

## Peculiarities of dynamics of Dirac fermions associated with zero-mass lines

Timur Tudorovskiy and Mikhail I. Katsnelson

*Institute for Molecules and Materials, Radboud University of Nijmegen, Heyendaalseweg 135, 6525AJ Nijmegen, The Netherlands*

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Zero-mass lines result in the appearance of linear dispersion modes for Dirac fermions. These modes play an important role in various physical systems. However, a Dirac fermion may not precisely follow a single zero-mass line, due to either tunneling between different lines or centrifugal forces. Being shifted from a zero-mass line the Dirac fermion acquires mass which can substantially influence its expected “massless” behavior. In the paper we calculate the energy gap caused by the tunneling between two zero-mass lines and show that its opening leads to the delocalization of linear dispersion modes. The adiabatic bending of a zero-mass line gives rise to geometric phases. These are the Berry phase, locally associated with a curvature, and a new phase, resulting from the mass square asymmetry in the vicinity of a zero-mass line.

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### I. INTRODUCTION

Zero-energy states are well known for the one-dimensional eigenvalue problem for the Dirac particle with a spatially dependent mass. Such a problem naturally arises in various contexts, both in condensed matter and in high-energy physics.<sup>1</sup> These are supersymmetric quantum mechanics,<sup>2</sup> fractional charge,<sup>3</sup> and solitons in polyacetylene.<sup>4</sup> For the monotonous mass distribution the wave function corresponding to the zero-energy state is localized in the vicinity of the point where the mass vanishes.

Let us consider the two-dimensional case. Assuming that the mass depends on a single variable only, say  $y$ , we come back to the one-dimensional problem. However, in two dimensions the motion is allowed not only along the  $y$  axis, but also along perpendicular to it the  $x$  axis. If the mass vanishes at the point  $y_0$ , the line  $(x, y_0)$  is the zero-mass line (ZML). Our two-dimensional problem reduces to the one-dimensional problem completely if we assume that the particle does not move along the ZML. For nonzero values of the momentum  $p_x$  along the ZML the energy of the particle is given by the linear relation  $E = \pm p_x$ , that is, the one-dimensional zero-energy state becomes a linear dispersion mode (LDM) in two dimensions.

LDMs naturally appear as edge states for inverted band semiconductors<sup>5</sup> as well as for a certain model of the quantum Hall effect.<sup>6</sup> Though single-particle LDMs in quantum Hall regime might not be a sufficient description for conventional two-dimensional electron gas,<sup>7</sup> this description seems to be adequate for the case of narrow graphene ribbons.<sup>8</sup> Another recent example where the ZMLs are essentially involved are given by topologically protected edge states in CdTe/HgTe/CdTe topological insulators,<sup>9–13</sup> graphene on boron nitride,<sup>14,15</sup> and chemically functionalized graphene,<sup>16–18</sup> where for an inhomogeneous functionalization the mass term can in general change its sign. LDMs arise in gapped bilayer graphene<sup>19–21</sup> and chiral  $p$ -wave superconductors.<sup>22</sup> Changing a width of CdTe/HgTe/CdTe quantum well or applying a gate voltage one can also create the ZML in bulk. Though our consideration will be formally applicable in all these cases, for the sake of definiteness we assume in this paper that LDMs relate to edge states in two-dimensional topological insulators.

Along a single straight ZML the linear dispersion mode propagates only in one direction. This unidirectional edge mode is very similar to the unidirectional edge states in conventional semiconductors placed in the high-magnetic field. It is well known that these states support the quantum Hall current.<sup>23</sup> The existence of unidirectional channels in topological insulators in zero-magnetic field is referred to, by analogy with the quantum Hall effect, as quantum spin Hall effect.<sup>9,24,25</sup>

From the physical point of view a mass square landscape in a topological insulator forms a waveguide around the ZML (see Fig. 1). LDM in such a waveguide corresponds to zero transversal momentum, that is, to the rest in the direction transversal to the ZML. This stands in stark contrast to the conventional Schrödinger particle in a soft-walls waveguide, where the lowest transversal energy is positive.

LDMs in topological insulators are known to be topologically protected against scattering by nonmagnetic impurities.<sup>13</sup> Topological protection results from the spatial separation of states traveling in different directions, thus the backscattering should be attenuated by the probability to tunnel through the bulk of a sample. This peculiarity is very similar to the behavior of conducting states in a quantum Hall effect. In an ideal system at zero temperature it is the tunneling between edge states which determines an accuracy of quantum Hall plateaus.

The assumption about the spatial separation of states traveling in different directions does not take into account peculiarities of quantum tunneling. Indeed, let us consider a ribbon with two parallel ZMLs at its opposite edges and assume that mass does not depend on the variable along the line. One can expect that the crossing of straight lines corresponding to a linear dispersion turns into the avoided crossing. Obviously this effect does not depend on the symmetry: A transition from a crossing to an avoided crossing caused by tunneling is generic.

Due to the gap opening the LDM cone turns into two branches, almost linear at large momenta. Let us consider an upper branch. For large negative momenta the wave function is localized at one edge of the sample and for large positive momenta it is localized at the other edge. Since the wave function smoothly depends on the longitudinal momentum, we should conclude that at a certain longitudinal momentum

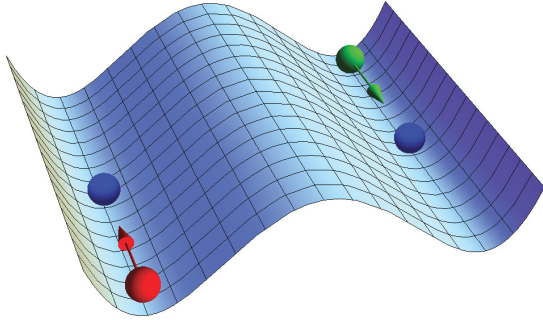


FIG. 1. (Color online) Particles corresponding to the LDM on the mass square surface. Red and green spheres illustrate edge states with large longitudinal momenta. They are localized at a single zero-mass line. Blue spheres illustrate states with small longitudinal momenta. Such states are localized at both ZMLs simultaneously.

amplitudes of the wave function are comparable at both ZMLs. At this point states traveling in different directions are not spatially separated and the topological protection may break (see Fig. 1). In Sec. III we illustrate this effect by an example of a symmetric mass distribution.

The tunneling between ZMLs has been studied experimentally.<sup>26</sup> In this work the differential conductance measurements in the integer quantum Hall regime between two parallel edge channels were reported. Theoretically<sup>27</sup> one considered the tunneling between quantum Hall edge states via the Landau-Zener-like mechanism: It was assumed that the main contribution comes from the vicinity of a point where edge channels are the closest to each other.

Topological protection of a single ZML should lead to a substantial change of dynamical properties of the LDM. Indeed, let us consider an adiabatically bent ZML. For a conventional Schrödinger particle the bending implies a geometric potential, proportional to the square of the curvature at every point.<sup>28,29</sup> This barrier results in the appearance of a trapped mode and backscattering. Both effects are forbidden for the LDM due to the topological protection. In Sec. IV we show that the bending of the ZML, indeed, tends only to the emergence of geometric phases.

## II. LINEAR DISPERSION MODES AND ZERO-MASS LINES

We assume that the dynamics of charge carriers is governed by the Hamiltonian

$$\hat{H} = \sigma_x \hat{p}_x + \sigma_y \hat{p}_y + \sigma_z m(y), \quad (1)$$

where the mass  $m(y)$  vanishes along the line  $y = 0$ . Here we put  $\hbar = v = 1$ , where  $v$  is the Fermi velocity. For a given energy  $E$  the wave function  $\Psi$  obeys the equation  $\hat{H}\Psi = E\Psi$ . We write  $\Psi = e^{ip_x x} \chi(y)$ ,

$$[\sigma_x p_x + \sigma_y \hat{p}_y + \sigma_z m(y)]\chi(y) = E\chi(y). \quad (2)$$

In a matrix form (2) reads

$$\begin{pmatrix} m - E & p_x - \partial_y \\ p_x + \partial_y & -m - E \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0. \quad (3)$$

Let us now sum up equations in (3) and subtract the first one from the second. Then we find

$$\begin{pmatrix} p_x - E & \partial_y + m \\ \partial_y - m & p_x + E \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0, \quad (4)$$

where we introduced the notations  $\eta_1 = (\chi_1 + \chi_2)/\sqrt{2}$ ,  $\eta_2 = (\chi_1 - \chi_2)/\sqrt{2}$  (see the Appendix for details). We can reduce (4) to scalar Schrödinger equations<sup>5</sup>

$$[-\partial_y^2 + m(y)^2 + m'(y)]\eta_1 = \lambda\eta_1, \quad (5)$$

$$[-\partial_y^2 + m(y)^2 - m'(y)]\eta_2 = \lambda\eta_2, \quad (6)$$

where  $E^2 = p_x^2 + \lambda$ . Functions  $\eta_1$  and  $\eta_2$  are not independent. Connection formulas read

$$(E + p_x)\eta_2 = (m - \partial_y)\eta_1, \quad (7)$$

$$(E - p_x)\eta_1 = (m + \partial_y)\eta_2.$$

Equations (5) and (6) can be written in the form

$$(m + \partial_y)(m - \partial_y)\eta_1 = \lambda\eta_1, \quad (8)$$

$$(m - \partial_y)(m + \partial_y)\eta_2 = \lambda\eta_2.$$

Multiplying the first equation in (8) by  $\eta_1$ , the second by  $\eta_2$ , and integrating over  $y$  we find

$$\lambda \|\eta_1\|^2 = \|(m - \partial_y)\eta_1\|^2 \geq 0, \quad (9)$$

$$\lambda \|\eta_2\|^2 = \|(m + \partial_y)\eta_2\|^2 \geq 0,$$

whence  $\lambda \geq 0$  since  $\eta_1$  and  $\eta_2$  cannot vanish simultaneously. In (9) we denoted

$$\|\eta_i\|^2 = \int_{-\infty}^{\infty} \eta_i^2(y) dy. \quad (10)$$

Let us consider a case when  $m$  monotonously depends on  $y$ , say  $m'(y) > 0$ . Then Eqs. (4) comprise a LDM for which  $E = -p_x$ . It is clear that  $\eta_1$  should be identically zero since (5) cannot have a zero eigenvalue if the potential is always positive. From (4) for  $\eta_1 = 0$  we have  $E = -p_x$  and

$$\eta_2(y) = \exp\left[-\int_0^y dy' m(y')\right]. \quad (11)$$

## III. TWO ZERO-MASS LINES

Let us consider a mass distribution

$$m(y) = y^2 - a^2. \quad (12)$$

It mimics a sample with two parallel ZMLs at  $y = \pm a$ . The infinitely large values  $m(y)$  in the limit  $y \rightarrow \pm\infty$  can be treated as two edges of the sample. Effective potentials

$$v_1(y) = m(y)^2 + m'(y) = (y^2 - a^2)^2 + 2y, \quad (13)$$

$$v_2(y) = m(y)^2 - m'(y) = (y^2 - a^2)^2 - 2y, \quad (14)$$

entering Eqs. (5) and (6) in this case correspond to tilted double wells (see Fig. 2).

It is easy to see that  $\lambda = 0$  does not belong to the spectrum of Eq. (6). Indeed, the exact solution (11) exponentially decays when  $y \rightarrow \infty$  and exponentially grows when  $y \rightarrow -\infty$ . Due to the conservation of Wronskian another linear

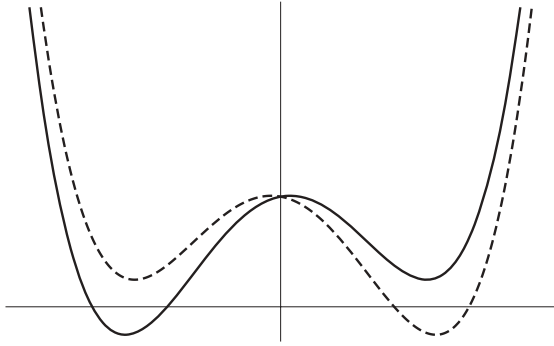


FIG. 2. Tilted double well potentials. Solid and dashed lines correspond to  $v_1$ ,  $v_2$ , respectively. An existence of the upturned well can lead to the Arago effect: Focusing in a shadow region.

independent solution of (6) exponentially grows when  $y \rightarrow \infty$  and exponentially decays when  $y \rightarrow -\infty$ . Similarly we prove that  $\lambda = 0$  does not belong to the spectrum of Eq. (5).

Nonzero  $\lambda$  leads to avoided crossing of branches  $E = \pm p_x$ , corresponding to well separated ZMLs (see Fig. 3). The dispersion relation reads  $E_n^\pm(p_x) = \pm\sqrt{p_x^2 + \lambda_n}$ . The splitting between branches  $E_0^\pm(p_x)$  is equal to  $2\sqrt{\lambda_0}$ , where  $\lambda_0$  is the lowest eigenvalue of the Schrödinger equations (5) and (6). We put  $p_x = 0$  and take the branch corresponding to  $E_0(0) = p_y = \sqrt{\lambda_0}$ . Equations (7) give

$$\begin{aligned} (m + \partial_y)\tilde{\eta}_2(y) &= p_y\tilde{\eta}_1(y), \\ (m - \partial_y)\tilde{\eta}_1(y) &= p_y\tilde{\eta}_2(y), \end{aligned} \quad (15)$$

where  $\tilde{\eta}_1(y)$ ,  $\tilde{\eta}_2(y)$  correspond to  $p_x = 0$ . If  $\tilde{\eta}_1(y)$ ,  $\tilde{\eta}_2(y)$  are known then the functions  $\eta_1(y)$ ,  $\eta_2(y)$  for  $p_x \neq 0$  can be reconstructed as

$$\eta_1(y) = (E + p_x)\tilde{\eta}_1(y), \quad \eta_2(y) = p_y\tilde{\eta}_2(y). \quad (16)$$

Let us multiply the first equation in (15) by  $\tilde{\eta}_1(y)$ , the second one by  $\tilde{\eta}_2(y)$ , and subtract the second result from the first one. We obtain  $\partial_y[\tilde{\eta}_1(y)\tilde{\eta}_2(y)] = p_y[\tilde{\eta}_1^2(y) - \tilde{\eta}_2^2(y)]$ . Integrating the last equality over  $y$  from minus to plus infinity we find  $\|\tilde{\eta}_1\|^2 = \|\tilde{\eta}_2\|^2$ , which suggests that  $\chi_{1,2}(y) =$

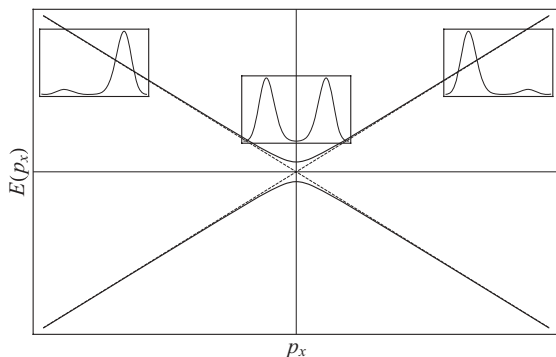


FIG. 3. The schematic view of the dispersion relation and the corresponding wave function reconstruction (insets). One sees that at zero longitudinal momenta the wave function is not localized at a single ZML.

$[\tilde{\eta}_1(y) \pm \tilde{\eta}_2(y)]/\sqrt{2}$  are not localized at a certain ZML at zero longitudinal momentum  $p_x$ , but rather have comparable amplitudes at both ZMLs. In contrast to the double well problem (see Ref. 30, p. 183) this effect is not a consequence of a spatial symmetry. The delocalization at zero longitudinal momentum may destroy the topological protection against disorder.

Let us now look how the localization appears at nonzero longitudinal momenta. For  $|p_x| \gg p_y$  we find  $E + p_x \gg p_y$  if  $p_x > 0$  and  $E + p_x \ll p_y$  if  $p_x < 0$ . Thus for large positive longitudinal momenta  $\chi_{1,2}$  are localized at one ZML and for large negative longitudinal momenta these functions are localized at another ZML.

For symmetric mass distribution (12) we can obtain an analytic estimation for the spitting. Let us change  $y$  to  $-y$  in (15). Taking into account that  $m(y)$  is an even function we find

$$\begin{aligned} (m - \partial_y)\tilde{\eta}_2(-y) &= p_y\tilde{\eta}_1(-y), \\ (m + \partial_y)\tilde{\eta}_1(-y) &= p_y\tilde{\eta}_2(-y). \end{aligned}$$

Thus we conclude  $\tilde{\eta}_1(-y) = \tilde{\eta}_2(y)$ . Let us multiply the first equation in (15) by  $\tilde{\eta}_1(y)$  and integrate over  $y$ . We obtain

$$\begin{aligned} p_y\|\tilde{\eta}_1\|^2 &= \int_{-\infty}^{\infty} dy \tilde{\eta}_2(-y)(m + \partial_y)\tilde{\eta}_2(y) \\ &= 2 \int_0^{\infty} dy \tilde{\eta}_2(-y)(m + \partial_y)\tilde{\eta}_2(y) + \tilde{\eta}_2^2(0) \\ &= 2p_y \int_0^{\infty} dy \tilde{\eta}_1^2(y) + \tilde{\eta}_2^2(0). \end{aligned} \quad (17)$$

Since the function  $\tilde{\eta}_1$  obeys the Schrödinger equation (5) with potential (13) for small  $p_y$  it remains exponentially small for any  $y > 0$  except the vicinity of the point  $y = a$ . In this vicinity  $\tilde{\eta}_1$  grows due to the focusing in a shadow region (an effect similar to the Arago spot, Fig. 2), but nevertheless remains small. Therefore we can neglect an integral in the last equality in (17) and write

$$p_y\|\tilde{\eta}_2\|^2 \simeq \tilde{\eta}_2^2(0), \quad (18)$$

where (11) was used to approximate  $\tilde{\eta}_2$  in the region  $y > 0$ . Using (11) again we find

$$p_y \int_0^{\infty} dy \exp\left[-2 \int_0^y dy' m(y')\right] = 1. \quad (19)$$

The integral in (19) can be computed using the Laplace method. It gives

$$p_y \sqrt{\frac{\pi}{a}} \exp\left[-2 \int_0^a dy' m(y')\right] = 1, \quad (20)$$

where

$$p_y = \sqrt{\frac{a}{\pi}} \exp\left[2 \int_0^a dy' m(y')\right] = \sqrt{\frac{a}{\pi}} \exp(-4a^3/3). \quad (21)$$

Thus for the splitting we obtain

$$2|p_y| = 2\sqrt{\frac{a}{\pi}} \exp(-4a^3/3). \quad (22)$$

Though the explicit estimation (22) of the splitting holds only for the specific symmetric mass distribution (12), one can

expect that for a generic case the splitting in the LDM for a system with two ZMLs is proportional to

$$\exp \left[ - \int_{a_1}^{a_2} |m(y)| dy \right], \quad (23)$$

where  $a_1 < a_2$  are positions of ZMLs. In contrast to the famous double well problem, the ground state  $\lambda > 0$  of (5) and (6) is still determined by the tunneling since for any  $m(y)$  every well treated apart generates a zero eigenenergy. For the double well the semiclassical degeneracy persists in the symmetric situation only.

#### IV. BENT ZERO-MASS LINE

Let us consider a bent ZML given by  $\{x, y\} = \mathbf{R}(\tau)$ , where  $\tau$  is a natural parameter, that is,  $|\mathbf{R}'(\tau)| = 1$ . In the vicinity of this line we introduce new variables  $\tau, \xi$  by the equality  $\{x, y\} = \mathbf{R}(\tau) + \xi \mathbf{n}(\tau)$ , where  $\mathbf{n}$  is a unit normal vector on the curve at the point  $\tau$ . In curvilinear coordinates (1) reads

$$H = - \frac{i \sigma \mathbf{R}'(\tau)}{1 - \xi k(\tau)} \frac{\partial}{\partial \tau} - i \sigma \mathbf{n}(\tau) \frac{\partial}{\partial \xi} + \sigma_z m, \quad (24)$$

where  $k(\tau) = -\langle \mathbf{R}', \mathbf{n}' \rangle$  is the curvature at the point  $\tau$  and  $\sigma \mathbf{b} = \sigma_x b_x + \sigma_y b_y$  for a vector  $\mathbf{b} = \{b_x, b_y\}$ . Since the Jacobian

$$J = \frac{D(x, y)}{D(\tau, \xi)} = 1 - k(\tau) \xi \quad (25)$$

is not unity, we introduce a new wave function

$$\tilde{\Psi} = \sqrt{1 - k(\tau) \xi} \Psi, \quad (26)$$

which in curvilinear coordinates has a ‘‘conventional’’ normalization condition:

$$\int_V d\tau d\xi (\tilde{\Psi}^\dagger \tilde{\Psi}) = 1. \quad (27)$$

Then the stationary Dirac equation reads  $\hat{H} \tilde{\Psi} = E \tilde{\Psi}$ ,

$$\hat{H} = \frac{\sigma \mathbf{R}'(\tau)}{1 - \xi k(\tau)} \hat{p}_\tau - i \sigma \mathbf{n}(\tau) \frac{\partial}{\partial \xi} + \sigma_z m - \frac{i k \sigma \mathbf{n}(\tau)}{2[1 - \xi k(\tau)]} - \frac{i \sigma \mathbf{R}'(\tau) \xi k'(\tau)}{2[1 - \xi k(\tau)]^2}.$$

In the case  $|m'_\tau| \ll |m'_\xi|$  we find  $\hat{H} \simeq \hat{H}_0 + \hat{H}_1$ ,

$$\hat{H}_0 = \sigma \mathbf{R}'(\tau) \hat{p}_\tau - i \sigma \mathbf{n}(\tau) \frac{\partial}{\partial \xi} + \sigma_z m, \quad (28)$$

$$\hat{H}_1 = \sigma \mathbf{R}'(\tau) \xi k(\tau) \hat{p}_\tau - \frac{ik}{2} \sigma \mathbf{n}(\tau). \quad (29)$$

In the adiabatic approximation the effective dynamics is one dimensional along the ZML. It is governed by the effective scalar Hamiltonian  $\hat{L} \simeq \hat{L}_0 + \hat{L}_1$ . The symbol<sup>31</sup>  $L_0(p_\tau, \tau)$  of  $\hat{L}_0$  is an eigenvalue of the problem<sup>32</sup>

$$\left[ \sigma \mathbf{R}'(\tau) p_\tau - i \sigma \mathbf{n}(\tau) \frac{\partial}{\partial \xi} + \sigma_z m \right] \chi(p_\tau, \tau) = L_0(p_\tau, \tau) \chi(p_\tau, \tau). \quad (30)$$

Using the notations  $\mathbf{n} = \{n_1, n_2\}$ ,  $\mathbf{R}' = \{n_2, -n_1\}$  we obtain

$$\begin{pmatrix} m - L_0 & p_\tau - \partial_\xi \\ p_\tau + \partial_\xi & -m - L_0 \end{pmatrix} \begin{pmatrix} \tilde{\chi}_1 \\ \tilde{\chi}_2 \end{pmatrix} = 0, \quad (31)$$

where  $\tilde{\chi}_1 = (n_2 - i n_1) \chi_1$ . After the replacement  $\tau \rightarrow x, \xi \rightarrow y$  (31) coincides with (4). From the expression<sup>32</sup>

$$L_1 = \langle \chi^\dagger H_1 \chi \rangle_\xi + i \left\langle \chi^\dagger \frac{\partial L_0}{\partial \tau} \frac{\partial \chi}{\partial p_\tau} \right\rangle_\xi - i \left\langle \chi^\dagger \frac{\partial H_0}{\partial p_\tau} \frac{\partial \chi}{\partial \tau} \right\rangle_\xi \quad (32)$$

we find

$$L_1 = - \frac{i}{2} \frac{\partial^2 L_0}{\partial p_\tau \partial \tau} + \langle \tilde{\chi}_1 \tilde{\chi}_2 \rangle_\xi k(\tau) p_\tau - \frac{k p_\tau}{2 L_0}. \quad (33)$$

Here  $\langle \cdot \rangle_\xi$  means the integration over  $\xi$ . To pass from (32) to (33) we have chosen  $\tilde{\chi} = \{\tilde{\chi}_1, \tilde{\chi}_2\}$  to be a real function and used the following equalities:  $\chi^\dagger(\sigma \mathbf{n}) \chi = 0$ ,  $\langle \chi^\dagger \chi'_{p_\tau} \rangle_\xi = 0$ ,  $\chi^\dagger(\sigma \mathbf{R}') \chi = 2 \tilde{\chi}_1 \tilde{\chi}_2$ ,  $\chi^\dagger(\sigma \mathbf{R}') \partial_\tau \chi = (\partial_\tau - ik) \tilde{\chi}_1 \tilde{\chi}_2$ ,  $\langle \tilde{\chi}_1 \tilde{\chi}_2 \rangle_\xi = p_\tau / 2 L_0$ , and  $(L_0)'_{p_\tau} = p_\tau / L_0$ .

The solution  $\psi$  of the effective longitudinal equation  $\hat{L} \psi = E \psi$  reads<sup>31</sup>

$$\psi(\tau) = \sqrt{\left| \frac{L_0}{p_\tau} \right|} e^{i\theta}, \quad (34)$$

$$\theta = \int p_\tau d\tau - \int \left[ L_0 \langle \tilde{\chi}_1 \tilde{\chi}_2 \rangle_\xi k(\tau) - \frac{1}{2} \right] d\tau. \quad (35)$$

It relates to the solution of the Dirac equation as  $\tilde{\Psi} = \chi \psi$ . From the definition of curvature we have  $k(\tau) d\tau = -\langle \mathbf{R}', d\mathbf{n} \rangle$ . Introducing the angle  $\phi$  between  $\mathbf{R}'$  and the  $x$  axis we find from the last equality  $k(\tau) d\tau = d\phi$ . This gives

$$\theta = \int p_\tau d\tau - \int L_0 \langle \tilde{\chi}_1 \tilde{\chi}_2 \rangle_\xi k(\tau) d\tau + \frac{\Delta\phi}{2}, \quad (36)$$

where  $\Delta\phi$  is the total rotation of the tangent vector.

From (34) it is easy to find the quantization rule. For a closed ZML without turning points (see Fig. 4) we obtain

$$\frac{1}{2\pi} \oint p_\tau d\tau - \frac{E}{2\pi} \oint \langle \tilde{\chi}_1 \tilde{\chi}_2 \rangle_\xi k(\tau) d\tau = n - \frac{w}{2}, \quad (37)$$

where  $p_\tau = \pm \sqrt{E^2 - \lambda(\tau)}$ ,  $\lambda(\tau) = L_0^2(p_\tau = 0, \tau)$ , and  $w$  is the winding number indicating how many times the tangent vector turns around a fixed point. For a curve without intersections  $w = 1$ .

For the LDM  $L_0 = E = -p_\tau$  provided that  $m'_\xi > 0$  and (33) gives

$$L_1 = \langle \tilde{\chi}_1 \tilde{\chi}_2 \rangle_\xi k(\tau) p_\tau + \frac{k}{2}. \quad (38)$$

The first term in (38) describes the energy gain due to the displacement of the Dirac fermion from the ZML caused by the

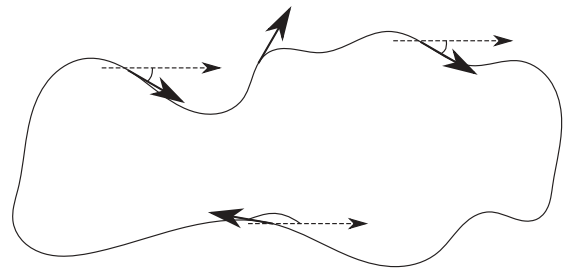


FIG. 4. Illustration of the closed ZML. Solid arrows show the tangent vectors, dashed arrows correspond to the  $x$  axis.



confinement asymmetry. This energy gain is the consequence of the centrifugal force. One sees that in the considered approximation the centrifugal force itself does not shift the particle since its energy is assumed to be relatively small. For a higher energy one can use the approach developed in Ref. 33.

The term  $k/2$  in (38) is a geometric potential associated with a curvature. It is well known<sup>28,29</sup> that for a Schrödinger particle in a bent waveguide the geometric potential is negative and proportional to  $k^2$ . It always leads to the formation of curvature induced bound states. On the contrary, for the LDM the geometric potential results in the appearance of a geometric phase only. Indeed, the effective longitudinal wave function  $\psi$  (34) reads

$$\psi(\tau) = e^{i\theta},$$

$$\theta = -E\tau - E \int \langle \tilde{\chi}_1 \xi \chi_2 \rangle_{\xi} k(\tau) d\tau + \frac{\Delta\phi}{2}. \quad (39)$$

The absence of trapped states for the Dirac fermion can be seen as a manifestation of the Klein tunneling<sup>34,35</sup> for the massless LDM. A similar effect was already found in Ref. 36 for a mode with a linear dispersion in a bent graphene ribbon.

For the LDM quantization condition (37) can be simplified to give

$$-\frac{El}{2\pi} - \frac{E}{2\pi} \oint \langle \tilde{\chi}_1 \xi \chi_2 \rangle_{\xi} k(\tau) d\tau = n - \frac{w}{2}. \quad (40)$$

Here  $l$  is the length of the closed ZML. From (40) we obtain the semiclassical spectral series  $E_n$ :

$$E_n = -\frac{2\pi}{l} \left( n - \frac{w}{2} \right) + \frac{2\pi n}{l^2} \oint \langle \tilde{\chi}_1 \xi \chi_2 \rangle_{\xi} k(\tau) d\tau. \quad (41)$$

Half-integer numbers in the first term of (41) can be seen as a manifestation of the Berry phase for the massless Dirac fermion. Indeed, along the ZML a particle described by LDM remains massless, therefore the conventional argument that such a particle acquires the Berry phase equal to  $\pi w$  can be repeated.

Using the results of Sec. II for a single ZML we find  $\chi_1 = -\chi_2 = \eta_2/\sqrt{2}$ , where  $\eta_2$  differs from (11) by a normalization factor

$$N(\tau) = \left\{ \int_{-\infty}^{\infty} \exp \left[ -2 \int_0^{\xi} d\xi' m(\tau, \xi') \right] d\xi \right\}^{-1/2}. \quad (42)$$

This gives

$$\langle \tilde{\chi}_1 \xi \chi_2 \rangle_{\xi} = -\frac{N^2(\tau)}{2} \int_{-\infty}^{\infty} \xi \exp \left[ -2 \int_0^{\xi} d\xi' m(\tau, \xi') \right] d\xi. \quad (43)$$

More precisely, the integration in (42) and (43) should be performed between finite limits lying in a sufficiently large

vicinity of the ZML, which, on the other side, does not contain any other ZML. We completely neglected tunneling effects, thus the constructed LDM should be considered as an asymptotic of the eigenfunction of the Dirac equation in the given vicinity of the ZML. The nonzero value of (43) means that the Dirac particle is slightly shifted from the ZML due to the local asymmetry of the confinement. Indeed, if  $m$  is an odd function of  $\xi$  the expression (43) vanishes. This happens since the confining potential  $m^2(\tau, \xi)$  in this case is symmetric with respect to the ZML.

## V. CONCLUSION

In the paper we studied the LDM dynamics of charge carriers in topological insulators. We have shown that the wave function of a charge carrier is localized along a single ZML only at large longitudinal momenta. At small longitudinal momenta the wave function has comparable amplitudes at both edges of the sample, which may affect the topological protection. We found that the curvature of a bent ZML forms a geometric potential, which however does not lead to an appearance of trapped modes due to the Klein tunneling.

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## APPENDIX: ON THE EFFECTIVE SCHRÖDINGER EQUATION

In this Appendix we would like to clarify the origin of the transformation performed to pass from (3) to (4). Let us write Eq. (2) in the form  $\sigma \hat{\pi} \chi(y) = E \chi(y)$ , where  $\hat{\pi} = \{p_x, \hat{p}_y, m(y)\}$ . We can square this equation using the equality

$$(\sigma \hat{\pi})^2 = \hat{\pi}^2 + i\sigma_x[\pi_y, \pi_z] + i\sigma_y[\pi_z, \pi_x] + i\sigma_z[\pi_x, \pi_y] \quad (A1)$$

valid for any noncommuting operators  $\hat{\pi}_x$ ,  $\hat{\pi}_y$ , and  $\hat{\pi}_z$ . We find

$$[p_x^2 + \hat{p}_y^2 + m(y)^2 + \sigma_x m'(y)] \chi = E^2 \chi. \quad (A2)$$

The last equation comprises the only matrix  $\sigma_x$ , which does not depend on  $y$ . We write

$$\chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \eta_1 + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \eta_2 \quad (A3)$$

and obtain (5) and (6).

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