), which encompassed many generalized type systems, were developed, the similarity of the PTS application, abstraction, and product rules, and rules of illative systems of combinatory logic or lambda calculus (ICLs) such as those of Bunder [3] and [11] and Aczel [1] was still apparent.

There were, however, many differences. The most important was that PTSs have judgments of the form

\[ x_1 : A_1, \ldots, x_n : A_n \vdash M : B \]  

(1)

where in each statement \( N : C \), \( C \) and \( N \) are “pseudoterms”. ICLs' judgments take the form

\[ X_1, \ldots, X_n \vdash Y \]  

(2)
where $X_1, \ldots, X_n$ and $Y$ are combinatory or $\lambda$-terms.

In a paper preliminary to this one (Bunder and Dekkers [12]), we aimed to overcome at least the minor differences that were part of the gap between PTSs and ICLs by developing variants of PTSs with ICL-like properties. One difference is that in PTSs the “context”, $x_1 : A_1, \ldots, x_n : A_n$, in (1) must be a sequence whose statements are introduced by one of two rules. For ICLs, the context $X_1, \ldots, X_n$ in (2) is an arbitrary set of terms. Also in ICLs, terms can be replaced by terms that are $\beta$- or $\beta\eta$-equal to them; for PTSs such substitutions are restricted. We introduced “set based PTSs” (SPTSs) which allow sets of judgments as contexts and unrestricted substitution of $\beta$-equality.

Also in PTSs, the abstraction rule has a stronger restriction than normally in ICLs. In [12] we introduced “abstraction altered PTSs” (APTSs) with an ICL-like abstraction rule. Finally we introduced SAPTSs which incorporated both changes.

We then showed, under what conditions and for which PTSs, PTS judgments were equivalent to SPTS, APTS, and SAPTS judgments. These new PTS-variants have independent interest in that they show that for many PTSs many of the rules can be relaxed. These more flexible PTS-variants are also closer to their formulas as type interpretations.

In this paper we show that, for each PTS, APTS, SPTS, and SAPTS, there is a corresponding ICL. That corresponding to a SAPTS is closest to the ICL used in the foundations of mathematics by Church, Curry, and their followers. (For details see Curry, Hindley, and Seldin [15], Bunder [13], or some of the series of papers in this journal which included Bunder [3] and [4].)

In each case, we show under what conditions the PTSs, APTSs and so on are equivalent to their ICL counterparts. Such equivalences hold for most standard PTSs, for example, the Calculus of Constructions. It was surprising that it was possible to extract from a large number of ICL-judgments of the form (2), variables $x_1, \ldots, x_n$, types $A_1, \ldots, A_n$, $B$ and a term $M$ to give equivalent PTS-judgments of the form (1).

The new ICL-systems are shown in the top face of the cube in Figure 1 on the next page. The equivalences between the type and illative systems are shown by the lines joining them. Restrictions to some equivalences, the PTS being normalizing (n), a $\delta$-equal version of a judgment being provable only ($\beta$), the contexts being “legal” ($L$) and the condition ($*$), valid for many, but not all PTSs, are shown. The numbers of the theorems proving the results are also indicated.

### 2 The Pure Type Systems

Given a class of variables $V = \{x, y, z, \ldots, x_1, x_2, \ldots\}$ and a class of constants $C = \{c_1, c_2, \ldots\}$ we have the following definition.

**Definition 2.1** The class of pseudoterms $\mathcal{T}$ is given by

$$\mathcal{T} = V[C[(\Pi V : \mathcal{T}, \mathcal{T})](\lambda V : \mathcal{T}, \mathcal{T})(\mathcal{T}, \mathcal{T})].$$

If $x \in V$ and $t_1, t_2 \in \mathcal{T}$, $(\lambda x : t_1, t_2)$ is interpreted as the $\lambda$-abstraction of $t_2$ with respect to the variable $x$ of type $t_1$ and $(\Pi x : t_1, t_2)$ is interpreted as the class (or type) of all generalized functions from $t_1$ to $t_2$, where $t_2$ may be dependent on the argument $x$ of the function. In $(\Pi x : t_1, t_2)$, $x$ is bound just as in $(\lambda x : t_1, t_2)$. $\text{FV}(t)$ will denote the set of free variables of $t$. 

Definition 2.2

1. If $M$ and $A$ are pseudoterms, $M : A$ is a statement.
2. $\Gamma$ is a context if it is a sequence of statements $\langle x_1 : A_1, \ldots, x_n : A_n \rangle$ where $x_1, \ldots, x_n \in V$. We will let $\text{FV}(\Gamma)$ be the set of free variables of the pseudoterms in $\Gamma$.
3. If $\Gamma$ is a context and $M$ and $A$ are pseudoterms then $\Gamma \vdash M : A$ is a judgment.

Definition 2.3 (Pure Type Systems, PTSs)

1. The specification of a PTS consists of a triple $S = (\delta, \mathcal{A}, \mathcal{R})$ where $\delta$ is a subclass of $\mathcal{C}$ called the sorts, $\mathcal{A}$ is a class of statements of the form $(c : s)$, and $\mathcal{R}$ is a subclass of $\delta \times \delta \times \delta$.
2. A Pure Type System (PTS) $\lambda S = \lambda (\delta, \mathcal{A}, \mathcal{R})$ determined by the specification $\mathcal{S} = (\delta, \mathcal{A}, \mathcal{R})$ is defined as follows. Statements and contexts are as in Definition 2.2. The notion of type derivation, written as $\Gamma \vdash^\lambda S M : A$ (or just $\Gamma \vdash M : A$) is defined by the following postulates.
2.1 The PTS postulates

(axioms) \( \{ \} \vdash c : s \) where \( c : s \in A \)

(start rule) \( \Gamma \vdash A : s \)

(x : A \vdash x : A \) where \( s \in S \) and \( x \notin FV(\Gamma) \)

(weakening rule) \( \Gamma \vdash M : A \)

(\( x \) : \( B \) \vdash M : A \)\( x \) / \( \notin FV(\Gamma) \)

(product rule) \( \Gamma \vdash A : s_1 \)

(\( x \) : \( A \) \vdash B : s_2 \)

(\( x \) / \( \notin FV(\Gamma) \))\( s_1, s_2, s_3 \in R \)

(abstraction rule) \( \Gamma, x : A \vdash M : B \)

(\( \lambda x : A. M : (\Pi x : A. B) : s \)

(\( s \in S \))

(application rule) \( \Gamma \vdash M : (\Pi x : A. B) \)

(\( \Gamma \vdash N : A \)

(\( \Gamma \vdash (M \cdot N) : B[x := N] \)

(conversion rule) \( \Gamma \vdash M : A \)

(\( \Gamma \vdash B : s \)

(\( A =_{\beta} B \)

(\( s \in S \))

A pseudoterm \( A \) is legal in a PTS if, for some \( \Gamma \) and \( B \), \( \Gamma \vdash A : B \) or \( \Gamma \vdash B : A \) in that PTS.

2.2 The SPTS postulates

For an SPTS the above axioms and the start, weakening, and conversion rules are replaced by

(axioms) \( \Delta \vdash_S c : s \) if \( c : s \in A \)

(start) \( \Delta \vdash_S M : A \) if \( M : A \in \Delta \)

(conversion) \( \Delta \vdash_S M : A \)

(\( A =_{\beta} M \)

(\( A =_{\beta} A \)

(\( \Delta \vdash_S N : B \)

where \( \Delta \) is an arbitrary set of statements \( P : C \), rather than a sequence of statements \( x : C \) formed using the start and weakening rules.

The remaining SPTS postulates are those of PTSs with \( \vdash_S \) for \( \vdash \) and with each \( \Gamma \) (which we use for sequences) replaced by \( \Delta \) (which we use for sets). The SPTS product and abstraction rules also require the restriction \( x \notin FV(\Delta, A) \) which is derivable for PTSs.

2.3 The APTS postulates

These are as for the PTS postulates except that \( \vdash_A \) is used for \( \vdash \) and the abstraction rule is replaced by

(abstraction) \( \Gamma, x : A \vdash_A M : B \)

(\( \Gamma \vdash_A A : s \)

(\( \Gamma \vdash_A (\lambda x : A. M : (\Pi x : A. B) \)

where \( s \in S \) and \( (+) \exists s_2, s_3 \left[ (s, s_2, s_3) \in R \land \forall D(B =_{\beta} D \in C \Rightarrow (D : s_2) \in A) \right] \).
Note that this varies slightly from the (+) in [12] which is

\[(+)^{\circ} \exists s_2, s_3 \ [(s, s_2, s_3) \in R \ & (B \in C \Rightarrow (B : s_2) \in A)] \]

We will denote this “old” system, with \((+)^{\circ}\), by \(A^\circ\)PTS. Note also that \(\Gamma \vdash_a A : s\) is actually derivable whenever \(\Gamma, x : A \vdash M : B\) is, but we will retain it here as it is required for SAPTSs.

### 2.4 The SAPTS postulates

These use \(\vdash_{SA}\) and have the alterations of the SPTSs and of the APTSs. We summarize below the results from [12] with minor variations due to the change from \((+)^{\circ}\) to (+). We first need some definitions.

**Definition 2.4** If \(\Gamma\) is the context \((x_1 : A_1, \ldots, x_n : A_n)\), \(S(\Gamma)\) is the set \(\{x_1 : A_1, \ldots, x_n : A_n\}\).

**Definition 2.5** A set \(\Delta\) is \(S\)-legal in an SPTS if \(\Delta \vDash \{x_1 : A_1, \ldots, x_n : A_n\}\) and

(i) \((\forall i, j)\ [1 \leq i < j \leq n \Rightarrow x_i \neq x_j]\);
(ii) \((\forall i)\ [1 \leq i \leq n \Rightarrow (\exists s_i \in R)\{x_1 : A_1, \ldots, x_{i-1} : A_{i-1} \vdash S A_i : s_i\}]\);
(iii) \((\forall i)\ [1 \leq i \leq n \Rightarrow x_i, \ldots, x_n \notin \text{FV}(A_i)]\).

**Definition 2.6** A context \(\Gamma\) is \(A\)-legal if for some \(M\) and \(B\), \(\Gamma \vdash_a M : B\).

**Definition 2.7** A set \(\Delta\) is \(SA\)-legal if (i), (ii), and (iii) of Definition 2.5 hold with \(\vdash_{SA}\) for \(\vdash_S\).

**Definition 2.8**

(i) \(\delta_1\) is the set generated by

(a) \(c : s \in A \Rightarrow s \in \delta_1\),
(b) \(s : s' \in A \Rightarrow s \in \delta_1\),
(c) \(s_1, s_2 \in \delta_1 \& (s_1, s_2, s) \in R \Rightarrow s \in \delta_1\);

(ii) \(\delta_2 = \{s_2 \in \delta_2 \& \exists (s_1, s_2, s) \in R\}\);
(iii) \(\delta_3 = \{s_3 \exists (s_1, s_2) \in \delta_1[(s_1, s_2, s_3) \in R]\}\).

The definition of \(\delta_1\) above varies from, but is equivalent to, that in [12]. This we show in Theorem 4.27.

**Condition 2.9 (⋆)** The condition (⋆) is defined as

\[\forall s_1 \in \delta_1 \forall s_2 \in \delta_2 \{((s_3 \in \delta) s_2 : s \in A \lor s_2 \in \delta_3) \Rightarrow \exists s_3 (s_1, s_2, s_3) \in R\} \]

**Theorem 2.10** For any PTS and SPTS with the same specification,

(i) \(\Gamma \vdash M : A \Rightarrow S(\Gamma)\ is-S-legal & S(\Gamma) \vdash S M : A\);
(ii) \(\Delta \vdash S M : A \& \Delta is-S-legal \Rightarrow \exists \Gamma\),
\(M', A'[\Delta \vDash_\beta S(\Gamma) & M =_\beta M' \& A =_\beta A' \& \Gamma \vdash M' : A']\).

**Theorem 2.11** For any PTS and APTS with the same specification,

(i) \(\Gamma \vdash P : C \Rightarrow \Gamma \vdash_a P : C\);
(ii) if (⋆) holds, then \(\Gamma \vdash_a P : C \Rightarrow \Gamma \vdash P : C\).
Theorem 2.12 For any APTS and SAPTS with the same specification,
(i) $\Gamma \vdash_A M : A \Rightarrow S(\Gamma) \text{ is SA-legal} \& S(\Gamma) \vdash_{SA} M : A$;
(ii) if (*) holds, $\Delta$ is SA-legal $\& \Delta \vdash_{SA} M : A \Rightarrow \exists \Gamma,
M', A'[\Delta \Rightarrow_{\beta} S(\Gamma) \& M \Rightarrow_{\beta} M' \& A \Rightarrow_{\beta} A' \& \Gamma \vdash_{A} M' : A']$.

Theorem 2.13 For any SPTS and SAPTS with the same specification,
(i) $\Delta \vdash_S M : A \& \Delta \text{ is S-legal} \Rightarrow \Delta \vdash_{SA} M : A \& \Delta \text{ is SA-legal} ;
(ii) if (*) holds, $\Delta \vdash_{SA} M : A \& \Delta \text{ is SA-legal} \Rightarrow \Delta \vdash_S M : A \& \Delta \text{ is S-legal}.$

2.5 Comments Theorems 2.10, 2.11, 2.12(ii), and 2.13(ii) and their proofs are the same as, or only slight variations of, due to the change from $(+^\circ)$ to $(+)$, Theorems 5.4, 8.1, and 8.3, 9.6(ii), and 9.8($\Leftarrow$) in [12]. The proof of Theorem 2.12(i) is by a simple induction on the derivation of $\Gamma \vdash_{A} M : A$. Theorem 2.13(i) follows from Theorems 2.10(ii), 2.11(i), 2.12(i), and conversion.

In order to illustrate the kinds of proofs required we present, in Section 5 below, the proofs of Theorems 2.11(ii) and 2.12(i). Note that in Theorems 2.12(i) and 2.13(i) condition (*) is not needed; this generalizes Theorems 9.6(i) and 9.8 ($\Rightarrow$) in [12].

To relate APTSs to $A^o$PTSs we have the following theorem.

Theorem 2.14 For any APTS and $A^o$PTS with the same specification,
(i) $\Gamma \vdash_{A} M : B \Rightarrow \Gamma \vdash_{A^o} M : B$;
(ii) if (*) holds, $\Gamma \vdash_{A^o} M : B \Rightarrow \Gamma \vdash_A M : B$.

Proof (i) This is obvious as the systems are identical except that abstraction for $A^o$PTS has a weaker version of $(+)$. (ii) If $\Gamma \vdash_{A^o} M : B$ and (*) hold, then by Theorem 8.5 of [12] (the $A^o$PTS version of Theorem 2.11(ii)) we have $\Gamma \vdash M : B$ and by Theorem 2.11(i) we have $\Gamma \vdash_A M : B$. □

3 Illative Systems

For each PTS, SPTS, APTS, and SAPTS that we will set up a system of illative lambda calculus, combinatory logic could easily have been used instead. We will refer to all of these illative systems as ICLs.

Definition 3.1 The class of pseudoterms $T$ is given by

$$T = V \mid C \mid GT(\lambda V.T) \mid \lambda V.T \mid TT$$

where $V$ and $C$ are as in Section 2. $G$ is an illative constant that corresponds to $\Pi$; it is related (see Notation 3.6) to the restricted generality used by Church and Curry.

Each ICL will have a specification $(\delta, [A], R)$ where $\delta \subseteq C$ is a set of sorts and $R$ a set of triples of sorts, as for PTSSs. $[A]$ is the set of axioms of the form $sc$, where $c : s$ is an axiom of the PTS specified by $(\delta, A, R)$. 
3.1 The IS postulates

(axioms) \( \Gamma \vdash s X \) where \( s \in S \)

(start rule) \( \Gamma \vdash s X \) where \( s \in S \) and \( x \not\in FV(\Gamma, X) \)

(weakening rule) \( \Gamma \vdash X \) \( \Gamma \vdash s Y \) where \( s \in S \) and \( x \not\in FV(\Gamma, Y) \)

(product rule) \( \Gamma \vdash X \) \( \Gamma \vdash s Y \) where \( s \in S \) and \( x \not\in FV(\Gamma, Y) \)

(abstraction rule) \( \Gamma \vdash s X \) where \( s \in S \)

(application rule) \( \Gamma \vdash X \) \( \Gamma \vdash X \) \( \Gamma \vdash s Y \) where \( s \in S \)

(conversion rule) \( \Gamma \vdash X \) \( \Gamma \vdash s Y \) where \( s \in S \)

Notation 3.2 Systems such as these were called separated systems in Curry, Hindley, and Seldin [15]. Most ICLs in the literature are not separated as they have the SIS and SAIS (conversion) rule below (possibly with \( \beta\eta \)-equality).

3.2 The SIS postulates These are as above with \( \vdash_{SI} \) for \( \vdash_{I} \) and \( \Delta \) for \( \Gamma \), except that (axioms) and the (start), (weakening), and (conversion) rules are replaced by

(axioms) \( \Delta \vdash_{SI} X \) if \( X \in [A] \)

(start) \( \Delta \vdash_{SI} X \) if \( X \in \Delta \)

(conversion) \( \Delta \vdash_{SI} X \) \( \Delta =_{\beta} \Delta' \) \( X =_{\beta} Y \) \( \Delta' \vdash_{SI} Y \).

The product and abstraction rules require the condition \( x \not\in FV(\Delta, X) \). In all the postulates, \( \Delta \) is an arbitrary set of terms.

3.3 The AIS postulates These are the IS-postulates with \( \vdash_{AI} \) for \( \vdash_{I} \) and (abstraction) replaced by

(abstraction) \( \Gamma, X \vdash_{AI} Y Z \) \( \Gamma \vdash_{AI} s X \) \( \Gamma \vdash_{AI} G X(\lambda x.Y) \)

where

\[ (+) \exists s_2, s_3 \left[ (s, s_2, s_3) \in \mathcal{R} \land \forall U(V =_{\beta} U \in \mathcal{C} \implies s_2 U \in [A]) \right]. \]

(\( \Gamma \vdash_{AI} s X \) is actually derivable if \( \Gamma, X \vdash_{AI} Y Z \) is, but we retain it in the rule as it is required for SAISs.)
3.4 The SAIS postulates  These have \( \vdash_{\text{SAI}} \) for \( \vdash \) and include the changed postulates from the SISs and AISs, where now in the abstraction rule \( Y \) is not \( \beta \)-equal to an abstract. \( \beta \)-equality is, in general, not decidable unless the terms involved have normal form. The normalization property, which says that all pseudoterms in a valid judgment have normal form, is one of our conditions for the equivalence of SAISs and PTSs (see Theorem 9.2).

Definition 3.3  A context \( \Gamma \) is \((A)I\)-legal if for some \( X \), \( \Gamma \vdash_{(A)I} X \).

Lemma 3.4  A context \( \Gamma \) is \((A)I\)-legal \( \iff \) \( \Gamma \equiv \langle X_1 x_1, \ldots, X_n x_n \rangle \) for some terms \( X_1, \ldots, X_n \) and variables \( x_1, \ldots, x_n \), where

(i) \( \forall i, j [1 \leq i < j \leq n \Rightarrow x_i \neq x_j] \);

(ii) \( \forall i [1 \leq i \leq n \Rightarrow \exists s_i \in \delta [X_1 x_1, \ldots, X_{i-1} x_{i-1} \vdash_{(A)I} s_i X_i]] \);

(iii) \( \forall i [1 \leq i \leq n \Rightarrow x_i, \ldots, x_n \notin FV(X_i)] \).

Proof  \((\Rightarrow)\) By induction on the derivation of \( \Gamma \vdash_{(A)I} X \) in Definition 3.3.

\((\Leftarrow)\) By the start rule from (i) and (ii) for \( i = n \).

\[ \square \]

Definition 3.5  A set of statements \( \Delta \) is said to be S\((A)I\)-legal, in an S\((A)IS\), if \( \Delta \equiv \{ X_1 x_1, \ldots, X_n x_n \} \) and (i), (ii), and (iii) of Lemma 3.4 hold with \( \vdash_{S(A)I} \) for \( \vdash_{(A)I} \).

Notation 3.6  Illative systems were first set up by Church and Curry using the symbol Curry denoted by \( \Xi \), instead of the \( G \) used above.

\[ G = \lambda xyz.\Xi x(\lambda u.(yu)(zu)). \]

The SAIS application rule follows, using this definition, from Curry’s (and Church’s) \( \Xi \)-elimination rule:

\[ \Delta \vdash \Xi XY \quad \Delta \vdash XU \]

\[ \frac{}{\Delta \vdash YU} \]

Their \( \Xi \) introduction rules,

Curry:

\[ \Delta, Xu \vdash Yu \quad u \notin FV(\Delta, XY) \]

\[ \frac{}{\Delta \vdash \Xi XY} \]

Church:

\[ \Delta, Xu \vdash Yu \quad \Delta \vdash XV \quad u \notin FV(\Delta, XY) \]

\[ \frac{}{\Delta \vdash \Xi XY} \]

led to inconsistency.

The SAIS abstraction rule follows from the \( \Xi \) introduction rule of Bunder [11] and [3],

\[ \Delta, Xu \vdash Yu \quad \Delta \vdash LX \quad u \notin FV(\Delta, XY) \]

\[ \frac{}{\Delta \vdash \Xi XY} \]

with \( L \equiv s \).

Notation 3.7  \( \exists s \in \delta \) will often be abbreviated to \( \exists s \) and \( \forall s \in \delta \) to \( \forall s \).

4 Lemmas and Definitions for PTSs, ISs, APTSs, and AISs

In Lemmas 4.1–4.11 we quote several well-known lemmas for PTSs from Barendregt [2] and others from [12], all without proofs.

Lemma 4.1 (Free Variable Lemma)  Let \( x_1 : A_1, \ldots, x_n : A_n \vdash B : C \). Then

(i) the \( x_1, \ldots, x_n \) are all distinct;
Lemma 4.2 (Start Lemma) If $\Gamma$ is a legal context then
(i) $(c : s) \in A \Rightarrow \Gamma \vdash c : s$,
(ii) $(x : A) \in \Gamma \Rightarrow \Gamma \vdash x : A$.

Lemma 4.3 (Substitution Lemma) $\Gamma, x : A \vdash B : C$ & $\Gamma \vdash D : A \Rightarrow \Gamma \vdash B[x := D] : C[x := D]$.

Lemma 4.4 (Correctness of Contexts Lemma) If $x_1 : A_1, \ldots, x_n : A_n \vdash M : A$ then for each $i, 1 \leq i \leq n$, there is an $s_i \in \Delta$ such that the derivation of $x_1 : A_1, \ldots, x_n : A_n \vdash M : A$ contains a derivation of $x_1 : A_1, \ldots, x_{i-1} : A_{i-1} \vdash A_i$.

Lemma 4.5 (Thinning Lemma) If $\Gamma$ and $\Gamma'$ are legal contexts and $\Gamma \subseteq \Gamma'$ then $\Gamma \vdash M : A \Rightarrow \Gamma' \vdash M : A$.

Lemma 4.6 (Combining Contexts Lemma) If $\Gamma_1$ and $\Gamma_2$ are legal contexts and $\text{fv}(\Gamma_1) \cap \text{fv}(\Gamma_2) = \emptyset$ then $\Gamma_1, \Gamma_2$ is a legal context.

Lemma 4.7 (Sharpened Generation Lemma) If $\Gamma \vdash P : B$ then
(i) $P \equiv c \in \mathcal{E} \Rightarrow (c : B) \in A \lor \exists B'[B =_\beta B' \land (c : B') \in A \lor \exists s[\Gamma \vdash B : s]]$;
(ii) $P \equiv x \in \forall \Rightarrow (x : B) \in \Gamma \lor \exists B'[B =_\beta B' \land (x : B') \in \Gamma \lor \exists s[\Gamma \vdash B : s]]$;
(iii) $P \equiv (\Pi x : A.C) \Rightarrow \exists s_1, s_2, s_3[\Gamma \vdash A : s_1 \land \Gamma, x : A \vdash C : s_2 \land (s_1, s_2, s_3) \in \mathcal{R} \land (B \equiv s_3 \lor (B =_\beta s_3 \land (\exists s[\Gamma \vdash B : s])]$);
(iv) $P \equiv (\lambda x : A.M) \Rightarrow \exists s, s_3[\Gamma \vdash (\Pi x : A.C) : s_3 \land \Gamma, x : A \vdash M : C \land (B \equiv \Pi x : A.C \lor (B =_\beta \Pi x : A.C \land \exists s[\Gamma \vdash B : s])])$;
(v) $P \equiv MN \Rightarrow \exists A, C[\Gamma \vdash M : (\Pi x : A.C) \land \Gamma \vdash N : A \land (B \equiv C[x := N] \lor (B =_\beta C[x := N] \land \exists s[\Gamma \vdash B : s])]$.

and in each case the deductions without an explicit $x : A$ in the context are shorter than that of $\Gamma \vdash P : B$.

Lemma 4.8 (Correctness of Types Lemma)
$$\Gamma \vdash M : A \Rightarrow \exists s[A \equiv s \lor \Gamma \vdash A : s].$$

Lemma 4.9 (Subject Reduction Lemma)
$$\Gamma \vdash M : A, \Gamma \rightarrow_\beta \Gamma', M \rightarrow_\beta M', A \rightarrow_\beta A' \Rightarrow \Gamma' \vdash M' : A'.$$

Lemma 4.10
$$\Gamma =_\beta \Gamma' \land \Gamma' \text{ is legal} \land \Gamma \vdash M : A \Rightarrow \Gamma' \vdash M : A.$$

Lemma 4.11 If $\Gamma \vdash M : A$ then at least one of
(i) $A \in \mathcal{E},$
(ii) $\exists s[\Gamma \vdash A : s \land (\exists s'[s : s' \in A] \lor s \in \mathcal{S})].$
In Lemmas 4.12–4.22 we have similar lemmas for ISs with similar proofs.

**Lemma 4.12 (Free Variable Lemma for ISs)** Let $X_1x_1, \ldots, X_nx_n \vdash Y$. Then

(i) the $x_1, \ldots, x_n$ are all distinct;
(ii) $FV(Y) \subseteq \{x_1, \ldots, x_n\}$;
(iii) $FV(X_i) \subseteq \{x_1, \ldots, x_{i-1}\}$ for $i \leq n$.

**Lemma 4.13 (Start Lemma for ISs)** If $\Gamma$ is an $I$-legal context, then

(i) $s_c \in [A] \Rightarrow \Gamma \vdash s_c$,
(ii) $Xx \in \Gamma \Rightarrow \Gamma \vdash Xx$.

**Lemma 4.14 (Substitution Lemma for ISs)**

**Lemma 4.15 (Correctness of Contexts Lemma for ISs)** If $X_1x_1, \ldots, X_nx_n \vdash Y$ then for each $i$, 1 ≤ $i$ ≤ $n$, there is an $s_i \in S$ such that the derivation of $X_1x_1, \ldots, X_nx_n \vdash Y$ contains a derivation of $X_1x_1, \ldots, X_{i-1}x_{i-1} \vdash s_i X_i$.

**Lemma 4.16 (Thinning Lemma for ISs)** If $\Gamma$ and $\Gamma'$ are $I$-legal contexts and $\Gamma \subseteq \Gamma'$ then $\Gamma \vdash Y \Rightarrow \Gamma' \vdash Y$.

**Lemma 4.17 (Combining Contexts Lemma for ISs)** If $\Gamma_1$ and $\Gamma_2$ are $I$-legal contexts and $FV(\Gamma_1) \cap FV(\Gamma_2) = \emptyset$ then $\Gamma_1, \Gamma_2$ is an $I$-legal context.

**Lemma 4.18 (Sharpened Generation Lemma for ISs)** If $\Gamma \vdash_1 Y Z$, then

(i) $Z \equiv c \in \mathcal{C} \Rightarrow Y Z \in [A] \vee \exists T[Y \equiv_\beta T & T Z \in [A] \& \exists s[\Gamma \vdash s Y]]$;
(ii) $Z \equiv x \in \mathcal{V} \Rightarrow Y x \in \Gamma \vee \exists T[Y \equiv_\beta T & T x \in \Gamma \& \exists s[\Gamma \vdash s Y]]$;
(iii) $Z \equiv GU(\lambda x V) \Rightarrow \exists s_1, s_2, s_3[\Gamma \vdash s_1 U & \Gamma, U x \vdash s_2 V & (s_1, s_2, s_3) \in \mathcal{R} & (Y \equiv s_3 \vee (Y \equiv_\beta s_3 \& \exists s[\Gamma \vdash s Y]))]$;
(iv) $Z \equiv \lambda x T \Rightarrow \exists U, V, s_3[\Gamma \vdash s_3 (GU(\lambda x V)) & \Gamma, U x \vdash V T & (Y \equiv GU(\lambda x V) \vee (Y \equiv_\beta GU(\lambda x V) \& \exists s[\Gamma \vdash s Y]))]$;
(v) $Z \equiv RT \Rightarrow \exists U, V[\Gamma \vdash GU(\lambda x V) R & \Gamma \vdash_1 U T & (Y \equiv V[x := T] \vee (Y \equiv_\beta V[x := T] \& \exists s[\Gamma \vdash s Y]))]$ where the derivations without an explicit $U x$ in the context are shorter than that of $\Gamma \vdash_1 Y Z$.

**Lemma 4.19 (Correctness of Types Lemma for ISs)** $\Gamma \vdash UV \Rightarrow \exists s[U \equiv s \vee \Gamma \vdash s U]$.

**Lemma 4.20 (Subject Reduction Lemma for ISs)** $\Gamma \vdash UV, \Gamma \Rightarrow _\beta \Gamma', U \Rightarrow _\beta U', V \Rightarrow _\beta V' \Rightarrow \Gamma' \vdash_1 U' V'$.

**Lemma 4.21** If $\Gamma \vdash UV$ then at least one of

(i) $U \in \mathcal{C}$,
(ii) $\exists s[\Gamma \vdash s U & (\exists s'[s' s \in [A]] \text{ or } s \in S_3)]$.

We give three extra lemmas.

**Lemma 4.22** If $\Gamma \vdash X$ then $X \equiv UV$ for some $U$ and $V$, where $U$ is not $\beta$-equal to an abstract.
Proof. $X \equiv UV$ follows by induction on the derivation of $\Gamma \vdash_1 X$. By the Correctness of Types Lemma, the Generation Lemma, and Subject Reduction for ISs, $U$ is not $\beta$-equal to an abstract. \hfill $\square$

Remark 4.24. By Lemma 4.23 we see that the Subject Reduction Lemma for ISs can be strengthened to

$$\Gamma \vdash_1 X, \Gamma \Rightarrow \Gamma', X \Rightarrow \beta X' \Rightarrow \Gamma' \vdash_1 X'.$$

Lemma 4.25 (Sharpened Generation Lemma for APTSs). If $\dot{\Gamma} \vdash_\Lambda P : B$, then (i), (ii), (iii), and (v) of Lemma 4.7 hold with $\vdash$ replaced by $\vdash_\Lambda$ and also,

(iv) $P \equiv (\lambda x : A.M) \Rightarrow \exists C, s [\dot{\Gamma} \vdash_\Lambda C : s \& \Gamma, x : A \vdash_\Lambda M : C]
\& (B \equiv \Pi x : A.C \vee (B \equiv_\beta \Pi x : A.C \& \exists s' [\dot{\Gamma} \vdash_\Lambda B : s'])].$

Proof. As for Lemma 4.7. \hfill $\square$

Lemma 4.26 (Sharpened Generation Lemma for AISs). If $\dot{\Gamma} \vdash_{aI} YZ$ then (i), (ii), (iii), and (v) of Lemma 4.18 hold with $\vdash$ replaced by $\vdash_{aI}$ and also,

(iv) $Z \equiv \lambda x.T \Rightarrow \exists U, V, s [\dot{\Gamma} \vdash_{aI} sU \& \Gamma, Ux \vdash_{aI} VT \& (Y \equiv GU(\lambda x.V) \vee (Y \equiv_\beta GU(\lambda x.V) \& \exists s' [\dot{\Gamma} \vdash_{aI} s' Y])).$

We now prove a theorem which is of importance for PTSs in general and which is also useful in later proofs.

Theorem 4.27

(i) $S_I$ is the set of inhabited sorts, that is, those sorts $s$ such that $\exists \dot{\Gamma}, M[\dot{\Gamma} \vdash M : s]$.

(ii) For every PTS with specification $(\delta, A, R)$ there is a PTS with specification $(\delta_1, A, R \cap \delta_1^3)$ that is equivalent in the sense that it has the same valid judgments.

Proof. We prove

$\exists \dot{\Gamma}, M, A[s \text{ appears in the statement } \dot{\Gamma} \vdash M : A]$ (1) \Rightarrow $s \in \delta_1$ (2) \Rightarrow $\exists \dot{\Gamma}', N[\dot{\Gamma}' \vdash N : s]$ (3) \Rightarrow (1).

(1) $\Rightarrow$ (2) By induction on the derivation of

$$\dot{\Gamma} \vdash M : A.$$ (4)

If (4) is an axiom then $M : A$ is $c : s$ or $s : s'$, so (a) or (b) of Definition 2.8(i) holds. If (4) comes by a product rule with $A \equiv s$, (c) holds. In all other cases the result holds by the induction hypothesis.

(2) $\Rightarrow$ (3) By induction on the derivation of (2). If this is by Definition 2.8(i)(a), the result holds; if it is by (b) it holds by a start rule. If this is by (c) we have $s_1, s_2 \in \delta_1$ and $(s_1, s_2, s) \in \mathcal{R}$ and by the induction hypothesis

$$\Gamma_1 \vdash N_1 : s_1$$
$$\Gamma_2 \vdash N_2 : s_2$$

where we can assume $FV(\Gamma_1) \cap FV(\Gamma_2) = \emptyset$. 


By Lemmas 4.5 and 4.6 we have
\[ \Gamma_1, \Gamma_2, \vdash N_1 : s_1 \]
\[ \Gamma_1, \Gamma_2 \vdash N_2 : s_2, \]
and by weakening
\[ \Gamma_1, \Gamma_2, x : N_1 \vdash N_2 : s_2 \]
where \( x \notin FV(\Gamma_1, \Gamma_2, N_1, N_2) \). A product rule now gives (3).

(3) \( \Rightarrow \) (1) is obvious.

(2) \( \Leftrightarrow \) (3) now establishes (i).

(1) \( \Leftrightarrow \) (2) shows that if \( s \in \delta - \delta_1 \), it cannot appear in any valid judgment. Hence only the sorts in \( S_1 \) and the triples in \( R \cap \delta_3 \) can be used to derive any valid judgment. This establishes (ii).

\[ \square \]

**Theorem 4.28** Theorem 4.27 holds for ISs if \( \vdash M : s \) in (i) is replaced by \( \Gamma \vdash_1 sM \).

**Proof** Analogous to that of Theorem 4.27. \( \square \)

5 Proof of Theorems 2.11(ii) and 2.12(i)

**Proof of Theorem 2.11(ii)** For a PTS and an APTS with the same specification and such that (\ast) holds, we prove
\[ \Gamma \vdash_A P : C \Rightarrow \Gamma \vdash P : C \]
by induction on the derivation of \( \Gamma \vdash_A P : C \). The only nontrivial case is where \( \Gamma \vdash_A P : C \) is obtained by the abstraction rule from
\[ \Gamma, x : A \vdash_\Lambda M : B & \Gamma \vdash_A A : s \]
where (\(+\)) holds for \( s \) and \( B, P \equiv \lambda x : A.M \) and \( C \equiv \Pi x : A.B \). By the induction hypothesis we have
\[ \Gamma, x : A \vdash M : B & \Gamma \vdash A : s. \]
We only need to show
\[ \exists s_3 \in \delta [\Gamma \vdash \Pi x : A.B : s_3]. \]
Lemma 4.11 applied to \( \Gamma, x : A \vdash M : B \) yields that we have at least one of
(i) \( B \in \mathcal{C} \),
(ii) \( \Gamma, x : A \vdash B : s_2 & (\exists s'[s_2 : s' \in A\] or \( s_2 \in \delta_3) \).
In case (i) we get from (\(+\))
\[ (\exists s_2, s_3)[(s, s_2, s_3) \in \mathcal{R} & \Gamma, x : A \vdash B : s_2], \]
so \( \Gamma \vdash \Pi x : A.B : s_3 \).

In case (ii), we get from (\ast), \( \exists s_3(s, s_2, s_3) \in \mathcal{R} \) (note that \( s \in \delta_1 \) by (\(+\)), so \( \Gamma \vdash \Pi x : A.B : s_3. \) \( \square \)
Proof of Theorem 2.12(i) We want $\Gamma \vdash_A M : A \Rightarrow S(\Gamma)$ is SA-legal & $S(\Gamma) \vdash_{SA} M : A$. We let $\Gamma \equiv \langle x_1 : A_1, \ldots, x_n : A_n \rangle$ and proceed by induction on the derivation of $\Gamma \vdash_A M : A$.

Case axiom Now $\Gamma \equiv (\ ), S(\Gamma) = \emptyset$ and is SA-legal, and $S(\Gamma) \vdash_{SA} M : A$.

Case start $\Gamma \equiv \Gamma^-, x_n : A_n, M \equiv x_n, A \equiv A_n$, and $\Gamma \vdash_A M : A$ is obtained from $\Gamma^- \vdash_A A_n : s$. By the induction hypothesis we have $S(\Gamma^-)$ is SA-legal and $S(\Gamma^-) \vdash_{SA} A_n : s$. Also $x_n \neq x_i$ and $x_n \notin FV(A_i)$ for $1 \leq i < n$, so $S(\Gamma)$ is SA-legal. $M : A \in S(\Gamma)$, hence $S(\Gamma) \vdash_{SA} M : A$.

Case weakening $\Gamma \equiv \Gamma^-, x_n : A_n$, and (1) is obtained from $\Gamma^- \vdash_A M : A$, $\Gamma \vdash_A A_n : s$. We have as above that $S(\Gamma)$ is legal. By the induction hypothesis we have $S(\Gamma^-) \vdash_{SA} M : A$. One easily proves the Thinning Lemma for SAPTSs:

$\Delta \vdash_{SA} N : B$, $\Delta \subseteq \Delta' \Rightarrow \Delta' \vdash_{SA} N : B$.

From this we get $S(\Gamma) \vdash M : A$.

Other Cases If $\Gamma \vdash_A M : A$ is obtained by one of the other rules, we find by the induction hypothesis applied to one of the premises from which $\Gamma \vdash_A M : A$ is obtained that $S(\Gamma)$ is S-legal. In each case $S(\Gamma) \vdash_{SA} M : A$ follows when the induction hypothesis is applied to the premises. □

6 Relations between Illative Systems

In each theorem and lemma in this and later sections we assume that the systems used have the same specification.

Theorem 6.1

$\Gamma \vdash_{I} X \Rightarrow S(\Gamma)$ is SI-legal and $S(\Gamma) \vdash_{SI} X$.

Proof By an easy induction on the derivation of $\Gamma \vdash_{I} X$, similar to the proof of Theorem 2.10(i). □

We now will prove a sort of converse:

$\Delta \vdash_{SI} X$, $\Delta$ SI-legal $\Rightarrow \exists \Gamma, Y[\Delta =_\beta S(\Gamma) \& X =_\beta Y \& \Gamma \vdash_{I} Y]$. The proof of this statement is very similar to the proof of Theorem 5.4(ii) in [12]. We first prove two lemmas, similar to Lemmas 5.2 and 5.3 in that paper, with similar proofs.

Lemma 6.2 If $\Delta \vdash_{SI} X$, $\Delta =_\beta S(\Gamma)$ where $\Gamma$ is I-legal, then there exists $Y =_\beta X$ such that $\Gamma \vdash_{I} Y$.

Proof By induction on the derivation of (1).

Case axiom $X \in [A]$. Now (2) follows by the Start Lemma for ISs.

Case start $X \in \Delta =_\beta S(\Gamma)$. Now $X =_\beta Y$ for some $Y \in \Gamma$ and by the Start Lemma for ISs we get $\Gamma \vdash_{I} Y$. 

Case product \( X \equiv s_3(GU(\lambda x. V)) \), \((s_1, s_2, s_3) \in R\), and (1) is obtained from
\[
\Delta \vdash_{S_1} s_1 U, \quad \text{and} \quad \Delta, U x \vdash_{S_1} s_2 V. \tag{3}
\]
\[
\Delta \vdash_{S_1} s_1 U'. \tag{4}
\]
By the induction hypothesis for (3), Lemma 4.23, and Subject Reduction for ISs, we get for some \( U' =_\beta U \),
\[
\Gamma \vdash_{S_1} s_1 U'. \tag{5}
\]
So \( \Gamma, U' x \) is I-legal. As \( \Delta, U x =_\beta S(\Gamma, U' x) \) we have by the induction hypothesis for (4), Lemma 4.23, and Subject Reduction
\[
\Gamma, U' x \vdash_{S_1} s_2 V'. \tag{6}
\]
where \( V' =_\beta V \).

By the product rule we get from (5) and (6),
\[
\Gamma \vdash_{S_1} s_3(GU' (\lambda x. V')). \tag{7}
\]
This is (2) with \( Y \equiv s_3(Gu'(\lambda x. V')) \).

Case abstraction \( X \equiv GU (\lambda x. V)(\lambda x. W) \) and (1) is obtained from
\[
\Delta, U x \vdash_{S_1} V W, \quad \text{and} \quad \Delta \vdash_{S_1} s(GU(\lambda x. V)). \tag{8}
\]
By the induction hypothesis applied to (8) and Subject Reduction for ISs we get
\[
\Gamma \vdash_{S_1} s(GU' (\lambda x. V'))
\]
for some \( U' =_\beta U, V' =_\beta V \), and hence by Lemma 4.18,
\[
\Gamma \vdash_{S_1} s_{1} U' \quad \text{and} \quad \Gamma, U' x \vdash_{S_1} s_{2} V'.
\]
Hence, also by Lemma 4.18, \( U' \) and \( V' \), so also \( U \) and \( V \), are not \( \beta \)-equal to abstracts.

From \( \Gamma \vdash_{S_1} s_{1} U' \) we get that \( \Gamma, U' x \) is I-legal. Hence we get by the induction hypothesis applied to (7),
\[
\Gamma, U' x \vdash_{S_1} V'' W'' \text{ where } V'' =_\beta V, W'' =_\beta W.
\]
By Church-Rosser and Subject Reduction finally we get
\[
\Gamma, U_1 x \vdash_{S_1} V_1 W_1, \quad \Gamma \vdash_{S_1} s(GU_1(\lambda x. V_1))
\]
where \( U_1 =_\beta U, V_1 =_\beta V \) and \( W_1 =_\beta W \).

We conclude (2) by the abstraction rule for ISs.

Case application \( X \equiv (V[x := R])(WR) \) and (1) is obtained from
\[
\Delta \vdash_{S_1} GU (\lambda x. V) W, \quad \text{and} \quad \Delta \vdash_{S_1} U R. \tag{9}
\]
\[
\Delta \vdash_{S_1} U R'. \tag{10}
\]
Similar to the previous case, now also using Lemma 4.19, we get that \( U \) is not \( \beta \)-equal to an abstract. We get
\[
\Gamma \vdash_{S_1} GU' (\lambda x. V') W' \quad \text{and} \quad \Gamma \vdash_{S_1} U' R'
\]
for some \( U' =_\beta U, V' =_\beta V, W' =_\beta W, \text{ and } R' =_\beta R \). Hence by application
\[
\Gamma \vdash (V'[x := R'])(W'R').
\]
\( \square \)

Lemma 6.3 If \( \Delta \) is SI-legal for a given SIS, then there is a context \( \Gamma \), legal for the IS, with the same specification, such that \( \Delta =_\beta S(\Gamma) \).
Proof By induction on the number $n$ in Definition 3.5. If $n = 1$ then $\Delta =_\beta \{Xx\}$, where $\vdash_1 sX$ and $x \notin FV(X)$.

By Lemma 6.2 and Subject Reduction for ISs there is an $X' =_\beta X$ such that $\vdash_1 sX'$. Thus $\Delta =_\beta \{X'x\}$ and as by a start rule $X'x \vdash_1 X'x$, we have that $X'x$ is legal.

If $n > 1$ we have $\Delta =_\beta \{X_1 x_1, \ldots, X_n x_n\}$ where (i), (ii), and (iii) of Lemma 3.4 hold. It follows that $\{X_1 x_1, \ldots, X_{n-1} x_{n-1}\}$ is also SI-legal and, by the induction hypothesis, that there is a legal context $\Gamma'$ such that $\{X_1 x_1, \ldots, X_{n-1} x_{n-1}\} =_\beta S(\Gamma')$.

Now by Lemma 6.2, Lemma 3.4(ii) with $i = n$, and Subject Reduction we have $\Gamma' \vdash n s_n X_n$ where $X_n =_\beta X'_n$. So $\Gamma', X'_n x_n$ is legal. As $\Delta =_\beta S(\Gamma', X'_n x_n)$ we have the required result. □

Theorem 6.4

\[ \Delta \vdash_1 X, \Delta SI-legal \Rightarrow \exists \Gamma', Y[\Delta =_\beta S(\Gamma') \& X =_\beta Y \& \Gamma \vdash_1 Y]. \]

Proof This immediately follows from Lemmas 6.2 and 6.3. □

Theorem 6.5

\[ \Gamma \vdash_1 X \Rightarrow \Gamma \vdash_{AI} X. \]

Proof By induction on the derivation of $\Gamma \vdash_1 X$ (similar to the proof of Theorem 8.1 in [12]). All cases are obvious except where $\Gamma \vdash_1 X$ comes by the abstraction rule from

\[
\begin{align*}
\Gamma, Yx & \vdash UV, \text{ and} \\
\Gamma & \vdash_1 s_3(GY(\lambda x.U)),
\end{align*}
\]

and $X \equiv GY(\lambda x.U)(\lambda x.V)$.

By (12) and the Sharpened Generation Lemma for ISs there is a triple $(s_1, s_2, s_3) \in R$ such that

\[ \Gamma \vdash_1 s_1 Y, \]

and $\Gamma, Yx \vdash_1 s_2 U$, where the derivation of (13) is shorter than that of $\Gamma \vdash_1 X$. If $U =_\beta R \in \mathcal{C}$, then the Sharpened Generation Lemma for ISs gives $s_2 R \in [A]$. Hence (+) holds for $U$ and $s_1$. By the induction hypothesis applied to (11) and (13) we have

\[ \Gamma, Yx \vdash_{AI} UV \& \Gamma \vdash_{AI} s_1 Y \]

which, given (+), gives $\Gamma \vdash_{AI} GY(\lambda x.U)(\lambda x.V)$. □

Theorem 6.6 If (\#) holds,

\[ \Gamma \vdash_{AI} X \Rightarrow \Gamma \vdash_1 X. \]

Proof Similar to the proof of Theorem 2.11(ii) in Section 5. We now use Lemma 4.22 instead of Lemma 4.11. In the last lines of the proof of Theorem 2.11(ii) we had that $s \in \delta_1$ because of $\Gamma \vdash A : s$, hence by (+), $s \in \delta_1 = \{s \in \delta_1 | s_2, s_3 \in [R]\}$. Now $\Gamma \vdash sU$ for some $U$, hence $s \in \delta_1$ by Theorem 4.28 and so again $s \in \delta_1$.

Now we will prove that if (\#) holds then

\[ \Delta \vdash_{SAI} X, \Delta SAI-legal \Rightarrow \exists \Gamma', Y[\Delta =_\beta S(\Gamma') \& X =_\beta Y \& \Gamma \vdash_{AI} Y]. \]

As in the proof of Theorem 6.4 this will follow from two lemmas. These are similar to Lemmas 6.2 and 6.3. □
Lemma 6.7  If
\[ \Delta \vdash_{\text{SAI}} X, \]  \tag{14}
\[ \Delta =_{\beta} S(\Gamma), \text{ where } \Gamma \text{ is AI-legal and } (*) \text{ holds, then there exists a } Y =_{\beta} X \text{ such that } \Gamma \vdash_{\text{AI}} Y. \]

Proof  As (*) holds, by Theorems 6.5 and 6.6,
\[ \Gamma \vdash_{I} Z \iff \Gamma \vdash_{\text{AI}} Z. \]
Hence IS properties such as Subject Reduction also hold for AISs. Now the proof of this lemma is the same as that of Lemma 6.2 except as follows.

Case abstraction  \[ X \equiv GU(\lambda x. V)(\lambda x. W) \] and (14) is obtained from
\[ \Delta, U \vdash_{\text{SAI}} VW \land \Delta \vdash_{\text{SAI}} sU \]
and
\[ (+) \exists s_2, s_3 [(s, s_2, s_3) \in R \land \forall T (V =_{\beta} T \in C \Rightarrow s_2 T \in \{A\})] \]
where \( V \) is not \( \beta \)-equal to an abstract. The proof of the case is now similar to the proof of the abstraction case in Lemma 6.2. \( \Box \)

Lemma 6.8  If (*) holds, then
\[ \Delta \text{ SAI-legal } \Rightarrow \Delta =_{\beta} S(\Gamma) \text{ for some AI-legal } \Gamma. \]

Proof  As in Lemma 6.7, properties such as Subject Reduction hold for AISs. Therefore the proof of this lemma is the same as the proof of Lemma 6.3. \( \Box \)

Theorem 6.9  If (*) holds, then
\[ \Delta \vdash_{\text{SAI}} X, \Delta \text{ SAI-legal } \Rightarrow \exists \Gamma, Y \left[ \Delta =_{\beta} S(\Gamma) \land X =_{\beta} Y \land \Gamma \vdash_{\text{AI}} Y \right]. \]

Proof  Directly from Lemmas 6.7 and 6.8. \( \Box \)

Now we are going to prove
\[ \Gamma \vdash_{\text{AI}} X \Rightarrow S(\Gamma) \text{ is SAI-legal and } S(\Gamma) \vdash_{\text{SAI}} X. \]
In the SAIS postulates of 3.4 we have the condition that in the abstraction rule \( Y \) is not \( \beta \)-equal to an abstract, we need to show this holds automatically for AISs.

Lemma 6.10  If \( \Gamma \vdash_{\text{AI}} X \) then \( X \equiv YZ \) where \( Y \) is not \( \beta \)-equal to an abstract.

Proof  Consider the AIS, \( \omega \), with the same \( A \) and \( S \) as the one considered here, but with \( R \) replaced by \( \delta^3 \). Then (*) holds for \( \omega \) and
\[ \Gamma \vdash_{\omega} X. \]

Now by Theorem 6.6,
\[ \Gamma \vdash_{I} X, \]
and by Lemma 4.23 we have that \( X \equiv YZ \) where \( Y \) is not \( \beta \)-equal to an abstract. \( \Box \)

Theorem 6.11  \[ \Gamma \vdash_{\text{AI}} X \Rightarrow S(\Gamma) \text{ is SAI-legal and } S(\Gamma) \vdash_{\text{SAI}} X. \]

Proof  By induction on the derivation of \( \Gamma \vdash_{\text{AI}} X \). In the abstraction case we use Lemma 6.10. \( \Box \)
Theorem 6.12

\( \Delta \vdash_{SI} X & \Delta \) is SI-legal \( \Rightarrow \Delta \vdash_{SAI} X & \Delta \) is SAI-legal.

**Proof**  By Theorems 6.4, 6.5, and 6.11 and the conversion rule for SAISs. \( \square \)

Theorem 6.13

If \((\ast)\) holds,

\( \Delta \vdash_{SAI} X & \Delta \) is SAI-legal \( \Rightarrow \Delta \vdash_{SI} X & \Delta \) is SI-legal.

**Proof**  By Theorems 6.9, 6.6, and 6.1 and the conversion rule for SISs. \( \square \)

In Sections 7 and 8 we link PTS variants and our illative systems, and in Section 9 we give a link between PTSs and SAISs, the systems closest to the ICLs in the literature.

### 7 From Type Systems to Illative Systems

We define a translation \([\ ]\) of pseudoterms of PTSs into ICL pseudoterms in the following way.

**Definition 7.1**  \([\ ]\)

\[
\begin{align*}
[x] & \equiv x \\
[c] & \equiv c \\
[MN] & \equiv [M][N] \\
[\lambda x : A.M] & \equiv \lambda x.[M] \\
[\Pi x : A.B] & \equiv G[A](\lambda x.[B]) \\
[M : A] & \equiv [A][M]
\end{align*}
\]

If \( B \) is a set or sequence of judgments \( M_i : A_i \), for some values of \( i \), then \([B]\) is the set or sequence of judgments \([A_i][M_i]\), for the same values of \( i \).

We need some lemmas about the translation \([\ ]\).

**Lemma 7.2**  If \( B \) and \( N \) are pseudoterms,

\([B][x := [N]] \equiv [B][x := N]\).

**Proof**  By induction on \( B \). \( \square \)

**Lemma 7.3**

(i) If \( M \rightarrow_\beta N \), then \([M] \rightarrow_\beta [N]\).

(ii) If \( M =_\beta N \), then \([M] =_\beta [N]\).

**Proof**  (i) It is sufficient to prove this for a single \( \beta \)-contraction. Let \((\lambda x : A.P)Q\) be the part of \( M \) that reduces to \( P[x := Q] \) in \( N \) then

\[
\begin{align*}
[(\lambda x : A.P)Q] & \equiv (\lambda x.[P])([Q]) \rightarrow_\beta [P][x := [Q]] \\
& \equiv [P[x := Q]]
\end{align*}
\]

by Lemma 7.2.

Any remaining parts of \( M \) are identical to the remaining parts of \( N \) and so any remaining parts of \([M]\) are identical to the remaining parts of \([N]\). Hence \([M] \rightarrow_\beta [N]\).

(ii) If \( M =_\beta N \) there is a \( P \) such that \( M \rightarrow_\beta P \) and \( N \rightarrow_\beta P \). By (i), \([M] \rightarrow_\beta [P]\) and \([N] \rightarrow_\beta [P] \) so \([M] =_\beta [N]\). \( \square \)
Lemma 7.4  If $A$ is a pseudoterm and $[A] \Rightarrow \beta A$, then there is a pseudoterm $B$ such that $X \equiv [B]$ and $A \Rightarrow \beta B$.

Proof  It is enough to prove this for a single $\beta$-contraction. We do this by induction on $A$.

Case 1  $A \equiv \lambda x : C.M$, $[A] \equiv \lambda x.[M]$ and $X \equiv \lambda x.Y$ where $[M] \Rightarrow \beta Y$. By the induction hypothesis there is a pseudoterm $N$ such that $Y \equiv [N]$ and $M \Rightarrow \beta N$. Then $B \equiv \lambda x : C.N$.

Case 2  $A \equiv \Pi x : D.C$, $[A] \equiv G(D)(\lambda x.[C])$ and $X \equiv GU(\lambda x.V)$ where $[D] \Rightarrow \beta U$ and $V \equiv [C]$ or $[D] \equiv U$ and $[C] \Rightarrow \beta V$. By the induction hypothesis there is an $E$ such that $D \Rightarrow \beta E$ and $U \equiv [E]$ or $C \Rightarrow \beta E$ and $V \equiv [E]$. Thus the lemma holds with $B \equiv \Pi x : E.C$ or $\Pi x : D.E$.

Case 3  $A \equiv CD$ and $X \equiv UV$ where $[C] \Rightarrow \beta U$ or $[D] \Rightarrow \beta V$. By the induction hypothesis there is an $E$ such that $D \Rightarrow \beta E$ and $[E] \equiv V$ or $C \Rightarrow \beta E$ and $[E] \equiv U$. Thus $B \equiv CE$ or $ED$.

Case 4  $A \equiv (\lambda x : C.M)N$ where $X \equiv [M][x := [N]]$. Then $X \equiv [M[x := N]]$ by Lemma 7.2, so $B \equiv M[x := N]$.

Note that in this proof we use that $G$ is primitive, hence not an abstract.

We can now prove that $[\ ]$ translations of valid (S)(A)PTS judgments are valid in the corresponding ICL.

Theorem 7.5  Let $X$ denote $\emptyset$, $S$, or $A$, then

$$\Gamma \vdash_X M : A \Rightarrow [\Gamma] \vdash_X [A][M].$$

Proof  By straightforward induction on the derivation of $\Gamma \vdash_X M : A$.

Theorem 7.6  If $(\ast)$ holds,

$$\Delta \vdash_{SA} M : A & \Delta is SA-legal \Rightarrow [\Delta] \vdash_{SA} [A][M] & [\Delta] is SA-legal.$$  

Proof  By Theorems 2.12(ii), 7.5, 6.11, and SAI conversion applied to the legality of $\Delta$ and to $\Delta \vdash_{SA} M : A$.

8 From Illative Systems to Type Systems

Definition 8.1 (of $\sim$)  If $A_1$ and $A_2$ are pseudoterms,

$$A_1 \Rightarrow \beta A_2 \Rightarrow A_1 \sim A_2,$$

$$A_i \Rightarrow \beta \Pi x_i : B_i \ldots \Pi x_n : B_n, s_i (i = 1, 2) \Rightarrow A_1 \sim A_2.$$

Lemma 8.2

$$\Gamma \vdash P_1 : B_1, \quad \Gamma \vdash P_2 : B_2$$

$$B_1 \sim B_2, [P_1] \equiv [P_2], P_1, P_2 in normal form \quad \Rightarrow P_1 \equiv P_2.$$

Proof  By induction on the structure of $P_1$.

Cases 1 and 2  $P_1 \equiv x$ and $P_1 \equiv c$ are trivially okay.

Case 3  $P_1 \equiv \lambda x : A_1.M_1$.

$[P_1] \equiv [P_2]$ so $P_2 \equiv \lambda x : A_2.M_2$. By the Generation Lemma for PTSs we get

$$B_i \Rightarrow \beta \Pi x : A_i.C_i, \quad \Delta, x : A_i \vdash M_i : C_i \quad (i = 1, 2).$$
Lemma 8.3 If $P_1$ and $P_2$ are pseudo SA-terms in normal form then
\[ [P_1] =_\beta [P_2] \Rightarrow [P_1] \equiv [P_2]. \]

Proof By Church-Rosser and Lemma 7.4. \[ \square \]

Lemma 8.4
\[
\begin{align*}
\Gamma \vdash P_1 : B_1, & \quad \Gamma \vdash P_2 : B_2 \\
B_1 \sim B_2, [P_1] =_\beta [P_2] & \quad \Rightarrow \quad P_1 =_\beta P_2.
\end{align*}
\]
Proof Let $Q_i$ be the normal form of $P_i$. Then $[Q_1] =_{\beta} [Q_2]$ and by Lemmas 8.2 and 8.3, we get $Q_1 =_{\beta} Q_2$ and so $P_1 =_{\beta} P_2$. □

Lemma 8.5 If $\Gamma_1 \vdash A, \Gamma_2 \vdash N : B$ and $[\Gamma_1] =_{\beta} [\Gamma_2]$, then $\Gamma_1 =_{\beta} \Gamma_2$.

Proof By induction on the length of $\Gamma_1$. Let

\[
\Gamma_1 \equiv x_1 : A_1, \ldots, x_n : A_n, \quad \text{and} \quad \Gamma_2 \equiv x_1 : B_1, \ldots, x_n : B_n.
\]

By Lemma 4.4, we have $s_1, s_2 \in S$ such that

\[
x_1 : A_1, \ldots, x_{n-1} : A_{n-1} \vdash A_n : s_1
\]

and

\[
x_1 : B_1, \ldots, x_{n-1} : B_{n-1} \vdash B_n : s_2.
\]

So by the induction hypothesis, if $[\Gamma_1] =_{\beta} [\Gamma_2], x_1 : A_1, \ldots, x_{n-1} : A_{n-1} =_{\beta} x_1 : B_1, \ldots, x_{n-1} : B_{n-1}$.

By Lemma 4.10,

\[
x_1 : A_1 \ldots x_{n-1} : A_{n-1} \vdash B_n : s_2,
\]

and by Lemma 8.4, $A_n =_{\beta} B_n$, that is, $\Gamma_1 =_{\beta} \Gamma_2$. □

Theorem 8.6 If in the IS corresponding to a normalizing PTS,

\[
\Gamma \vdash I UV,
\]

then there exist $\Gamma_1, A, M$ such that $\Gamma \equiv [\Gamma_1], U \equiv [A], V \equiv [M]$, and

\[
\Gamma_1 \vdash M : A.
\]

Proof By induction on the derivation of (1).

Case 1 (Axiom) $\vdash I sc$. This case is trivially okay.

Case 2 (Start)

\[
\frac{\Gamma^- \vdash I sU}{\Gamma^-, Ux \vdash I Ux}
\]

where $\Gamma \equiv \Gamma^-, Ux$ and $V \equiv x$. By the induction hypothesis

\[
\Gamma^- \vdash A : B
\]

where

\[
\Gamma^- = [\Gamma_1^-], s = [B] \text{ and } U \equiv [A], \text{ hence } B \equiv s, \text{ so } \Gamma^-_1 \vdash A : s.
\]

Now by start $\Gamma^-_1, x : A \vdash x : A$ which is (2).

Case 3 (Weakening)

\[
\frac{\Gamma^- \vdash I UV \quad \Gamma^- \vdash I sY}{\Gamma^-, Yx \vdash I UV}
\]

where $\Gamma \equiv \Gamma^-, Yx$. By the induction hypothesis we have

\[
\Gamma_2 \vdash M : A, \Gamma_3 \vdash C : B
\]

where

\[
\Gamma^- = [\Gamma_2] = [\Gamma_3], V \equiv [M], U \equiv [A], Y \equiv [C] \text{ and, as above, } B \equiv s.
\]

By Lemmas 8.5 and 4.10, $\Gamma_2 \vdash C : s$ and by a weakening rule we have (2) with $\Gamma_1 \equiv \Gamma_2, x : C$. 
Case 4 (Conversion)

\[ \Gamma \vdash U \quad \Gamma \vdash sY \quad Y =_{\beta} U \]

\[ \Gamma \vdash YV \]

By the induction hypothesis we get similarly to above,

\[ \Gamma_1 \vdash M : B, \Gamma_1 \vdash A : s \text{ where } \Gamma \equiv [\Gamma_1], U \equiv [B], V \equiv [M], \text{ and } Y \equiv [A]. \]

By Lemmas 4.8, 7.4, and 8.4, we get \( A =_{\beta} B \). Hence by conversion \( \Gamma_1 \vdash M : A \) which is (2).

Case 5 (Application)

\[ \Gamma \vdash GU(\lambda x. V)T \quad \Gamma \vdash UR \]

\[ \Gamma \vdash (V[x := R]) (TR) \]

We get similarly to above

\[ \Gamma_1 \vdash N : B, \quad \Gamma_1 \vdash P : C \]

\[ \Gamma \equiv [\Gamma_1], T \equiv [N], GU(\lambda x. V) \equiv [B], U \equiv [C], \text{ and } R \equiv [P]. \]

By Lemmas 4.8 and 4.7(iii), \( \Gamma_1 \vdash E : s \), for some \( s \), so we have \( \Gamma_1 \vdash P : C, C =_{\beta} E \) (by 4.8 and 8.4).

Hence by conversion

\[ \Gamma_1 \vdash P : E. \]

Hence \( \Gamma_1 \vdash NP : F[x := P] \) which is (2).

Case 6 (Abstraction)

\[ \Gamma, Yx \vdash UV \quad \Gamma, Yx \vdash s(GY(\lambda x. U)) \]

\[ \Gamma \vdash GY(\lambda x. U)(\lambda x. V) \]

By the induction hypothesis we get, similarly to above,

\[ \Gamma_1, x : B \vdash N : C, \quad \Gamma_1 \vdash \Pi x : D.E : s \]

where

\[ \Gamma \equiv [\Gamma_1], Y \equiv [B], U \equiv [C], V \equiv [N], Y \equiv [D], \text{ and } U \equiv [E]. \]

By Lemmas 4.8 and 8.4 we get \( B =_{\beta} D \) and \( C =_{\beta} E \).

By Lemma 4.10 and the Generation Lemma we get

\[ \Gamma_1, x : D \vdash N : C. \]

Now \( C =_{\beta} E \) and \( \Gamma_1, x : D \vdash E : s' \) for some \( s' \). So by conversion

\[ \Gamma_1, x : D \vdash N : E. \]

Hence by abstraction

\[ \Gamma_1 \vdash \lambda x : D.N : \Pi x : D.E \]

which is (2).

Case 7 (Product)

\[ \Gamma, Yx \vdash s_2Z \quad \Gamma \vdash s_1Y \]

\[ \Gamma \vdash s_3(GY(\lambda x. Z)) \]

where \( (s_1, s_2, s_3) \in R \). We get

\[ \Gamma_1, x : B \vdash C : s_2, \quad \Gamma_1 \vdash D : s_1 \]
where
\[
\Gamma \equiv [\Gamma_1], \quad Y \equiv [B], \quad Z \equiv [C], \quad \text{and} \quad Y \equiv [D].
\]
By Lemma 8.4 we get \(B =_\beta D\), hence \(\Gamma_1, x : D \vdash C : s_2\) by Lemma 4.10.
So by product,
\[
\Gamma_1 \mid \Pi x : D. C : s_3,
\]
which is (2).

**Theorem 8.7** For SISs and SPTSs such that the corresponding PTS is normalizing, if \(\Delta\) is SI-legal and
\[
\Delta \vdash_{SI} X,
\]
then there exist \(\Delta_1, A,\) and \(M\) such that
\[
\Delta =_\beta [\Delta_1], \quad X =_\beta [A][M], \quad \Delta_1 \text{S-legal, and } \Delta_1 \vdash_S M : A.
\]
**Proof** From Theorems 6.4, 8.6, and 2.10(i).

**Theorem 8.8** For AISs and APTSs such that the corresponding PTS is normalizing and satisfies (\(*\)), if
\[
\Gamma \vdash_{AI} X,
\]
then there exist \(\Gamma_1, A,\) and \(M\) such that
\[
\Gamma \equiv [\Gamma_1], \quad X \equiv [A][M], \quad \text{and} \quad \Gamma_1 \vdash_A M : A.
\]
**Proof** From Theorems 6.6, 8.6, and 2.11(ii).

**Theorem 8.9** For SAISs and SAPTSs such that the corresponding PTS is normalizing and satisfies (\(*\)), if \(\Delta\) is SAI-legal and
\[
\Delta \vdash_{SAI} X,
\]
then there exist \(\Delta_1, A,\) and \(M\) such that
\[
\Delta =_\beta [\Delta_1], \quad X =_\beta [A][M], \quad \Delta_1 \text{SA-legal, and } \Delta_1 \vdash_{SA} M : A.
\]
**Proof** From Theorems 6.9, 8.8, and 2.12(i).

### 9 Linking PTSs and SAISs

We are now able to link PTSs to SAISs, the systems closest to the illative systems in the literature.

**Theorem 9.1**
\[
\Gamma \vdash M : A \Rightarrow [\Gamma] \vdash_{SAI} [A][M].
\]
**Proof** By Theorems 7.5, 6.5, and 6.11.

**Theorem 9.2**
\[
\Delta \vdash_{SAI} X \Rightarrow \exists \Gamma, M, A \ [S(\Gamma) =_\beta \Delta, [A][M] =_\beta X \& \Gamma \vdash M : A],
\]
provided \(\Delta\) is SAI-legal, the PTS is normalizing, and (\(*\)) holds.
**Proof** By Theorems 6.9, 6.6, and 8.6.
Illative systems of combinatory logic such as those of Bunder [11], [3], [4], and [7], the later “Frege Structures” of Aczel [1], and the version of the Calculus of Constructions in Coquand [14] and Seldin [17] are slightly more general than the SAIs that we have developed here in that in

\[ \Delta, Xx \vdash YZ \]

in the abstraction rule, \( Y \) may be an abstract and in that (+) need not hold. Some have additional definitions and postulates such as conversion with \( \beta\eta \)-equality. Still by Theorem 9.1, the translation of any valid PTS judgment is valid in these illative systems. By Theorem 9.2, a subclass of the theorems of these illative systems can be translated back into PTSs.

It was thought that setting up the above two links between PTSs and SAISs would allow a transfer of properties from one to the other. We will examine the most important such property, that of consistency.

### 10.1 SAIS consistency

The original illative systems of Church and Curry were inconsistent in the strong sense that every term (including an arbitrary variable or \( \lambda \)-term) was provable. Some later systems that included a class of propositions \( H \) were inconsistent in the weaker sense (see Bunder [5], [6], and [9] and Bunder and Meyer [10]) that all propositions were provable. This was expressed as \( \vdash \Xi HI \), which can be translated into \( \vdash GH(\lambda x y)(\lambda x x) \) and, by Definition 7.1, with \( * \) for \( H \), into \( \vdash (\lambda x : * x) : (\Pi x : *.y.y) \).

By the Sharpened Generation Lemma (4.7) this is not a valid judgment of a PTS, so by Theorem 9.2, SAISs are consistent in the strong sense that not all propositions are provable, if the corresponding PTS is normalizing and satisfies \((*)\).

In \( (\Pi x : *.y.y) \) is, by the Correctness of Types Lemma (4.8), not even a possible type, so it seems that \( \Xi HI \) cannot be represented in a SAIS given normalization and \((*)\).

In many ICLs in the literature, however, it is important to have \( \Xi HI \) as a proposition so that negation can be defined by \( \sim X \equiv X \supset \Xi HI \). Also, in these, the properties of intuitionistic implication and negation are derived from the postulates for \( \Xi \) (or \( G \)) using either the definition \( H \equiv \lambda x.L(\lambda y.x) \) or \( L \equiv FUH \) for some \( U \) and the axiom \( \vdash LH \). \( \vdash LH \) is the counterpart to \( \vdash * \) \( \square \) a standard PTS axiom, but to have sorts defined in terms of other sorts and having types that are abstracts is not possible in PTSs or in SAISs. Hence a gap remains between SAISs and the illative systems in the literature.

### 10.2 PTS consistency

A PTS is inconsistent if, for some \( M \),

\[ \vdash M : (\Pi x : * x), \]

that is, if \( M \) is a proof that every proposition (element of \( * \)) is a theorem.

If this were valid we would have, in the corresponding SAIS,

\[ \vdash G * (\lambda x.x)[M] \]

or

\[ *_x \vdash x([M]x). \]

If \( * \) is interpreted as \( H \), the class of propositions, this is unprovable (and in fact ill formed as was the translation of SAIS inconsistency into PTSs). However, if \( * \) is
interpreted instead as a class of sets and the term $[M]$ as a choice function, the result is in fact true!

10.3 Why the mismatch? The reason for the mismatch is, of course, that in a PTS only the type is considered as a proposition of predicate calculus, whereas in illative systems the translation of the term and the type, that is, a whole statement is considered as a proposition. Despite this we have seen that the postulates of PTSs and (S)(A)ISs are remarkably similar and in fact equivalent, modulo legality, $\beta$-equality, and ($*$).

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